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Bijjective link between Chapoton's new intervals and bipartite planar maps

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Abstract

In 2006, Chapoton defined a class of Tamari intervals called “new intervals” in his enumeration of Tamari intervals, and he found that these new intervals are equi-enumerated with bipartite planar maps. We present here a direct bijection between these two classes of objects using a new object called “degree tree”. Our bijection also gives an intuitive proof of an unpublished equi-distribution result of some statistics on new intervals given by Chapoton and Fusy.

1 Introduction

On classical Catalan objects, such as Dyck paths and binary trees, we can define the famous *Tamari lattice*, first proposed by Dov Tamari [Tam62]. This partial order was later found woven into the fabric of other more sophisticated objects. A notable example is diagonal coinvariant spaces [BPR12, BCP], which have led to several generalizations of the Tamari lattice [BPR12, PRV17], and also incited the interest in intervals in such Tamari-like lattices. Recently, there is a surge of interest in the enumeration [Cha06, BMFPR11, CP15, FPR17] and the structure [BB09, Fan17, Cha18] of different families of Tamari-like intervals. In particular, several bijective relations were found between various families of Tamari-like intervals and planar maps [BB09, FPR17, Fan18]. The current work is a natural extension of this line of research.

In [Cha06], other than counting Tamari intervals, Chapoton also introduced a subclass of Tamari intervals called *new intervals*, which are irreducible elements in a grafting construction of intervals. Definitions of these objects and related statistics

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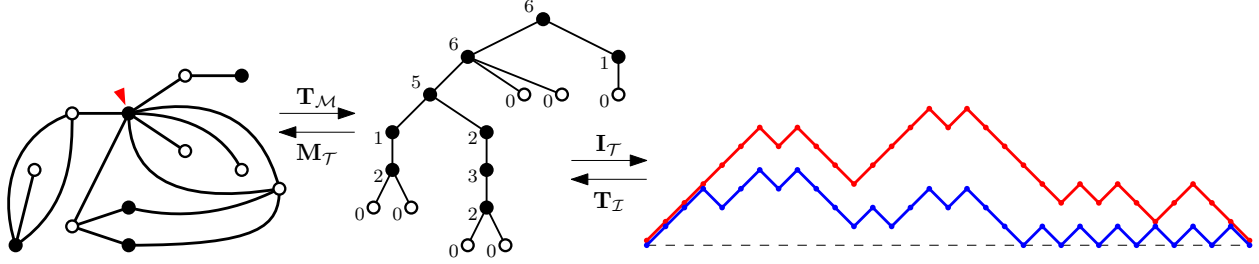


Figure 1: Our bijections between bipartite planar maps, degree trees and new intervals

are postponed to the next section. The number of new intervals in the Tamari lattice of order $n \geq 2$ was given in [Cha06], which equals

$$\frac{3 \cdot 2^{n-2}(2n-2)!}{(n-1)!(n+1)!}.$$

This is also the number of bipartite planar maps with $n-1$ edges. Furthermore, in a more recent unpublished result of Chapoton and Fusy (see [Fus17] for details), a symmetry in three statistics on new intervals was observed, then also proven by identifying the generating function of new intervals recording these statistics with that of bipartite planar maps recording the number of black vertices, white vertices and faces, three statistics well-known to be equi-distributed. These results strongly hint a bijective link between the two classes of objects.

In this article, we give a direct bijection between new intervals and bipartite planar maps (see Figure 1) explaining the results above. Our bijection can also be seen as a generalization of a bijection on trees given in [JS15] in the study of random maps. We have the following theorem, with statistics defined in the next section.

Theorem 1.1. *There is a bijection \mathbf{I}_M from the set \mathcal{I}_{n+1} of new intervals of size $n+1$ to the set \mathcal{M}_n of bipartite planar maps with n edges for every $n \geq 0$, with \mathbf{M}_T its inverse, such that, for a bipartite planar map M and $I = \mathbf{I}_M(M)$, which is a new interval, we have*

$$\begin{aligned} \mathbf{white}(M) &= \mathbf{c}_{00}(I), & \mathbf{black}(M) &= \mathbf{c}_{01}(I) \\ \mathbf{face}(M) &= 1 + \mathbf{c}_{11}(I), & \mathbf{outdeg}(M) &= \mathbf{rcont}(I) - 1. \end{aligned}$$

This bijection is intermediated by a new family of objects called *degree trees*, and was obtained in the spirit of some previous work of the author [FPR17, Fan18]. Our bijection was inspired and extending another bijection given in [JS15] between plane trees, which can be seen as bipartite planar maps.

Although the symmetry between statistics in new intervals is already known, our bijection captures this symmetry in an intuitive way, thus also opens a new door to the structural study of new intervals via bipartite maps and related objects. It is particularly interesting to see what natural involutions on bipartite maps, such as switching black and white in the coloring, induce on new intervals via our bijections.

In the rest of this article, we first define the related objects and statistics in Section 2. Then we show a bijection between bipartite planar maps and degree trees in

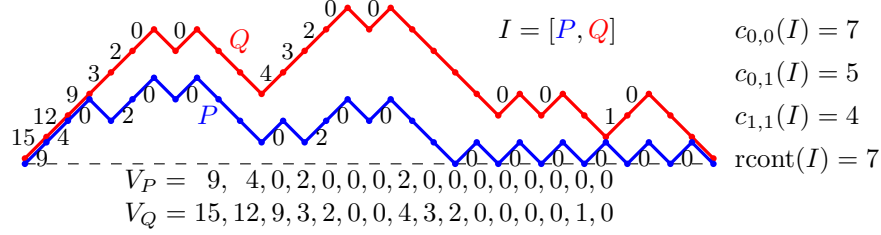


Figure 2: An example of Chapoton’s new interval with bracket vectors for both paths and related statistics.

Section 3, then a bijection between degree trees and new intervals in Section 4. We conclude by some remarks on the study of symmetries in new intervals in Section 5.

2 Preliminaries

A *Dyck path* P is a lattice path composed by up steps $u = (1, 1)$ and down steps $d = (1, -1)$, starting from the origin, ending on the x -axis while never falling below it. A *rising contact* of P is an up step of P on the x -axis. A non-empty Dyck path has at least one rising contact, which is the first step. We can also see a Dyck path P as a word in the alphabet $\{u, d\}$ such that all prefixes have more u than d . The *size* of a Dyck path is half its length. We denote by \mathcal{D}_n the set of Dyck paths of size n .

We now define the Tamari lattice, introduced in [Tam62], as a partial order on \mathcal{D}_n using a characterization in [HT72]. Given a Dyck path P seen as a word, its i^{th} up step u_i *matches* with a down step d_j if the factor P_i of P strictly between u_i and d_j is also a Dyck path. It is clear that there is a unique match for every u_i . We define the *bracket vector* V_P of P by taking $V_P(i)$ to be the size of P_i . The *Tamari lattice* of order n is the partial order \preceq on \mathcal{D}_n such that $P \preceq Q$ if and only if $V_P(i) \leq V_Q(i)$ for all $1 \leq i \leq n$. See Figure 2 for an example. A *Tamari interval* of size n can be viewed as a pair of Dyck paths $[P, Q]$ of size n with $P \preceq Q$.

In [Cha06], Chapoton defined a subclass of Tamari intervals called “new intervals”. Originally defined on pairs of binary trees, this notion can also be defined on pairs of Dyck paths (see [Fus17]). The example in Figure 2 is also a new interval. Given a Tamari interval $[P, Q]$, it is a *new interval* if and only if the following conditions hold:

- (i) $V_Q(1) = n$;
- (ii) For all $1 \leq i \leq n$, if $V_Q(i) > 0$, then $V_P(i) \leq V_Q(i + 1)$.

We denote by \mathcal{I}_n the set of new intervals of size $n \geq 1$.

We now define several statistics on new intervals. Given a Dyck path P of size n , its *type* $\text{Type}(P)$ is defined as a word w such that, if the i -th up step u_i is followed by an up step in P , then $w_i = 1$, otherwise $w_i = 0$. Since the last up step is always followed by a down step, we have $w_n = 0$. Note that our definition here is slightly

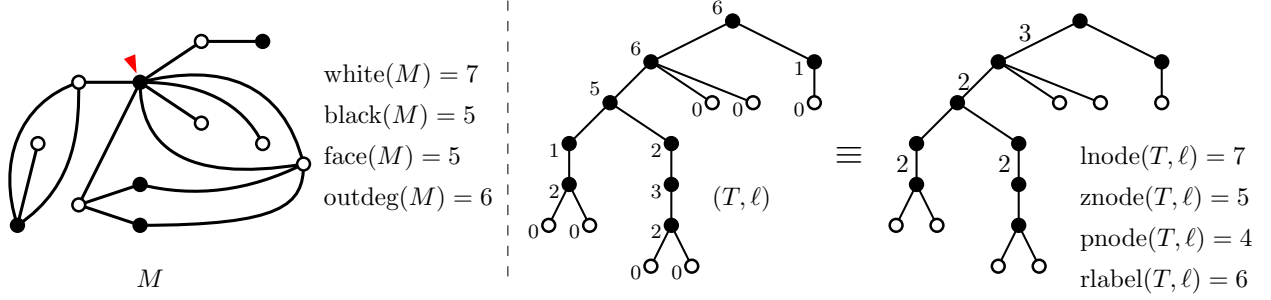


Figure 3: Left: an example of bipartite map. Right: an example of degree trees and the corresponding edge labels (zeros are omitted). Both with related statistics.

different from that in, e.g., [FPR17], where the last letter is not taken into account. Given a new interval $I = [P, Q] \in \mathcal{I}_n$, if $\text{Type}(P)_i = 1$ and $\text{Type}(Q)_i = 0$, then we have $V_P(i) > 0$ and $V_Q(i) = 0$, violating the condition for Tamari interval. Therefore, we have only three possibilities for $(\text{Type}(P)_i, \text{Type}(Q)_i)$. We define $\mathbf{c}_{00}(I)$ (resp. $\mathbf{c}_{01}(I)$ and $\mathbf{c}_{11}(I)$) to be the number of indices i such that $(\text{Type}(P)_i, \text{Type}(Q)_i) = (0, 0)$ (resp. $(0, 1)$ and $(1, 1)$). We also define $\mathbf{rcont}(I)$ to be the number of rising contacts of the lower path P in $I = [P, Q]$. Figure 2 also shows such statistics in the example. We define the generating function $F_{\mathcal{I}} \equiv F_{\mathcal{I}}(t, x; u, v, w)$ of new intervals as

$$F_{\mathcal{I}}(t, x; u, v, w) = \sum_{n \geq 1} t^n \sum_{I \in \mathcal{I}_n} x^{\mathbf{rcont}(I)-1} u^{\mathbf{c}_{00}(I)} v^{\mathbf{c}_{01}(I)} w^{\mathbf{c}_{11}(I)}. \quad (1)$$

We note that the power of x of the contribution of a new interval I is $\mathbf{rcont}(I) - 1$.

For the other side of the bijection, a *bipartite planar map* M is a drawing of a bipartite graph (in which all edges link a black vertex to a white one) on the plane, defined up to continuous deformation, such that edges intersect only at their ends. Edges in M cut the plane into *faces*, and the *outer face* is the infinite one. The *size* of M is its number of edges. In the following, we only consider *rooted* bipartite planar maps, which have a distinguished corner c called the *root corner* of the outer face on a black vertex, which is called the *root vertex*. See the left part of Figure 3 for an example. We denote by \mathcal{M}_n the set of (rooted) bipartite planar maps of size n . We allow the bipartite planar map of size 0, which consists of only one black vertex.

We also define some natural statistics on bipartite planar maps. For M a bipartite planar map, we denote by $\mathbf{black}(M)$, $\mathbf{white}(M)$ and $\mathbf{face}(M)$ the number of black vertices, white vertices and faces respectively. We also denote by $\mathbf{outdeg}(M)$ the *half-degree* of the outer face, i.e., half its number of corners. We take the convention that the outer face of the one-vertex map is of degree 0. These statistics are also illustrated in the left part of Figure 3. We define the generating function $F_{\mathcal{M}} \equiv F_{\mathcal{M}}(t, x; u, v, w)$ of bipartite planar maps enriched with these statistics by

$$F_{\mathcal{M}} \equiv F_{\mathcal{M}}(t, x; u, v, w) = \sum_{n \geq 0} t^n \sum_{M \in \mathcal{M}_n} x^{\mathbf{outdeg}(M)} u^{\mathbf{black}(M)} v^{\mathbf{white}(M)} w^{\mathbf{face}(M)}. \quad (2)$$

It is well known that $\mathbf{black}(M)$, $\mathbf{white}(M)$, $\mathbf{face}(M)$ are jointly equi-distributed in

\mathcal{M}_n , meaning that $F_{\mathcal{M}}$ is symmetric in u, v, w . This can be seen with the bijection between bipartite maps and bicubic maps by Tutte [Tut63], or with rotation systems of bipartite maps (see [LZ04, Chapter 1]).

To describe our bijection, we propose an intermediate class of objects called “degree trees”. An example is given in the right part of Figure 3. The meaning of this name will be clear in the description of our bijection. We can also see degree trees as a variant of description trees introduced by Cori, Jacquard and Schaeffer in [CJS97]. A *degree tree* is a pair (T, ℓ) , where T is a plane tree, and ℓ is a labeling function defined on nodes of T such that

- If v is a leaf, then $\ell(v) = 0$;
- If v is an internal node with k children v_1, v_2, \dots, v_k , then $\ell(v) = k - a + \ell(v_1) + \ell(v_2) + \dots + \ell(v_k)$ for some integer a with $0 \leq a \leq \ell(v_1)$.

We observe that the leftmost child of a node v is special when computing $\ell(v)$. This is different from the case of description trees. The size of a degree tree (T, ℓ) is the number of edges. We denote by \mathcal{T}_n the set of degree trees (T, ℓ) of size n .

Given a degree tree (T, ℓ) , we can replace ℓ by a labeling function on *edges*. More precisely, for an internal node v , we label its leftmost descending edge by the value of a used in the computation of $\ell(v)$, and all other edges by 0. We denote this edge labeling function by ℓ_{Λ} . It is clear that, given T , the mapping $\ell \mapsto \ell_{\Lambda}$ is an injection. Given ℓ_{Λ} , we can easily recover ℓ using its definition with the value $a = \ell_{\Lambda}(v)$ when computing $\ell(v)$.

We also define several natural statistics on degree trees, illustrated in Figure 3, using its edge labeling. Let (T, ℓ) be a degree tree with ℓ_{Λ} the corresponding edge labeling, and v a node in T . If v is a leaf, then it is called a *leaf node*. Otherwise, let e be the leftmost descending edge of v . If $\ell_{\Lambda}(e) = 0$, then v is a *zero node*, otherwise it is a *positive node*. We denote by $\mathbf{lnode}(T, \ell)$, $\mathbf{znode}(T, \ell)$ and $\mathbf{pnode}(T, \ell)$ the number of leaf nodes, zero nodes and positive nodes in (T, ℓ) respectively. For $T \in \mathcal{T}_n$, we have $\mathbf{lnode}(T, \ell) + \mathbf{znode}(T, \ell) + \mathbf{pnode}(T, \ell) = n + 1$. We also define the statistic \mathbf{rlabel} by taking $\mathbf{rlabel}(T, \ell) = \ell(r)$ with r the root of T .

Lemma 2.1. *Let (T, ℓ) be a degree tree, and ℓ_{Λ} the corresponding edge labeling. We have*

1. *If v has m descendants, then we have $\ell(v) = m - \sum_{e \in T_v} \ell_{\Lambda}(e)$, where T_v is the subtree induced by v ;*
2. *$\ell(v)$ is positive, and we have $\ell(v) = 0$ if and only if v has no descendant.*

Proof. The first point can be seen through induction on tree size. It holds clearly for the tree with no edge. Let T be a tree of size n , and v its root. Since the subtrees induced by each v_i have sizes strictly less than n , by induction hypothesis, we only need to check the condition on v . Let v_1, \dots, v_k be the descendants of v , and e_i the edge linking v_i and v . From the definition of ℓ we have

$$\ell(v) = k - \ell_{\Lambda}(e_1) + \ell(v_1) + \dots + \ell(v_k).$$

To show that $\ell(v) = m - \sum_e \ell_\Lambda(e)$, we must account for all descendants and all edges in T_v . However, those in one of the subtrees induced by some v_i are already accounted in $\ell(v_i)$. What remain are the nodes v_1, \dots, v_k , which are accounted by k , and the edges e_1, \dots, e_k , which are accounted by $-\ell_\Lambda(e_1)$, as $\ell_\Lambda(e_i) = 0$ for all $i > 1$. We thus conclude the induction.

The second point can also be proved by induction on tree size. It is clearly correct when T is the tree with no edge, and for the induction step, we observe that

$$\ell(v) = k + (\ell(v_1) - \ell_\Lambda(e_1)) + \ell(v_2) + \dots + \ell(v_k) \geq k \geq 1,$$

since $\ell(v_i) \geq 0$ by induction hypothesis and $0 \leq \ell_\Lambda(e_1) \leq \ell(v_1)$ by the definition of ℓ_Λ . □

3 Degree trees and bipartite maps

Our bijection from bipartite maps to new intervals is relayed by degree trees, in which the related statistics are transferred in an intuitive way. We now start by the bijection from maps to trees.

3.1 From bipartite maps to degree trees

It is well known that plane trees with n nodes in which k of them are leaves are counted by Narayana numbers (*cf.* [Drm15]). In [JS15], Janson and Stefánsson described a bijection between such plane trees and plane trees with n nodes in which k of them are of even depth, providing yet another interpretation of Narayana numbers. We now introduce a bijection between bipartite planar maps and degree trees, which can be seen as a generalization of the bijection in [JS15].

We first define a transformation \mathbf{T}_M from \mathcal{M}_n to \mathcal{T}_n for all n . Let $M \in \mathcal{M}_n$. If $n = 0$, we define $\mathbf{T}_M(M)$ to be the tree with one node. Otherwise, we perform the following exploration procedure to obtain a tree T with a labeling ℓ_Λ on its edges. In this procedure, we distinguish edges in M , which will be deleted one by one, and edges in T that we add. We start from the root vertex, with the edge next to the root corner in clockwise order as the pending edge. Suppose that the current vertex is u and the pending edge is e_M , which is always in M . We repeat two steps, *advance* and *prepare*, until termination. Roughly, in the advance step we modify edges in M and T and update the current vertex and the pending edge, and then in the prepare step we fix potential problems. The advance step comes in the following cases illustrated in Figure 4:

- (A1) If e_M is a bridge to a vertex v of degree 1, then we delete e_M in M and add $e_T = e_M$ in T . The new current vertex is $u' = u$, and we define $\ell_\Lambda(e_T) = 0$.
- (A2) If e_M is a bridge to a vertex v of degree at least 2, let e_1 be the edge adjacent to v next to e_M in clockwise order, and w the other end of e_1 . We draw a new edge e_T

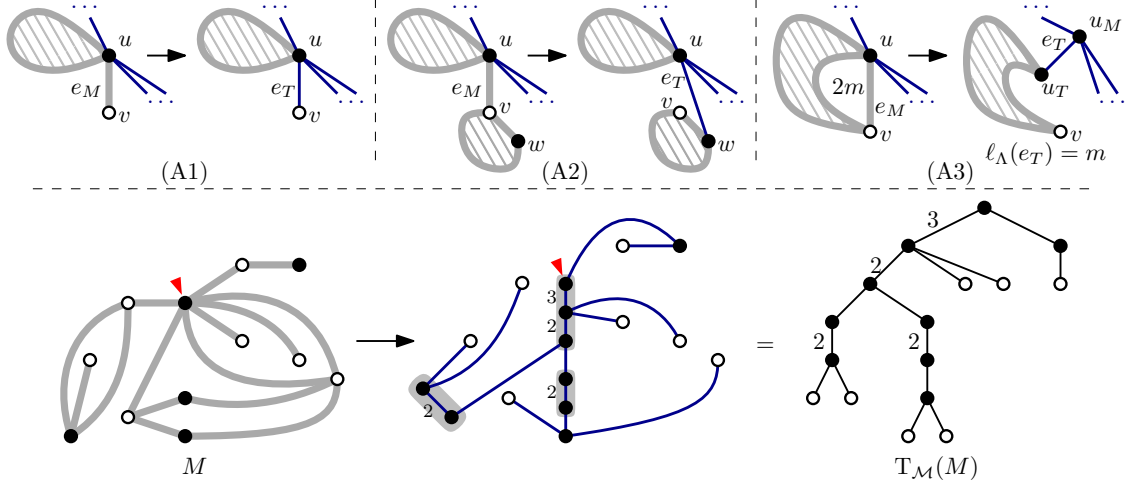


Figure 4: Cases in the advance step of $\mathbf{T}_{\mathcal{M}}$ and an example of the bijection $\mathbf{T}_{\mathcal{M}}$. Nodes in the same shaded pack come from the same vertex in the map.

in T from u to w such that e_M, e_1, e_T form a face with u, v, w in counter-clockwise order. The next current vertex is $u' = w$. We delete e_M , and define $\ell_{\Lambda}(e_T) = 0$.

- (A3) If e_M is not a bridge, we split u into u_M and u_T , with u_T taking all edges in T and u_M taking the rest. We add a new edge e_T in T from u_M to u_T . Since e_M is not a bridge, by planarity, it is between the outer face and a face of degree $2m$ with $m > 0$. We define $\ell_{\Lambda}(e_T) = m$ and delete e_M . The next current vertex is $u' = u_T$.

In the prepare step, let u' be the new current vertex, which is adjacent to the new edge e_T . The next pending edge is the next remaining edge in M starting from e_T in the clockwise order around u' . If no such edge exists, we backtrack in the tree T until finding a vertex u'' with such an edge e''_M , and we set u'' as the current vertex, and e''_M the pending edge. If no such vertex exists, the procedure terminates, and we shall obtain a tree T with an edge label function ℓ_{Λ} . We define $\mathbf{T}_{\mathcal{M}}(M)$ as the degree tree (T, ℓ) , with ℓ the node labeling corresponding to ℓ_{Λ} . See Figure 4 for an example of $\mathbf{T}_{\mathcal{M}}$. The bijection in [JS15] is simply $\mathbf{T}_{\mathcal{M}}$ applied to a plane tree, where Case (A3) never applies, and the degree tree (T, ℓ) obtained has $\ell_{\Lambda} = 0$ for all edges.

We now prove that $\mathbf{T}_{\mathcal{M}}(M)$ is well-defined. We start by describing the structure of the map in intermediate steps. The *leftmost branch* of a tree is the path starting from the root node and taking the leftmost descending edge at each node till a leaf.

Lemma 3.1. *Let $M \in \mathcal{M}_n$ and $T = \mathbf{T}_{\mathcal{M}}(M)$. Let M_i^+ be the map after the i -th prepare step, with u_i the current vertex and e_i the pending edge. We denote by T_i the partially constructed T in M_i^+ , and M_i that of the remaining of M . Clearly T_i and M_i form a partition of edges in M_i^+ .*

For every i , T_i is a tree, and M_i^+ is T_i with connected components of M_i attached to the left of nodes on the leftmost branch of T_i , one component to only one vertex, with u_i the

deepest such vertex and e_i its first edge in M_i in clockwise order from the leftmost branch of T_i .

Proof. We proceed by induction on i . The case $i = 0$ is trivial. We now suppose that the induction hypothesis holds for i , and we prove that it also holds for $i + 1$. Suppose that the component of M_i attached to u_i is $M_{i,*}$, then e_i is in $M_{i,*}$. For the $(i + 1)$ -st advance step, we have three possibilities.

- **Case (A1):** e_i links u_i to a node v_i of degree 1. The advance step then turns e_i into an edge in T_{i+1} . It is clear that T_{i+1} is also a tree, and other components of M_i are still in M_{i+1} and attached to the same vertices, except $M_{i,*}$, which becomes empty if only e_i is in it, or is turned into $M_{i+1,*}$ with e_i deleted otherwise. In the latter case, since v_i was of degree 1, the deletion of e_i does not disconnect $M_{i+1,*}$, thus $M_{i+1,*}$ is still attached to u_i . Either way, all components of M_{i+1} are still attached to T_{i+1} on the leftmost branch. Then in the prepare step, either $M_{i+1,*}$ is not empty, and we have $u_{i+1} = u_i$, with e_{i+1} the next edge in clockwise order of e_i , or it is empty, and we backtrack on the leftmost branch until finding a vertex with a component of M_{i+1} attached, which is also the last one in the preorder of T_{i+1} , and e_{i+1} is the next edge in the clockwise order of the last backtracking edge. Therefore, by induction hypothesis, e_{i+1} is also the first one in M_{i+1} starting from any edge of u_{i+1} in T_{i+1} .
- **Case (A2):** e_i links u_i to a node v_i of degree at least 2, and e_i is a bridge in M_i , thus also in $M_{i,*}$. The removal of e_i breaks $M_{i,*}$ into two parts, $M_{i+1,1}$ attached to u_i , and $M_{i+1,2}$ containing v_i . Let $e_{T,i}$ be the edge added to T_{i+1} in the advance step, linking u_i to a node w_i . By construction, w_i is in $M_{i+1,2}$, therefore not in T_i by induction hypothesis. Thus, T_{i+1} is a tree, and the newly separated component $M_{i+1,2}$ is attached to T_{i+1} by w_i . All other components of M_i remains in M_{i+1} and attached to T_{i+1} . Then in the prepare step, since $M_{i+1,2}$ is not empty, we have $u_{i+1} = w_i$, and e_{i+1} the first edge of w_i in $M_{i+1,2}$ in clockwise order, starting from e_i linking w_i to its parent u_i .
- **Case (A3):** e_i is not a bridge in M_i . The remaining $M_{i+1,*}$ of $M_{i,*}$ after the removal of e_i is still connected. Let $e_{T,i}$ be the edge added to T_{i+1} in the advance step, linking u_i to a node w_i . By construction, $M_{i+1,*}$ is attached to w_i . We verify the conditions on u_{i+1} and e_{i+1} with the same reasoning as in Case (A2).

As the induction hypothesis is valid in all cases, we conclude the proof. □

We now prove that trees obtained in $\mathbf{T}_{\mathcal{M}}$ are degree trees.

Proposition 3.2. *Given $M \in \mathcal{M}_n$ a bipartite map of size n , the tree $(T, \ell) = \mathbf{T}_{\mathcal{M}}(M)$ is a degree tree of size n .*

Proof. From Lemma 3.1, we know that the whole procedure of $\mathbf{T}_{\mathcal{M}}$ does not stop before consuming all n edges in M , and T is a tree. Therefore, T is a tree of size n .

Let ℓ_{Δ} be the edge labeling obtained in the procedure of $\mathbf{T}_{\mathcal{M}}$. The labels in ℓ_{Δ} are all positive by construction. We also observe that $\ell_{\Delta}(e) > 0$ for an edge $e \in T$ implies

that e links a node u to its leftmost child, as only Case (A3) has the possibility of $\ell_\Lambda(e) > 0$, and the new edge e_T added in that case becomes the leftmost descending edge of u after the duplication. We now only need to prove that the node labeling ℓ corresponding to ℓ_Λ satisfies the conditions of degree trees.

We now define a labeling ℓ' on nodes of T . By Lemma 3.1, the first time a node u is explored on T , there is a component of some remaining edges in M attached to u , which is itself a planar map. We denote by M_u this planar map. We define $\ell'(u)$ to be half of the degree of the outer face of M_u . We now prove that $\ell(u) = \ell'(u)$ by induction on the size of the subtree induced by u . For the base case, u is a leaf, and $\ell(u) = 0 = \ell'(u)$. When u is an internal node with children u_1, \dots, u_k from left to right, by induction hypothesis, we have $\ell(u_i) = \ell'(u_i)$ for all i . Now, for $i \geq 2$, the node u_i is produced by Case (A1) or (A2), thus are linked by bridges to u in M_u . The contribution of such u_i to $\ell'(u)$ is thus $\ell'(u_i) + 1$. For u_1 , by checking all cases, its contribution to $\ell'(u)$ is $\ell'(u_1) + 1 - \ell_\Lambda(e_1)$, where e_1 is the edge between u and u_1 . The only case that needs attention is Case (A3), where a face of degree $2\ell_\Lambda(e_1)$ is merged with the outer face by the removal of e_1 , increasing the degree of the outer face by $2\ell_\Lambda(e_1) - 2$. Therefore, the degree of the outer face of the part attached to u leading to u_1 before the exploration of u_1 is the correct value $2\ell'(u_1) + 2 - 2\ell_\Lambda(e_1)$. We thus have

$$\ell'(u) = \sum_{i=1}^k (\ell'(u_i) + 1) - \ell_\Lambda(e_1) = k - \ell_\Lambda(e_1) + \sum_{i=1}^k \ell(u_i) = \ell(u).$$

We thus conclude by induction that $\ell = \ell'$. Then, since the degree of the outer face of a planar bipartite map is at least 2, we have $\ell(u_1) - \ell_\Lambda(e_1) \geq 0$ for each edge e_1 from a node u to its first child u_1 . Hence, (T, ℓ) satisfies the conditions of degree trees. \square

The transformation \mathbf{T}_M transfers some statistics from \mathcal{M}_n to \mathcal{T}_n as follows.

Proposition 3.3. *Given $M \in \mathcal{M}_n$, let $(T, \ell) = \mathbf{T}_M(M)$. We have*

$$\begin{aligned} \mathbf{white}(M) &= \mathbf{inode}(T, \ell), & \mathbf{black}(M) &= \mathbf{znode}(T, \ell), \\ \mathbf{face}(M) &= 1 + \mathbf{pnode}(T, \ell), & \mathbf{outdeg}(M) &= \mathbf{rlabel}(T, \ell). \end{aligned}$$

Proof. Since in \mathbf{T}_M we only walk on black vertices, all leaves in T are from white vertices, which are never split. Hence $\mathbf{white}(M) = \mathbf{inode}(T, \ell)$. Then at each occurrence of Case (A3), we lost a face but gain a positive node in T , thus $\mathbf{face}(M) = 1 + \mathbf{pnode}(T, \ell)$, with 1 for the outer face. Now for $\mathbf{black}(M) = \mathbf{znode}(T, \ell)$, we note that a new black vertex in M is reached only in Case (A2), which leads to a zero edge. For $\mathbf{outdeg}(M)$, we notice $\mathbf{outdeg}(M) = n - \sum_f \deg(f)/2$, summing over all internal faces f of M . However, by the bijection, we have $\sum_f \deg(f)/2 = \sum_{e \in T} \ell_\Lambda(e)$, and we conclude by Lemma 2.1(1) applied to the root. \square

3.2 From degree trees to bipartite maps

We now define a transformation \mathbf{M}_T from \mathcal{T}_n to \mathcal{M}_n , which is precisely the inverse of \mathbf{T}_M . Let $(T, \ell) \in \mathcal{T}_n$ and $\ell_\Lambda = \Lambda(\ell)$. We now perform the following procedure that

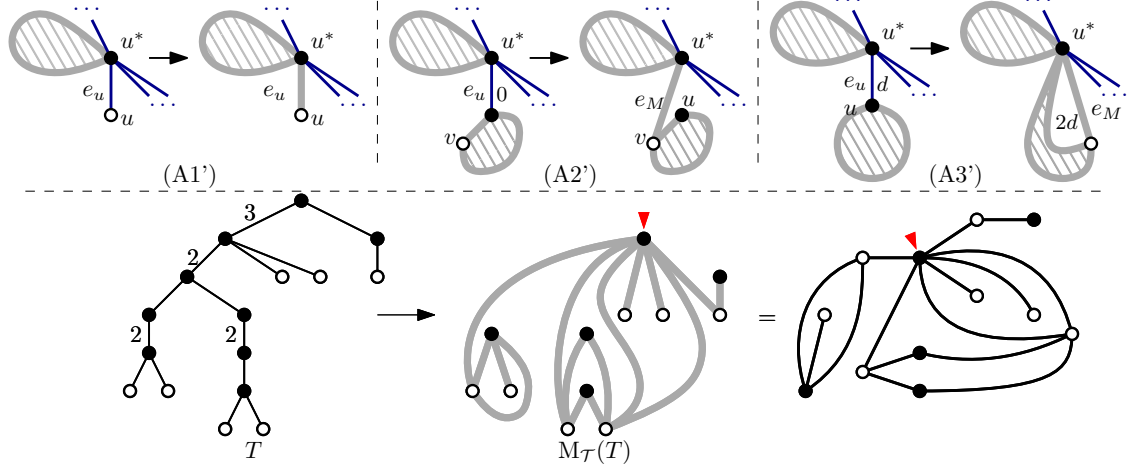


Figure 5: Cases in the procedure of $\mathbf{M}_{\mathcal{T}}$, and an example of $\mathbf{M}_{\mathcal{T}}$

deals with nodes in T in postorder (*i.e.*, first visit the subtrees induced by children from left to right, then the parent). For each node u , let u^* be its parent and e_u the edge between u and u^* . By construction, when we deal with u , its induced subtree has already been dealt with, transformed into a bipartite planar map M_u attached to u . We have three cases, illustrated in Figure 5.

- **Case (A1')**: If u is a leaf, then we delete e_u from T and add it to M .
- **Case (A2')**: If u is not a leaf but $\ell_{\Lambda}(e_u) = 0$, let e' be the edge next to e_u around u in counterclockwise order, and v the other end of e' . As M_u is bipartite, $v \neq u$. We add a new edge e_M from u^* to v such that the triangle formed by e_u, e', e_M has vertices u^*, u, v in clockwise order, without any edge inside. We then delete e_u .
- **Case (A3')**: If $\ell_{\Lambda}(e_u) > 0$, let d be the degree of the outer face of M_u . If $2\ell_{\Lambda}(e_u) > d$, then the procedure fails. Otherwise, we start from the corner of M_u to the right of e_u and walk clockwise along edges for $2\ell_{\Lambda}(e_u) - 1$ times to another corner, and we connect the two corners by a new edge e_M in M , making a new face of degree $2\ell_{\Lambda}(e_u)$. The component remains planar and bipartite. We finish by contracting e_u .

In the end, we obtain a planar bipartite map M with the same root corner as T . We define $\mathbf{M}_{\mathcal{T}}(T, \ell) = M$. We see that (A1'), (A2') and (A3') are exactly the opposite of (A1), (A2), (A3) in the definition of $\mathbf{T}_{\mathcal{M}}$.

We first show that the procedure above never fails, thus $\mathbf{M}_{\mathcal{T}}$ is always well-defined. It follows easily that we always have bipartite planar maps from $\mathbf{M}_{\mathcal{T}}$.

Proposition 3.4. *Given (T, ℓ) a degree tree, for a node $u \in T$, let M_u be the map obtained in the procedure of $\mathbf{M}_{\mathcal{T}}(T, \ell)$ from the subtree T_u induced by u . Then the degree of the outer face of M_u is $2\ell(u)$, and the procedure never fails.*

Proof. We use induction on the size of the subtree T_u . It clearly holds when u is a leaf. Suppose that u is an internal node. Let u_1, \dots, u_k be its children from left to right. Since every edge e_i linking u_i to u must be in Case (A1') or (A2') for $i \geq 2$, the contribution of the part M_{u_i} to the degree of the outer face is $2 + 2\ell(u_i)$ by induction hypothesis. If e_1 linking u_1 to u is also a bridge, then the contribution is $2 + 2\ell(u_1)$. Otherwise, we are in Case (A3'), in which we create a new face of degree $2\ell_\Lambda(e)$, where ℓ_Λ is the corresponding edge labeling. We never fail in this case, since by the definition of Λ , we have $0 \leq \ell_\Lambda(e) \leq \ell(u_1)$. Therefore, M_{u_1} has an outer face of degree $2\ell(u_1) + 2 - 2\ell_\Lambda(e)$. The degree of the outer face of M_u is thus

$$2\ell(u_1) + 2 - 2\ell_\Lambda(e) + \sum_{i=2}^k (2 + 2\ell(u_i)) = 2\ell(u).$$

We thus conclude the induction. \square

Proposition 3.5. *For (T, ℓ) a degree tree, $M = \mathbf{M}_T(T, \ell)$ is a bipartite planar map.*

Proof. Planarity is easily checked through the definition of \mathbf{M}_T . Faces in M are only created in Case (A3'), thus all of even degree. Since M is planar, every cycle of edges can be seen as a gluing of faces, which are all of even degree. Therefore, the cycle obtained is always of even length, meaning that M is bipartite. \square

It is also clear that \mathbf{M}_T is the inverse of \mathbf{T}_M .

Proposition 3.6. *The transformation \mathbf{T}_M is a bijection from \mathcal{M}_n to \mathcal{T}_n , with \mathbf{M}_T its inverse.*

Proof. By Proposition 3.2, we only need to prove that $\mathbf{T}_M \circ \mathbf{M}_T = \text{id}_T$ and $\mathbf{M}_T \circ \mathbf{T}_M = \text{id}_M$.

For $\mathbf{T}_M \circ \mathbf{M}_T = \text{id}_T$, it is clear that the operations in cases of \mathbf{M}_T are reverted by those in \mathbf{T}_M , and by Lemma 3.1, the degree tree is constructed node by node in reverse postorder in \mathbf{T}_M . We thus have $\mathbf{T}_M \circ \mathbf{M}_T = \text{id}_T$.

To show that $\mathbf{M}_T \circ \mathbf{T}_M = \text{id}_M$, we only need to check that they are applied exactly in the reverse order, and there is only one possibility for reversing operations in each case of \mathbf{T}_M . The first point is again ensured by Lemma 3.1. For the second point, the only case to check is Case (A3). To revert operation in this case, we need to create a new face of given degree by cutting the outer face with an edge. By planarity, there is only one way to proceed, which is that of Case (A3) in \mathbf{M}_T . We thus conclude that \mathbf{M}_T is indeed the inverse of \mathbf{T}_M , and they are all bijections. \square

4 Degree trees and new intervals

We now present the bijective link between degree trees and new intervals, which also gives a combinatorial explanation of the conditions of new intervals in terms of trees.

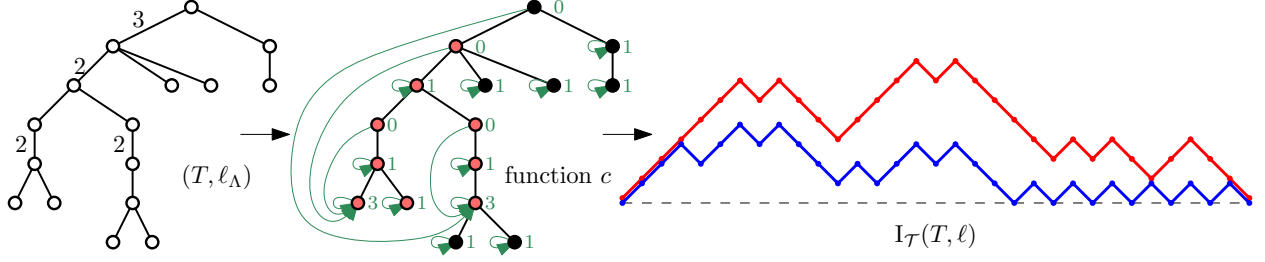


Figure 6: Example of the bijection $\mathbf{I}_{\mathcal{T}}$ on a degree tree represented by its edge labeling. The middle shows the certificate of each node.

4.1 From degree trees to new intervals

Given $(T, \ell) \in \mathcal{T}_n$, let ℓ_{Λ} be the corresponding edge labeling. We define a transformation $\mathbf{I}_{\mathcal{T}}$ by constructing a pair of Dyck paths $[P, Q]$ from (T, ℓ) . We take $Q = uQ'd$, where Q' comes from the classical bijection between plane trees and Dyck paths by doing a traversal of T in *preorder* (parent first, then subtrees from left to right), recording the evolution of depth. For P , we first assign to every node a *certificate*, and we define a *certificate function* c on T as in [Fan18, FPR17]. We process all nodes in T in the *reverse preorder*, initially colored black. At the step for a node v , if v is a leaf, then its certificate is itself. Otherwise, let e be the leftmost descending edge of v . We then visit nodes after v in preorder, and color each visited black node by red. We stop at the node w just before the $(\ell_{\Lambda}(e) + 1)$ -st black node, and the certificate of v is w . When $\ell_{\Lambda}(e) = 0$, we take $w = v$. Now, we take $c(w)$ to be the number of nodes with w as certificate. With the function c , the path P is given by concatenation of $ud^{c(v)}$ for all nodes v in preorder. We then define $\mathbf{I}_{\mathcal{T}}(T, \ell) = [P, Q]$. An example of $\mathbf{I}_{\mathcal{T}}$ is given in Figure 6.

To prove that $\mathbf{I}_{\mathcal{T}}(T, \ell)$ is a new interval, we start by some properties of certificates.

Lemma 4.1. *Let (T, ℓ) be a degree tree of size n and ℓ_{Λ} the corresponding edge labeling. For a node $v \in T$, let w be the certificate of v . Then either $w = v$, or w is a descendant of v in the leftmost subtree T_* of v . In the latter case, w is not the last node of T_* in preorder.*

Proof. Let v_1, v_2, \dots, v_{n+1} be the nodes in T in preorder. We prove our statement for all v_i by reverse induction on i . It is clear that the last node v_n in preorder is a leaf, hence its certificate is itself. The base case is thus valid.

For the induction step, suppose that all v_j 's with $i < j \leq n$ satisfy the induction hypothesis. If v_i is a leaf, then the induction hypothesis holds for i . We now suppose that v_i has at least one child. Let T_* the subtree induced by the left-most child v_* of v_i , and e_* the edge linking v_* to v_i . If v_* is a leaf, then $\ell_{\Lambda}(e_*) = 0$ and the induction hypothesis is clearly correct. We suppose that v_* is not a leaf. We consider the coloring just before the step for v_i . Since nodes in T_* come after v_i in the preorder, their processing only changes color of nodes in T_* by induction hypothesis. Therefore, there are $\sum_{e \in T_*} \ell_{\Lambda}(e)$ red nodes in T_* . By Lemma 2.1(1), there are thus $(\ell(v_*) + 1)$ black nodes in T_* , where the extra 1 accounts for v_* itself, which is never red after its

process step. Since $\ell_\Lambda(e) \leq \ell(v_*) + 1$, the $(\ell(v_*) + 1)$ -st black node starting from v_* must be in T_* . Hence, the certificate of v_i is either v_i or in T_* , and cannot be the last node in T_* . We thus conclude the induction. \square

Lemma 4.2. *Let (T, ℓ) be a degree tree, and v, v' two distinct nodes in T with w, w' their certificates respectively. Suppose that v precedes v' in the preorder. Then w cannot be strictly between v' and w' in the preorder. Furthermore, if $v' \neq w'$, then $w \neq v'$.*

Proof. We only need to consider the case $v \neq w$ and $v' \neq w'$, as other cases are trivial. In the coloring process, since v precedes v' in the preorder, v' is treated before v . By construction, in the coloring process, after the step for v' , the nodes between v' to w' (excluding v' but including w') are all colored red. Therefore, in the process step for v , the visit will not stop strictly between v' and w' , nor at v' , as such a stop requires a succeeding black node. Hence, w is not strictly between v' and w' , and $w \neq v'$. \square

Note that in the lemma above, we can have $w = v'$ when $v' = w'$.

Proposition 4.3. *Let $(T, \ell) \in \mathcal{T}_n$ be a degree tree of size n . The pair of Dyck paths $[P, Q] = \mathbf{I}_{\mathcal{T}}(T, \ell)$ is a new interval in \mathcal{I}_{n+1} .*

Proof. Let v_1, v_2, \dots, v_{n+1} be the nodes in T (including the root) in preorder, and T_i the subtree induced by v_i for $1 \leq i \leq n+1$. We now prove that both P and Q are Dyck paths, with a combinatorial interpretation of their bracket vector V_P and V_Q . From the construction of Q , it is clear that Q is a Dyck path, and we have $V_Q(i) = |T_i|$, where $|T_i|$ is the size of T_i (i.e., the number of edges).

For P , from the construction of P and Lemma 4.1, a node that gives an up step never comes after its certificate that gives a down step, meaning that there are at least as many up steps as down steps in any prefix of P , making it a Dyck path. To compute $V_P(i)$, we consider v_i , its certificate w_i , and the subword P' of P that comes from the nodes from v_i to w_i (both v_i and w_i included). If v_i is a leaf or $v_i = w_i$, it is clear that $P' = ud^{c(v_i)}$ and $V_P(i) = 0$. Otherwise, we consider a node v_j strictly between v_i and w_i in the preorder of T , in which case we can write $P' = ud^{c(v_i)}P''ud^{c(w_i)}$. Firstly, let w_j be the certificate of v_j , then by Lemma 4.2, w_j cannot come strictly after w_i . Thus in P' there are more down steps than up steps. Secondly, by Lemma 4.2, no node has v_i as certificate, implying that $c(v_i) = 0$. Thirdly, also by Lemma 4.2, if v_j is a certificate of a node, then this node must be strictly between v_i and v_j , already contributing an up step to P'' . Therefore, in any prefix of P'' , there are at least the same number of up steps than down steps. We then have the i -th up step in P generated by v_i matches with one of the down steps in P' (by the first point), but not those in P'' or induced by v_i itself (by the second and the third point), therefore it matches with a down step generated by w_i . Since v_{i+1} is the first child of v_i . By Lemma 4.1, w_i is in the subtree induced by v_{i+1} , but not the last node, implying $V_P(i) \leq |T_{i+1}|$.

We now compare V_P and V_Q . It is clear that $V_Q(1) = n$. If $V_Q(i) = 0$, then v_i is a leaf, and we have $V_P(i) = 0 \leq V_Q(i)$. If $V_Q(i) > 0$, then v_i has descendants, and we

have $V_P(i) \leq |T_{i+1}| = V_Q(i+1) < V_Q(i)$ in this case. Therefore, the pair $[P, Q]$ is not only a Tamari interval, but also a new interval. It is clear from the construction of P and Q that they are Dyck paths of size $n + 1$. \square

We also have the following property of the new interval obtained from a given degree tree via $\mathbf{I}_{\mathcal{T}}$.

Proposition 4.4. *For a degree tree (T, ℓ) with ℓ_{Λ} the corresponding edge labeling, let $I = [P, Q] = \mathbf{I}_{\mathcal{T}}(T, \ell)$. For an internal node $v \in T$, let e be the edge linking v to its leftmost child v' , and $r = \ell_{\Lambda}(e)$. Let P_v be the subpath of P strictly between the up step contributed by v in P and its matching down step. Then the number of rising contacts in P_v as a Dyck path is r .*

Proof. Let w be the certificate of v . The subpath P_v comes from the contributions of nodes from v' to w , while deleting extra down steps from w due to potentially other nodes preceding v in preorder taking up w as certificate.

By Lemma 4.2, no node preceding v in preorder has its certificate strictly between v and w , and the certificate of nodes from v' to w cannot be strictly after w in the preorder. Therefore, P_v is totally determined by the relation of certificates for nodes from v' to w , which is known when the coloring process gets v treated. In that step, exactly r black nodes are colored red, denoted by v_1, v_2, \dots, v_r in the preorder. Let w_1, \dots, w_r be their certificates respectively.

First we prove that, for $1 \leq i \leq r$, the subpath of P_v contributed by nodes from v_i to w_i , denoted by P_i , is a Dyck path with one rising contacts. This is again due to Lemma 4.2, making the certificates of nodes strictly between v_i and w_i to be between v_i and w_i (can be equal to w_i). Thus P_i has the same number of up steps and down steps. Since the up step from a node always comes before the down step from its certificate, P_i is a Dyck path. There is no other rising contact of P_i , because the up step from v_i is matched by the last down step from w_i .

Now, clearly we have $v_1 = v'$, as v' is the node next to v in preorder, thus treated in the coloring process just before v , but the treatment always leave v_1 black. Now, at the step of v_1 in the coloring process, w_1 is the red node just before a black node in preorder. This black node cannot come after v_2 , as it would entail v_2 being red in the step for v , but not before v_2 either, as it would still be black in the step for v , violating the definition of v_2 . The same argument applies to all v_i , thus the next node of w_i in preorder is v_{i+1} for $1 \leq i \leq r - 1$. We now consider w_r . The node v_+ next to w_r in preorder must be black at the step for v_r , and remains black through all treatments for nodes till v_1 . Therefore, v_+ must come strictly after w , and we can only have $w = w_r$. We thus conclude that every node from v' to w is between some pair of v_i and w_i . Therefore, we can write $P_v = P_1 \cdots P_r$, and we conclude that the number of rising contacts in P_v is indeed r . \square

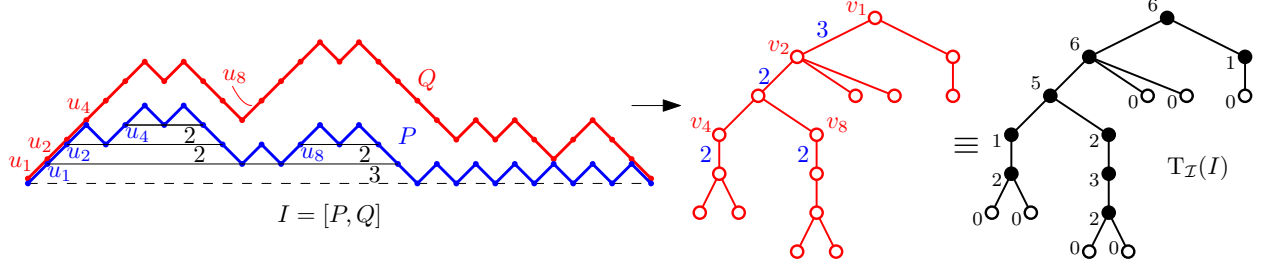


Figure 7: Example of the bijection \mathbf{T}_I on a new interval $I = [P, Q]$

4.2 From new intervals to degree trees

We now define a transformation \mathbf{T}_I for the reverse direction. Let $I = [P, Q] \in \mathcal{I}_{n+1}$ be a new interval. Since $V_Q(1) = n$, we can write $Q = uQ'd$. We first construct a plane tree T of size n from Q' using again the classical bijection. Now, let v_1, \dots, v_{n+1} be the nodes of T in preorder. We note that $V_Q(i)$ is the size of the subtree induced by v_i , which is equal to the number of descendants of v_i . We now define the edge labeling ℓ_Λ of T . If e is the left-most descending edge of v_i , then we take $\ell_\Lambda(e)$ the number of rising contacts in P_i , where P_i is the subpath of P strictly between the i -th up step and its matching down step. Otherwise, we take $\ell_\Lambda(e) = 0$. We define $\mathbf{T}_I(I) = (T, \ell)$, with ℓ the node labeling corresponding to ℓ_Λ . An example of \mathbf{T}_I is given in Figure 7. We first show that (T, ℓ) is indeed a degree tree.

Proposition 4.5. *Let $I = [P, Q] \in \mathcal{I}_{n+1}$, then $(T, \ell) = \mathbf{T}_I(I)$ is a degree tree of size n .*

Proof. Let ℓ_Λ be the edge labeling obtained when applying \mathbf{T}_I to I . We start by the following property of ℓ_Λ . Suppose that e' is an edge linking the j -th node v_j in T to its leftmost child v_{j+1} , and T_{j+1} is the subtree induced by v_{j+1} . We know that $\ell_\Lambda(e')$ is the number of rising contacts in P_j , where P_j is the subpath of P strictly between the j -th up step and its matching down step. In other words, $\ell_\Lambda(e')$ is the number of up steps in P_j that starts at the same height (y -coordinate) as the upper end of the j -th up step in P . Since in this case we have $V_Q(j) > 0$ as v_j is not a leaf, by the condition of new intervals, we have $V_P(j) \leq V_Q(j+1)$. Since up steps in P_j comes from descendants of v_j , and $V_Q(j+1)$ is the number of descendants of v_{j+1} , which are the first descendants of v_j in preorder, we conclude that all up steps in P contributing to $\ell_\Lambda(e')$ are from nodes in T_{j+1} , but not the last one in preorder.

From the construction, it is clear that the sizes match, and we only need to show that, for any edge e linking an internal node v to its leftmost child v_* , we have $\ell_\Lambda(e) \leq \ell(v_*)$. Let m be the number of descendants of v_* , and T_* is the subtree induced by v_* . The property above means that nodes whose up steps contributed to $\ell_\Lambda(e)$ or $\ell_\Lambda(e')$ for any $e' \in T_*$ must be in T_* , but not the last one in preorder. It is clear that every up step can only contribute to $\ell_\Lambda(e')$ for at most one e' . We thus have

$$m \geq \ell_\Lambda(e) + \sum_{e' \in T_*} \ell_\Lambda(e').$$

We deduce $\ell_\Lambda(e) \leq \ell(v_*)$ using the same argument as for Lemma 2.1(1). \square

Some natural statistics are transferred from new intervals to degree trees via $\mathbf{T}_{\mathcal{I}}$.

Proposition 4.6. *Given $I = [P, Q] \in \mathcal{I}_{n+1}$, let $(T, \ell) = \mathbf{T}_{\mathcal{I}}(I)$. We have*

$$\begin{aligned} \mathbf{c}_{00}(I) &= \mathbf{inode}(T, \ell), & \mathbf{c}_{01}(I) &= \mathbf{znode}(T, \ell), \\ \mathbf{c}_{11}(I) &= \mathbf{pnode}(T, \ell), & \mathbf{rcont}(I) &= 1 + \mathbf{rlabel}(T, \ell). \end{aligned}$$

Proof. Let v_i be the i -th node of T in preorder. By the definition of $\mathbf{T}_{\mathcal{I}}$, the node v_i is a leaf if and only if $\text{Type}(Q)_i = 0$. Hence, $\mathbf{c}_{00}(I) = \mathbf{inode}(T, \ell)$. Moreover, if v_i is an internal node, then $\text{Type}(P)_i = 0$ if and only if $\ell_\Lambda(e_i) = 0$, where e_i is the leftmost descending edge of v_i , and ℓ_Λ the edge labeling corresponding to ℓ . We thus conclude for $\mathbf{c}_{01}(I) = \mathbf{znode}(T, \ell)$ and $\mathbf{c}_{11}(I) = \mathbf{pnode}(T, \ell)$. For $\mathbf{rcont}(I)$, we observe that rise contacts come from up steps not contributing to the edge labeling ℓ_Λ , meaning that $\mathbf{rcont}(I) = n + 1 - \sum_{e \in T} \ell_\Lambda(e)$. By applying Lemma 2.1(1) to the root, we have $\mathbf{rlabel}(T, \ell) = n - \sum_{e \in T} \ell_\Lambda(e)$, therefore $\mathbf{rcont}(I) = 1 + \mathbf{rlabel}(T, \ell)$. \square

Using Proposition 4.4, we check that $\mathbf{I}_{\mathcal{T}}$ and $\mathbf{T}_{\mathcal{I}}$ are bijections.

Proposition 4.7. *For any $n \geq 0$, the transformation $\mathbf{I}_{\mathcal{T}}$ is a bijection from \mathcal{T}_n to \mathcal{I}_{n+1} , with $\mathbf{T}_{\mathcal{I}}$ its inverse.*

Proof. By Propositions 4.3 and 4.5, we only need $\mathbf{T}_{\mathcal{I}} \circ \mathbf{I}_{\mathcal{T}} = \text{id}_{\mathcal{T}}$ and $\mathbf{I}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{I}} = \text{id}_{\mathcal{I}}$.

For $\mathbf{T}_{\mathcal{I}} \circ \mathbf{I}_{\mathcal{T}} = \text{id}_{\mathcal{T}}$, let $(T, \ell) \in \mathcal{T}_n$ and $I = [P, Q] = \mathbf{I}_{\mathcal{T}}(T, \ell)$. Now we consider $(T', \ell') = \mathbf{T}_{\mathcal{I}}(I)$. It is clear from the definition of $\mathbf{T}_{\mathcal{I}}$ and $\mathbf{I}_{\mathcal{T}}$ that $T = T'$. We now show that $\ell = \ell'$, which is equivalent to $\ell_\Lambda = \ell'_\Lambda$, where ℓ_Λ (resp. ℓ'_Λ) is the edge labeling corresponding to ℓ (resp. ℓ'). Let e be an edge in T . We only need to consider the case where e links a node v to its leftmost child v' . Suppose that v is the i -th node in the preorder of T . Let P_i be the subpath of P between the i -th up step and its matching down step. Now by Proposition 4.4 and the definition of $\mathbf{T}_{\mathcal{I}}$, the number of rising contacts in P_i is equal to both $\ell_\Lambda(e)$ and $\ell'_\Lambda(e)$, making $\ell_\Lambda(e) = \ell'_\Lambda(e)$, thus $\ell = \ell'$. We conclude that $\mathbf{T}_{\mathcal{I}} \circ \mathbf{I}_{\mathcal{T}} = \text{id}_{\mathcal{T}}$.

For $\mathbf{I}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{I}} = \text{id}_{\mathcal{I}}$, let $I = [P, Q] \in \mathcal{I}_{n+1}$ and $(T, \ell) = \mathbf{T}_{\mathcal{I}}(I)$. We take ℓ_Λ the edge labeling corresponding to ℓ . Now we consider $I' = [P', Q'] = \mathbf{I}_{\mathcal{T}}(I)$. Again, it is clear that $Q = Q'$, and we only need to show that $P = P'$. For $1 \leq i \leq n + 1$, let P_i (resp. P'_i) be the subpath of P (resp. P') strictly between the i -th up step and its matching down step, and e_i the edge linking the i -th node in the preorder of T to its leftmost child. By the definition of $\mathbf{T}_{\mathcal{I}}$ and Proposition 4.4, there are $\ell_\Lambda(e_i)$ rising contacts in both P_i and P'_i for every i . However, suppose that P (resp. P') leads to a plane tree T_P (resp. $T_{P'}$) via the classical bijection. Since the number of rising contacts in P_i (resp. P'_i) is the degree of the $(i + 1)$ -st node in the preorder of T_P (resp. $T_{P'}$), we know that the degrees of nodes in T_P and $T_{P'}$ in preorder are the same. This leads to $T_P = T_{P'}$, meaning that $P = P'$. We thus conclude that $\mathbf{I}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{I}} = \text{id}_{\mathcal{I}}$. \square

5 Symmetries and structure

With the bijections in Section 3 and 4, we construct the following bijections between new intervals and bipartite maps, which is our main result.

Proof of Theorem 1.1. We take $\mathbf{I}_{\mathcal{M}} = \mathbf{I}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{M}}$ and $\mathbf{M}_{\mathcal{I}} = \mathbf{M}_{\mathcal{T}} \circ \mathbf{T}_{\mathcal{I}}$. Their validity is from Proposition 3.6 and 4.7. The equalities of statistics come from Proposition 3.3 and 4.6. \square

The symmetry between the statistics **white**, **black** and **face** on bipartite maps is then transferred to new intervals.

Corollary 5.1. *The generating functions $F_{\mathcal{I}}$ and $F_{\mathcal{M}}$ are related by*

$$tF_{\mathcal{M}} = wF_{\mathcal{I}}.$$

In particular, the series $wF_{\mathcal{I}}$ is symmetric in u, v, w .

Proof. The equality is a direct translation of Theorem 1.1 in generating functions. The symmetry of $wF_{\mathcal{I}}$ comes from that of $F_{\mathcal{M}}$. \square

Remark 1. For $M \in \mathcal{M}$, let D be the multiset of half-degrees of internal faces of M . From the definition of $\mathbf{T}_{\mathcal{M}}$, the multiset D is also the multiset of non-zero edge labels of $\mathbf{T}_{\mathcal{M}}(D)$. Now, let $[P, Q] = \mathbf{I}_{\mathcal{M}}(M)$. From the definition of $\mathbf{T}_{\mathcal{I}}$, the multiset of non-zero labels of $\mathbf{T}_{\mathcal{M}}(D)$ is also that of the number of rising contacts of subpaths of P between matching steps. We can thus refine Corollary 5.1 by this multiset D . Such refinement is particularly interesting in the domain of maps. We can enrich $F_{\mathcal{M}}$ by an infinity of variables $(p_k)_{k \geq 1}$, with p_k marking internal faces of half-degree k . Such enriched version is particularly nice and has deep link with factorization of the symmetric group and other objects. See [FC16] for more details. It would be interesting to see how results on refined enumeration of bipartite maps can be transferred to new intervals.

As mentioned before, the symmetry of $\mathbf{c}_{00}, \mathbf{c}_{01}, \mathbf{c}_{11}$ in new intervals was already known to Chapoton and Fusy, and a proof relying on generating functions was outlined in [Fus17], which makes use of recursive decompositions of new intervals [Cha06, Lemma 7.1] and bipartite planar maps. Our bijective proof can be seen as direct version of this recursive proof, in the sense that $\mathbf{T}_{\mathcal{I}}$ and $\mathbf{T}_{\mathcal{M}}$ are canonical bijections of these recursive decompositions. Details will be given in a follow-up article.

Since there are bijections for bipartite maps that permute black vertices, white vertices and faces arbitrarily, there should be an isomorphic symmetry structure hidden in new intervals via bijections. If we regard new intervals as pairs of binary trees, it is easy to see that there is an involution consisting of exchanging the two trees in the pair while taking their mirror images. This involution exchanges the statistics \mathbf{c}_{00} and \mathbf{c}_{11} , corresponding to **white** and **face** in bipartite planar maps. The structural study of these symmetries under our bijections is the subject of a follow-up article.

Furthermore, there is another class of combinatorial objects called β -(0,1) trees, which are description trees for bicubic planar maps in bijection with bipartite maps [CJS97, CS03]. An involution on these trees is given in [CKdM15], which may be related to symmetries we mentioned above.

However, as a precaution for all structural study, we should note that our bijections are subjected to various choices taken in their definitions. For instance, in the definition of the bijection $T_{\mathcal{M}}$ from bipartite maps to degree trees, in the case (A2), if we fix an integer $k \geq 0$, and construct the new edge e_T in the partial tree by connecting to the k -th black corner on the outer face in clockwise order, and changing the definition of M_T accordingly, we will have a bijection parameterized by k , which is different for all k . There is thus an infinity of bijections compatible with all results in this article. Therefore, it is possible that the bijections defined here may not preserve some wanted structure between related objects, but a similar bijection does.

As pointed out by an anonymous reviewer, the notion of degree tree bears similarities to that of “grafting trees” defined in [Pon19], which is in turn closely related to description trees of type (1,1) in [CS03] and closed flows on forests [CCP14, Fan18]. Since our degree tree can be seen as a special case of description trees of type (1,1), it would be interesting to see how the bijections extend to the general Tamari intervals and corresponding maps. For instance, the anonymous reviewer also observed that, elements of our bijection from degree trees to new intervals can be used to construct a direct bijection from grafting trees to general Tamari intervals.

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