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The Physics of the one-dimensional nonlinear Schrödinger equation in fiber optics: Rogue waves, modulation instability and self-focusing phenomena

François Copie, Stéphane Randoux, Pierre Suret^{*}

Univ. Lille, CNRS, UMR 8523 - PhLAM - Physique des Lasers Atomes et Molécules, Lille F-59000, France

ABSTRACT

We review the different dynamical mechanisms leading to the emergence of coherent structures in physical systems described by the integrable one-dimensional nonlinear Schrödinger equation (1DNLSE) in the focusing regime. In this context, localized and coherent structures are very often associated to rogue wave events. We focus on one-dimensional optical experiments and in particular on (single mode) optical fibers experiments. In the focusing regime of 1DNLSE, the so-called modulation instability (MI), arising from nonlocal perturbation of the plane waves, is the most common phenomenon. Alongside the standard MI, other mechanisms are responsible for the emergence of rogue waves. We classify the different scenarios by considering those induced by small perturbations of unstable stationary state (the plane waves) and the ones arising from the self-focusing of large pulses without any perturbation. In the former case, the perturbations can be local, global, random or deterministic. In the latter case, the self-focusing dynamics can be observed both with isolated pulses or with large initial fluctuations of the optical power. We review the dynamics of emergence of localized structures in all these different scenarios.

1. Introduction

The one-dimensional nonlinear Schrödinger equation (1DNLSE) is a universal equation in Physics for several fundamental reasons. First, it describes *at leading order* the propagation of nonlinear waves in various fields of Physics such as fluids, nonlinear optics or Bose-Einstein condensates (BEC) [1,2]. Secondly, since the 1DNLSE is an exactly integrable partial differential equation (PDE), it can be studied in the framework of the Inverse Scattering Transform (IST) also called nonlinear Fourier transform [3–8]. IST provides exact analytical solutions of the 1DNLSE such as the well-known *Solitons* that collide elastically [9].

In the focusing regime, the 1DNLSE exhibits a rich variety of dynamical phenomena that find their origin in the process of *Modulational Instability (MI)* or *Benjamin-Feir Instability*. It is an ubiquitous mechanism in nature that has been extensively studied in the last five decades [10–16]. It has been observed for example in deep water waves [17], in BEC [18] or in nonlinear fiber optics in the anomalous dispersion regime [19]. MI is often seen as the exponential amplification of a weakly-modulated monochromatic carrier wave. The simplest picture of MI is therefore the interaction between a strong carrier monochromatic wave at a frequency ω and two small sidebands at frequencies $\omega \pm \Omega$. In the early stage of the MI, the growth rate (or gain) of the sidebands is easily computed as a function of the sidebands frequency and exhibits a maximum that roughly corresponds to the balance between (third order) nonlinear effect and linear effect (group velocity dispersion) [19]. For a monochromatic carrier wave at ω perturbed by two small sidebands at $\omega \pm \Omega$, the nonlinear stage of MI, *i.e.* the long-term spatio-temporal dynamics is given by *breathers*-like solutions of 1DNLSE [20]. In particular, the Akhmediev breather (AB) is often referenced as the exact solution of MI [21–25].

AB is a fundamental example of the innumerable exact solutions of 1DNLSE in the focusing regime. Together with Peregrine solitons (PS) and Kuznetsov-Ma solitons, it belongs to the family of the solitons on finite background (SFB) [26–30]. These breather

^{*} Corresponding author.

E-mail address: Pierre.Suret@univ-lille.fr (P. Suret).

solutions of the 1DNLSE can emerge spontaneously from the propagation of a perturbed modulationally unstable continuous wave (CW) background [21–23,25,31]. It has also been demonstrated that the PS can emerge from the nonlinear propagation of a large isolated pulse [32]. Because breathers are localized in space and time and because the 1DNLSE describes at leading order unidirectional wave packets propagating in deep water, they are often considered as prototypes of Rogue Waves [25,28,33,34].

Rogue Waves (RWs) or ‘freak waves’ have been first studied in the context of Oceanography [35,36]. From the general point of view, RWs are extremely large amplitude waves but their precise definition depends on the context and it can vary from one scientific community to the other. In the ocean, it is difficult to measure the statistics of these rare extreme events and a definition with a threshold is often used: RWs correspond to waves having a surface elevation greater than two times the so-called significant wave height [35]. When the statistics can be accurately measured, RWs can be studied using a statistical approach. In this case, RWs are defined as extremely large amplitude waves occurring more frequently than expected from the normal law [37,38]. A common approach is then to compare the probability density function (PDF) of a given variable with the Gaussian statistics and to look for ‘L-shaped’ or ‘heavy-tailed’ PDF [38].

Since the pioneering work of Solli et al. in 2007 [39], RWs have been studied in various contexts in optics [40–46] such as optical cavities [47], supercontinuum generation in fibers [39,48–51], partially coherent waves propagation in fibers [52–56], semiconductor lasers [42], mode-locked fiber lasers [57], laser filamentation [58] and Raman fiber amplifiers or lasers [59,60]. As pointed out in Ref. [61], there is no obvious analogy between most of the optical experiments on extreme events and oceanography. However the direct correspondence between nonlinear optics and hydrodynamics provided by the 1DNLSE allows one-to-one comparison between optics and hydrodynamics in some specific cases [31,55]. In order to build a rigorous analogy between hydrodynamics and optics, statistics of the water surface elevation must be compared with the statistics of the electric field (and not the power).

It is important to note that in numerous papers, the terminology ‘Rogue Wave’ has been extended to any localized solution of PDEs or to any extreme event phenomena. As a matter of fact, ‘RW’ often does not refer to oceanic rogue waves. The focusing 1DNLSE plays a central role in this general question of RW [37,38,45,46,61].

In the following, we review recent works devoted to the emergence of *complex and coherent structures* in experimental systems well described by the focusing 1DNLSE in the focusing regime. We will focus on nonlinear optical waves systems and in particular on optical fibers. MI and RWs phenomena have been the topic of a large number of extensive books and reviews in the past few years [61–67]. The aim of this paper is to review different mechanisms leading to the emergence of temporally or spatially localized structures by adopting an original perspective. We here consider five general forms of initial conditions presented in Fig. 1 that have been studied experimentally and which illustrate two distinct phenomena that lead to the generation of such structures. On one side, harmonic, noisy and localized perturbations of the plane wave solution can trigger the MI processes while on another side, isolated coherent pulses and partially coherent fields might experience self-focusing dynamics. From a different viewpoint, noisy perturbations of plane wave and partially coherent fields may be gathered under the common designation of *random initial conditions* whereas localized perturbations of plane wave and coherent pulses can both be associated with *local mechanisms*. Then remain harmonic perturbations of the plane wave that are strongly linked to the formation of fundamental breathers and higher-order solutions of the 1DNLSE. We propose in this paper to bridge various works done on this topic covering all these scenarii.

In Section 2, we introduce the normalized 1DNLSE and the concept of MI. Seminal experiments on MI are described and we review the experimental work on breathers generation and on the observations that have been made over the last decade.

In Section 3, we describe the scenario where the initial plane wave background is influenced by a localized perturbation. We also describe dynamical features arising in the semiclassical limit of the 1DNLSE, when intense light pulses propagate in optical fibers. We briefly describe the universal scenario of the regularization of the gradient catastrophe that leads to the local formation of the Peregrine soliton.

In Section 4, we review recent works devoted to the emergence of RW when initial conditions are noisy. The phenomena arising from the nonlinear propagation of random waves in systems described by 1DNLSE correspond to the *integrable turbulence*. This terminology has been first introduced by Zakharov [68] and integrable turbulence is now a growing field of research both from the theoretical [68–75] and from the experimental [52,53,55,56,76,77] points of view. In Section 4, we compare two cases: the MI of a

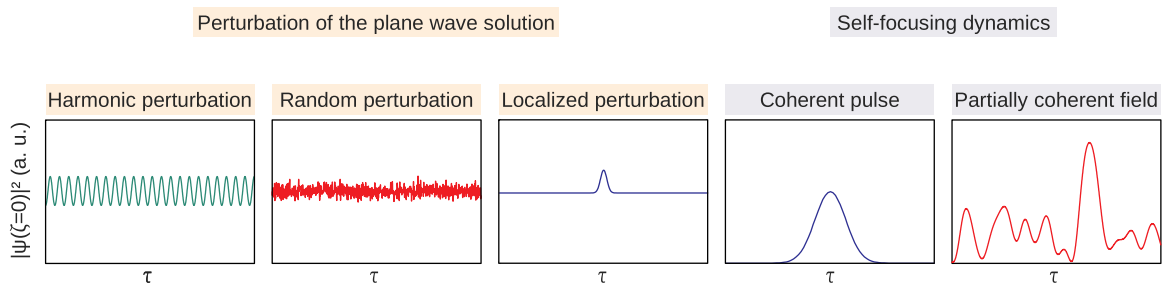


Fig. 1. Schematics of several initial conditions leading to the emergence of localized coherent structures in the framework of the 1DNLSE. The case of harmonic perturbations and the associated fundamental solutions are considered in Section 2 while random perturbations and partially coherent field dynamics are treated in Section 4. Experiments involving either localized perturbations of cw field or isolated coherent pulses are reviewed in Section 3. .

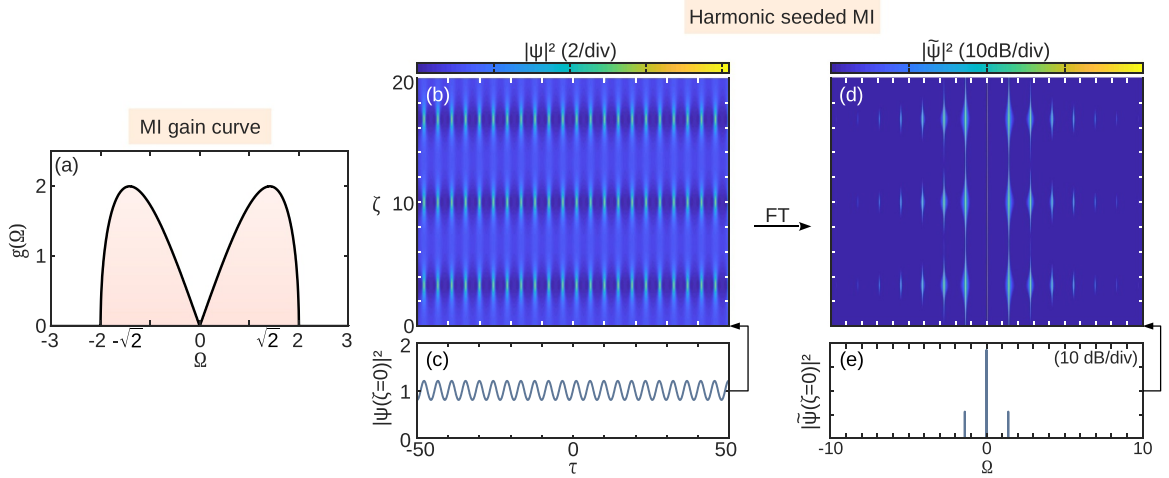


Fig. 2. Modulation instability gain and dynamical evolution. (a) MI gain curve $g(\Omega)$ calculated from the linear stability analysis of Eq. (1). (b) 2D color plot of the spatio-temporal evolution of $|\psi|^2$ where MI is initially triggered by the sine modulation of (c). Three recurrences are observed. (d, e) Same as (b, c) in the spectral domain. The initial spectrum (e) exhibits weak narrow peaks at $\Omega = \pm\sqrt{2}$ symmetric with respect to the pump at $\Omega = 0$.

plane wave initially perturbed by noise and the propagation of partially coherent waves having Gaussian statistics at initial stage.

2. 1DNLSE, Modulation instability and solutions of fundamental importance

2.1. The one-dimensional nonlinear Schrödinger equation and the modulation instability phenomenon

The one dimensional nonlinear Schrödinger equation describes the unidirectional propagation of the envelope of nonlinear waves under the slowly varying envelope approximation in a great variety of fields including optics. In particular, it is very well suitable for the description of experiments realized with single mode fibers, (*i.e.* fibers supporting a single transverse distribution of intensity). In its adimensional form, the 1DNLSE reads :

$$i \frac{\partial \psi}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 \psi}{\partial \tau^2} + |\psi|^2 \psi = 0 \tag{1}$$

where ζ and τ are the dimensionless distance and time respectively, and ψ is proportionnal to the slow varying amplitude of the electric field. In the present form, it assumes anomalous GVD which is necessary for the occurrence of MI of plane waves described by the integrable 1DNLSE. Equation (1) accurately encompasses the dynamics of nonlinear optical systems over a wide range of situations. However, a tremendous amount of studies have reported experiments where the integrable 1DNLSE fails to describe the behaviour of the system because of additional effects such as dissipation or higher order effects that influence the wave dynamics. In this review we focus on dynamical features that are well described by Eq. (1). Applying standard tools of linear stability analysis, it is found that the plane wave solutions of Eq. (1) are modulationally unstable with respect to harmonic perturbations. Accordingly, sinusoidal perturbations detuned from the pump frequency by a quantity Ω initially experience exponential amplification with growth rate $g(\Omega) = |\Omega|(\Omega_c - \Omega^2)^{1/2}$ with $\Omega_c = 2$. The curve $g(\Omega)$ represented in Fig. 2(a) takes the characteristic form of symmetric lobes around $\Omega = 0$ with a maximum at $\Omega = \pm\sqrt{2}$. To illustrate the type of dynamics that MI can trigger in the 1DNLSE we present in Fig. 2(b) the typical spatio-temporal evolution of $|\psi|^2$ (which is related to the intensity in optical experiments) obtained by numerically integrating Eq. (1) for an initial condition consisting of a CW field perturbed by a sine-modulation at the frequency of maximum MI gain ($\Omega = \sqrt{2}$) (see Fig. 2(c)). This scenario results in a dynamics exhibiting a recurrent behaviour where AB-like structures are formed periodically (3 times in this case). In the frequency domain, maximum compression points correspond to the broadest spectra as can be seen in the 2D color plot of Fig. 2(d).

2.2. Spectral signatures of noise-induced modulation instability in experiments

Modulation Instability is one of the most ubiquitous phenomena in the physics of nonlinear waves. Interestingly, various theoretical investigations of this phenomenon have been realised independently in the 60s in very distinct branches of nonlinear physics [16]. Among those is the field of electromagnetic waves where self-modulation has been studied not so long after the first demonstration of laser radiation [78]. Nearly 20 years later, MI of short pulses was identified in an optical fiber experiment via the recording of symmetric sidebands in the averaged optical spectrum and oscillations of the autocorrelation trace (see Fig. 3(a)) [79]. After this seminal paper, MI in fiber optics has been the subject of countless studies for many reasons including its role in the generation of supercontinuum [80,81], parametric amplifiers [82,83], trains of pulses [84] as well as for being supposedly a key mechanism in the

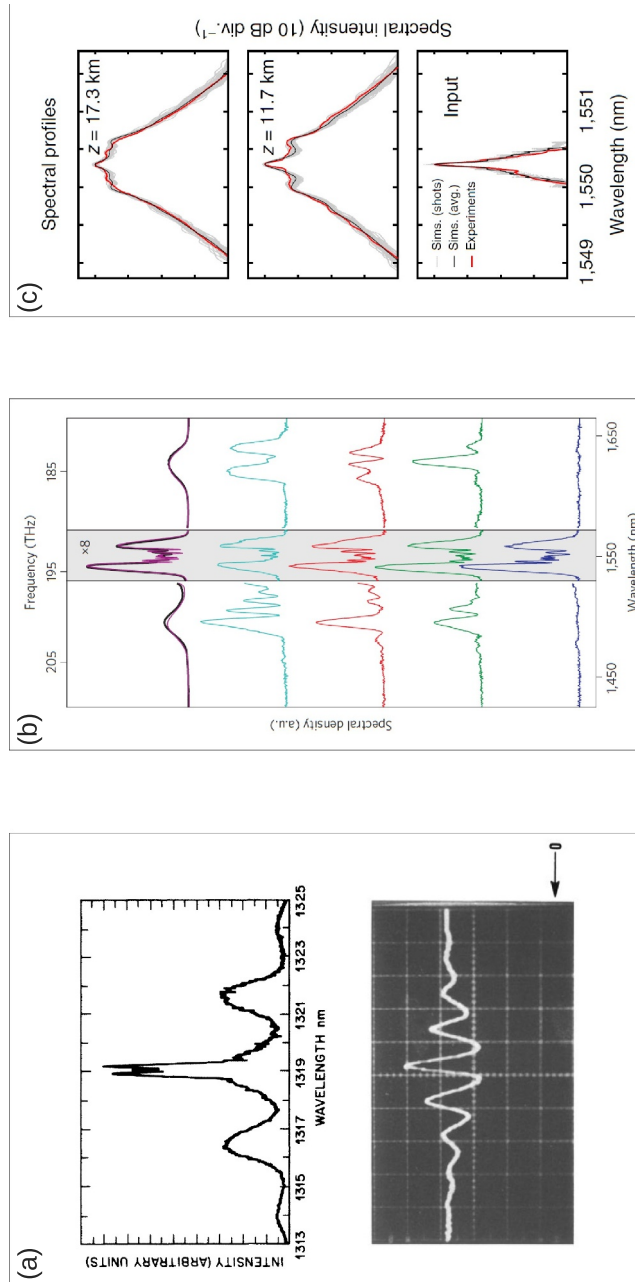


Fig. 3. Observations of Modulation Instability spectrum in fiber optics experiments. (a) Upper panel: average optical spectrum. Lower panel: Autocorrelation trace exhibiting symmetric oscillations with a period of 2.2 ps. Pulse duration was 100 ps. (b) Average (upper trace) and single-shot (all other traces) spectra acquired using the DFT method. Strong shot-to-shot fluctuations are recorded. Pulse duration was 3 ps. (c) Experimental and simulated spectra at different stages of the MI process. Phase modulation was applied to a cw laser to suppress Brillouin scattering. (a) is reproduced from Ref. [79], (b) from Ref. [85] and (c) from Ref. [54] with permission.

formation of optical rogue waves [39,40,61]. MI signatures in experiments have been limited to the average power spectrum until 2012 when *instantaneous* spectra of short picosecond pulses were first recorded via a frequency-to-time conversion technique [85]. Strong shot-to-shot fluctuations were observed that confirmed the stochastic nature of the MI process (see Fig. 3(b)). Quite recently, MI of cw laser in optical fiber has been investigated in real-time in the temporal domain thanks to a time-lens system and we show in Fig. 3(c) the average spectral recordings that were reported to complete this brief overview [54]. The two latter studies are described in more details in Section 4.2 that focusses on single-shot measurements of MI.

2.3. Breathers solutions and experiments

Apart from MI, a broad class of analytical solutions of the 1DNLSE referred to as solitons on finite background (SFB) or breathers have been identified [22–24]. Among those are the Akhmediev breather (AB, periodic in τ , localised in ζ), the Kuznetsov-Ma soliton (KM, localised in τ , periodic in ζ) or the Peregrine soliton (PS, localised both in τ and ζ). For instance, in the spatio-temporal diagram of Fig. 2(b) the evolution of the field up to $\zeta = 6$ resembles that of the AB. Accurate description at longer propagation distance has recently been obtained in the framework of finite gap theory [86]. Interestingly in the case of more realistic (noisy) initial conditions it appears that structures close to the first order analytical solutions (AB, KM soliton, PS) emerge locally. Also, the interplay between multiple breathers, the existence of higher order breather solutions and MI result in very rich dynamical scenarii [34,87].

Regarding breather solutions, although signatures of such structures can be observed in the randomly seeded MI process [34,88], their clear observation requires a careful design of the initial condition in experiment. This is done by tailoring a complex field to be launched at the input of the system that is as close as possible to the analytical form of the solution. In practice, a CW laser is used in conjunction with phase and intensity modulators to generate equally spaced comb lines in the spectrum (*i.e.* a periodic waveform in the temporal domain). Then a programmable optical filter can be used to address each of the frequency component's amplitude and phase. At the output of the fiber, several detection methods are used such as, optical spectrum analysis (OSA), optical sampling (OSO) or frequency resolved optical gating (FROG). Using this experimental protocol and other techniques, many studies have been realised showing the controlled generation of the PS [33,89–91] (see Fig. 9(b) and 6(b)), KM soliton [26] and AB [29,92]. An interesting representation of the latter lies in the phase space where they take the form of homoclinic loops also called separatrices [22,23,93,94]. For example, a system whose initial state lies upon the AB separatrix will follow a closed trajectory in the phase plane that returns to its initial state, as imposed by the localised character of the solution in ζ . However, any deviation from that ideal (unstable) trajectory push the system into periodic orbits associated to the well known Fermi-Pasta-Ulam-Tsingou recurrence [86,95–97]. The system then exhibits locally and periodically AB-like evolutions. The exact dynamics of the system is thus extremely dependent on the initial condition and especially on the relative phase between the injected Fourier modes. Recent experimental works have addressed this sensitivity to the initial condition in both fiber optic [98,99] and photorefractive crystal [100] platforms (see Fig. 4). By adjusting the phase detuning, the authors managed to set their system on states that are on one side or the other of the homoclinic loop. In the case of outer orbits - with respect to the separatrix - an apparent symmetry is broken characterised by the occurrence of out of phase breathers from one recurrence to the other as it has been observed similarly in hydrodynamical experiments [101].

Beyond the *simple* MI process triggered by a weak harmonic perturbation of a cw field, more complex spatio-temporal dynamics have been predicted in the case of low frequencies of the perturbation with respect to the MI typical frequency. In particular, weak modulations having their fundamental frequency together with their second harmonic falling within the MI gain bandwidth (see Fig. 2(a)) are known to lead to higher-order MI where deterministic pulse splitting sequences are generated [87,102].

Even though this review paper is limited to phenomena described by the scalar 1DNLSE which already exhibits an extremely rich phenomenology, we point out that the range of parameters accessible in experiments allows to investigate even more complex regimes of MI. For instance, the interplay between polarisation states of light is responsible for the so-called vectorial MI [103–112] whereas higher order dispersion effects might extend the scalar MI to the normal GVD regime [113,114]. The recirculation of the optical field in a resonant fiber loop has also been found to enrich the MI process [115–117] and still attracts considerable interest for its association to dissipative solitons and frequency comb generation. Also, Raman scattering often significantly impacts the nonlinear dynamics of fiber optic systems by breaking the integrability of the 1DNLSE. This effect can be modeled in various manners and plays a crucial role for instance in the generation of optical supercontinuum [19,27,118–120].

3. Local mechanisms

3.1. Local perturbation of a plane wave

The nonlinear stage of MI in the 1DNLSE with periodic boundary conditions is described in terms of a homoclinic structure [22,93] characterized by two qualitatively different families of doubly-periodic solutions, through which the perturbation is cyclically amplified and back-converted to the background. These two families are separated by the so-called Akhmediev breather, which represents the separatrix of the homoclinic structure, featuring a single cycle of conversion and back-conversion [98,100].

However the dynamics of MI is strikingly different when the initial constant background is not influenced by periodic perturbations but by localized perturbations. In this case, it has been shown very recently that the developpement of the coherent structures strongly depends on the soliton content of the perturbation which can be defined from the inverse scattering transform (IST) [121–125]. In Section 5, we briefly discuss the IST which represents a well-established technique for solving nonlinear partial differential equations of integrable nature such as the 1DNLSE or the Korteweg-de-Vries equation. IST is based on a transformation

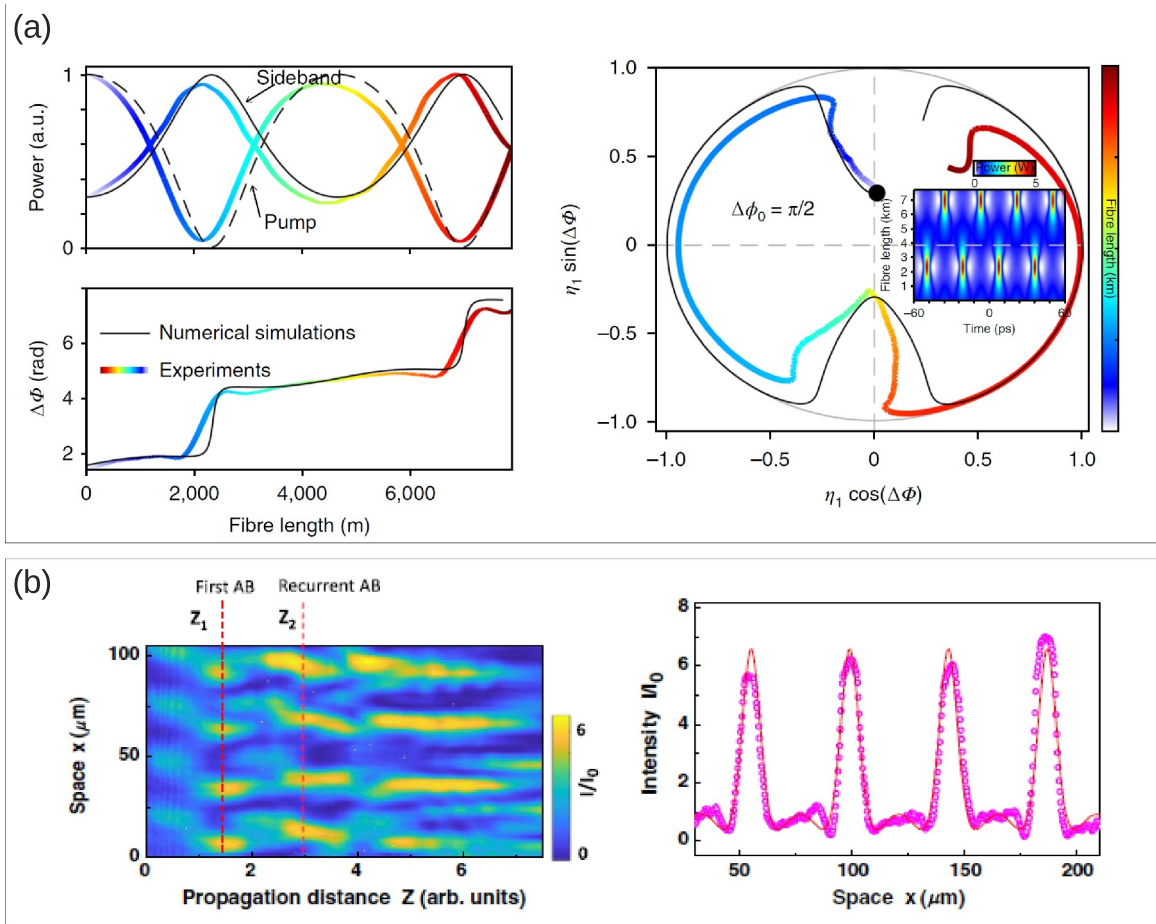


Fig. 4. Observations of Akhmediev breathers recurrences (a) Fiber optic experiment, longitudinal evolution of the power of the pump and signal in the MI band (upper left panel) and of their relative phase (lower left panel). Right panel, same data represented in the phase plane showing broken symmetry of the recurrence. (b) Experiment in a photorefractive crystal, evolution of an initial spatial modulation exhibiting 3 recurrences (left panel) and cross-section of the intensity at the first maximum compression point along with a fit of the AB solution (right panel). (a) is reproduced from Ref. [98] and (b) from Ref. [100] with permission.

between the space-time domain and a nonlinear spectral domain. The IST spectrum consists of two parts : a discrete part which corresponds to the solitons and a continuous part which corresponds to the dispersive waves.

- local perturbation without soliton content

Using the IST for the focusing 1DNLSE with non-zero background [126,127], it was shown in Ref. [124] that there exist generic perturbations of the constant background for which no discrete spectrum is present, and MI is mediated by the continuous spectrum of the associated scattering problem. The nonlinear dynamics of MI is characterized by a long time ‘hyperbolic’ scenario, where a universal (*i.e.* not depending on the shape of the initial localized perturbation at leading order) nonlinear oscillatory structure develops and expands in time with finite speed, giving rise to a characteristic spatio-temporal wedge. In sharp contrast with MI scenarios involving the formation of breathers, this scenario involves the formation of a symmetric expanding nonlinear wave structure described by the modulated elliptic solution of the 1DNLSE [123]. The modulation provides a gradual transition from a fundamental soliton (realized only asymptotically as a local approximation of the solution) resting at the center to small-amplitude dispersive waves propagating away from the center with linear group velocity. This universal modulated elliptic solution of the 1DNLSE saturating the MI was first obtained in Ref. [125] in the framework of the Whitham modulation theory [128].

The asymptotic behavior described in Ref. [124,125] has been shown to represent a rather general feature of modulationally unstable media. Indeed the fact that an initially-localized perturbation evolves into an expanding nonlinear oscillatory structure represents a universal feature that also arises in a broad class of wave systems governed by nonlinear evolution equations more complex than the 1DNLSE [130]. The question of MI induced by localized perturbations has been investigated in recent optical fiber experiments reported in Ref. [129]. Using a recirculating fiber loop as experimental platform, the space-time evolution of bright and dark localized perturbations has been observed (see Fig. 5(a)). Experimental results have demonstrated the robustness to noise and to small dissipation of the expanding modulated solution theoretically found in Refs. [123–125]. Note that the observed oscillatory structures are closely linked to other oscillating structures observed from an evolving step in power [131,132].

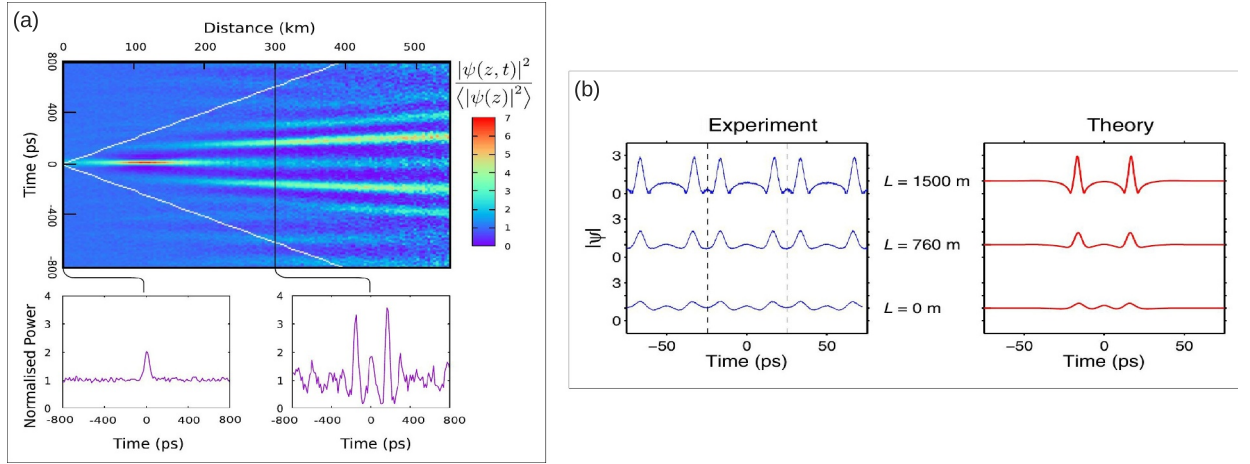


Fig. 5. Observations of the propagation of localised perturbations and superregular breathers in fiber optics experiments. (a) Experimental results showing the space-time evolution of a modulationally unstable plane wave perturbed at the initial stage by a localised bright (positive) peak. (b) Superregular breather solutions of the 1DNLSE observed in optical fiber experiments. (a) is reproduced from Ref. [129] and (b) from Ref. [31] with permission from the authors.

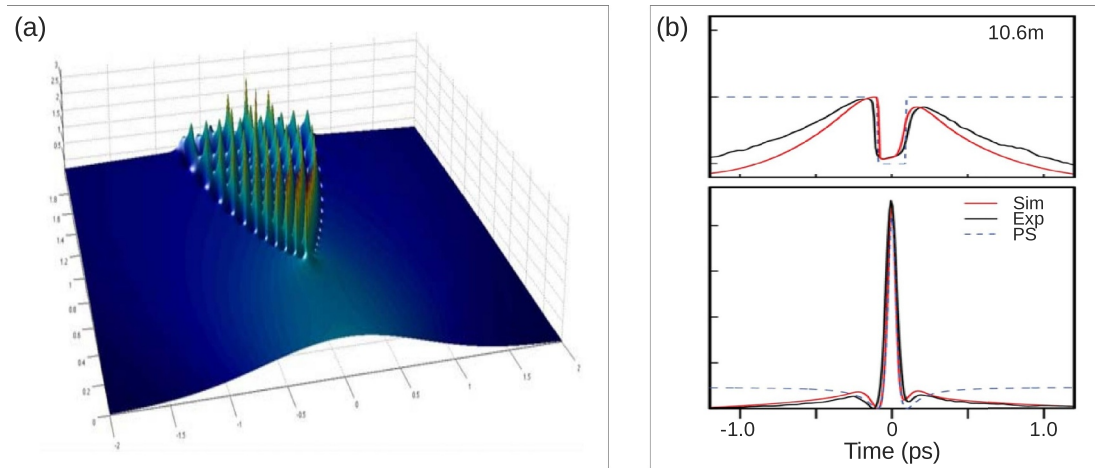


Fig. 6. Self-focusing evolution of pulses showing nonlinear localisation of power. (a) Numerical simulation in the semi-classical (zero dispersion) limit of the focusing 1DNLSE. Regularization of the local gradient catastrophe occurs via the emergence of a PS. (b) Optical experiment showing the local emergence of the PS as a result of the self-focusing propagation of a light pulse inside a fiber (a) is reproduced from Ref. [32] and (b) from Ref. [91] with permission from the authors.

- local perturbation with soliton content

In another approach considered for the description of localized perturbations of the plane background, MI was described in terms of particular pairs of breathers with opposite velocities, termed superregular breathers [69,122,133], which may eventually superpose at initial time in such a way to yield a small oscillating perturbation of the plane wave background. These superregular breathers, which have a pure discrete spectrum and no continuous spectrum in the 1DNLSE associated scattering problem, have been observed in optical and hydrodynamical experiments reported in Ref. [31] (see Fig. 5(b)). It has been recently shown that breathing solitons on finite background that are associated with the discrete IST spectrum may coexist with the expanding nonlinear wave structure driven by the continuous IST spectrum [121]. From this perspective, 1DNLSE simulations have revealed various scenarios that involve the expanding nonlinear wave structure developing in a characteristic wedge together with breathers which lie inside or outside the wedge [121,134]. Note that localized wave structures termed breather wave molecules have been also recently observed in Ref. [135].

3.2. Pulse propagation in the strongly nonlinear regime : Local emergence of Peregrine soliton

As discussed in Section 2, it is well known that the Peregrine soliton (PS) is a breather solution of the 1DNLSE that may appear in the development of plane wave MI. However mathematical studies have shown that the PS in fact appears more generally during the

nonlinear localisation of high power pulses in the semi-classical (zero dispersion) limit of the focusing 1DNLSE (see Fig. 6(a)) [32].

When high power light pulses propagate inside an optical fiber in a regime where light propagation is described by the focusing 1DNLSE, the initial stage of the pulse evolution is dominated by nonlinear effects and linear dispersive effects play a negligible role. This regime where self-phase modulation drives the pulse dynamics is characterized by a self-steepening of the phase and amplitude profiles that goes together with a strong broadening of the Fourier spectrum of the pulse. The gradual steepening leads to the formation of a gradient catastrophe when the derivatives of the phase and amplitude profiles go to infinity. At this point, dispersive effects come into play and the gradient catastrophe is regularized via the local generation of breathers.

It has been shown in Ref. [32] that the mechanism of dispersive regularization of a gradient catastrophe leads in a universal way to the formation of a PS. This has been confirmed by optical fiber experiments reported in Ref. [91] (see Fig. 6(b)). Careful measurements of the phase and power profiles of a light pulse propagating inside an optical fiber have shown that nonlinear temporal compression in the 1DNLSE yields to the local emergence of the PS.

4. Random initial conditions

4.1. Statistical properties : Noise-driven modulation instability versus partially coherent waves

Integrable turbulence refers to the complex spatio-temporal dynamics arising when a field having some initial random properties propagates in a system described by an integrable PDE [52,53,55,56,68,69,73–77,136–138]. In integrable systems, Fermi-Pasta-Ulam recurrences are expected [95,98,100,139]. It is therefore somehow surprising that the long-term evolution of integrable turbulence is characterized by the existence of a steady state for the statistical properties of the waves. Note that the existence of this “steady statistical state” has been demonstrated in numerical simulations, *i.e.* at finite propagation distance [52,71,76,138]. At long propagation distance, the integrability is always broken in experiments or in numerical simulations. Up to now, the existence of this stationary state has not been rigorously proved and is still an open question in pure Mathematics. However, it is important to note that the stationary states found in numerical simulations are robust and do not depend on the numerical integration scheme used for example.

A striking feature of integrable turbulence is the dependence of these stationary statistical properties on the statistical properties of the initial conditions. Two different initial conditions have been mainly studied in the focusing regime of 1DNLSE: (i) Plane wave perturbed by small noise and (ii) partially coherent waves having initially a Gaussian statistics. The typical spatio-temporal evolution and the PDF of the power $|\psi|^2$ of the stationary state are represented in Fig. 7. Note that the MI of cnoidal waves has also been investigated in Ref. [72].

The first situation (i) is commonly called the *noise-driven MI*. It may be observed in natural environment when a quasi-monochromatic wave packet is slightly perturbed by noise. *The main remarkable result is that the single point statistics of the field ψ is Gaussian at long-propagation distance* [71]. This corresponds to an exponential distribution of the power $|\psi|^2$ (see Fig 7(c)).

The second case (ii) corresponds to the nonlinear propagation of a field initially made of the linear superposition of numerous independent waves with random phases (*i.e.* independent Fourier components) [52,53,55,56,76,140]. Because of the central limit theorem, the single-point statistics of the field ψ is Gaussian. This kind of field is commonly found in natural conditions when the nonlinearity is initially weak. For example, in optics, partially coherent waves are found in spatial speckles [141,142], in some strongly multimode lasers [52] or in spontaneous emission light [53,56]. *The main remarkable result is that the single-point statistics of the field ψ becomes heavy tailed (non Gaussian) at long-propagation distance* [52,76] (see Fig 7(f)).

There is an apparent paradox in the two cases (noise-driven MI and partially coherent waves): at long-distance of propagation, noisy plane waves lead to Gaussian statistics whereas Gaussian statistics lead to heavy-tailed PDF. Actually, these results hold only for single-point statistics. The field found in the long-term evolution of the noise-driven MI is *not* a Gaussian process because the multiple points statistics are not Gaussian. This has been recently demonstrated theoretically and experimentally by using the autocorrelation of the power (the $g^{(2)}(\tau) = \langle P(t)P(t+\tau) \rangle / \langle P(t) \rangle^2$ function) [143]. In the noise-driven MI, the autocorrelation of the power exhibits remarkable oscillations around unity whereas $g^{(2)}(\tau) \geq 0$ for any thermal (partially coherent) light.

The two cases described here have been recently studied in optical experiments and in particular in optical fibers systems (see Secs. 4.2 and 4.3).

4.2. Single-shot observations of modulation instability in optical fiber experiments

Optical fibers represent a favorable tabletop laboratory to investigate the Physics of the focusing 1DNLSE. In particular, MI has been widely studied the last decades in this context [19,79]. However, it is only recently that the single-shot observation of MI spontaneously emerging from noise has been reported.

Shot-to-shot optical spectra have been first recorded using the so-called dispersive Fourier transform (DFT) (see Fig. 3(b)) [85]. In this work, the authors investigate MI induced by the nonlinear propagation of picosecond pulses in optical fibers. In this regime, the spectral width of the pump laser (\sim THz) is narrower than the typical spectral width of the MI gain (\sim 10 THz). As a consequence, the MI process is driven by the natural noise. In these pulsed experiments, correlations among spectral components are investigated. These experiments demonstrate in particular anti-bunched arrangement in both the spectral and temporal domains.

In experiments reported in Ref. [85], access to shot-to-shot spectra was enabled because the pump pulses were temporally isolated. Note that in these experiments based on DFT, the temporal dynamics of the pulses could not be recorded. In CW regime, the optical spectrum is generally only defined as an averaged quantity (the spectral power density) and DFT cannot be used. However,

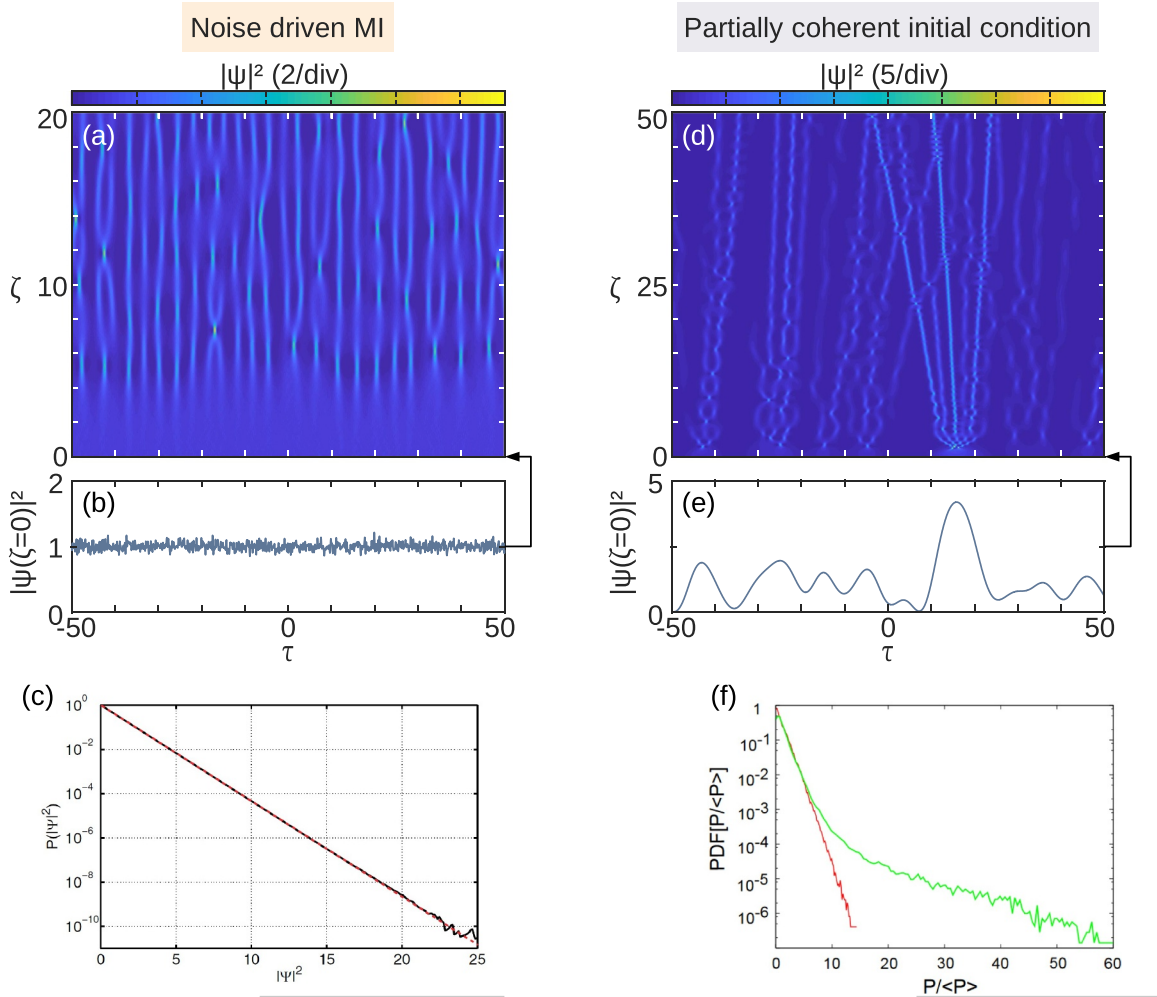


Fig. 7. Comparison between noise seeded modulation instability and partially coherent initial field. (a, b) Typical 2D color plots of the spatio-temporal evolution of $|\psi|^2$ when the initial condition is either a slightly perturbed cw field (b) or a strongly modulated, partially coherent one (e). (c, f) Respective probability density functions extracted from literature. In (c) the solid black line refers to numerical simulations and the dashed red line to exponential statistics. In (f) the red line refers to the input field (exponential statistics) and the green line to the output field (heavy-tailed). (c) is reproduced from Ref. [71] and (f) from Ref. [52] with permission. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the single-shot observation of light having picosecond time scales can be achieved by using temporal imaging techniques [144–147]. Although temporal imaging techniques have been demonstrated thirty years ago, it is only very recently that they have been applied to the study of nonlinear random waves [53,54,56]. Two experimental results based on temporal-imaging techniques have been published in 2016: one devoted to partially coherent waves [53] and one devoted to the noise-driven MI [54].

In [54], the authors investigate the MI from the propagation of a CW field in optical fibers using the so-called time-lens technique. The ultrafast dynamics of MI (with picosecond time scale) is recorded in single-shot at the output of the optical fiber thanks to a commercial time-lens [54]. The dynamics of the optical power is recorded within 25–50 ps-long temporal windows. This enabled the first experimental study of the precise shape of the coherent structures emerging in MI driven by noise together with their statistics (see Fig. 8(a)). In these experiments, the averaged power launched into the optical fiber is $\langle P \rangle \sim 1$ W corresponding to MI time scales of 5 ps.

Very recently, another strategy has been used to investigate noise-driven MI in optical fibers [143]. Low optical power ~ 40 mW corresponding to MI time scales of 100 ps are used. In this low power regime, the optical power dynamics can be therefore straightforwardly recorded using ultrafast photodiodes and oscilloscope. In order to compensate the large value of the nonlinear length $L_{NL} \sim 25$ km, a recirculating fiber loop is used. The losses are compensated by Raman amplification and the signal is recorded at each round trip. This work provides two awaited observations. First, the spatio-temporal dynamics similar to the one displayed in Fig. 7(a) is recorded for the first time in optical fiber experiments (see Fig. 8(b)). Secondly, the experimental observations confirm the remarkable statistical properties that have been predicted using numerical simulations in Ref. [71]: the probability density function (PDF) of the optical power at long distance of propagation is exponential, thus corresponding to a Gaussian statistics of the field and

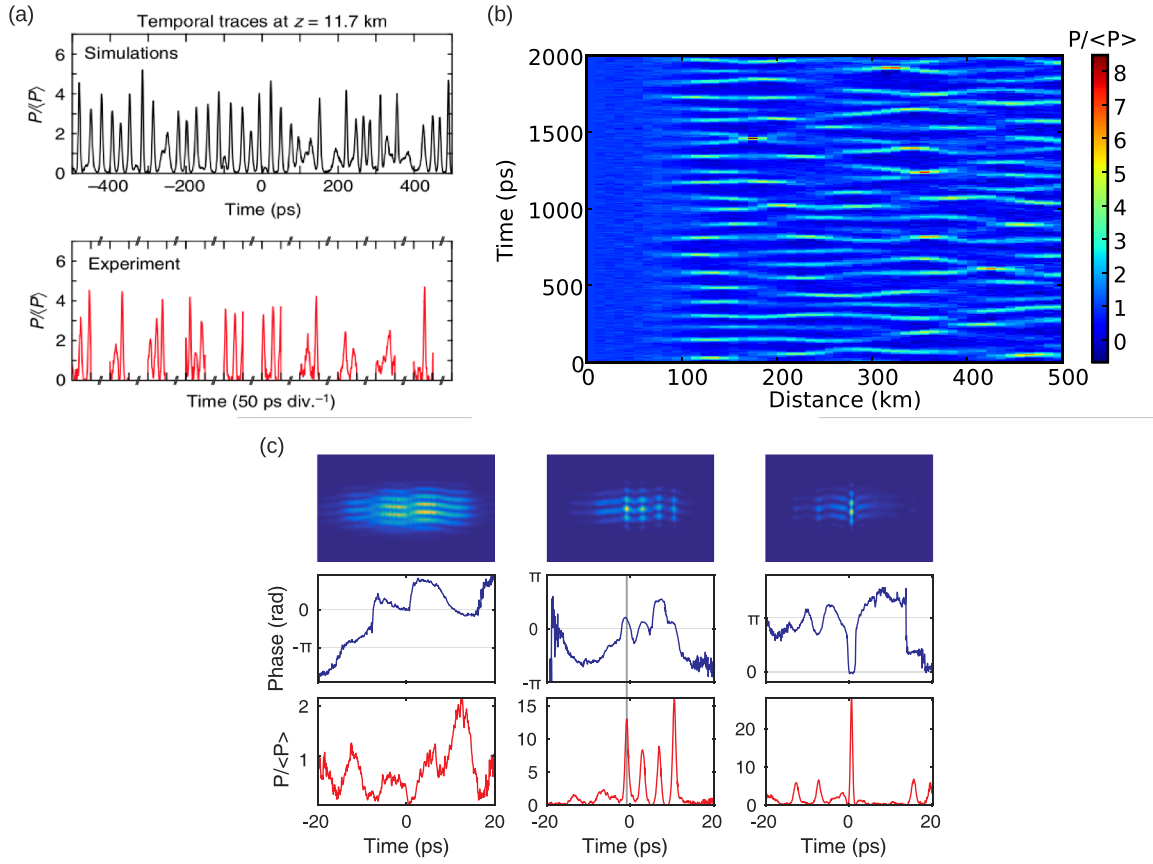


Fig. 8. Ultrafast single-shot measurements of nonlinear random waves in optical fibers. (a) Modulation instability driven by noise recorded using a time lens system. lower panel: typical dynamics of the optical power observed at the output of the fiber. (b) Spatio-temporal evolution of the intensity of noise-driven modulation instability experimentally recorded through the use of a recirculating fiber loop with a loss compensation scheme. (c) Partially coherent waves recorded using heterodyne time microscope. Left column: initial random light (filtered spontaneous emission). Central and right columns: typical phase and intensity dynamics observed at the output of the fiber. In the third column, note the coherent structures similar locally to the Peregrine soliton (with the typical π -jump of the phase) embedded in the random fluctuations. The first row represents the raw data recorded by the camera in the heterodyne time microscope. (a) is reproduced from Ref. [54], (b) from Ref. [143] and (c) from Ref. [56] with permission.

the transient evolution of the statistics is characterized by damped oscillations. Moreover, as explained above, the autocorrelation of optical power ($g^{(2)}(\tau)$) exhibits remarkable oscillations that characterize successive bunched and anti-bunched arrangements in the temporal domain. The autocorrelation of power thus allows to distinguish the linear superposition of waves (providing purely Gaussian process) from the nonlinear stage of MI (Gaussian single-point statistics of the field but oscillatory structures of $g^{(2)}(\tau)$ around unity).

The fundamental question of MI has first been raised by surface water gravity waves experiments [14,15]. Nowadays, it still represents an active field of experimental research in optics [54,85,143,148] and in hydrodynamics [149]. However, the statistical properties of the noise-driven MI are not yet theoretically understood. The very recent single-shot observations performed by using time lens [54] or recirculating fiber loop [143] have started a new era in the study of the noise-driven modulation instability.

4.3. Single-shot observations of partially coherent waves in optical fiber experiments

The emergence of heavy-tailed statistics of uni-directional wave packets in the presence of third order nonlinearity have been demonstrated in water tank experiments reported for example in Ref. [150]. In hydrodynamics, the surface elevation dynamics is easily measured with accuracy. In optics involving visible or near infrared light, the measurement of the wave envelope can be technically challenging.

In experiments performed in the spatial domain, the profile of the beam is simply recorded with cameras. One-dimensional nonlinear speckles have been investigated in “spatial” experiments [141]. At leading order, these experiments are described by 1DNLSE where the transverse coordinate x replace the time τ in Eq. (1). The transverse profile is measured using a simple camera. In these experiments, a speckle is first produced by a laser beam going through a random diffuser and then propagates in a nonlinear

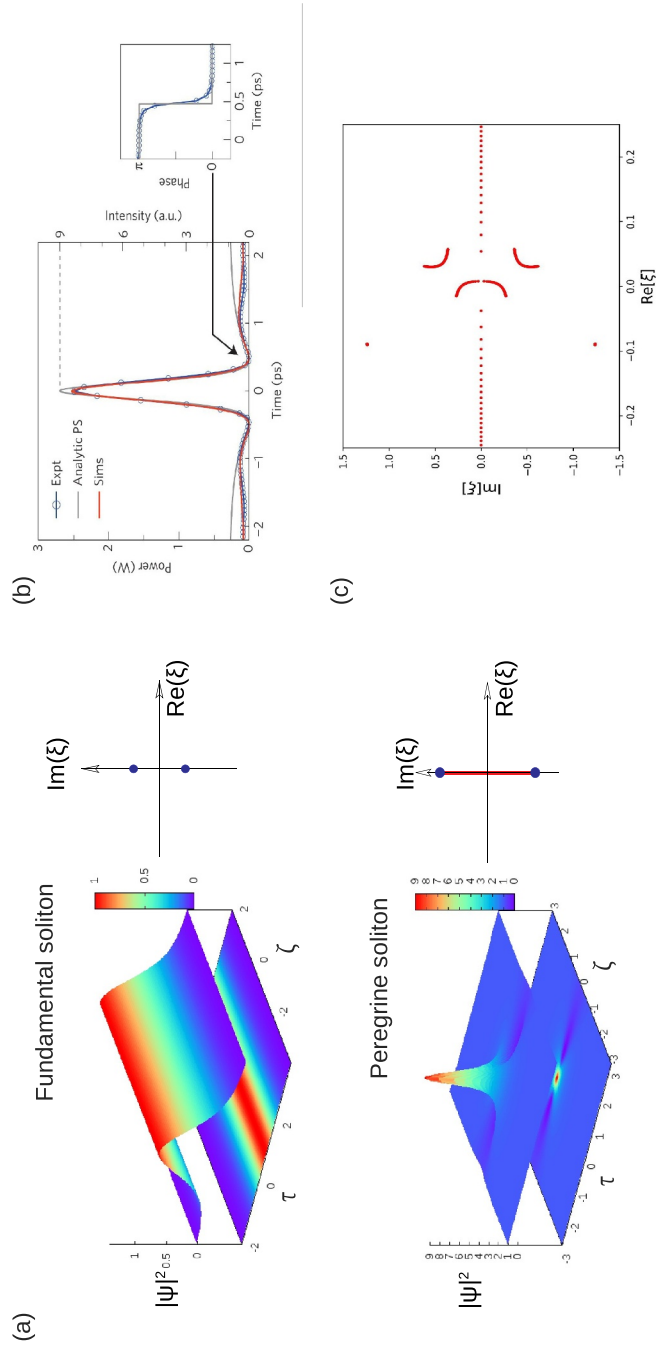


Fig. 9. Analytical results from the IST theory. (a) Spatio-temporal evolution (left) and spectral portraits (right) of the fundamental soliton (upper panel) and the Peregrine soliton (lower panel). The red lines represent branchcuts and the blue points represent complex conjugate double points. (b) Power and phase profiles of the Peregrine soliton recorded in optical fiber experiments. (c) Spectral portrait of the Peregrine soliton experimentally observed in Ref. [33], see Ref. [159] for details. (a) are reproduced from Ref. [160], (b) from Ref. [33] and (c) from Ref. [159] with permission.

focusing $Al_{0.18}Ga_{0.82}As$ planar waveguide. These experiments reveal heavy-tailed statistics of the optical power. The context of this work is Hanbury Brown and Twiss interferometry but the results also contribute to the field of integrable turbulence (random initial conditions launched into a system described by 1DNLSE) even if this terminology is not mentioned explicitly. Note that in [141], the authors also investigate defocusing (thermal) nonlinearity.

Let us emphasize that there is a conceptual difference between these spatial experiments and temporal experiments performed in single-mode fibers. These spatial experiments correspond to the nonlinear diffraction of a speckle having a finite extension. Theoretically, this corresponds to *zero-boundary conditions* (BCs). By using zero BCs, the 'standard' IST can be used to describe the experiments [142]. One of the major consequence of zero BC is the separation of solitons having different (transverse) velocities at long distance of propagation and the separation of (linear) dispersive waves from solitons. This must be distinguished from *non-zero boundary conditions* such as periodic BCs. In this case, solitons and dispersive waves interact for ever.

In single-mode fiber experiments (such as the noise-driven experiments described in Section 4.2), the physical time plays the role of the space whereas the propagation distance is the evolution variable (see Section 2). The duration of CW experiments must not be confused with the window of observation (of time lens system for example). CW experiments can be arbitrary long and can be switched on for several minutes for example. As a consequence, in a similar way to periodic BCs with large window in simulations, the solitons cannot separate. The theoretical description of IST with periodic BCs enters within the framework of finite-gap theory [9,20,151]).

Fundamental experiments devoted to partially coherent waves have been recently performed in single-mode fibers [52,53,55,56]. In these experiments performed in the temporal domain, the initial conditions are equivalent to the speckle in spatial experiments. The light launched into single-mode fibers is typically emitted by a spontaneous emission source which is filtered and amplified. As explained in the Section 4.1, the statistics of the electric field emitted by such a source is Gaussian because the light field is composed of a linear superposition of numerous independent Fourier components with random phases [152,153]. This corresponds to an exponential distribution of the optical power.

In optical fiber experiments with partially coherent waves, the typical timescale is similar to the one of MI, *i.e.* the picosecond (for an average power ~ 1 Watt at $\lambda \sim 1550$ nm in standard single mode fiber SMF-28). As for the MI process, breakthrough in the study of partially coherent waves has been recently achieved owing to the development ultrafast measurement techniques. First The PDF of the optical power has been accurately measured using asynchronous optical sampling method in 2015 [52]. This measurement technique is based on the sum frequency generation obtained from the interaction between the studied random light and femtosecond pulses. As expected in the focusing regime, the measured statistics is heavy-tailed at the output of the fiber in the focusing regime. The dynamics of the optical power of partially coherent waves has then been recorded using a new kind of time microscope [53]. In this temporal imaging device, the time is encoded onto the horizontal axis of a camera. By using a monochromatic reference in the time microscope, the phase of the ultrafast nonlinear random waves has also been measured (see Fig. 8(c) and Ref. [56] for details). In this 'heterodyne' time microscope (and its variant SEAHORSE), the time is encoded onto the horizontal axis of a camera whereas the phase is encoded onto the vertical axis (position of fringes of interferences, see upper panels in Fig. 8(c)).

Note that two relevant parameters influence the MI process: the optical power of the monochromatic wave and the amplitude of the noise. In experiments with partially coherent waves launched into a given fiber, the two controlled parameters are (i) the average power and (ii) the timescale of the input light. The wave system is strongly nonlinear when the initial time scale is large compared to the time scale of solitons (semiclassical limit of 1DNLSE). In the experiments of Fig. 8(c), the time scale of light fluctuations at the input of the fiber was ~ 10 ps while the time scale characterizing coherent structures observed at the output of the fiber was typically 1 ps. These experiments have revealed an important mechanism underlying the emergence of rogue waves in nonlinear random waves. The analysis of the data revealed that a structure very similar to the Peregrine soliton emerges locally from some of the large random humps as shown in the right panels of Fig. 8(c) (and Refs. [53,56]). Note the signature of the π -jump of the phase at times where optical power falls to zero. Numerical simulations performed with the experimental parameters also reveal this phenomenon. This emergence of the Peregrine soliton embedded in random waves is driven by the regularization of gradient catastrophe rigorously demonstrated in the framework of the zero-dispersion limit of 1DNLSE with one isolated pulse (see Section 3.2 and Ref. [32]). Recently, it has been demonstrated numerically that this mechanism corresponds maximum deviation of the statistics from Gaussianity [154].

5. Conclusion and perspectives

We have reviewed recent studies devoted to the physical phenomena described by the 1DNLSE and observed in optical fiber experiments. Despite the fact that the MI phenomenon has been studied for more than five decades, the different scenarii of self-focusing phenomena continues to attract a significant amount of experimental and theoretical researches. While the so-called modulation instability has been discovered in the 60s [15], several new dynamical mechanisms of fundamental importance have been identified only recently. In particular, the regularization of gradient catastrophe plays a crucial role for large or powerful pulses propagating in optical fibers [32,91]. Moreover, the modulation instability induced by a local perturbation may be ruled by different mechanisms, depending on the soliton content of the perturbation, and is the subject of active theoretical and experimental researches [121,122,124,129,133,134,155].

Each type of these mechanisms leads to the emergence of coherent structures. Many of these coherent structures, in particular the breathers solutions of 1DNLSE, are considered as prototypes of rogue waves, *i.e.* localized extreme events emerging along the propagation. The emergence of these structures has been extensively studied and this topic still represents a subject of active research. The advances in ultrafast measurement [53,54,56] and the development of recirculating fiber loop [129,143] have recently open new

exciting perspectives of research: now, phase and amplitude of subpicosecond dynamics can be recorded in single-shot and the spatiotemporal dynamics along the propagation can also be recorded in single-shot.

One of the most challenging open questions regarding the physics 1DNLSE is the understanding of the statistical properties of nonlinear random waves. In particular, there is no existing theory describing the propagation of random waves in integrable systems. Nowadays it is understood that the question of noise-driven MI enters within the framework of integrable turbulence [68]. Integrable turbulence designates an incoherent nonlinear wave ensemble (including solitons, small-amplitude dispersive waves) in an integrable system. The determination of the statistical properties of this ensemble is one of the long-standing problems in nonlinear wave dynamics [69,76]. The question of the nonlinear evolution of statistical ensembles of random waves having large fluctuations at initial time also enters within the framework of integrable turbulence. Experiments and numerical simulations have revealed that heavy- (resp. low-) tailed deviations from the initial statistics are observed in focusing (resp. defocusing) regime [76]. It is now clear that the nonlinear evolution of such random waves at short time can be approached within the semiclassical description of the 1DNLSE [77,137].

The natural theoretical tool to investigate integrable turbulence is the inverse scattering transform (IST). The IST is a well-established technique for solving nonlinear partial differential equations of integrable nature such as the 1DNLSE or the Korteweg-de Vries equation. The method of IST, developed for 1DNLSE by Zakharov and Shabat in Ref. [156], is based on a transformation between the spacetime domain and a nonlinear spectral domain, in which the nonlinear evolution of some given input field reduces to a linear problem. The first step of the IST method is the so-called Direct Scattering Transform (DST), which gives the scattering data of a given initial field. The scattering data consists of the spectrum itself and a set of 'scattering coefficients'. Crucially, the spectrum is preserved during the nonlinear evolution of the wave system while the evolution of the remaining scattering data is found to be particularly simple. The evolved scattering data are then transformed back to the physical spacetime domain by solving a (linear) inverse problem. The whole procedure can be viewed as a nonlinear counterpart of the classical Fourier transform method for solving linear partial differential equations. Thus, the DST can be regarded as a nonlinear Fourier transform that provides a spectral signature of the nonlinear modes composing the field under consideration (see Ref. [157,158] for some recent applications of this concept in the nonlinear optics context).

If the input field is periodic in space and/or in time [20], the traditional IST method does not apply and its modification, called the Finite Gap Theory (FGT), represents the appropriate framework for analyzing the content of the field in terms of nonlinear modes [17,20,86]. These nonlinear modes are characterized by Zakharov-Shabat spectra made of finite numbers of bands representing segments in the complex spectral plane. Within the FGT, the multi-phase NLSE solutions are characterized by a genus, calculated as $N - 1$, where N is the number of spectral bands. Physically, the genus characterizes the number of degrees of freedom (*i.e.* the number of fundamental oscillatory modes, or phases) within the nonlinear periodic or quasi-periodic solution for the envelope of the plane wave [17]. Mathematically, the genus of a finite-band NLSE solution represents the genus of the hyperelliptic Riemann surface, on which this solution is defined in terms of theta functions [17]. From the viewpoint of the FGT, the plane wave solution of the NLSE is classified as a regular genus 0 solution while the fundamental soliton represents a degenerate genus 1 solution so that its FGT spectrum consists of two complex conjugate double spectral points (degenerate bands), the counterparts of the discrete spectrum in the Zakharov-Shabat scattering problem with decaying potentials. The standard solitons on finite background (AB, KM soliton, and PS) all are degenerate genus 2 solutions so that their spectral portraits are made of a single vertical band in the spectral plane, complemented by two complex conjugate double points (that can be viewed as degenerate bands) [8,136]. The spectral portraits of the fundamental soliton and of the PS that are determined from the resolution of the Zakharov-Shabat problem are shown in Fig. 9(a).

Recently, numerical tools of nonlinear spectral analysis have been applied to optical signals recorded in fiber experiments. For example, it has been shown that the PS observed in optical fiber experiments does not represent a degenerate genus 2 solution of the 1DNLSE but a more complex non-degenerate genus 4 solution [159]. The implementation of the IST analysis of experimental data has been made possible thanks to the single shot measurement of the phase and amplitude of the optical field. The use of IST as a tool of data analysis will probably become essential in future experimental studies of 1DNLSE Physics in optical fibers. From a practical point of view, the IST method has been shown to represent a promising approach to overcoming transmission limitations in fiber communication channels by encoding information in the nonlinear IST spectrum [158,161–165]. Recently, tools of nonlinear spectral analysis have also been used to investigate some properties of various kinds of optical wave fields [159,166,167].

The theoretical description of the statistical features typifying nonlinear random waves in (quasi-) integrable systems at long evolution time still represents an open question both from the physical and mathematical point of view. One of the important perspective of research is the use of the concept of soliton gases [168]. Inspired by this approach and based on IST theory, a model of the asymptotic stage of the noise-induced MI based on N -soliton solutions of the integrable focusing 1DNLSE has been recently proposed. These N -soliton solutions are bound states of strongly interacting solitons having a specific distribution of the IST eigenvalues together with random phases [169].

Finally, it is important to note that the studies reviewed above are devoted to the Physics of the integrable 1DNLSE. Of course, experiments are never exactly integrable and can only approach integrability. As a consequence, the study of the influence of high order effects and of losses is of fundamental importance. In optical fibers, both linear effects (high order dispersion, linear dissipation) and nonlinear effects (stimulated Brillouin and Raman scattering...) may strongly influence the dynamics. While all these effects have been widely investigated in various contexts [19], *e.g.* supercontinuum generation [80], their influence, as perturbations of integrable turbulence, is essentially unexplored and represents an important question for the future investigations of this field of research.

Declarations of Competing Interest

None

Acknowledgments

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