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► **To cite this version:**

Samy Skander Bahoura. A compactness result for solutions to an elliptic system on the annulus.. 2020. ⟨hal-02441253v2⟩

**HAL Id: hal-02441253**

**<https://hal.science/hal-02441253v2>**

Preprint submitted on 3 Feb 2020

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# A COMPACTNESS RESULT FOR SOLUTIONS TO AN ELLIPTIC SYSTEM ON THE ANNULUS.

SAMY SKANDER BAHOURA

ABSTRACT. We give blow-up behavior for solutions to an elliptic system with Dirichlet condition. Also, we have a compactness result for this elliptic system with Lipschitz condition on the annulus.

**Mathematics Subject Classification:** 35J60 35B44 35B45 35B50

Keywords: blow-up, elliptic system, linear operators, Dirichlet condition, a priori estimate, annulus, boundary, Lipschitz condition, Pohozaev-Rellich identity.

## 1. INTRODUCTION AND MAIN RESULTS

We set  $\Delta = \partial_{11} + \partial_{22}$  on the annulus centered at the origin with radii 1 and 1/2 of  $\mathbb{R}^2$ .

We consider the following system:

$$(P) \begin{cases} -\Delta u - \epsilon \cdot \epsilon(x)(x \cdot \nabla u) = V e^v & \text{in } \Omega \subset \mathbb{R}^2, \\ -\Delta v + \epsilon \cdot \epsilon(x)(x \cdot \nabla v) = W e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Here:

We denote by  $C(1)$  and  $C(1/2)$  the unit circle and the circle of radius 1/2 respectively.

$\Omega = A(0, 1/2, 1)$  is an annulus of center 0 and radii 1/2, 1.

We assume that:

$\Omega$  is an annulus of exterior circle the unit circle and interior circle the circle of radius 1/2.

We assume that:

$\epsilon(x) \equiv 0$  in a neighborhood of the unit circle  $C(1)$  and  $\epsilon(x) \equiv 1$  in a neighborhood of the circle of radius 1/2.

$$0 \leq \epsilon \leq 1, \quad 0 \leq V \leq b_1 < +\infty, \quad e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega),$$

$$0 \leq W \leq b_2 < +\infty, \quad e^v \in L^1(\Omega) \text{ and } v \in W_0^{1,1}(\Omega),$$

When  $u = v$  and  $\epsilon = 0$ , the above system is reduced to an equation which was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-18], one can find some existence and compactness results, also for a system.

For the case  $\epsilon = 0$ , in [12], Dupaigne-Farina-Sirakov proved (by an existence result of Montenegro, see [17]) that the solutions of the above system when  $V$  and  $W$  are constants can be extremal and this condition imply the boundedness of the energy and directly the compactness. Note that in [11], if we assume (in particular) that  $\nabla \log V$  and  $\nabla \log W$  and  $V > a > 0$  or  $W > a' > 0$  and  $V, W$  are nonnegative and uniformly bounded then the energy is bounded and we have a compactness result.

Note that in the case of one equation (and  $\epsilon = 0$ ), we can prove by using the Pohozaev identity that if  $+\infty > b \geq V \geq a > 0$ ,  $\nabla V$  is uniformly Lipschitzian that the energy is bounded when  $\Omega$  is starshaped. In [15] Ma-Wei, using the moving-plane method showed that this fact is true for all domain  $\Omega$  with the same assumptions on  $V$ . In [11] De Figueiredo-do O-Ruf extend this fact to a system by using the moving-plane method for a system.

More generally, we have not a global compactness to the previous problem with one equation, perhaps we need more information on  $V$  to conclude to the boundedness of the solutions. When  $\nabla \log V$  is Lipschitz function and  $\epsilon = 0$ , Chen-Li and Ma-Wei see [7] and [15], showed that we have a compactness on all the open set. The proof is via the moving plane-Method of Serrin and Gidas-Nirenberg. Note that in [11] and  $\epsilon = 0$  we have the same result for this system when  $\nabla \log V$  and  $\nabla \log W$  are uniformly bounded. We will see below that for a system we also have a compactness result when  $V$  and  $W$  are Lipschitzian and  $\epsilon \in (0, 1)$ .

Now consider the case of one equation and  $\epsilon = 0$ . In this case our equation have nice properties.

If we assume  $V$  with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafir see [18], that, if  $(u_i)_i, (V_i)_i$  are two sequences of functions solutions of the previous equation without assumption on the boundary and,  $0 < a \leq V_i \leq b < +\infty$ , then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose  $(V_i)_i$  uniformly Lipschitzian with  $A$  the Lipschitz constant, then,  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$ , see [5].

Here we are interested by the case of an elliptic system. First, we give the behavior of the blow-up points on the boundary, for this general elliptic system, and in the second time we have a proof of compactness of the solutions to this elliptic system with Lipschitz condition.

Here, we write an extention of Brezis-Merle Problem (see [6]) to a elliptic system:

**Problem.** Suppose that  $V_i \rightarrow V$  and  $W_i \rightarrow W$  in  $C^0(\bar{\Omega})$ , with,  $0 \leq V_i$  and  $0 \leq W_i$ . Also, we consider two sequences of solutions  $(u_i), (v_i)$  of  $(P)$  relatively to  $(V_i), (W_i)$  such that,

$$\int_{\Omega} e^{u_i} dx \leq C_1, \quad \int_{\Omega} e^{v_i} dx \leq C_2,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C_3?$$

and,

$$\|v_i\|_{L^\infty} \leq C_4?$$

In this paper we give a characterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when  $V_i$  and  $W_i$  are uniformly Lipschitzian and close to constants. For the behavior of the blow-up points on the boundary, the following condition are enough,

$$0 \leq V_i \leq b_1, \quad 0 \leq W_i \leq b_2,$$

The conditions  $V_i \rightarrow V$  and  $W_i \rightarrow W$  in  $C^0(\bar{\Omega})$  are not necessary.

But for the proof of the compactness for the system, we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A_i \rightarrow 0, \quad \|\nabla W_i\|_{L^\infty} \leq B_i \rightarrow 0.$$

Our main result are:

**Theorem 1.1.** *Assume that  $\max_{\Omega} u_i \rightarrow +\infty$  and  $\max_{\Omega} v_i \rightarrow +\infty$  Where  $(u_i)$  and  $(v_i)$  are solutions of the probleme (P) with  $\epsilon_i \rightarrow 0$ , and:*

$$0 \leq V_i \leq b_1, \quad \text{and} \quad \int_{\Omega} e^{u_i} dx \leq C_1, \quad \forall i,$$

and,

$$0 \leq W_i \leq b_2, \quad \text{and} \quad \int_{\Omega} e^{v_i} dx \leq C_2, \quad \forall i,$$

then; after passing to a subsequence, there is a finction  $u$ , there is a number  $N \in \mathbb{N}$  and  $N$  points  $x_1, x_2, \dots, x_N \in \partial\Omega$ , such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi + \sum_{j=1}^N \alpha_j \varphi(x_j), \quad \alpha_j \geq 4\pi,$$

for any  $\varphi \in C^0(\partial\Omega)$ , and,

$$u_i \rightarrow u \quad \text{in} \quad C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

$$\int_{\partial\Omega} \partial_\nu v_i \varphi \rightarrow \int_{\partial\Omega} \partial_\nu v \varphi + \sum_{j=1}^N \beta_j \varphi(x_j), \quad \beta_j \geq 4\pi,$$

for any  $\varphi \in C^0(\partial\Omega)$ , and,

$$v_i \rightarrow v \quad \text{in} \quad C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

In the following theorem, we have a proof for the global a priori estimate which concern the problem (P).

**Theorem 1.2.** *Assume that  $(u_i), (v_i)$  are solutions of (P) relatively to  $(V_i), (W_i)$  with the following conditions:*

$$\epsilon_i \rightarrow 0, \quad 0 \leq V_i \leq b_1, \quad \|\nabla V_i\|_{L^\infty} \leq A_i \rightarrow 0, \quad \text{and} \quad \int_{\Omega} e^{u_i} \leq C_1,$$

$$0 \leq W_i \leq b_2, \quad \|\nabla W_i\|_{L^\infty} \leq B_i \rightarrow 0, \quad \text{and} \quad \int_{\Omega} e^{v_i} \leq C_2,$$

We have,

$$\|u_i\|_{L^\infty} \leq C_3(b_1, b_2, (\epsilon_i), (A_i), (B_i), C_1, C_2, \Omega),$$

and,

$$\|v_i\|_{L^\infty} \leq C_4(b_1, b_2, (\epsilon_i), (A_i), (B_i), C_1, C_2, \Omega),$$

To prove Theorem 1.2, we argue by contradiction and use Theorem 1.1.

## 2. PROOF OF THE THEOREMS

### Proof of theorem 1.1:

We have:

$$u_i, v_i \in W_0^{1,1}(\Omega).$$

Since  $e^{u_i} \in L^1(\Omega)$  by the corollary 1 of Brezis-Merle's paper (see [6]) we have  $e^{v_i} \in L^k(\Omega)$  for all  $k > 2$  and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

And,

We have:

$$v_i, u_i \in W_0^{1,1}(\Omega).$$

Since  $e^{v_i} \in L^1(\Omega)$  by the corollary 1 of Brezis-Merle's paper (see [6]) we have  $e^{u_i} \in L^k(\Omega)$  for all  $k > 2$  and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$v_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Also, we have by a duality theorem:

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \|\nabla v_i\|_{L^q} \leq C'_q, \quad \forall i \text{ and } 1 < q < 2.$$

For the blow-up which are inside the domain, we start by solving the equations:

$$\begin{cases} -\Delta w_i^1 = -\epsilon_i \cdot \epsilon(x)(x \cdot \nabla u_i) & \text{in } \Omega \subset \mathbb{R}^2, \\ w_i^1 = 0 & \text{in } \partial\Omega. \end{cases}$$

and,

$$\begin{cases} -\Delta w_i^2 = \epsilon_i \cdot \epsilon(x)(x \cdot \nabla v_i) & \text{in } \Omega \subset \mathbb{R}^2, \\ w_i^2 = 0 & \text{in } \partial\Omega. \end{cases}$$

By the duality theorem the functions  $w_i^1$  and  $w_i^2$  are uniformly bounded. Thus for the blow-up inside the domain we have the same work for a system without the linear terms  $\epsilon_i \cdot \epsilon(x)(x \cdot \nabla u_i)$  and  $\epsilon_i \cdot \epsilon(x)(x \cdot \nabla v_i)$ .

Since  $V_i e^{v_i}$  and  $W_i e^{u_i}$  are bounded in  $L^1(\Omega)$ , we can extract from those two sequences two subsequences which converge to two nonnegative measures  $\mu_1$  and  $\mu_2$ . (This procedure is similar to the procedure of Brezis-Merle, we apply corollary 4 of Brezis-Merle paper, see [6]).

If  $\mu_1(x_0) < 4\pi$ , by a Brezis-Merle estimate for the first equation, we have  $e^{u_i} \in L^{1+\epsilon}$  around  $x_0$ , by the elliptic estimates, for the second equation, we have  $v_i \in W^{2,1+\epsilon} \subset L^\infty$  around  $x_0$ , and, returning to the first equation, we have  $u_i \in L^\infty$  around  $x_0$ .

If  $\mu_2(x_0) < 4\pi$ , then  $u_i$  and  $v_i$  are also locally bounded around  $x_0$ .

Thus, we take a look to the case when,  $\mu_1(x_0) \geq 4\pi$  and  $\mu_2(x_0) \geq 4\pi$ . By our hypothesis, those points  $x_0$  are finite.

We will see that inside  $\Omega$  no such points exist. By contradiction, assume that, we have  $\mu_1(x_0) \geq 4\pi$ . Let us consider a ball  $B_R(x_0)$  which contain only  $x_0$  as nonregular point. Thus, on  $\partial B_R(x_0)$ , the two sequence  $u_i$  and  $v_i$  are uniformly bounded. Let us consider:

$$\begin{cases} -\Delta z_i = V_i e^{v_i} & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z_i = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

By the maximum principle, because we have  $w_i^1$  and  $w_i^2$ :

$$z_i \leq u_i + O(1)$$

and  $z_i \rightarrow z$  almost everywhere on this ball, and thus,

$$\int e^{z_i} \leq C_0 \int e^{u_i} \leq C,$$

and,

$$\int e^z \leq C.$$

but,  $z$  is a solution in  $W_0^{1,q}(B_R(x_0))$ ,  $1 \leq q < 2$ , of the following equation:

$$\begin{cases} -\Delta z = \mu_1 & \text{in } B_R(x_0) \subset \mathbb{R}^2, \\ z = 0 & \text{in } \partial B_R(x_0). \end{cases}$$

with,  $\mu_1 \geq 4\pi$  and thus,  $\mu_1 \geq 4\pi\delta_{x_0}$  and then, by the maximum principle in  $W_0^{1,1}(B_R(x_0))$ :

$$z \geq -2 \log |x - x_0| + C$$

thus,

$$\int e^z = +\infty,$$

which is a contradiction. Thus, there is no nonregular points inside  $\Omega$

Thus, we consider the case where we have nonregular points on the boundary, we use two estimates:

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C_1, \quad \int_{\partial\Omega} \partial_\nu v_i d\sigma \leq C_2,$$

and,

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \|\nabla v_i\|_{L^q} \leq C'_q, \quad \forall i \text{ and } 1 < q < 2.$$

We have the same computations, as in the case of one equation.

We consider a points  $x_0 \in \partial\Omega$  such that:

$$\mu_1(x_0) < 4\pi.$$

We consider a test function on the boundary  $\eta$  we extend  $\eta$  by a harmonic function on  $\Omega$ , we write the equation:

$$-\Delta((u_i - u)\eta) = (V_i e^{v_i} - V e^v)\eta + \nabla(u_i - u) \cdot \nabla\eta + \epsilon_i \cdot \epsilon(x)(x \cdot \nabla u_i)\eta = f_i$$

with,

$$\int |f_i| \leq 4\pi - \epsilon + o(1) < 4\pi - 2\epsilon < 4\pi,$$

$$-\Delta((v_i - v)\eta) = (W_i e^{u_i} - W e^u)\eta + \nabla(v_i - v) \cdot \nabla\eta - \epsilon_i \cdot \epsilon(x)(x \cdot \nabla v_i)\eta = g_i,$$

with,

$$\int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi,$$

or,

$$\int |g_i| \leq 4\pi - \epsilon + o(1) < 4\pi,$$

By the Brezis-Merle estimate, we have uniformly,  $e^{u_i} \in L^{1+\epsilon}$  around  $x_0$ , by the elliptic estimates, for the second equation, we have  $v_i \in W^{2,1+\epsilon} \subset L^\infty$  around  $x_0$ , and, returning to the first equation, we have  $u_i \in L^\infty$  around  $x_0$ .

We have the same thing if we assume:

$$\mu_2(x_0 = 0) < 4\pi,$$

Thus, if  $\mu_1(x_0) < 4\pi$  or  $\mu_2(x_0 = 0) < 4\pi$  or  $\mu_2(x_0 \neq 0) < 4\pi$ , we have for  $R > 0$  small enough:

$$(u_i, v_i) \in L^\infty(B_R(x_0) \cap \bar{\Omega}).$$

By our hypothesis the set of the points such that:

$$\mu_1(x_0) \geq 4\pi, \quad \mu_2(x_0 = 0) \geq 4\pi, \quad \text{or,} \quad \mu_2(x_0 \neq 0) \geq 4\pi,$$

is finite, and, outside this set  $u_i$  and  $v_i$  are locally uniformly bounded. By the elliptic estimates, we have the  $C^1$  convergence to  $u$  and  $v$  on each compact set of  $\bar{\Omega} - \{x_1, \dots, x_N\}$ .

For the blow-up on the boundary: the boundary contain two connected components. By the maximum principle and without loss of generality one can assume  $\partial_\nu u_i \geq 0$  and  $\partial_\nu v_i \geq 0$ .

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure  $\mu_1$  such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \mu_1(\varphi), \quad \forall \varphi \in C^0(\partial\Omega).$$

We take an  $x_0 \in \partial\Omega$  such that,  $\mu_1(x_0) < 4\pi$ . For  $\epsilon > 0$  small enough set  $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$  on the unit disk or one can assume it as an interval. We choose a function  $\eta_\epsilon$  such that,

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a  $\tilde{\eta}_\epsilon$  such that,

$$\begin{cases} -\Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

**Remark:** We use the following steps in the construction of  $\tilde{\eta}_\epsilon$ :

We take a cutoff function  $\eta_0$  in  $B(0, 2)$  or  $B(x_0, 2)$ :

1- We set  $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$  in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart  $(f, \tilde{\Omega})$  with  $f(0) = x_0$  and we take  $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$  to have connected sets  $I_\epsilon$  and we take  $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$ . Because  $f, f^{-1}$  are Lipschitz,  $|f(x) - x_0| \leq k_2|x| \leq 1$  for  $|x| \leq 1/k_2$  and  $|f(x) - x_0| \geq k_1|x| \geq 2$  for  $|x| \geq 2/k_1 > 1/k_2$ , the support of  $\eta$  is in  $I_{(2/k_1)\epsilon}$ .

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take:  $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$  and  $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$ , we extend it by 0 outside  $f(B_1(0))$ . We have  $f(B_1(0)) = D_1(x_0)$ ,  $f(B_\epsilon(0)) = D_\epsilon(x_0)$  and  $f(B_\epsilon^+(0)) = D_\epsilon^+(x_0)$  with  $f$  and  $f^{-1}$  smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And,  $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$ , since  $f$  is Lipschitz. Here  $H_1$  is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} -\Delta \bar{\eta}_\epsilon = -\Delta \eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \bar{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finally we set  $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$ . Also, by the maximum principle and the elliptic estimates we have :

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with  $C_1$  depends on  $\Omega$ .

We use the following estimate, see [8],

$$\|\nabla v_i\|_{L^q} \leq C_q, \quad \|\nabla u_i\|_q \leq C_q, \quad \forall i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that,  $(v_i)$  converge weakly in  $W_0^{1,q}(\Omega)$ , almost everywhere to a function  $v \geq 0$  and  $\int_\Omega e^v < +\infty$  (by Fatou lemma). Also,  $V_i$  weakly converge to a nonnegative function  $V$  in  $L^\infty$ .

We deduce from the last estimate that,  $(u_i)$  converge weakly in  $W_0^{1,q}(\Omega)$ , almost everywhere to a function  $u \geq 0$  and  $\int_\Omega e^u < +\infty$  (by Fatou lemma). Also,  $W_i$  weakly converge to a nonnegative function  $W$  in  $L^\infty$ .

The function  $u, v$  are in  $W_0^{1,q}(\Omega)$  solutions of :

$$\begin{cases} -\Delta u = V e^v \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

And,

$$\begin{cases} -\Delta v = W e^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle's result, see [6], we have  $e^{ku} \in L^1(\Omega), k > 1$ . By the elliptic estimates, we have  $v \in C^1(\bar{\Omega})$ .

According to the corollary 1 of Brezis-Merle's result, see [6], we have  $e^{kv} \in L^1(\Omega), k > 1$ . By the elliptic estimates, we have  $u \in C^1(\bar{\Omega})$ .

For two vectors  $f$  and  $g$  we denote by  $f \cdot g$  the inner product of  $f$  and  $g$ .

We can write:

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{v_i} - V e^v)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon + \epsilon_i \cdot \epsilon(x)(x \cdot \nabla u_i)\eta_\epsilon. \quad (1)$$

$$-\Delta((v_i - v)\tilde{\eta}_\epsilon) = (W_i e^{u_i} - W e^u)\tilde{\eta}_\epsilon - 2\nabla(v_i - v) \cdot \nabla\tilde{\eta}_\epsilon - \epsilon_i \cdot \epsilon(x)(x \cdot \nabla v_i)\tilde{\eta}_\epsilon.$$

We use the interior estimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between  $\tilde{\eta}_\epsilon$  and  $u$ , we obtain,

$$\int_{\Omega} V e^v \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C' \epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (2)$$

We have,

$$\begin{cases} -\Delta u_i = V_i e^{v_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between  $u_i$  and  $\tilde{\eta}_\epsilon$  to have:

$$\int_{\Omega} V_i e^{v_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \rightarrow \mu_1(\eta_\epsilon) \leq \mu_1(J'_\epsilon) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all  $\epsilon > 0$  there is  $i_0 = i_0(\epsilon)$  such that, for  $i \geq i_0$ ,

$$\int_{\Omega} |(V_i e^{v_i} - V e^v)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let  $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$  and  $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$ ,  $\epsilon > 0$ . Then, for  $\epsilon$  small enough,  $\Sigma_\epsilon$  is hypersurface.

The measure of  $\Omega - \Omega_{\epsilon^3}$  is  $k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$ .

**Remark:** for the unit ball  $\bar{B}(0, 1)$ , our new manifold is  $\bar{B}(0, 1 - \epsilon^3)$ .

(Proof of this fact; let's consider  $d(x, \partial\Omega) = d(x, z_0)$ ,  $z_0 \in \partial\Omega$ , this imply that  $(d(x, z_0))^2 \leq (d(x, z))^2$  for all  $z \in \partial\Omega$  which it is equivalent to  $(z - z_0) \cdot (2x - z - z_0) \leq 0$  for all  $z \in \partial\Omega$ , let's consider a chart around  $z_0$  and  $\gamma(t)$  a curve in  $\partial\Omega$ , we have;

$(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$ , we have  $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$ , this imply that  $x = z_0 - s\nu_0$  where  $\nu_0$  is the outward normal of  $\partial\Omega$  at  $z_0$ )

With this fact, we can say that  $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}$ . It is sufficient to work on  $\partial\Omega$ . Let's consider a charts  $(z, D = B(z, 4\epsilon_z), \gamma_z)$  with  $z \in \partial\Omega$  such that  $\cup_z B(z, \epsilon_z)$  is cover of  $\partial\Omega$ . One can extract a finite cover  $(B(z_k, \epsilon_k))$ ,  $k = 1, \dots, m$ , by the area formula the measure of  $S \cap B(z_k, \epsilon_k)$  is less than a  $k\epsilon$  (a  $\epsilon$ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary.

We write,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx. \quad (5)$$

Step 2.1: Estimate of  $\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$ .

First, we know from the elliptic estimates that  $\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$ ,  $C_1$  depends on  $\Omega$

We know that  $(|\nabla u_i|)_i$  is bounded in  $L^q$ ,  $1 < q < 2$ , we can extract from this sequence a subsequence which converge weakly to  $h \in L^q$ . But, we know that we have locally the uniform convergence to  $|\nabla u|$  (by Brezis-Merle's theorem), then,  $h = |\nabla u|$  a.e. Let  $q'$  be the conjugate of  $q$ .

We have,  $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take  $f = 1_{\Omega-\Omega_{\epsilon^3}}$ , we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1, \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega-\Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for  $i \geq i_1(\epsilon)$ ,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3(k_1 \|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant  $C_1$  does not depend on  $\epsilon$  but on  $\Omega$ .

Step 2.2: Estimate of  $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$ .

We know that,  $\Omega_\epsilon \subset \subset \Omega$ , and (because of Brezis-Merle's interior estimates)  $u_i \rightarrow u$  in  $C^1(\Omega_{\epsilon^3})$ . We have,

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For  $\epsilon > 0$ , we have for  $i \in \mathbb{N}$ ,  $i \geq \max\{i_1, i_2, i_3\}$ ,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (7)$$

From (4) and (7), we have, for  $\epsilon > 0$ , there is  $i_3 = i_3(\epsilon) \in \mathbb{N}$ ,  $i_3 = \max\{i_0, i_1, i_2\}$  such that,

$$\int_{\Omega} |-\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \quad (8)$$

We choose  $\epsilon > 0$  small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with  $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \frac{\epsilon_0}{2}$ .

We can use Theorem 1 of [6] to conclude that there are  $q \geq \tilde{q} > 1$  such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|} dx \leq \int_{\Omega} e^{q|u_i - u|} \tilde{\eta}_\epsilon dx \leq C(\epsilon, \Omega).$$

where,  $V_\epsilon(x_0)$  is a neighborhood of  $x_0$  in  $\bar{\Omega}$ . Here we have used that in a neighborhood of  $x_0$  by the elliptic estimates,  $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$ .

Thus, for each  $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$  there is  $\epsilon_{x_0} > 0, q_{x_0} > 1$  such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i. \quad (9)$$

Now, we consider a cutoff function  $\eta \in C^\infty(\mathbb{R}^2)$  such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$-\Delta(v_i \eta) = W_i e^{u_i} \eta - 2\nabla v_i \cdot \nabla \eta - v_i \Delta \eta + \epsilon_i \cdot \epsilon(x)(x \cdot \nabla v_i) \eta.$$

Because,

$$e^{v_i} \in L^1, \quad \nabla v_i \in L^q, \quad q > 1, \text{ uniformly,}$$

By the elliptic estimates,  $(v_i)_i$  is uniformly bounded in  $L^\infty(V_\epsilon(x_0))$ . Finally, we have, for some  $\epsilon > 0$  small enough,

$$\|v_i\|_{C^{0,\theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

Now, we consider a cutoff function  $\eta \in C^\infty(\mathbb{R}^2)$  such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$-\Delta(u_i \eta) = V_i e^{v_i} \eta - 2\nabla u_i \cdot \nabla \eta - u_i \Delta \eta - \epsilon_i \cdot \epsilon(x)(x \cdot \nabla u_i) \eta.$$

By the elliptic estimates,  $(u_i)_i$  is uniformly bounded in  $L^\infty(V_\epsilon(x_0))$  and also in  $C^{0,\theta}$  norm.

If we repeat this procedure another time, we have a boundedness of  $(u_i)_i$  and  $(v_i)_i$  in the  $C^{1,\theta}$  norm, because they are bounded in  $W^{2,q} \subset W^{1,q^*}$  norms with  $2q/(2-q) = q^* > 2$ .

We have the same computations and conclusion if we consider a regular point  $x_0 = 0$  for the measure  $\mu_2$  with  $\mu_2(\{0\}) < 4\pi$ .

We have proved that, there is a finite number of points  $\bar{x}_1, \dots, \bar{x}_m$  such that the sequence  $(u_i)_i$  and  $(v_i)_i$  are locally uniformly bounded (in  $C^{1,\theta}$ ,  $\theta > 0$ ) in  $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$ .

Proof of theorem 1.2:

Here, we use the Pohozaev-Rellich identity for a system around each blow-up point ( $\nu$  is the outward normal), see [16].

We have:

$$u_i, v_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}), k > 2.$$

Let's consider  $B_\epsilon = B_\epsilon(y_0) = \Omega_\epsilon^1$  a neighborhood of the blow-up point  $y_0$ . Then we have:

$$\begin{aligned} & \int_{B_\epsilon} [\Delta u_i(x \cdot \nabla v_i) + \Delta v_i(x \cdot \nabla u_i)] dx = \\ & = \int_{\partial B_\epsilon} [(x \cdot \nabla u_i)(\nu \cdot \nabla v_i) + (x \cdot \nabla v_i)(\nu \cdot \nabla u_i) - (x \cdot \nu)(\nabla u_i \cdot \nabla v_i)] d\sigma \end{aligned}$$

and after integration by parts for the equations of our system;

$$\begin{aligned} & \int_{B_\epsilon} [\Delta u_i(x \cdot \nabla v_i) + \Delta v_i(x \cdot \nabla u_i)] dx = \int_{B_\epsilon} -[V_i e^{v_i}(x \cdot \nabla v_i) + W_i e^{u_i}(x \cdot \nabla u_i)] dx = \\ & = \int_{B_\epsilon} [2(V_i e^{v_i} + W_i e^{u_i}) + (x \cdot \nabla V_i) e^{v_i} + (x \cdot \nabla W_i) e^{u_i}] dx - \int_{\partial B_\epsilon} (x \cdot \nu)(V_i e^{v_i} + W_i e^{u_i}) d\sigma. \end{aligned}$$

Finally,

$$\begin{aligned} & \int_{B_\epsilon} [2(V_i e^{v_i} + W_i e^{u_i}) + (x \cdot \nabla V_i) e^{v_i} + (x \cdot \nabla W_i) e^{u_i}] dx - \int_{\partial B_\epsilon} (x \cdot \nu)(V_i e^{v_i} + W_i e^{u_i}) d\sigma = \\ & = \int_{\partial B_\epsilon} [(x \cdot \nabla u_i)(\nu \cdot \nabla v_i) + (x \cdot \nabla v_i)(\nu \cdot \nabla u_i) - (x \cdot \nu)(\nabla u_i \cdot \nabla v_i)] d\sigma. \end{aligned}$$

The boundary of the annulus contain two connected components. For the first component the unit circle  $C(1)$ ,  $\epsilon(x) \equiv 0$ , we use a conformal map, to move a part of the unit circle into a part of an axis, and after we apply the previous Pohozaev-Rellich identity to have:

a) On  $C(1)$  after using a conformal map,  $x \cdot \nu = 0$  and  $u_i = v_i = 0$  on the axis:

$$\int_{B_\epsilon} V_i e^{v_i} + \int_{B_\epsilon} W_i e^{u_i} dx = O(\epsilon) + o(1),$$

However

$$\int_{B_\epsilon} V_i e^{v_i} + \int_{B_\epsilon} W_i e^{u_i} dx = \int_{\partial B_\epsilon} \partial_\nu u_i d\sigma + \int_{\partial B_\epsilon} \partial_\nu v_i d\sigma = \alpha_1 + \beta_1 + O(\epsilon) \rightarrow \alpha_1 + \beta_1 > 0,$$

which a contradiction.

b) On  $C(1/2)$ :

On the second connected component  $C(1/2)$ , we use directly the Pohozaev identity on a small neighborhood of a nonregular point  $y_0$  (obtained by charts around  $y_0$ ), we multiply, by  $x \cdot \nabla u_i$  and  $x \cdot \nabla v_i$ , and we integrate by parts, we obtain (here  $\nu = -2x$  on  $C(1/2)$  and  $u_i = v_i = 0$  on  $C(1/2)$ ):

$$\begin{aligned} 2\left(\int_{\Omega_\epsilon^1} V_i e^{v_i} + \int_{\Omega_\epsilon^1} W_i e^{u_i}\right) + \int_{C(1/2) \cap B_\epsilon(y_0)} -(-2\|x\|^2)(\partial_\nu u_i)(\partial_\nu v_i) d\sigma + \\ + \int_{C(1/2) \cap B_\epsilon(y_0)} -2(-\|x\|^2)(V_i + W_i) = \\ = \left| \int_{\Omega_\epsilon^1} (x \cdot \nabla V_i) e^{v_i} + (x \cdot \nabla W_i) e^{u_i} \right| + O(\epsilon). \end{aligned}$$

The terms of the previous left hand side are non-negatives.

We tend  $i \rightarrow +\infty$  and then  $\epsilon \rightarrow 0$ ,  $\nabla V_i \rightarrow 0$  and  $\nabla W_i \rightarrow 0$  to obtain:

$$\lim_{\epsilon \rightarrow 0} \lim_{i \rightarrow +\infty} \int_{\Omega_\epsilon^1} (V_i e^{v_i} + W_i e^{u_i}) = 0,$$

however:

$$\int_{\Omega_\epsilon^1} (V_i e^{v_i} + W_i e^{u_i}) dx = \int_{\partial \Omega_\epsilon^1} \partial_\nu u_i d\sigma + \int_{\partial \Omega_\epsilon^1} \partial_\nu v_i d\sigma + O(\epsilon) + o(1) \rightarrow \alpha_1 + \beta_1 > 0,$$

a contradiction.

**Remark:** Usually in the Pohozaev-Rellich identity we multiply the equations by  $(x - y_0) \cdot \nabla u_i$  and  $(x - y_0) \cdot \nabla v_i$ , around the blow-up point  $y_0$ , but here because we have the terms  $x \cdot \nabla u_i$  and  $x \cdot \nabla v_i$ , one can remove them only if we multiply the equations of the system by these terms and not  $(x - y_0) \cdot \nabla u_i$  and  $(x - y_0) \cdot \nabla v_i$ . This is a reason why we assume that  $\nabla V_i \rightarrow 0$ ,  $\nabla W_i \rightarrow 0$ , to have the integrals which contain those terms close to 0.

## REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.
- [2] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.
- [3] Bartolucci, D. A "sup+Cinf" inequality for Liouville-type equations with singular potentials. Math. Nachr. 284 (2011), no. 13, 1639-1651.
- [4] Bartolucci, D. A 'sup+Cinf' inequality for the equation  $-\Delta u = V e^u / |x|^{2\alpha}$ . Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), no. 6, 1119-1139
- [5] H. Brezis, Y.Y. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [6] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of  $-\Delta u = V(x)e^u$  in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).
- [7] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.
- [8] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L1. J. Math. Soc. Japan 25 (1973), 565-590.
- [9] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in  $\mathbb{R}^2$ . Commun. Anal. Geom. 6, No.1, 1-19 (1998).

- [10] D.G. De Figueiredo, P.L. Lions, R.D. Nussbaum, A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations, J. Math. Pures et Appl., vol 61, 1982, pp.41-63.
- [11] D.G. De Figueiredo, J. M. do O. B. Ruf. Semilinear elliptic systems with exponential nonlinearities in two dimensions. Adv. Nonlinear Stud. 6 (2006), no. 2, 199-213.
- [12] Dupaigne, L. Farina, A. Sirakov, B. Regularity of the extremal solutions for the Liouville system. Geometric partial differential equations, 139-144, CRM Series, 15, Ed. Norm., Pisa, 2013.
- [13] YY. Li, I. Shafrir. Blow-up analysis for solutions of  $-\Delta u = Ve^u$  in dimension two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.
- [14] YY. Li. Harnack Type Inequality: the method of moving planes. Commun. Math. Phys. 200,421-444 (1999).
- [15] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.
- [16] E. Mitidieri. A Rellich type identity and applications. Comm. P.D.E. 18. issue 1-2, 1993, pp 125-151.
- [17] Montenegro. M. Minimal solutions for a class of elliptic systems. Bull. London. Math. Soc. 37 (2005), no. 3, 405-416.
- [18] I. Shafrir. A sup+inf inequality for the equation  $-\Delta u = Ve^u$ . C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

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