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Abstract We propose a $\mathbb{P}_1$ finite-element scheme with mass-lumping for a model of two incompressible and immiscible phases in a porous media flow. We prove the dissipation of the free energy and the existence of a solution to the nonlinear scheme. We also present numerical simulations to illustrate the behavior of the scheme.

Key words: Two-phase porous media flows, energy stable finite-elements

MSC (2010)**: 65M60, 65M12, 35K65**

1 Immiscible two-phase flows in porous media

We are interested in the numerical approximations of the equations governing an immiscible incompressible two-phase flow in a porous medium. Let $\Omega \subset \mathbb{R}^d$ ($d = 2,3$) be an open bounded polyhedral subset with Lipschitz boundary condition and let $t_\mathrm{f} > 0$ be an arbitrary finite time horizon. Then the conservation of the wetting (subscript w) and non-wetting phases (subscript n) are given by

$$\phi \partial_t s_\alpha - \nabla \cdot (\eta_\alpha(s_\alpha) \Lambda \nabla p_\alpha) = q_\alpha(s_\alpha), \qquad \alpha \in \{\mathrm{w}, \mathrm{n}\}, \tag{1}$$

where the unknowns are the phase saturations s_α, which satisfy

$$s_\mathrm{n} + s_\mathrm{w} = 1, \tag{2}$$

Clément Cancès
Inria, Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
e-mail: clement.cances@inria.fr

Flore Nabet
CMAP, Ecole polytechnique, CNRS, I.P. Paris, 91128 Palaiseau, France
e-mail: flore.nabet@polytechnique.edu

and the phase pressures p_α. The porosity $\phi \in (0,1)$ is given, as well as the intrinsic permeability Λ, which is assumed to be symmetric and uniformly elliptic. The mobility $\eta_\alpha : [0,1] \to \mathbb{R}$ is assumed to be continuous and strictly increasing, with $\eta_\alpha(0) = 0$ and $\eta_\alpha(s) > 0$ if $s > 0$. They are extended to the whole $\mathbb{R}$ by $\eta_\alpha(s) = 0$ if $s < 0$ and $\eta_\alpha(s) = \eta_\alpha(1)$ if $s > 1$. The sources q_α are such that

$$q_\alpha(\mathbf{x}, s_\alpha) = q_{\text{inj}}(\mathbf{x}) \frac{\eta_\alpha(c_\alpha)}{\eta_{\text{w}}(c_{\text{w}}) + \eta_{\text{n}}(c_{\text{n}})} - q_{\text{sink}}(\mathbf{x}) \frac{\eta_\alpha(s_\alpha)}{\eta_{\text{w}}(s_{\text{w}}) + \eta_{\text{n}}(s_{\text{n}})}, \tag{3}$$

where $c_{\text{w}} \in (0,1]$ and $c_{\text{n}} = 1 - c_{\text{w}}$ is the prescribed composition of the injected mixture, and where $q_{\text{inj}}, q_{\text{sink}} \in L^\infty(\Omega)$ are nonnegative, bounded, and such that $\int_\Omega q_{\text{inj}} = \int_\Omega q_{\text{sink}}$. The phase pressures are linked by the capillary pressure relation

$$p_{\text{n}} - p_{\text{w}} = \gamma(s_{\text{n}}), \tag{4}$$

where $\gamma \in L^1(0,1)$ is strictly increasing, nonnegative, and blows up as s_{n} tends to 1. This function is extended for $s < 0$ by $\gamma(s) = \gamma(0) + 2s$. We further assume that $s \mapsto \eta_{\text{w}}(1-s)\gamma(s) \in L^\infty(0,1)$ and $s \mapsto \eta_{\text{w}}(1-s)\gamma'(s) \in L^1(0,1)$. These assumptions are satisfied by the usual models of the literature (see for instance [1]). The system is complemented with no-flux boundary conditions and initial conditions $s_\alpha^{\text{ini}} \in L^\infty(\Omega; [0,1])$ that are compatible with (2). Note that since $\gamma \in L^1(0,1)$, then $\Gamma : s \mapsto \int_0^s \gamma(a)\mathrm{d}a$ is bounded on $[0,1]$. The phase pressures being only defined up to a constant, we enforce additionally that $\int_\Omega p_{\text{n}} = 0$.

Multiplying (1) by p_α, summing over $\alpha \in \{\text{n}, \text{w}\}$, integrating over Ω, and using (2) and (4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \phi \Gamma(s_{\text{n}}) + \int_\Omega \sum_{\alpha \in \{\text{n},\text{w}\}} \eta_\alpha(s_\alpha) \Lambda \nabla p_\alpha \cdot \nabla p_\alpha = \int_\Omega \sum_{\alpha \in \{\text{n},\text{w}\}} q_\alpha(s_\alpha) p_\alpha. \tag{5}$$

Following [6], we define the global pressure P by $P = p_{\text{n}} - r(s_{\text{n}})$ with $r : s_{\text{n}} \mapsto \int_0^{s_{\text{n}}} \frac{\eta_{\text{w}}(1-a)}{\eta_{\text{n}}(a)+\eta_{\text{w}}(1-a)} \gamma'(a)\mathrm{d}a$. The definition of P yields

$$\sum_\alpha \eta_\alpha(s_\alpha) |\nabla p_\alpha|^2 = (\eta_{\text{n}}(s_{\text{n}}) + \eta_{\text{w}}(s_{\text{w}})) |\nabla P|^2 + \frac{\eta_{\text{n}}(s_{\text{n}}) \eta_{\text{w}}(s_{\text{w}})}{\eta_{\text{n}}(s_{\text{n}}) + \eta_{\text{w}}(s_{\text{w}})} |\nabla \gamma(s_{\text{n}})|^2. \tag{6}$$

In view of the particular form (3) of the source terms,

$$\sum_{\alpha \in \{\text{n},\text{w}\}} q_\alpha(s_\alpha) p_\alpha \le \left(q_{\text{inj}} - q_{\text{sink}}\right) \left(P + r(s_{\text{n}})\right) + q_{\text{sink}} k(s_{\text{n}}), \tag{7}$$

with $k(s_{\text{n}}) = \frac{\eta_{\text{w}}(1-s_{\text{n}})}{\eta_{\text{w}}(1-s_{\text{n}})+\eta_{\text{n}}(s_{\text{n}})} \gamma(s_{\text{n}})$. Since $\eta_{\text{w}}(1-\cdot)\gamma' \in L^1(0,1)$ and $\eta_{\text{w}}(1-\cdot)\gamma \in L^\infty(0,1)$, both r and k are bounded on $(0,1)$. Moreover, the extensions outside $(0,1)$ of η_α and γ ensure that for all $\varepsilon > 0$, there exists C_ε such that

$$|s| + |k(s)| + |r(s)| \le \varepsilon \Gamma(s) + C_\varepsilon, \quad \forall s \in (-\infty, 1). \tag{8}$$

Combining (7) with (6) in (5) together with the uniform ellipticity of Λ, $\eta_{\rm n}(s)+\eta_{\rm w}(1-s)\geq\delta>0$ for all s, and (8) we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi\Gamma(s_{\rm n})+\int_\Omega\left(|\nabla P|^2+\sum_{\alpha\in\{\rm n,w\}}\eta_\alpha(s_\alpha)|\nabla p_\alpha|^2\right)\leq C. \tag{9}$$

This estimate is enough to establish the existence of a weak solution. In this paper, our goal is to show that this stability is still encoded in very natural numerical schemes. For the sake of simplicity, we present our analysis in the framework of $\mathbb{P}_1$ finite-elements with mass-lumping, but our approach can be extended to a wide family of schemes having the structure highlighted in [3, Section 3]. We show here how to transpose estimate (9) to the discrete setting and to infer the existence of discrete solutions therefrom. A full convergence study will be carried out in a forthcoming contribution. While deeply inspired from [7], the goal of this paper is to exploit more finely the energy estimate which allows to relax some stringent conditions on the anisotropy, on the mesh and the non-linearities presented in [7].

2 An energy stable finite-element scheme

We study the problem (1)–(4) using a $\mathbb{P}_1$ conforming finite-element scheme with mass-lumping for the space discretization. Let $\mathscr{T}$ be a conforming simplicial discretization of Ω. We denote by $T\in\mathscr{T}$ a simplex, $\mathscr{V}_\mathscr{T}$ is the set of all the vertices $\mathbf{a}$ and $\mathscr{V}_T\subset\mathscr{V}_\mathscr{T}$ the set of the $(d+1)$ vertices $\mathbf{a}_0,\cdots,\mathbf{a}_d$ of the simplex T. We also denote by $V_h=\{u_h\in C(\overline{\Omega}):u_h|_T \text{ is affine for all } T\in\mathscr{T}\}$ the usual conforming $\mathbb{P}_1$ finite-element space corresponding to the mesh $\mathscr{T}$ and by $(\varphi_\mathbf{a})_{\mathbf{a}\in\mathscr{V}_\mathscr{T}}$ the basis of V_h. In order to deal with the mass-lumping procedure, for any vertex $\mathbf{a}\in\mathscr{V}_\mathscr{T}$, we define the set $\mathfrak{s}_\mathbf{a}$, the boundary $\partial\mathfrak{s}_\mathbf{a}$ of which being defined by the hyperplanes joining the centers of mass of the simplices, edges (and faces if $d=3$) sharing $\mathbf{a}$ as a vertex. We can now define the functional space $X_h:=\{u\in L^\infty(\Omega):u|_{\mathfrak{s}_\mathbf{a}} \text{ is constant for all } \mathbf{a}\in\mathscr{V}_\mathscr{T}\}$, and the linear mappings $\pi_X:C(\overline{\Omega})\to X_h$ and $\pi_V:C(\overline{\Omega})\to V_h$ by $\pi_\ell u(\mathbf{a})=u(\mathbf{a})$, for any $\mathbf{a}\in\mathscr{V}_\mathscr{T}$, for any $u\in C(\overline{\Omega})$, $\ell=X,V$. In order to lighten the notations, for any $u_h\in V_h$ we write $\pi_X u_h=\overline{u}_h$. We will use the following Poincaré inequality that can be established as in [2]: there exist C_1, $C_2>0$ depending only on the mesh regularity such that for any $u_h\in V_h$,

$$\left\|\overline{u}_h-\frac{1}{|\Omega|}\int_\Omega u_h\right\|_{L^2(\Omega)}\leq C_1\left\|u_h-\frac{1}{|\Omega|}\int_\Omega u_h\right\|_{L^2(\Omega)}\leq C_2\|\nabla u_h\|_{L^2(\Omega)}. \tag{10}$$

Before detailing the numerical scheme, we have to define the discrete tensor field $\Lambda_h:\Omega\to\mathbb{R}^{d\times d}$ almost everywhere by $\Lambda_h(x):=\Lambda_T:=\frac{1}{|T|}\int_T\Lambda$ if $x\in T$. From there, we define the matrix $\mathbf{A}_T:=(\alpha_{i,j}^T)_{1\leq i,j\leq d}\in\mathbb{R}^{d\times d}$ by

$$\alpha_{i,j}^T = \alpha_{j,i}^T := \int_T \Lambda_T \nabla \varphi_{\mathbf{a}_i} \cdot \nabla \varphi_{\mathbf{a}_j}$$

and for any $u_h, v_h \in V_h$ one has,

$$\int_T \Lambda_T \nabla u_h \cdot \nabla v_h = \begin{pmatrix} v_{\mathbf{a}_1} - v_{\mathbf{a}_0} \\ \dots \\ v_{\mathbf{a}_d} - v_{\mathbf{a}_0} \end{pmatrix} \cdot \mathbf{A}_T \begin{pmatrix} u_{\mathbf{a}_1} - u_{\mathbf{a}_0} \\ \dots \\ u_{\mathbf{a}_d} - u_{\mathbf{a}_0} \end{pmatrix}. \tag{11}$$

Following [4], we can prove that there exists $C_3 > 0$ depending on the regularity of the mesh and on the anisotropy ratio of Λ and $C_4 > 0$ depending, in addition, on d such that that for any $T \in \mathscr{T}$ the matrix $\mathbf{A}_T$ satisfies

$$\mathrm{cond}_2(\mathbf{A}_T) \le C_3 \ \text{ and } \ \sum_{i=1}^d \left(\sum_{j=1}^d |\alpha_{i,j}^T| \right) (u_{\mathbf{a}_i})^2 \le C_4 \mathbf{u} \cdot \mathbf{A}_T \mathbf{u}, \ \forall \mathbf{u} = (u_{\mathbf{a}_i}) \in \mathbb{R}^d. \tag{12}$$

We are now in a position to give the numerical scheme using a backward Euler scheme for the time discretization. Let $(t^n)_{n=0,\cdots,N}$ be a partition of the interval $[0, t_\mathrm{f}]$ and for $n = 1, \cdots, N$ we denote by $\tau_n = t^n - t^{n-1}$ the time step. We define the discrete initial data by $s_{\alpha,h}^0 := \sum_{\mathbf{a} \in \mathscr{V}_\mathscr{T}} s_{\alpha,\mathbf{a}}^0 \varphi_\mathbf{a} \in V_h$ with $s_{\alpha,\mathbf{a}}^0 = \frac{1}{|\mathfrak{s}_\mathbf{a}|} \int_{\mathfrak{s}_\mathbf{a}} s_\alpha^{\mathrm{ini}}$.

Let $s_\alpha^{n-1} \in V_h$ be given, we search for $s_\alpha^n, p_\alpha^n \in V_h$ such that for any $v_{\alpha,h} \in V_h$ with $\alpha = (\mathrm{n}, \mathrm{w})$ one has,

$$\phi \int_\Omega \frac{\overline{s}_{\alpha,h}^n - \overline{s}_{\alpha,h}^{n-1}}{\tau_n} \overline{v}_{\alpha,h} + \int_\Omega \eta_{\alpha,h}^n \Lambda_h \nabla p_{\alpha,h}^n \cdot \nabla v_{\alpha,h} = \int_\Omega \overline{q}_\alpha(\overline{s}_{\alpha,h}^n) \overline{v}_{\alpha,h}, \tag{13a}$$

$$s_{\mathrm{n},h}^n + s_{\mathrm{w},h}^n = 1, \tag{13b}$$

$$p_{\mathrm{n},h}^n - p_{\mathrm{w},h}^n = \gamma_{\mathrm{n},h}^n, \tag{13c}$$

$$\int_\Omega p_{\mathrm{n},h}^n = 0. \tag{13d}$$

We have denoted by $\eta_{\alpha,h}^n = \pi_V \eta(s_{\alpha,h}^n)$, $\gamma_{\mathrm{n},h}^n = \pi_V \gamma(s_{\mathrm{n},h}^n)$ and,

$$\overline{q}_\alpha(\overline{s}_{\alpha,h}^n) = \overline{q}_{\mathrm{inj}} \frac{\eta_\alpha(c_\alpha)}{\eta_\mathrm{w}(c_\mathrm{w}) + \eta_\mathrm{n}(c_\mathrm{n})} - \overline{q}_{\mathrm{sink}} \frac{\eta_\alpha(\overline{s}_{\alpha,h}^n)}{\eta_\mathrm{w}(\overline{s}_{\mathrm{w},h}^n) + \eta_\mathrm{n}(\overline{s}_{\mathrm{n},h}^n)}.$$

Mimicking the continuous case, we define the discrete global pressure and we can obtain the discrete counterpart of (6).

Proposition 1 *Let $s_{\alpha,h}^n, p_{\alpha,h}^n \in V_h$ be a solution to the scheme* (13)*. Then there exists $C_5 > 0$ depending on the regularity of the mesh, on the anisotropy ratio of Λ, on δ and d such that*

$$\int_\Omega \Lambda_h \nabla P_h \cdot \nabla P_h \le C_5 \left(\int_\Omega \eta_{\mathrm{n},h}^n \Lambda_h \nabla p_{\mathrm{n},h}^n \cdot \nabla p_{\mathrm{n},h}^n + \int_\Omega \eta_{\mathrm{w},h}^n \Lambda_h \nabla p_{\mathrm{w},h}^n \cdot \nabla p_{\mathrm{w},h}^n \right),$$

where $P_h^n = p_{\mathrm{n},h}^n - \pi_V r(s_{\mathrm{n},h}^n) \in V_h$.

Proof We define the functions

$$f_{\mathrm{n}}(s)=\frac{\eta_{\mathrm{n}}(s)}{\eta_{\mathrm{n}}(s)+\eta_{\mathrm{w}}(1-s)} \text{ and } f_{\mathrm{w}}(s)=\frac{\eta_{\mathrm{w}}(1-s)}{\eta_{\mathrm{n}}(s)+\eta_{\mathrm{w}}(1-s)}.$$

Then, noting that $f_{\mathrm{n}}+f_{\mathrm{w}}=1$ and using equation (13c), for any $T\in\mathscr{T}$ and for any vertices $\mathbf{a}_0,\mathbf{a}_i\in\mathscr{V}_T$, there exists $s_i^n\in[\min(\mathbf{a}_0,\mathbf{a}_i),\max(\mathbf{a}_0,\mathbf{a}_i)]$ such that,

$$P^n_{\mathbf{a}_0}-P^n_{\mathbf{a}_i}=f_{\mathrm{n}}(s_i^n)\left(p^n_{\mathrm{n},\mathbf{a}_0}-p^n_{\mathrm{n},\mathbf{a}_i}\right)-f_{\mathrm{w}}(s_i^n)\left(p^n_{\mathrm{w},\mathbf{a}_0}-p^n_{\mathrm{w},\mathbf{a}_i}\right).$$

Since η_α is strictly increasing, for any $T\in\mathscr{T}$ with $\mathbf{a}_0,\cdots,\mathbf{a}_d$ as vertices

$$\eta^n_{\alpha,T}:=\frac{1}{d+1}\sum_{i=0}^{d}\eta_\alpha(s^n_{\alpha,\mathbf{a}_i})\geq\frac{1}{d+1}\max_{\mathbf{x}\in T}\eta_\alpha(\mathbf{x})\geq\frac{1}{d+1}\eta_\alpha(s_i^n). \tag{14}$$

Thus using that $f_{\mathrm{n}},f_{\mathrm{w}}\leq 1$ and $\eta_{\mathrm{n}}(s)+\eta_{\mathrm{w}}(1-s)\geq\delta>0$ we obtain,

$$\frac{\delta}{2(d+1)}\sum_{i=0}^{d}\left|P^n_{\mathbf{a}_0}-P^n_{\mathbf{a}_i}\right|^2\leq\eta^n_{\mathrm{n},T}\sum_{i=0}^{d}\left|p^n_{\mathrm{n},\mathbf{a}_0}-p^n_{\mathrm{n},\mathbf{a}_i}\right|^2+\eta^n_{\mathrm{w},T}\sum_{i=0}^{d}\left|p^n_{\mathrm{w},\mathbf{a}_0}-p^n_{\mathrm{w},\mathbf{a}_i}\right|^2.$$

Since for any v_1,v_2,w satisfying $|v_1|^2+|v_2|^2\geq\mathrm{cond}_2(\mathbf{A}_T)|w|^2$ one has

$$v_1\cdot\mathbf{A}_Tv_1+v_2\cdot\mathbf{A}_Tv_2\geq w\cdot\mathbf{A}_Tw,$$

we use equality (11) associated with the fact that the condition number of $\mathbf{A}_T$ is bounded, cf. (12). Then summing the resulting estimate over $T\in\mathscr{T}$ and noting that the Lagrange vertex-quadrature formula is exact on $\mathbb{P}_1$ (see [5, Remark 2.2]) we obtain the claim. □

Proposition 2 *Let $s^{n-1}_{\alpha,h}\in V_h$ be given and $s^n_\alpha,p^n_\alpha\in V_h$ be a solution to the scheme* (13)*. There exists $C_6>0$ depending on the data of the continuous problem but neither on the mesh $\mathscr{T}$ or nor the time step τ_n such that,*

$$\phi\int_\Omega\Gamma(\bar{s}^n_{\mathrm{n},h})+\tau_n\sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}}\int_\Omega\eta^n_{\alpha,h}\Lambda_h\nabla p^n_{\alpha,h}\cdot\nabla p^n_{\alpha,h}+\tau_n\int_\Omega\nabla P^n_h\cdot\nabla P^n_h \leq C_6\left(1+\phi\int_\Omega\Gamma(\bar{s}^{n-1}_{\mathrm{n},h})\right).$$

Proof Let us choose $v_{\alpha,h}=p^n_{\alpha,h}$ as test function in equation (13a) and then add the resulting equations. Then, since Γ is convex, thanks to relation (13c) we obtain

$$\phi\int_\Omega\Gamma(\bar{s}^n_{\mathrm{n},h})+\sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}}\tau_n\int_\Omega\eta^n_{\alpha,h}\Lambda_h\nabla p^n_{\alpha,h}\cdot\nabla p^n_{\alpha,h} \leq\phi\int_\Omega\Gamma(\bar{s}^{n-1}_{\mathrm{n},h})+\tau_n\int_\Omega\sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}}\overline{q}_\alpha(\bar{s}^n_{\alpha,h})\overline{p}^n_{\alpha,h}. \tag{15}$$

As for the continuous case, one has

$$\sum_{\alpha\in\{\mathrm{n,w}\}} \overline{q}_\alpha(\overline{s}^n_{\alpha,h})\overline{p}^n_{\alpha,h} \le \left(\overline{q}_{\mathrm{inj}} - \overline{q}_{\mathrm{sink}}\right)\left(\overline{P}^n_h + r(\overline{s}^n_{\mathrm{n},h})\right) + \overline{q}_{\mathrm{sink}} k(\overline{s}^n_{\mathrm{n},h}). \tag{16}$$

Using the definition of the discrete global pressure P^n_h and equation (13d), combined with the discrete Poincaré inequality (10) and (8) give

$$\begin{aligned}\|\overline{P}^n_h\|_{L^1(\Omega)} &\le C_2|\Omega|^{1/2}\|\nabla P^n_h\|_{L^2(\Omega)} + \int_\Omega \left|r(\overline{s}^n_{\mathrm{n},h})\right| \\ &\le \varepsilon\|\nabla P^n_h\|^2_{L^2(\Omega)} + \varepsilon\left\|\Gamma(\overline{s}^n_{\mathrm{n},h})\right\|_{L^1(\Omega)} + C_\varepsilon.\end{aligned} \tag{17}$$

Since $q_{\mathrm{inj}}, q_{\mathrm{sink}} \in L^\infty(\Omega)$, the use of the above inequality and of (8) in (16) leads to

$$\int_\Omega \sum_{\alpha\in\{\mathrm{n,w}\}} \overline{q}_\alpha(\overline{s}^n_{\alpha,h})\overline{p}^n_{\alpha,h} \le \varepsilon\|\nabla P^n_h\|^2_{L^2(\Omega)} + \varepsilon\left\|\Gamma(\overline{s}^n_{\mathrm{n},h})\right\|_{L^1(\Omega)} + C_\varepsilon \tag{18}$$

whatever $\varepsilon > 0$. Using (18) together with Proposition 1 in (15) provides the expected bound. □

Thanks to equations (13b) and (13c) we see that the saturations and the pressures of the wetting and non-wetting phases are linked. Thus we can choose the pressure of the wetting phase and the capillary pressure as main unknowns. Choosing $v_{\alpha,h} = \varphi_{\mathbf{a}}$ as test functions in equations (13a) we can rewrite the scheme (13) as a nonlinear system of $2\#\mathscr{V}_\mathscr{T}$ algebraic equations $\mathscr{F}^n((\gamma(s^n_{\mathrm{n},\mathbf{a}}), p^n_{\mathrm{w},\mathbf{a}})_{\mathbf{a}\in\mathscr{V}_\mathscr{T}}) = 0$. Since $\gamma(1) = +\infty$, the function $\mathscr{F}^n$ is continuous but non uniformly continuous. However, we prove in the following lemma that this situation is avoided for a solution to the scheme (13).

Proposition 3 *Let $s^{n-1}_{\alpha,h} \in V_h$ be such that $\int_\Omega \overline{s}^{n-1}_{\mathrm{w},h} \ge 0$ and $s^n_{\alpha,\mathrm{h}}, p^n_{\alpha,\mathrm{h}} \in V_h$ be a solution the scheme* (13)*. There exists $\sigma_{\tau_n,\mathscr{T}}, \varepsilon_{\tau_n,\mathscr{T}} > 0$ depending on the data of the continuous problem, $\mathscr{T}$, τ_n and $s^{n-1}_{\mathrm{n},h}$ such that,*

$$-\sigma_{\tau_n,\mathscr{T}} \le s^n_{\mathrm{n},\mathbf{a}} \le 1 - \varepsilon_{\tau_n,\mathscr{T}}, \quad \forall \mathbf{a} \in \mathscr{V}_\mathscr{T}.$$

Proof First of all, thanks to the extension of γ for $s < 0$, the energy estimate given in Proposition 2 yields $\int_\Omega ((\overline{s}^n_{\mathrm{n},h})^-)^2 \le C^{n-1}$, which provides the lower bound.

Then we prove a bound on the pressure of the non-wetting phase $p_{\mathrm{n},h}$. Thanks to inequality (17) and the definition of P^n_h one has

$$\|\overline{p}^n_{\mathrm{n},h}\|_{L^1(\Omega)} \le C^{n-1} \quad \Rightarrow \quad |p^n_{\mathrm{n},\mathbf{a}}| \le \frac{C^{n-1}}{|\mathfrak{s}_\mathbf{a}|},\ \forall \mathbf{a} \in \mathscr{V}_\mathscr{T}. \tag{19}$$

Now, let us note that proving the upper bound is equivalent to proving that there exists $\gamma^\star_{\tau_n,\mathscr{T}}$ such that for any $\mathbf{a} \in \mathscr{V}_\mathscr{T}$, $\gamma(s_{\mathrm{n},\mathbf{a}}) \le \gamma^\star_{\tau_n,\mathscr{T}}$.

We choose $v_{\mathrm{w},h} = 1$ as test function in equation (13a), then since q_{inj} is nonnegative, $\eta_{\mathrm{n}}(s) + \eta_{\mathrm{w}}(1-s) \ge \delta > 0$ and $c_{\mathrm{w}} > 0$ (and so $\eta_{\mathrm{w}}(c_{\mathrm{w}}) > 0$), one has

$$s^n_{\mathrm{w},\mathbf{a}_\mathrm{i}} \geq \frac{1}{|\Omega|}\int_\Omega \overline{s}^n_{\mathrm{w},h} > \frac{1}{|\Omega|}\left(\int_\Omega \overline{s}^{n-1}_{\mathrm{w},h} - \frac{\tau_n}{\delta}\|q_{\mathrm{sink}}\|_{L^\infty(\Omega)}\int_\Omega \eta_\mathrm{w}(\overline{s}^n_{\mathrm{n},h})\right).$$

Note that we proved here by induction that $\int_\Omega \overline{s}^n_{\mathrm{w},h} \geq 0$. Since $s \mapsto (s+\eta_\mathrm{w}(s))^{-1}$ is Lipschitz, there exists $\mathbf{a}_\mathrm{i} \in \mathscr{V}_\mathscr{T}$ such that $s^n_{\mathrm{w},\mathbf{a}_\mathrm{i}} > 0$ that is there exists $\mathbf{a}_\mathrm{i} \in \mathscr{V}_\mathscr{T}$ such that $s^n_{\mathrm{n},\mathbf{a}_\mathrm{i}} < 1$.

Let $\mathbf{a}_\mathrm{f} \in \mathscr{V}_\mathscr{T}$ be arbitrary and $(\mathbf{a}_q)_{q=0,\cdots,\ell}$ be a path from $\mathbf{a}_\mathrm{i}$ to $\mathbf{a}_\mathrm{f}$. Let $q \in \{0,\cdots,\ell-1\}$. Using the property (12) of the matrix $\mathbf{A}_T$ and since the quadrature formula is exact on $\mathbb{P}_1$, Proposition 2 gives

$$\sum_{T\in\mathscr{T}} \eta^n_{\mathrm{w},T}\sum_{i=1}^d\left(\sum_{j=1}^d |\alpha^T_{i,j}|\right)\left(p^n_{\mathrm{w},h}(\mathbf{a}_i) - p^n_{\mathrm{w},h}(\mathbf{a}_0)\right)^2 \leq \frac{C_4C_6}{\tau_n}\left(1+\phi\int_\Omega \Gamma(\overline{s}^{n-1}_{\mathrm{n},h})\right).$$

We assume by induction that there exists $\varepsilon_{\tau_n,\mathscr{T}} > 0$ such that $s^n_{\mathrm{n},\mathbf{a}_q} < 1-\varepsilon_{\tau_n,\mathscr{T}}$ that is $s^n_{\mathrm{w},\mathbf{a}_q} > \varepsilon_{\tau_n,\mathscr{T}}$. Thus, if T is a simplex with $\mathbf{a}_q, \mathbf{a}_{q+1}$ as vertices, the definition (14) of $\eta^n_{\mathrm{w},T}$ yields $\eta^n_{\mathrm{w},T} \geq \frac{\eta(s^n_{\mathrm{w},\mathbf{a}_q})}{d+1} \geq \varepsilon'_{\tau_n,\mathscr{T}}$. Thanks to equations (13c) and (19) it follows that,

$$\left|\gamma(s^n_{\mathrm{n},\mathbf{a}_q}) - \gamma(s^n_{\mathrm{n},\mathbf{a}_{q+1}})\right| - \left|p^n_{\mathrm{n},\mathbf{a}_q} - p^n_{\mathrm{n},\mathbf{a}_{q+1}}\right| \leq C_{\tau_n,\mathscr{T}} \quad \Rightarrow \quad \gamma(s^n_{\mathrm{n},\mathbf{a}_{q+1}}) \leq \gamma^{\star\star}_{\tau_n,\mathscr{T}}.$$

We conclude the proof by induction along the path. □

The bound on the saturation associated with the definition (14) on $\eta^n_{\mathrm{w},T}$ yields $\eta^n_{\mathrm{w},T} \geq \eta_\mathrm{w}(\varepsilon_{\tau_n,\mathscr{T}})$. This, combined with the Poincaré inequality (10) and since $\gamma(s^n_{\mathrm{n},\mathbf{a}}) \leq \gamma(1-\varepsilon_{\tau_n,\mathscr{T}})$ for any $\mathbf{a} \in \mathscr{V}_\mathscr{T}$, allows us to obtain a discrete bound on the pressure.

Proposition 4 *There exists $p^\star_{\tau_n,\mathscr{T}} > 0$ depending on the data of the continuous problem, $\mathscr{T}$, τ_n and $s^{n-1}_{\mathrm{n},h}$ such that $\int_\Omega |p^n_{\mathrm{w},h}|^2 \leq p^\star_{\tau_n,\mathscr{T}}$.*

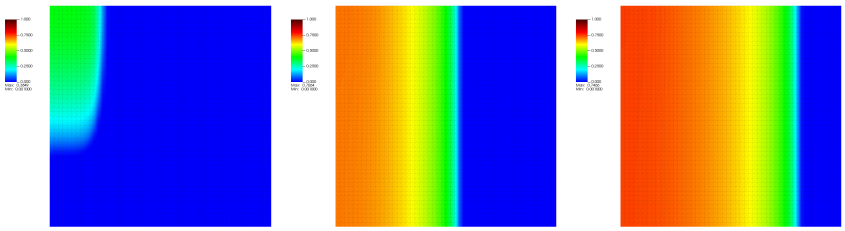
Thanks to the material introduced above, it is possible to prove the existence of a solution to the discrete problem using the topological degree theory.

Theorem 1 (Existence of a solution) *Let $s^{n-1}_{\mathrm{n},h} \in V_h$ be given, there exists at least one solution to the scheme* (13).

3 Numerical results

We present here numerical results obtained with the software FreeFem [8] in the two-dimension case by choosing as main unknowns the saturation of the non-wetting phase and the pressure of the wetting phase. To solve the nonlinear system we use a Newton method with a stopping criteria on the ℓ^∞-norm between two successive iterations. The computational domain is the unit square $\Omega =]0,1[^2$ and the mesh is made up of triangles whose mesh size is approximately equal to 0.028. The

final time is $t_\mathrm{f} = 0.015$ and the time step is constant $\tau_n = 10^{-3}$. We choose the porosity $\phi = 0.3$, the permeability tensor field $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}$ and $c_\mathrm{w} = 0.2$. For $s \in [0,1]$ we define the mobility functions by $\eta_\mathrm{n}(s) = s^2$ and $\eta_\mathrm{w}(s) = 2s$, the capillary pressure by $\gamma(s) = \frac{1}{\sqrt{1-s}}$ and the source functions are defined by $q_\mathrm{inj} = 40.1_{[0,0.2]\times[0.8,1]}$ and $q_\mathrm{sink} = 40.1_{[0.8,1]\times[0,0.2]}$. We plot in Figure 1 the approximate saturation of the non-wetting phase.



(a) $t = 0.002$ (b) $t = 0.01$ (c) $t = 0.015$

Fig. 1: Approximate saturation $s_{\mathrm{n},h}$ in Ω for different times t.

One observes from the outset of the simulation the influence on the injection well q_inj and of the anisotropy ratio in the longitudinal direction. Moreover we can see that the maximum does not exceed $c_\mathrm{n} = 0.8$.

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