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$\mathscr{L}$-splines as diffusive limits of dissipative kinetic models

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Abstract Dissipative kinetic models inspired by neutron transport are studied in a $(1+1)$-dimensional context: first, in the two-stream approximation, then in the general case of continuous velocities. Both are known to relax, in the diffusive scaling, toward a damped heat equation. Accordingly, it is shown that "uniformly accurate" $\mathscr{L}$-splines discretizations of this parabolic asymptotic equation emerge from well-balanced schemes involving scattering S-matrices for the kinetic models. Numerical tests confirm these theoretical findings.

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1 Introduction

1.1 Diffusive limit of isotropic scattering with adsorption

Consider a kinetic model, where $\sigma(x) > 0$ is the opacity and $\kappa(x)$ the damping,

$$\partial_t f + v\partial_x f = \sigma(x)\left(c(x)\int_{-1}^{1} f(t,x,v')\frac{\mathrm{d}v'}{2} - f\right), \qquad 0 < c := 1-\kappa < 1, \quad (1.1)$$

for $x \in \mathbb{R}$ and $v \in (-1,1)$. It is well known that, within a convenient rescaling of variables, it relaxes towards the damped heat equation,

$$\partial_t \rho + \sigma(x)\kappa(x)\rho = \partial_x\left(\frac{\partial_x \rho}{3\sigma(x)}\right), \qquad \text{or } = \frac{\partial_{xx}\rho}{3\sigma}, \text{ if } \sigma \text{ is a constant.} \quad (1.2)$$

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More precisely, let $0 < \varepsilon \ll 1$, with $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$, so that

$$\varepsilon \partial_t f^\varepsilon + v \partial_x f^\varepsilon = \frac{\sigma}{\varepsilon}\Big((1-\varepsilon^2\kappa)\langle f^\varepsilon\rangle - f^\varepsilon\Big), \qquad \langle f^\varepsilon\rangle := \int_{-1}^{1} f^\varepsilon(t,x,v')\frac{\mathrm{d}v'}{2}. \quad (1.3)$$

in which each power of ε can be balanced. The first three equations read,

$$\sigma(f_0 - \langle f_0\rangle) = 0, \quad (1.4)$$
$$v\partial_x f_0 + \sigma(f_1 - \langle f_1\rangle) = 0, \quad (1.5)$$
$$\partial_t f_0 + v\partial_x f_1 + \sigma(f_2 - \langle f_2\rangle) + \sigma\kappa\langle f_0\rangle = 0. \quad (1.6)$$

The operator $T = Id - \langle\cdot\rangle$ is a self-adjoint Fredholm operator on $L^2(-1,1)$, hence (1.4) means that $f_0 \in \mathrm{Ker}(T)$, so that $f_0 = f_0(t,x)$ is independent of v. Accordingly, (1.5) rewrites as an integral equation for $f_1(t,x,v)$,

$$T f_1 = -\frac{v}{\sigma}\partial_x f_0, \qquad -\frac{v}{\sigma}\partial_x f_0 \in \mathrm{Im}(T).$$

By the Fredholm alternative, $\mathrm{Im}(T) = \mathrm{Ker}(T)^\perp$, so f_1 exists because

$$\forall a(t,x) \in \mathrm{Ker}(T), \qquad \int_{-1}^{1} \frac{a(t,x)\,v}{\sigma(x)}\partial_x f_0\,\mathrm{d}v = \frac{a(t,x)\,\partial_x f_0}{\sigma(x)}\langle v\rangle = 0.$$

The solution f_1 is defined up to any element of $\mathrm{Ker}(T)$,

$$f_1(t,x,v) = -\frac{v}{\sigma(x)}\partial_x f_0 + C(t,x).$$

Yet, (1.6) rewrites as an integral equation for f_2, too:

$$\sigma(f_2 - \langle f_2\rangle) = -\partial_t f_0 - v\partial_x f_1 - \sigma\kappa\langle f_0\rangle.$$

Again, by the Fredholm alternative, a solution exists if its right-hand side belongs to $\mathrm{Ker}(T)^\perp$, hence is L^2-orthogonal to functions independent of v. This yields a compatibility condition for both the unknowns f_0, f_1,

$$\langle \partial_t f_0 + v\partial_x f_1 + \sigma\kappa f_0\rangle = 0,$$

which brings (1.2) with $f_0 = \rho$, because

$$\langle v\partial_x f_1\rangle = -\frac{1}{2}\int_{-1}^{1} v\partial_x\left(\frac{v\partial_x f_0 + C(t,x)}{\sigma(x)}\right)\mathrm{d}v = -\frac{1}{3}\partial_x\left(\frac{\partial_x f_0}{\sigma(x)}\right).$$

1.2 $\mathscr{L}$-splines emerging from kinetic well-balanced schemes

The type of numerical schemes we consider in order to approximate solutions of (1.1), or its simpler version (2.1) are well-balanced and heavily rely on discrete-ordinates along with so–called "S-matrices", as advocated in [7,12,15,16] and [11, Part II]. In a nutshell, given a (symmetric) quadrature rule on the velocity variable, ($K = 1$ corresponds to a two-stream approximation)

$$v_k : \ k \in \{-K, \dots, -1, 1, \dots, K\}, \quad 0 < v_1 < v_2 < \dots < v_K, \quad v_{-k} = -v_k,$$

we denote

$$\mathscr{V} = (v_1, \dots, v_K)^\top \in \mathbb{R}^K, \qquad \mathbb{V} := \mathrm{diag}(v_1, \dots, v_K, v_1, \dots, v_K) \in \mathscr{M}_{2K}(\mathbb{R}).$$

Corresponding weights $(\omega_k)_{k=1,\dots,K}$ may be given by a Gauss quadrature, so

$$\int_V \phi(v)dv \text{ is approximated by } \sum_{k=1}^{K} \omega_k(\phi(v_k) + \phi(-v_k)).$$

Since (1.1), (1.3) are linear, a methodology for deriving well-balanced schemes was given in [11, Part II] (see also [16]). It consists in locally solving steady-state equations of (1.1), say in the interval $x \in (x_{j-1}, x_j)$, where $x_j = j\Delta x$, with boundary conditions, $f(x_{j-1}, |v|), f(x_j, -|v|)$. These "incoming data" induce "outgoing fluxes", and both can be related by means of a so-called "scattering S-matrix", denoted $\mathscr{S}^{\varepsilon}_{j-\frac{1}{2}}$ and there holds:

$$\begin{pmatrix} \bar{f}_{j-\frac{1}{2}}(\mathscr{V}) \\ \bar{f}_{j-\frac{1}{2}}(-\mathscr{V}) \end{pmatrix} := \begin{pmatrix} f(x_j, \mathscr{V}) \\ f(x_{j-1}, -\mathscr{V}) \end{pmatrix} = \mathscr{S}^{\varepsilon}_{j-\frac{1}{2}} \begin{pmatrix} f_{j-1}(\mathscr{V}) \\ f_j(-\mathscr{V}) \end{pmatrix}. \tag{1.7}$$

Given a time-step $\Delta t > 0$, a time-marching scheme reads in standard notation,

$$\begin{aligned} \begin{pmatrix} f_j^{n+1}(\mathscr{V}) \\ f_{j-1}^{n+1}(-\mathscr{V}) \end{pmatrix} &= \begin{pmatrix} f_j^{n}(\mathscr{V}) \\ f_{j-1}^{n}(-\mathscr{V}) \end{pmatrix} - \frac{\Delta t}{\varepsilon \Delta x} \mathbb{V} \begin{pmatrix} f_j(\mathscr{V}) - \bar{f}_{j-\frac{1}{2}}(\mathscr{V}) \\ f_{j-1}(-\mathscr{V}) - \bar{f}_{j-\frac{1}{2}}(-\mathscr{V}) \end{pmatrix} \\ &= \begin{pmatrix} f_j^{n}(\mathscr{V}) \\ f_{j-1}^{n}(-\mathscr{V}) \end{pmatrix} (1 - \mathbb{V}\frac{\Delta t}{\varepsilon \Delta x}) + \mathbb{V}\frac{\Delta t}{\varepsilon \Delta x} \mathscr{S}^{\varepsilon}_{j-\frac{1}{2}} \begin{pmatrix} f_{j-1}(\mathscr{V}) \\ f_j(-\mathscr{V}) \end{pmatrix}. \end{aligned} \tag{1.8}$$

Yet, to handle the (possibly stiff) diffusive scaling (1.3) where ε can be significantly smaller than Δx (Asymptotic-Preserving property), the IMEX strategy consists in treating implicitly the (stiff) terms in $\frac{1}{\varepsilon}$ appearing in (1.8). Accordingly, assume the following decomposition of the S-matrix holds for any $\varepsilon > 0$,

$$\boxed{\mathscr{S}^{\varepsilon}_{j-\frac{1}{2}} = \mathscr{S}^{0}_{j-\frac{1}{2}} + \varepsilon \mathscr{S}^{1,\varepsilon}_{j-\frac{1}{2}},} \tag{1.9}$$

as originally proposed in [16], then (1.8)–(1.7) becomes

$$\begin{aligned} &\begin{pmatrix} f_j^{n+1}(\mathscr{V}) \\ f_{j-1}^{n+1}(-\mathscr{V}) \end{pmatrix} + \frac{\Delta t}{\varepsilon \Delta x} \mathbb{V} \left[\begin{pmatrix} f_j^{n+1}(\mathscr{V}) \\ f_{j-1}^{n+1}(-\mathscr{V}) \end{pmatrix} - \mathscr{S}^{0}_{j-\frac{1}{2}} \begin{pmatrix} f_{j-1}^{n+1}(\mathscr{V}) \\ f_{j}^{n+1}(-\mathscr{V}) \end{pmatrix} \right] \\ &\qquad = \begin{pmatrix} f_j^{n}(\mathscr{V}) \\ f_{j-1}^{n}(-\mathscr{V}) \end{pmatrix} + \frac{\Delta t}{\Delta x} \mathbb{V} \mathscr{S}^{1,\varepsilon}_{j-\frac{1}{2}} \begin{pmatrix} f_{j-1}^{n}(\mathscr{V}) \\ f_{j}^{n}(-\mathscr{V}) \end{pmatrix}. \end{aligned} \tag{1.10}$$

The corresponding scheme on macroscopic densities is recovered by means of,

$$\forall (j,n) \in \mathbb{Z} \times \mathbb{N}, \qquad \rho_j^n := \sum_{k=1}^{K} \omega_k (f_j^n(v_k) + f_j^n(-v_k)), \tag{1.11}$$

and is expected to be consistent with (1.2) up to $O(\varepsilon)$ terms. Indeed, it will be proved that, for both the two-stream approximation (2.1) and the continuous velocities model (1.1), the asymptotic discretization emerging from the limiting process is actually the "$\mathscr{L}$-splines" scheme derived in [13, Appendix] (for $f(x) \equiv 0$), which is "uniformly accurate" at steady-state, see *e.g.* [2,20]: its error estimates are uniform with respect to the viscosity parameter.

1.3 Plan of the paper

The paper proceeds by increasing difficulty: in §2, the two-stream approximation corresponding to $K = 1$ is studied. Steady-states are of exponential shape and the S-matrix is derived. Follows the well-balanced scheme for which various stability properties hold, see Lemma 2.2. The diffusive limit is handled in the next section, and the asymptotic-preserving property is proved in Proposition 2.1. Continuous velocities are treated in §3, which follows the same roadmap, despite more involved calculations. The expression of the S-matrix is taken from the book [11, §9.1–2], so that the diffusive rescaling is studied in §3.1, and the aforementioned decomposition is given in Proposition 3.1. The IMEX scheme (3.16) follows and it is checked that it is consistent with the $\mathscr{L}$-spline approximation (3.22) of the damped heat equation (1.2). Various numerical tests, involving both constant and position-varying parameters, are performed in §4 and conclusive remarks are given in §5.

2 Two-stream approximation

The simplest occurrence of (1.1) is the 1D "damped Goldstein-Taylor" model,

$$\begin{aligned} \partial_t f^\pm \pm \partial_x f^\pm &= \sigma \left(\frac{1-\kappa}{2}(f^+ + f^-) - f^\pm \right), \qquad \kappa \in (0,1), \\ &= \sigma \left(\frac{1-\kappa}{2} f^\mp - \frac{1+\kappa}{2} f^\pm \right). \end{aligned} \tag{2.1}$$

Its diffusive limit is easily obtained by defining, for $0 < \varepsilon \ll 1$,

$$\rho = f^+ + f^-, \qquad J = \frac{f^+ - f^-}{\varepsilon},$$

so that the rescaled version of (2.1),

$$\varepsilon \partial_t f^\pm \pm \partial_x f^\pm = \mp \frac{\sigma}{2\varepsilon}(f^+ - f^-) - \frac{\kappa \sigma \varepsilon}{2}(f^+ + f^-),$$

rewrites as a dissipative 2×2 macroscopic system,

$$\partial_t\rho+\partial_x J+\sigma\kappa\rho=0,\qquad \varepsilon^2\partial_t J+\partial_x\rho=-\sigma J,$$

and by passing formally to the limit $\varepsilon\to 0$, it comes

$$\partial_t\rho-\partial_x\left(\frac{\partial_x\rho}{\sigma}\right)+\sigma\kappa\rho=0. \tag{2.2}$$

2.1 S-matrix and well-balanced scheme

Consider the "locally frozen" stationary equations of (2.1),

$$\pm\partial_x f^{\pm}=\sigma_{j-\frac{1}{2}}\left(\frac{1-\kappa_{j-\frac{1}{2}}}{2}(f^{+}+f^{-})-f^{\pm}\right),\qquad x\in(x_{j-1},x_j),$$

where $x_j=j\Delta x$, $j\in\mathbb{Z}$, and $\sigma_{j-\frac{1}{2}},\kappa_{j-\frac{1}{2}}$ are either local averages or pointwise values of positive parameters $\sigma(x),\kappa(x)$. With no loss of generality, and for ease of reading, we carry all the computations with $\sigma_{j-\frac{1}{2}}\equiv 1$ because the general case can be deduced by locally changing variables,

$$\forall j\in\mathbb{Z},\qquad \Delta x\mapsto\sigma_{j-\frac{1}{2}}\Delta x. \tag{2.3}$$

Thanks to the simplicity of the model (2.1), we have:

Lemma 2.1 *Any steady-state of (2.1) is linear combination of "damped modes",*

$$f^{\pm}(x)=\frac{\exp(\pm x\sqrt{\kappa})}{1\pm\sqrt{\kappa}},\qquad \kappa>0.$$

Proof See Appendix A

Having at hand the steady-states of the two-stream model, the corresponding 2×2 S-matrix easily follows [12] (or [11, Chap. 9]), by $S_\kappa=\tilde{M}M^{-1}$, with

$$M=\begin{pmatrix}\frac{1}{1-\sqrt{\kappa}} & \frac{\exp(-\Delta x\sqrt{\kappa})}{1+\sqrt{\kappa}}\\ \frac{\exp(-\Delta x\sqrt{\kappa})}{1+\sqrt{\kappa}} & \frac{1}{1-\sqrt{\kappa}}\end{pmatrix},\qquad \tilde{M}=\begin{pmatrix}\frac{\exp(-\Delta x\sqrt{\kappa})}{1-\sqrt{\kappa}} & \frac{1}{1+\sqrt{\kappa}}\\ \frac{1}{1+\sqrt{\kappa}} & \frac{\exp(-\Delta x\sqrt{\kappa})}{1-\sqrt{\kappa}}\end{pmatrix},$$

so that,

$$
\begin{aligned}
S_\kappa &= \tilde{M} M^{-1} \\
&= \begin{pmatrix} \frac{\exp(-\Delta x\sqrt{\kappa})}{1-\sqrt{\kappa}} & \frac{1}{1+\sqrt{\kappa}} \\ \frac{1}{1+\sqrt{\kappa}} & \frac{\exp(-\Delta x\sqrt{\kappa})}{1-\sqrt{\kappa}} \end{pmatrix} \\
&\quad \times \frac{(1-\kappa)^2}{(1+\sqrt{\kappa})^2 - (1-\sqrt{\kappa})^2 \exp(-2\Delta x\sqrt{\kappa})} \begin{pmatrix} \frac{1}{1-\sqrt{\kappa}} & -\frac{\exp(-\Delta x\sqrt{\kappa})}{1+\sqrt{\kappa}} \\ -\frac{\exp(-\Delta x\sqrt{\kappa})}{1+\sqrt{\kappa}} & \frac{1}{1-\sqrt{\kappa}} \end{pmatrix} \\
&= \frac{\exp(\Delta x\sqrt{\kappa})(1-\kappa)^2/2}{(1+\kappa)\sinh(\Delta x\sqrt{\kappa}) + 2\sqrt{\kappa}\cosh(\Delta x\sqrt{\kappa})} \\
&\quad \times \begin{pmatrix} \frac{\exp(-\Delta x\sqrt{\kappa})}{1-\sqrt{\kappa}} & \frac{1}{1+\sqrt{\kappa}} \\ \frac{1}{1+\sqrt{\kappa}} & \frac{\exp(-\Delta x\sqrt{\kappa})}{1-\sqrt{\kappa}} \end{pmatrix} \begin{pmatrix} \frac{1}{1-\sqrt{\kappa}} & -\frac{\exp(-\Delta x\sqrt{\kappa})}{1+\sqrt{\kappa}} \\ -\frac{\exp(-\Delta x\sqrt{\kappa})}{1+\sqrt{\kappa}} & \frac{1}{1-\sqrt{\kappa}} \end{pmatrix} \\
&= \frac{(1-\kappa)^2/2}{(1+\kappa)\sinh(\Delta x\sqrt{\kappa}) + 2\sqrt{\kappa}\cosh(\Delta x\sqrt{\kappa})} \\
&\quad \times \begin{pmatrix} \frac{1}{(1-\sqrt{\kappa})^2} - \frac{1}{(1+\sqrt{\kappa})^2} & \frac{\exp(\Delta x\sqrt{\kappa})-\exp(-\Delta x\sqrt{\kappa})}{1-\kappa} \\ \frac{\exp(\Delta x\sqrt{\kappa})-\exp(-\Delta x\sqrt{\kappa})}{1-\kappa} & \frac{1}{(1-\sqrt{\kappa})^2} - \frac{1}{(1+\sqrt{\kappa})^2} \end{pmatrix},
\end{aligned}
$$

which means that,

$$
\boxed{S_\kappa = \frac{\begin{pmatrix} 2\sqrt{\kappa} & (1-\kappa)\sinh(\Delta x\sqrt{\kappa}) \\ (1-\kappa)\sinh(\Delta x\sqrt{\kappa}) & 2\sqrt{\kappa} \end{pmatrix}}{2\sqrt{\kappa}\cosh(\Delta x\sqrt{\kappa}) + (1+\kappa)\sinh(\Delta x\sqrt{\kappa})}.}
$$

Accordingly, with the preceding notations, $K = 1$ and

$$
v_{\pm 1} = \pm 1, \qquad \omega_1 = 1, \qquad \mathscr{V} = \{1\}, \qquad \mathbb{V} = \mathrm{Id}_{\mathbb{R}^2}. \tag{2.4}
$$

The scheme (1.8) rewrites simply,

$$
\begin{pmatrix} f_j^{+,n+1} \\ f_{j-1}^{-,n+1} \end{pmatrix} = \begin{pmatrix} f_j^{+,n} \\ f_{j-1}^{-,n} \end{pmatrix} (1 - \frac{\Delta t}{\Delta x}) + \frac{\Delta t}{\Delta x} S_\kappa \begin{pmatrix} f_{j-1}^{+,n+1} \\ f_j^{-,n+1} \end{pmatrix}. \tag{2.5}
$$

Lemma 2.2 *Under the CFL condition $\Delta t \leq \Delta x$, the scheme (2.5) preserves positivity, dissipates L^1 and L^∞ norms and is consistent with (2.1) as $\Delta x \to 0$.*

Remark 2.1 Using (2.3) in (2.5), one immediately deduces the scheme for the system (2.1) endowed with a varying opacity $\sigma(x)$: it suffices to change $\Delta x \mapsto \sigma_{j-\frac{1}{2}} \Delta x$, so as to get a j-dependent S-matrix $(S_\kappa)_{j-\frac{1}{2}}$, involving x-dependent values $\cosh(\sigma_{j-\frac{1}{2}} \Delta x \sqrt{\kappa_{j-\frac{1}{2}}})$, $\sinh(\sigma_{j-\frac{1}{2}} \Delta x \sqrt{\kappa_{j-\frac{1}{2}}})$.

Proof In order to check consistency with (2.1) when $0<\kappa<1$ and $\Delta x\to 0$, S_κ is split between the 2×2 identity matrix and a $O(\Delta x)$ remainder:

$$\begin{aligned}S_\kappa &= \begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{1}{2\sqrt{\kappa}\cosh(\Delta x\sqrt{\kappa})+(1+\kappa)\sinh(\Delta x\sqrt{\kappa})}\times\Bigg[\\ &\qquad\begin{pmatrix}2\sqrt{\kappa}(1-\cosh(\Delta x\sqrt{\kappa}))&0\\0&2\sqrt{\kappa}(1-\cosh(\Delta x\sqrt{\kappa}))\end{pmatrix}\\ &\qquad+\sinh(\Delta x\sqrt{\kappa})\begin{pmatrix}-(1+\kappa)&1-\kappa\\1-\kappa&-(1+\kappa)\end{pmatrix}\Bigg]\\ &=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\Delta x}{2+\Delta x(1+\kappa)}\begin{pmatrix}-(1+\kappa)&(1-\kappa)\\(1-\kappa)&-(1+\kappa)\end{pmatrix}+O(\kappa^2).\end{aligned}$$

When $\kappa\to 0$, the S-matrix of Goldstein-Taylor model [14] must be recovered:

$$S_\kappa\simeq\frac{1/\sqrt{\kappa}}{2+\Delta x}\begin{pmatrix}2\sqrt{\kappa}&(1-\kappa)\Delta x\sqrt{\kappa}\\(1-\kappa)\Delta x\sqrt{\kappa}&2\sqrt{\kappa}\end{pmatrix}\to\frac{1}{2+\Delta x}\begin{pmatrix}2&\Delta x\\\Delta x&2\end{pmatrix}.$$

Moreover, it is clear that the entries of S_κ are positive, and both its rows and columns add to a number smaller than one. Accordingly, the well-balanced scheme based on S_κ preserves positivity, along with L^1 and L^∞ bounds.

2.2 Diffusive limit and asymptotic-preserving scheme

In order to study the diffusive limit, it is customary to rescale,

$$\kappa\to\varepsilon^2\kappa,\qquad \Delta x\to\Delta x/\varepsilon,\qquad (\text{or } \sigma_{j-\frac{1}{2}}\Delta x\to\sigma_{j-\frac{1}{2}}\Delta x/\varepsilon),$$

so that, the corresponding S_κ^ε rewrites

$$\begin{aligned}S_\kappa^\varepsilon &= \frac{1}{(1+\varepsilon^2\kappa)\sinh(\Delta x\sqrt{\kappa})+2\varepsilon\sqrt{\kappa}\cosh(\Delta x\sqrt{\kappa})}\\ &\qquad\times\begin{pmatrix}2\varepsilon\sqrt{\kappa}&(1-\varepsilon^2\kappa)\sinh(\Delta x\sqrt{\kappa})\\(1-\varepsilon^2\kappa)\sinh(\Delta x\sqrt{\kappa})&2\varepsilon\sqrt{\kappa}\end{pmatrix}.\end{aligned}$$

For values $0\le\varepsilon\ll 1$, the S-matrix decomposes like (1.9),

$$\begin{aligned}S_\kappa^\varepsilon &= \begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{2\varepsilon\sqrt{\kappa}}{(1+\varepsilon^2\kappa)\sinh(\Delta x\sqrt{\kappa})+2\varepsilon\sqrt{\kappa}\cosh(\Delta x\sqrt{\kappa})}\times\Bigg[\\ &\qquad\begin{pmatrix}1&-\cosh(\Delta x\sqrt{\kappa})\\-\cosh(\Delta x\sqrt{\kappa})&1\end{pmatrix}\\ &\qquad-\varepsilon\sqrt{\kappa}\begin{pmatrix}0&\sinh(\Delta x\sqrt{\kappa})\\\sinh(\Delta x\sqrt{\kappa})&0\end{pmatrix}\Bigg]\\ &=\begin{pmatrix}0&1\\1&0\end{pmatrix}+\frac{2\varepsilon\sqrt{\kappa}}{\sinh(\Delta x\sqrt{\kappa})}\begin{pmatrix}1&-\cosh(\Delta x\sqrt{\kappa})\\-\cosh(\Delta x\sqrt{\kappa})&1\end{pmatrix}+O(\varepsilon^2),\end{aligned}\tag{2.6}$$

so that, when $\varepsilon\to 0$, the "$\mathscr{L}$-splines scheme" proposed in [13, §A] is recovered.

Proposition 2.1 *For (2.1), the IMEX discretization (1.10) simplifies into*

$$\begin{pmatrix} f_j^{+,n+1} \\ f_{j-1}^{-,n+1} \end{pmatrix} + \frac{\Delta t}{\varepsilon \Delta x} \begin{pmatrix} f_j^{+,n+1} - f_j^{-,n+1} \\ f_{j-1}^{-,n+1} - f_{j-1}^{+,n+1} \end{pmatrix} = \begin{pmatrix} f_j^{+,n} \\ f_{j-1}^{-,n} \end{pmatrix} + \frac{\Delta t}{\Delta x} \mathscr{S}^{1,\varepsilon}_{j-\frac{1}{2}} \begin{pmatrix} f_{j-1}^{+,n} \\ f_j^{-,n} \end{pmatrix}, \tag{2.7}$$

where $\mathscr{S}^{1,\varepsilon}_{j-\frac{1}{2}}$ is deduced from (2.6). Accordingly, in the limit $\varepsilon \to 0$, the macroscopic density $\rho_j^n = f_j^{+,n} + f_j^{-,n}$ evolves like,

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t \sqrt{\kappa}}{\Delta x \sinh(\sqrt{\kappa}\Delta x)} \Big(\rho_{j+1}^n - 2\cosh(\sqrt{\kappa}\Delta x)\rho_j^n + \rho_{j-1}^n \Big). \tag{2.8}$$

Proof Thanks to the decomposition (2.6) and (2.4), the general expression (1.10) yields (2.7). In order to check consistency with the asymptotic diffusion equation (2.2), it suffices to shift indexes in the second equation in (2.7),

$$\begin{aligned} \begin{pmatrix} f_j^{+,n+1} \\ f_j^{-,n+1} \end{pmatrix} + \frac{\Delta t}{\varepsilon \Delta x} \begin{pmatrix} f_j^{+,n+1} - f_j^{-,n+1} \\ f_j^{-,n+1} - f_j^{+,n+1} \end{pmatrix} &= \begin{pmatrix} f_j^{+,n} \\ f_j^{-,n} \end{pmatrix} \\ &+ \frac{2\Delta t}{\Delta x} \begin{pmatrix} \frac{\sqrt{\kappa}}{\sinh(\Delta x\sqrt{\kappa})} \big(f_{j-1}^{+,n} - \cosh(\Delta x \sqrt{\kappa}) f_j^{-,n} \big) \\ \frac{\sqrt{\kappa}}{\sinh(\Delta x\sqrt{\kappa})} \big(-\cosh(\Delta x \sqrt{\kappa}) f_j^{+,n} + f_{j+1}^{-,n} \big) \end{pmatrix} + O(\varepsilon), \end{aligned}$$

before adding, so stiff relaxation terms simplify. The asymptotic discretization (2.8) follows, because the (stiff) Maxwellian penalization terms produce,

$$\forall j, n \in \mathbb{Z} \times \mathbb{N}, \qquad f_j^{\pm,n} = \frac{\rho_j^n}{2} + O(\varepsilon),$$

One may follow Remark 2.1 and extend (2.7) to a general case where both parameters σ, κ are position-dependent. The asymptotic scheme (2.8) was previously derived by means of "$\mathscr{L}$-spline interpolation" in, *e.g.*, [13, Appendix]; it is "uniformly accurate" in the sense of [2] (see also [20] and [21, Chap. 9]).

2.3 Rigorous estimates for constant parameters

For constant, positive σ, κ, the scheme (2.7) rewrites as

$$\begin{aligned} \begin{pmatrix} 1 + \frac{\Delta t}{\varepsilon \Delta x} & -\frac{\Delta t}{\varepsilon \Delta x} \\ -\frac{\Delta t}{\varepsilon \Delta x} & 1 + \frac{\Delta t}{\varepsilon \Delta x} \end{pmatrix} \begin{pmatrix} f_j^{+,n+1} \\ f_j^{-,n+1} \end{pmatrix} &= \begin{pmatrix} f_j^{+,n} \\ f_j^{-,n} \end{pmatrix} + \\ &\frac{2\Delta t \sqrt{\kappa}}{\Delta x [(1+\varepsilon^2\kappa) s_\kappa + 2\varepsilon\sqrt{\kappa} c_\kappa]} \begin{pmatrix} f_{j-1}^{+,n} - (c_\kappa + \varepsilon \frac{s_\kappa}{\sqrt{\kappa}}) f_j^{-,n} \\ -(c_\kappa + \varepsilon \frac{s_\kappa}{\sqrt{\kappa}}) f_j^{+,n} + f_{j+1}^{-,n} \end{pmatrix}, \end{aligned}$$

with $c_\kappa = \cosh(\Delta x \sqrt{\kappa})$, $s_\kappa = \sinh(\Delta x \sqrt{\kappa})$, so that

$$\begin{aligned} \begin{pmatrix} f_j^{+,n+1} \\ f_j^{-,n+1} \end{pmatrix} = \frac{1}{1 + \frac{2\Delta t}{\varepsilon \Delta x}} \begin{pmatrix} 1 + \frac{\Delta t}{\varepsilon \Delta x} & \frac{\Delta t}{\varepsilon \Delta x} \\ \frac{\Delta t}{\varepsilon \Delta x} & 1 + \frac{\Delta t}{\varepsilon \Delta x} \end{pmatrix} \Bigg[\begin{pmatrix} f_j^{+,n} \\ f_j^{-,n} \end{pmatrix} + \\ \frac{2\Delta t \sqrt{\kappa}}{\Delta x [(1+\varepsilon^2\kappa) s_\kappa + 2\varepsilon\sqrt{\kappa} c_\kappa]} \begin{pmatrix} f_{j-1}^{+,n} - (c_\kappa + \varepsilon \frac{s_\kappa}{\sqrt{\kappa}}) f_j^{-,n} \\ -(c_\kappa + \varepsilon \frac{s_\kappa}{\sqrt{\kappa}}) f_j^{+,n} + f_{j+1}^{-,n} \end{pmatrix} \Bigg], \end{aligned}$$

and one can follow the roadmap of [14, §3] in order to produce rigorous bounds. Accordingly, to invert the implicit part of the IMEX scheme, let

$$a = 1 + \frac{\Delta t}{\varepsilon \Delta x}, b = \frac{\Delta t}{\varepsilon \Delta x}, c = \frac{2\Delta t\sqrt{\kappa}}{\Delta x[(1+\varepsilon^2\kappa)s_\kappa + 2\varepsilon\sqrt{\kappa}c_\kappa]}, d = c(c_\kappa + \varepsilon\frac{s_\kappa}{\sqrt{\kappa}}),$$

and it comes,

$$\begin{aligned}(a+b)f_j^{+,n+1} &= (a-bd)f_j^{+,n} + (b-ad)f_j^{-,n} + acf_{j-1}^{+,n} + bcf_{j+1}^{-,n},\\ (a+b)f_j^{-,n+1} &= (b-ad)f_j^{+,n} + (a-bd)f_j^{-,n} + bcf_{j-1}^{+,n} + acf_{j+1}^{-,n}.\end{aligned}$$

The sum of the incremental coefficients is

$$\begin{aligned}(a+b)(1-d+c) &= (a+b)\left(1 + c\left[1 - c_\kappa(1+\varepsilon\frac{\tanh(\Delta x\sqrt{\kappa})}{\sqrt{\kappa}})\right]\right)\\ &\le (a+b)\left(1 - \Delta t\frac{2\sqrt{\kappa}}{\Delta x s_\kappa}|c_\kappa - 1|\right) + O(\varepsilon)\\ &\le (a+b) \text{ for } \varepsilon \text{ small enough,}\end{aligned}$$

and the (parabolic) CFL restriction follows from,

$$b - ad \ge 0 \Leftrightarrow b(1-d) - d \ge 0, \text{ so } \frac{2\Delta t(\sqrt{\kappa} + O(\varepsilon))}{\Delta x \tanh(\Delta x\sqrt{\kappa})} \le 1. \tag{2.9}$$

The scheme preserves positivity under the similar restriction,

$$d \le \min(\frac{b}{a}, \frac{a}{b}) = \frac{b}{1+b}.$$

By adding the equations and summing over j,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}(f_j^{+,n+1} + f_j^{-,n+1}) &= (1-d)\sum_{j\in\mathbb{Z}}(f_j^{+,n} + f_j^{-,n}) + c\sum_{j\in\mathbb{Z}}(f_{j-1}^{+,n} + f_{j+1}^{-,n})\\ &= (1-d+c)\sum_{j\in\mathbb{Z}}(f_j^{+,n} + f_j^{-,n}) \le \sum_{j\in\mathbb{Z}}(f_j^{+,n} + f_j^{-,n}),\end{aligned}$$

because $d \ge c$ from the observation that $c_\kappa \ge 1$ and $s_\kappa \ge 0$. Hence we recover the dissipation property. Denoting $\Delta f_j = f_{j+1} - f_j$ and $TV(f) = \sum_{j\in\mathbb{Z}}\Delta x|\Delta f_j|$,

$$\begin{aligned}(a+b)|\Delta f_j^{+,n+1}| &\le (a-bd)|\Delta f_j^{+,n}| + (b-ad)|\Delta f_j^{-,n}| + ac|\Delta f_{j-1}^{+,n}| + bc|\Delta f_{j+1}^{-,n}|,\\ (a+b)|\Delta f_j^{-,n+1}| &\le (b-ad)|\Delta f_j^{+,n}| + (a-bd)|\Delta f_j^{-,n}| + bc|\Delta f_{j-1}^{+,n}| + ac|\Delta f_{j+1}^{-,n}|.\end{aligned}$$

Adding and summing, we deduce the TVD property for (2.7):

$$\boxed{TV(f^{+,n+1}) + TV(f^{-,n+1}) \le (1-d+c)(TV(f^{+,n}) + TV(f^{-,n})).}$$

Subtracting the equations and noticing that $a - b = 1$,

$$\begin{aligned}(a+b)(f_j^{+,n+1} - f_j^{-,n+1}) &= (1+d)(f_j^{+,n} - f_j^{-,n}) + c(f_{j-1}^{+,n} - f_{j+1}^{-,n})\\ &= (1+d)(f_j^{+,n} - f_j^{-,n}) + c(f_{j-1}^{+,n} - f_{j-1}^{-,n}) + c(f_{j-1}^{-,n} - f_{j+1}^{-,n})\end{aligned}$$

so that a summation and triangle inequality brings

$$(a+b)\sum_{j\in\mathbb{Z}}|f_j^{+,n+1}-f_j^{-,n+1}|\leq(1+d+c)\sum_{j\in\mathbb{Z}}|f_j^{+,n}-f_j^{-,n}|+2c\Delta xTV(f^{-,n}).$$

The stability condition implies

$$c\leq d\leq\frac{b}{1+b},\qquad a=1+b.$$

which yields,

$$\sum_{j\in\mathbb{Z}}|f_j^{+,n+1}-f_j^{-,n+1}|\leq\frac{1+3b}{(1+2b)(1+b)}\sum_{j\in\mathbb{Z}}|f_j^{+,n}-f_j^{-,n}|+\frac{2b\,\Delta xTV(f^{-,n})}{(1+2b)(1+b)}.$$

Leting $\varepsilon\to 0$, we get that $b=O(\frac{1}{\varepsilon})$, along with for well-prepared initial data,

$$\boxed{\sum_{j\in\mathbb{Z}}|f_j^{+,n+1}-f_j^{-,n+1}|=O(\varepsilon).}\tag{2.10}$$

Theorem 2.1 *Assume initial data $f^{\pm}(t=0,\cdot)\in L^1\cap BV(\mathbb{R})$, then under the CFL condition (2.9), the scheme (2.7) is TVD; moreover, if it is well-prepared,*

$$\|f^+(t=0,\cdot)-f^-(t=0,\cdot)\|_{L^1(\mathbb{R})}=O(\varepsilon),$$

then (2.10) holds for $n\in\mathbb{N}$.

Consistency with the asymptotic $\mathscr{L}$-spline scheme (2.8) follows from (2.10).

3 Continuous velocity model with adsorption

The next level of complexity is given by the following equation,

$$\partial_t f+v\partial_x f=\sigma\left(\frac{c}{2}\rho-f(t,x,v)\right),\qquad 0<c=1-\kappa<1,\tag{3.1}$$

for $x\in\mathbb{R}$, $v\in(-1,1)$. The construction of stationary solutions for (3.1) relies on Case's method of separation of variables (already evoked, see also Appendix A) and is carried out in [11, §9.1–2]. Here, we give more information about the Case's "normal modes" λ which satisfy,

$$1=\frac{c}{2}\int_{-1}^{1}\frac{\mathrm{d}v}{1+\lambda v}=\frac{c}{2}\int_0^1\frac{1}{1+\lambda v}+\frac{1}{1-\lambda v}\mathrm{d}v,$$

the last equality showing that if λ is solution, then so is $-\lambda$, too. The "discrete mode" is such that $1/|\lambda|\notin(0,1)$, and it corresponds to the smallest damping exponent $|\lambda|$ among all Case's modes. By expanding fractions to second order,

$$1\simeq\frac{c}{2}\int_0^1 2+2\lambda^2v^2\mathrm{d}v=c(1+\lambda^2/3),\qquad\frac{\lambda^2}{3}=\kappa.$$

In particular, the diffusive scaling implies that $\kappa \to \varepsilon^2\kappa$, so that $|\lambda| \to \varepsilon|\lambda|$. Accordingly, we look for stationary solutions under the form,

$$\bar{f}(x,v) = e^{-\lambda x}\phi_\lambda(v), \qquad \phi_\lambda(v) = \frac{1}{2}\frac{1-\kappa}{1-\lambda v}, \qquad \int_{-1}^{1}\phi_\lambda(v)\,dv = 1. \tag{3.2}$$

The normalization condition allows to compute eigenvalues λ; by symmetry, if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue. As a consequence,

$$(\lambda-\mu)\int_{-1}^{1} v\phi_\lambda(v)\phi_\mu(v)\,dv = 0. \tag{3.3}$$

Let us introduce a (symmetric) Gauss quadrature such that, for $k \in \{1,\dots,N\}$,

$$v_{-k} = -v_k, \qquad \omega_{-k} = \omega_k, \qquad \sum_{k=1}^{K}\omega_k = 1, \qquad \sum_{k=1}^{K}\omega_k {v_k}^2 = \frac{1}{3}. \tag{3.4}$$

Hence, the discrete versions of (3.2) and (3.3) read

$$\frac{1-\kappa}{2}\sum_{k=1}^{K}\omega_k\Big(\frac{1}{1-\lambda v_k}+\frac{1}{1+\lambda v_k}\Big) = 1, \tag{3.5}$$

and

$$\forall\lambda \neq \mu, \qquad \sum_{k=1}^{K}\omega_k v_k(\phi_\lambda(v_k)\phi_\mu(v_k) - \phi_\lambda(-v_k)\phi_\mu(-v_k)) = 0. \tag{3.6}$$

3.1 The $2K \times 2K$ scattering S-matrix

We introduce the diffusive scaling:

$$t \to \varepsilon^2 t, \qquad x \to \varepsilon x, \qquad \kappa \to \varepsilon^2\kappa.$$

Then the discrete eigenvalues are denoted λ^ε.

Lemma 3.1 *We have $\lambda_0^\varepsilon = \sqrt{3\kappa}\varepsilon + o(\varepsilon)$, as $\varepsilon \to 0$.*

Proof Expanding (3.5), as $\lambda \to 0$, we get

$$1 = \frac{1-\kappa\varepsilon^2}{2}\sum_{k=1}^{K}\omega_k\Big(\frac{1}{1-\lambda v_k}+\frac{1}{1+\lambda v_k}\Big) = (1-\kappa\varepsilon^2)\sum_{k=1}^{K}\omega_k(1+\lambda^2 v_k^2 + o(\lambda^2)).$$

Using (3.4) allows to conclude the proof.

We use the same notations as in [16]. In particular, the matrix in $\mathscr{M}_K(\mathbb{R})$

$$\frac{1}{1+\mathscr{V}\otimes\lambda} \overset{def}{=} \left(\frac{1}{1+v_k\lambda_\ell}\right)_{k,\ell}.$$

Truncating to the first $2K$ eigenmodes yields a quite general steady-state,

$$\bar{f}(x,v) = \sum_{\ell=0}^{K-1} \left(\frac{a_\ell\, e^{-\lambda_\ell^\varepsilon x/\varepsilon}}{1-\lambda_\ell^\varepsilon v} + \frac{b_\ell\, e^{\lambda_\ell^\varepsilon (x-\Delta x)/\varepsilon}}{1+\lambda_\ell^\varepsilon v} \right), \quad \lambda_\ell^\varepsilon \geq 0.$$

We denote the vector of so-called "normal eigenmodes"

$$0 < \lambda^\varepsilon := \left(\lambda_1^\varepsilon, \ldots, \lambda_{K-1}^\varepsilon\right)^\top,$$

and the matrix of "Case's eigenfunctions" in $\mathscr{M}_{K\times 2K}(\mathbb{R})$,

$$\Phi^{\varepsilon,\pm}(x) = \begin{pmatrix} \dfrac{e^{-\lambda^\varepsilon x/\varepsilon}}{1\mp \mathscr{V}\otimes\lambda^\varepsilon} & \dfrac{e^{-\lambda_0^\varepsilon x/\varepsilon}}{1\mp \mathscr{V}\otimes\lambda_0^\varepsilon} & \dfrac{e^{\lambda^\varepsilon (x-\Delta x)/\varepsilon}}{1\pm \mathscr{V}\otimes\lambda^\varepsilon} & \dfrac{e^{\lambda_0^\varepsilon (x-\Delta x)/\varepsilon}}{1\pm \mathscr{V}\otimes\lambda_0^\varepsilon} \end{pmatrix},$$

such that

$$\bar{f}(x,\pm\mathscr{V}) = \Phi^{\varepsilon,\pm}(x) \begin{pmatrix} a \\ b \end{pmatrix},$$

where,

$$a = (a_1, \ldots, a_{K-1}, a_0)^\top \text{ and } b = (b_1, \ldots, b_{K-1}, b_0)^\top.$$

It follows that,

$$\begin{pmatrix} \bar{f}(0,\mathscr{V}) \\ \bar{f}(\Delta x,-\mathscr{V}) \end{pmatrix} = \begin{pmatrix} \Phi^{\varepsilon,+}(0) \\ \Phi^{\varepsilon,-}(\Delta x) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \begin{pmatrix} \bar{f}(\Delta x,\mathscr{V}) \\ \bar{f}(0,-\mathscr{V}) \end{pmatrix} = \begin{pmatrix} \Phi^{\varepsilon,+}(\Delta x) \\ \Phi^{\varepsilon,-}(0) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus, the scattering matrix is given by

$$\mathscr{S}^\varepsilon = \widetilde{M}^\varepsilon (M^\varepsilon)^{-1}, \quad \text{with } \widetilde{M}^\varepsilon := \begin{pmatrix} \Phi^{\varepsilon,+}(\Delta x) \\ \Phi^{\varepsilon,-}(0) \end{pmatrix}, \quad M^\varepsilon := \begin{pmatrix} \Phi^{\varepsilon,+}(0) \\ \Phi^{\varepsilon,-}(\Delta x) \end{pmatrix}. \tag{3.7}$$

We first study the asymptotics of the scattering matrix when $\varepsilon \to 0$:

Proposition 3.1 *The scattering matrix* (3.7) *admits the decomposition*

$$\boxed{\mathscr{S}^\varepsilon = \begin{pmatrix} \mathbf{0}_K & \mathbf{I}_K - \zeta\gamma \\ \mathbf{I}_K - \zeta\gamma & \mathbf{0}_K \end{pmatrix} + \varepsilon B^0 + o(\varepsilon),} \tag{3.8}$$

where

$$B^0 := \frac{\sqrt{3\kappa}}{\sinh(\sqrt{3\kappa}\Delta x)} \begin{pmatrix} (2-\zeta\gamma)\mathscr{V}\beta^\top & -\cosh(\sqrt{3\kappa}\Delta x)(2-\zeta\gamma)\mathscr{V}\beta^\top \\ -\cosh(\sqrt{3\kappa}\Delta x)(2-\zeta\gamma)\mathscr{V}\beta^\top & (2-\zeta\gamma)\mathscr{V}\beta^\top \end{pmatrix}, \tag{3.9}$$

with notation : $\mathbf{I}_K, \mathbf{0}_K$ *are* $K\times K$ *identity and null matrices, respectively, and* $\mathbf{0}_{\mathbb{R}^K}, \mathbf{1}_{\mathbb{R}^K}$ *are* $\mathbb{R}^K$ *vectors of zeros and ones. Moreover,*

$$\zeta = \left(\frac{1}{1-\mathscr{V}\otimes\lambda^0}\right) - \left(\frac{1}{1+\mathscr{V}\otimes\lambda^0}\right) \in \mathscr{M}_{K\times K-1}(\mathbb{R}),$$

and $\gamma \in \mathscr{M}_{K-1\times K}(\mathbb{R})$, $\beta \in \mathbb{R}^K$ *are such that*

$$\gamma\left(\frac{1}{1-\mathscr{V}\otimes\lambda^0}\right) = \mathbf{I}_{K-1}, \qquad \gamma\mathbf{1}_{\mathbb{R}^K} = \mathbf{0}_{\mathbb{R}^{K-1}}, \tag{3.10}$$

$$\beta^\top\left(\frac{1}{1-\mathscr{V}\otimes\lambda^0}\right) = \mathbf{0}_{\mathbb{R}^{K-1}}^\top, \qquad \beta^\top\mathbf{1}_{\mathbb{R}^K} = 1. \tag{3.11}$$

We refer to [16, Appendix] for the existence of γ, β satisfying (3.10)– (3.11).

Proof As $\varepsilon \to 0$ we deduce that

$$\mathscr{S}^\varepsilon = \widetilde{M}^\varepsilon (M^\varepsilon)^{-1} \to \widetilde{M}^0 (M^0)^{-1},$$

where, thanks to Lemma 3.1,

$$\widetilde{M}^0 = \begin{pmatrix} \mathbf{0}_{K\times(K-1)} & e^{-\sqrt{3\kappa}\Delta x}\mathbf{1}_{\mathbb{R}^K} & \frac{1}{1+\mathscr{V}\otimes\lambda} & \mathbf{1}_{\mathbb{R}^K} \\ \frac{1}{1+\mathscr{V}\otimes\lambda} & \mathbf{1}_{\mathbb{R}^K} & \mathbf{0}_{K\times(K-1)} & e^{-\sqrt{3\kappa}\Delta x}\mathbf{1}_{\mathbb{R}^K} \end{pmatrix}, \tag{3.12}$$

$$M^0 = \begin{pmatrix} \frac{1}{1-\mathscr{V}\otimes\lambda} & \mathbf{1}_{\mathbb{R}^K} & \mathbf{0}_{K\times(K-1)} & e^{-\sqrt{3\kappa}\Delta x}\mathbf{1}_{\mathbb{R}^K} \\ \mathbf{0}_{K\times(K-1)} & e^{-\sqrt{3\kappa}\Delta x}\mathbf{1}_{\mathbb{R}^K} & \frac{1}{1-\mathscr{V}\otimes\lambda} & \mathbf{1}_{\mathbb{R}^K} \end{pmatrix}. \tag{3.13}$$

We may compute the inverse of M^0 thanks to γ and β introduced above,

$$(M^0)^{-1} = \begin{pmatrix} \begin{pmatrix} \gamma \\ \frac{1}{1-e^{-2\sqrt{3\kappa}\Delta x}}\beta^\top \end{pmatrix} & \begin{pmatrix} \mathbf{0}_{(K-1)\times K} \\ \frac{-1}{e^{\sqrt{3\kappa}\Delta x}-e^{-\sqrt{3\kappa}\Delta x}}\beta^\top \end{pmatrix} \\ \begin{pmatrix} \mathbf{0}_{(K-1)\times K} \\ \frac{-1}{e^{\sqrt{3\kappa}\Delta x}-e^{-\sqrt{3\kappa}\Delta x}}\beta^\top \end{pmatrix} & \begin{pmatrix} \gamma \\ \frac{1}{1-e^{-2\sqrt{3\kappa}\Delta x}}\beta^\top \end{pmatrix} \end{pmatrix}. \tag{3.14}$$

Computing, using the definition of ζ and (3.10)–(3.11), we get

$$\widetilde{M}^0 (M^0)^{-1} = \begin{pmatrix} \mathbf{0}_K & \mathbf{I}_K - \zeta\gamma \\ \mathbf{I}_K - \zeta\gamma & \mathbf{0}_K \end{pmatrix}.$$

Then,

$$B^\varepsilon := \frac{1}{\varepsilon}(\widetilde{M}^\varepsilon (M^\varepsilon)^{-1} - \widetilde{M}^0 (M^0)^{-1}) = \frac{1}{\varepsilon}(\widetilde{M}^\varepsilon - \widetilde{M}^0 (M^0)^{-1} M^\varepsilon)(M^\varepsilon)^{-1}.$$

We have $\mathscr{S}^\varepsilon = \widetilde{M}^0 (M^0)^{-1} + \varepsilon B^\varepsilon$: straightforward computations yield

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\widetilde{M}^\varepsilon - \widetilde{M}^0 (M^0)^{-1} M^\varepsilon) =$$
$$\begin{pmatrix} \mathbf{0}_{K\times(K-1)} & \sqrt{3\kappa}e^{-\sqrt{3\kappa}\Delta x}(2-\zeta\gamma)\mathscr{V} & \mathbf{0}_{K\times(K-1)} & -\sqrt{3\kappa}(2-\zeta\gamma)\mathscr{V} \\ \mathbf{0}_{K\times(K-1)} & \sqrt{3\kappa}(2-\zeta\gamma)\mathscr{V} & \mathbf{0}_{K\times(K-1)} & -\sqrt{3\kappa}e^{-\sqrt{3\kappa}\Delta x}(2-\zeta\gamma)\mathscr{V} \end{pmatrix}$$

Hence,

$$\lim_{\varepsilon\to 0} B^\varepsilon = \left(\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\widetilde{M}^\varepsilon - \widetilde{M}^0 (M^0)^{-1} M^\varepsilon)\right)(M^0)^{-1} = B^0,$$

where B^0 is defined in (3.9).

3.2 Asymptotic IMEX numerical scheme

Thanks to Proposition 3.1, we have at hand the expansion of the S-matrix depending on ε; then, the scheme for the dissipative kinetic equation reads,

$$\begin{pmatrix} f_j^{n+1}(\mathscr{V}) \\ f_{j-1}^{n+1}(-\mathscr{V}) \end{pmatrix} + \frac{\Delta t}{\varepsilon \Delta x}\mathbb{V}\begin{pmatrix} f_j^{n+1}(\mathscr{V}) - (\mathbf{I}_K - \zeta\gamma) f_j^{n+1}(-\mathscr{V}) \\ f_{j-1}^{n+1}(-\mathscr{V}) - (\mathbf{I}_K - \zeta\gamma) f_{j-1}^{n+1}(\mathscr{V}) \end{pmatrix}$$
$$= \begin{pmatrix} f_j^{n}(\mathscr{V}) \\ f_{j-1}^{n}(-\mathscr{V}) \end{pmatrix} + \frac{\Delta t}{\Delta x}\mathbb{V} B^{\varepsilon}\begin{pmatrix} f_{j-1}^{n}(\mathscr{V}) \\ f_{j}^{n}(-\mathscr{V}) \end{pmatrix}. \tag{3.15}$$

We want now to identify the limit as $\varepsilon \to 0$ of this scheme. We proceed as in [16] and rewrite the scheme in IMEX form

$$\boxed{\frac{1}{\varepsilon}\mathscr{R}^{\varepsilon}\begin{pmatrix} f_j^{n+1}(\mathscr{V}) \\ f_j^{n+1}(-\mathscr{V}) \end{pmatrix} = \begin{pmatrix} f_j^{n}(\mathscr{V}) \\ f_j^{n}(-\mathscr{V})) \end{pmatrix} + \frac{\Delta t}{\Delta x}\mathbb{V}\begin{pmatrix} B_1^{\varepsilon} f_{j-1}^{n}(\mathscr{V}) + B_2^{\varepsilon} f_j^{n}(-\mathscr{V}) \\ B_3^{\varepsilon} f_j^{n}(\mathscr{V}) + B_4^{\varepsilon} f_{j+1}^{n}(-\mathscr{V}) \end{pmatrix},} \tag{3.16}$$

where we use the notation $B^{\varepsilon} = \begin{pmatrix} B_1^{\varepsilon} & B_2^{\varepsilon} \\ B_3^{\varepsilon} & B_4^{\varepsilon} \end{pmatrix}$, and

$$\mathscr{R}^{\varepsilon} = \varepsilon \mathbf{I}_{2K} + \frac{\Delta t}{\Delta x}\mathbb{V}\begin{pmatrix} \mathbf{I}_K & \zeta\gamma - \mathbf{I}_K \\ \zeta\gamma - \mathbf{I}_K & \mathbf{I}_K \end{pmatrix}. \tag{3.17}$$

Solving this system amounts to inverting matrix $\mathscr{R}^{\varepsilon}$. ¿From (3.17),

$$\mathscr{R}^{\varepsilon} = \mathscr{R}^{0} + \varepsilon \mathbf{I}_{2K}, \quad \text{where } \mathscr{R}^{0} := \frac{\Delta t}{\Delta x}\mathbb{V}\begin{pmatrix} \mathbf{I}_K & \zeta\gamma - \mathbf{I}_K \\ \zeta\gamma - \mathbf{I}_K & \mathbf{I}_K \end{pmatrix}.$$

Assuming that f admits a Hilbert expansion $f = f^0 + \varepsilon f^1 + o(\varepsilon)$, it can be plugged into the scheme (3.16). Thus, by identifying each power of ε, we get

$$\mathscr{R}^{0}\begin{pmatrix} \{f^0\}_j^{n+1}(\mathscr{V}) \\ \{f^0\}_j^{n+1}(-\mathscr{V}) \end{pmatrix} = 0, \tag{3.18}$$

and at order 0 in ε,

$$\mathscr{R}^{0}\begin{pmatrix} \{f^1\}_j^{n+1}(\mathscr{V}) \\ \{f^1\}_j^{n+1}(-\mathscr{V}) \end{pmatrix} = \begin{pmatrix} \{f^0\}_j^{n}(\mathscr{V}) - \{f^0\}_j^{n+1}(\mathscr{V}) \\ \{f^0\}_j^{n}(-\mathscr{V}) - \{f^0\}_j^{n+1}(-\mathscr{V}) \end{pmatrix} \tag{3.19}$$
$$+ \frac{\Delta t\sqrt{3\kappa}}{\Delta x \sinh(\sqrt{3\kappa}\Delta x)}\mathbb{V}\begin{pmatrix} (2\mathbf{I}_K - \zeta\gamma)\mathscr{V}\beta^{\top}(\{f^0\}_{j-1}^{n}(\mathscr{V}) - \cosh(\sqrt{3\kappa}\Delta x)\{f^0\}_j^{n}(-\mathscr{V})) \\ (2\mathbf{I}_K - \zeta\gamma)\mathscr{V}\beta^{\top}(\{f^0\}_{j+1}^{n}(-\mathscr{V}) - \cosh(\sqrt{3\kappa}\Delta x)\{f^0\}_j^{n}(-\mathscr{V})) \end{pmatrix}.$$

¿From (3.6) with $\lambda = \lambda_0^{\varepsilon}$, and letting $\varepsilon \to 0$, we deduce that

$$\forall \ell \in \{1, \dots, K-1\}, \qquad \sum_{k=1}^{K} \omega_k v_k (\phi_{\lambda_\ell^0}(v_k) - \phi_{\lambda_\ell^0}(-v_k)) = 0. \tag{3.20}$$

Then, we can use the result in [16, Lemma B.1], which states that :

- $\mathrm{Ker}(\mathscr{R}^0) = \mathrm{Span}(\mathbf{1}_{\mathbb{R}^{2K}})$,
- $\mathrm{Im}(\mathscr{R}^0) = \Big\{ Z = (Z_1\ Z_2)^\top,\ Z_i \in \mathbb{R}^K \text{ such that } \sum_{k=1}^K \omega_k (Z_{1k} + Z_{2k}) = 0 \Big\}$.

We deduce from (3.18)

$$\{f^0\}_j^{n+1}(\pm\mathscr{V}) = \frac{\rho_j^{n+1}}{2}\mathbf{1}_{\mathbb{R}^K}, \qquad \rho_j^n = \sum_{k=1}^K \omega_k (f_j^n(v_k) + f_j^n(-v_k)). \tag{3.21}$$

Then, injecting (3.21) into (3.19) and using (3.11), we obtain

$$\begin{aligned}\mathscr{R}^0 \begin{pmatrix} \{f^1\}_j^{n+1}(\mathscr{V}) \\ \{f^1\}_j^{n+1}(-\mathscr{V}) \end{pmatrix} =& \frac{1}{2}\begin{pmatrix} (\rho_j^n - \rho_j^{n+1})\mathbf{1}_{\mathbb{R}^K} \\ (\rho_j^n - \rho_j^{n+1})\mathbf{1}_{\mathbb{R}^K} \end{pmatrix} \\ &+ \frac{\Delta t\sqrt{3\kappa}}{2\Delta x \sinh(\sqrt{3\kappa}\Delta x)} \mathbb{V} \begin{pmatrix} (2\mathbf{I}_K - \zeta\gamma)\mathscr{V}(\rho_{j-1}^n - \cosh(\sqrt{3\kappa}\Delta x)\rho_j^n) \\ (2\mathbf{I}_K - \zeta\gamma)\mathscr{V}(\rho_{j+1}^n - \cosh(\sqrt{3\kappa}\Delta x)\rho_j^n) \end{pmatrix},\end{aligned}$$

which admits a solution if and only if the right hand side belongs to $\mathrm{Im}(\mathscr{R}^0)$,

$$\begin{aligned}0 =& (\rho_j^n - \rho_j^{n+1}) \sum_{k=1}^K \omega_k \\ &+ \frac{\Delta t\sqrt{3\kappa}}{2\Delta x \sinh(\sqrt{3\kappa}\Delta x)} \Big(\rho_{j-1}^n + \rho_{j+1}^n - 2\cosh(\sqrt{3\kappa}\Delta x)\rho_j^n\Big) \sum_{k=1}^K \omega_k v_k [(2\mathbf{I}_K - \zeta\gamma)\mathscr{V}]_k\end{aligned}$$

Using (3.20), we deduce that

$$\sum_{k=1}^K \omega_k v_k [\zeta\gamma\mathscr{V}]_k = 0.$$

Hence, with (3.4), we conclude that the limit scheme reads

$$\boxed{\rho_j^{n+1} = \rho_j^n + \frac{\Delta t\sqrt{3\kappa}}{3\Delta x \sinh(\sqrt{3\kappa}\Delta x)} (\rho_{j-1}^n + \rho_{j+1}^n - 2\cosh(\sqrt{3\kappa}\Delta x)\rho_j^n).} \tag{3.22}$$

4 Numerical results

All the tests are performed on coarse grids, so as to emphasize robustness.

4.1 Two-stream model

4.1.1 Constant parameters

For $\sigma \equiv 1$ and $\kappa \equiv 0.1$, the consistency of the IMEX scheme (2.7) with the asymptotic one (2.8) is checked on Fig. 4.1. Maxwellian initial data were taken,

$$\forall x \in (0,1), \qquad f^\pm(x) = \exp(-20(x-0.5)^2)/4, \qquad \varepsilon = 10^{-5}, \tag{4.1}$$

along with 32 points for the computational grid. Outputs of both time-marching schemes at time $T = 0.1$ confirms the theoretical analysis in the former sections.

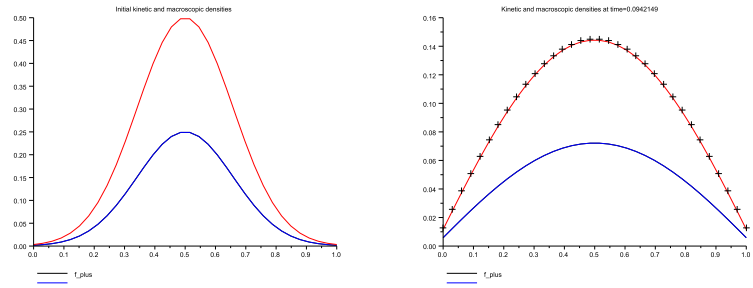

Fig. 4.1 Initial data (left) and approximations at $T = 0.1$ (right) for (4.1).

4.1.2 Position-varying parameters

Consider the two-stream model (2.1) for $x \in (0,1)$ with

$$\sigma(x) = 10(1 - 0.99\,\chi_{x\in(0.15,0.85)}), \qquad \kappa(x) = 0.01 + 0.98\,\chi_{x\in(0.25,0.75)}, \tag{4.2}$$

χ_A being the indicator of the set A, and initial data,

$$f^+(x) = 0.25\exp(-20(x-0.5)^2)/2, \qquad f^-(x) = 0.75\exp(-20(x-0.5)^2)/2.$$

The scheme (2.5) is iterated until numerical steady-state, see the results on Fig.

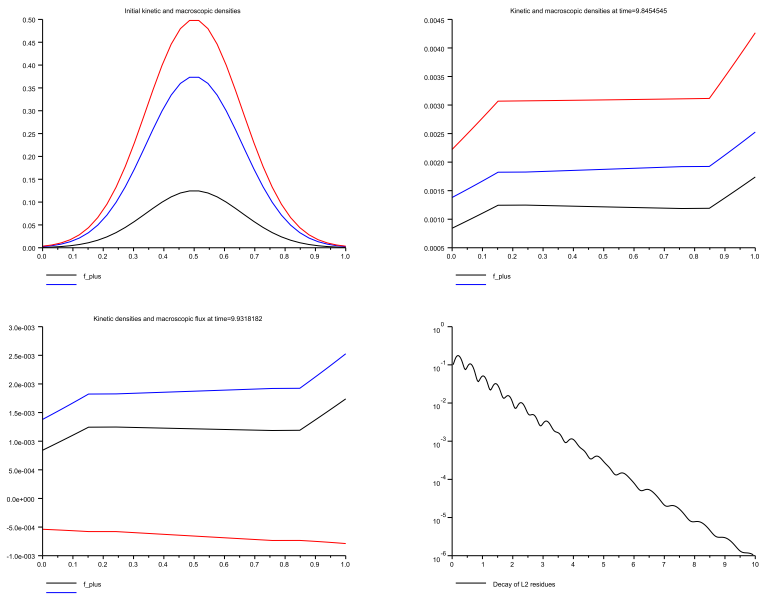

Fig. 4.2 Initial data, macroscopic densities and fluxes, residues (left to right, top to bottom) for (4.2).

4.2. Discontinuous parameters yield Lipschitz-continuous solutions which are well captured on a coarse grid with 32 equally spaced points.

4.2 Continuous velocities with Gaussian quadrature

4.2.1 Constant parameters

Yet, consider now the IMEX scheme (3.16) with $\sigma \equiv 1$, $\kappa \equiv 0.15$ and $\varepsilon = 10^{-5}$. The computational grid contains 32 equally-spaced points in x and $K = 8$ points in v retrieved from a Gauss-Legendre quadrature of $(-1,1)$. Initial data are

$$f(x,v) = \exp(-5v^2) \times (\exp(-35(x-0.45)^2) + \exp(-35(x+0.45)^2)). \tag{4.3}$$

Results at time $T = 0.2$ are displayed on Fig. 4.3. The macroscopic density ρ retrieved from (3.16) is very similar to the one produced by the asymptotic scheme (3.22), thus confirming the calculations made in §3.2.

4.2.2 Position-varying parameters

Hereafter, the situation in which the parameter κ is discontinuous,

$$\forall x \in (-1,1), \qquad \kappa(x) = 0.15 + 0.7\chi_{x>0}, \tag{4.4}$$

with identical initial data (4.3) is tackled. Results with 32 grid-points and $K = 8$ are displayed on Fig. 4.4. In particular, a satisfactory consistency with the asymptotic $\mathscr{L}$-spline scheme on ρ is observed for $\varepsilon = 10^{-5}$.

5 Conclusion and outlook

This article somehow completes the former paper [16]: numerical diffusive limits of (1+1)-dimensional kinetic models (1.1) are studied by rescaling well-balanced schemes relying on "scattering S-matrices", as advocated in [11, Part II]. The main difference being that here, the total mass isn't preserved in the time-evolution of the kinetic density, and this produces a "sink term" in the asymptotic parabolic approximation. Numerically, this raises an interesting question: given a kinetic model involving two distinct nonnegative parameters (here, κ and σ), does the so–called "asymptotic-preserving" property ask for consistency with the diffusive approximation (1.2) for all $\kappa, \sigma \in (0,1) \times \mathbb{R}^+$?

Clearly, the corresponding problem is stiff when $\sigma \gg 1$ and $\kappa \simeq 1$. Such an issue was already present in [16, §4], where the diffusive limit of a run-and-tumble kinetic model was studied for arbitrary strong velocity fields. The asymptotic parabolic scheme was the "Il'in/Scharfetter-Gummel" one, which is known to be uniformly accurate and can be related to "$\mathscr{L}$-spline interpolation", see *e.g.* [19,20,22,23]. Here, we get a similar situation as both (2.8) and (3.22) are "exponential-fit" discretizations of the damped heat equation. A challenging problem consists in addressing the same process in (2+2)-dimensional models, based on the numerical techniques proposed in [17] (relying on [5,6,10,24]). Robust and "truly two-dimensional" approximations of parabolic equations were previously studied in *e.g.* [3,4] (see again [20,23]).

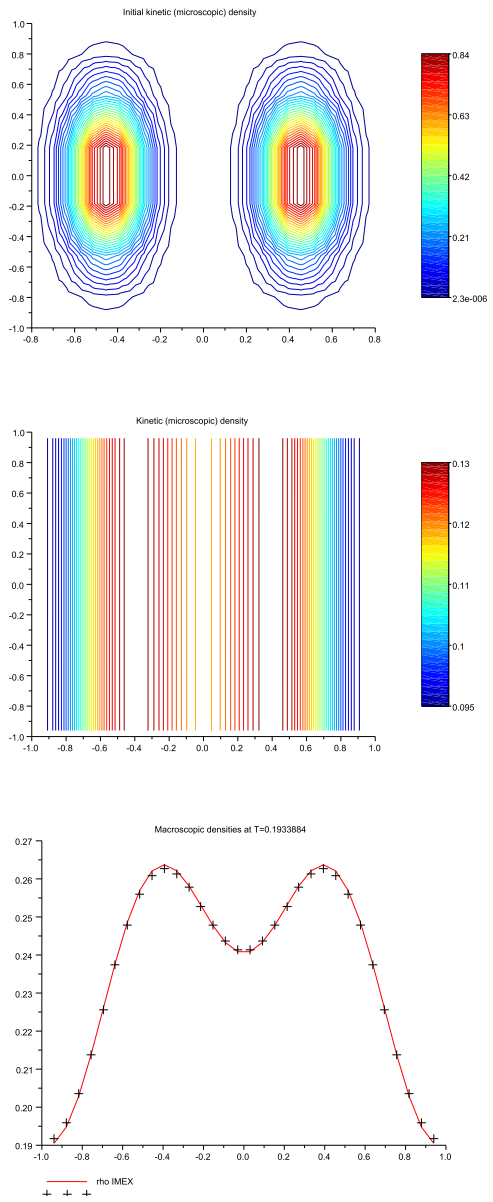

Fig. 4.3 Asymptotic-preserving property for (4.3).

Acknowledgments

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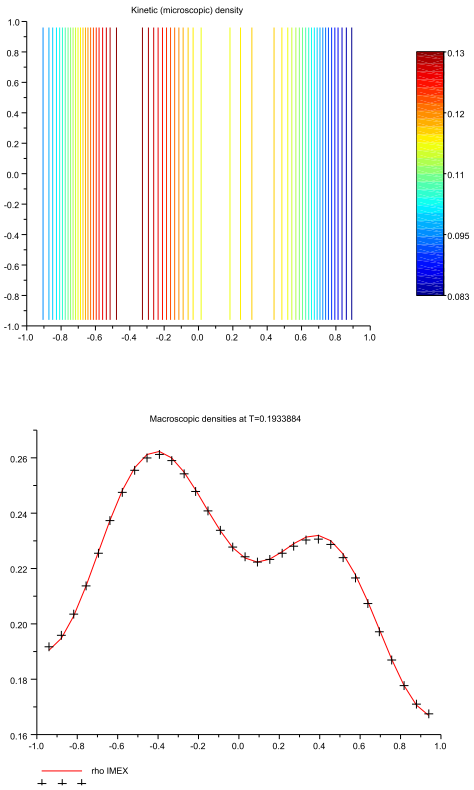

Fig. 4.4 Asymptotic-preserving property for discontinuous κ, see (4.4) with identical data.

A Steady-states of the two-stream approximation

There are at least three ways to derive the steady-states of (2.1), each one having its own interest. Besides the most direct one, relying on diagonalization, the method based on "Case's elementary solutions" [1,8, 18] is useful because it applies to more general 1D kinetic models. The most elaborate derivation may be the one in [5,6], as it is able to handle multi-dimensional problems, too. Let's review each of them:

– direct computations, for instance by diagonalizing the matrix,

$$\begin{aligned} A &= \begin{pmatrix} -(1+\kappa)/2 & (1-\kappa)/2 \\ -(1-\kappa)/2 & (1+\kappa)/2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+\sqrt{\kappa}}{1-\sqrt{\kappa}} & \frac{1-\sqrt{\kappa}}{1+\sqrt{\kappa}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{\kappa} & 0 \\ 0 & \sqrt{\kappa} \end{pmatrix} \frac{1}{4\sqrt{\kappa}} \begin{pmatrix} 1-\kappa & -(1-\sqrt{\kappa})^2 \\ \kappa-1 & (1+\sqrt{\kappa})^2 \end{pmatrix}, \end{aligned}$$

meaning that exponential modes $\exp(\pm x\sqrt{\kappa})$ are to be expected.

– "Caseology' [1,8]', consisting in separating variables in a 1D equation,

$$v\partial_x f(x,v) = \frac{c}{2}\int_{-1}^{1} f(x,v')\mathrm{d}v' - f, \qquad f(x,v) = \psi(x)\phi(v).$$

By normalizing, one gets the "constants of separation", $\lambda \in \mathbb{R}$,

$$\frac{\psi'(x)}{\psi(x)} = \lambda = \frac{1-\phi(v)}{v\phi(v)}, \qquad \frac{c}{2}\int_{-1}^{1}\phi(v)\mathrm{d}v = 1.$$

which yields automatically,

$$\psi(x) = \exp(\lambda x), \quad \phi(v) = \frac{1}{1+\lambda v}, \quad f(x,v) = \frac{\exp(\lambda x)}{1+\lambda v}.$$

It remains to restrict $f(x,v)$ to $v = \pm 1$, giving

$$1 = \frac{c}{2}\int_{-1}^{1} \phi(v)\mathrm{d}v = \frac{c}{2}\left(\frac{1}{1+\lambda} + \frac{1}{1-\lambda}\right), \qquad \lambda^2 = 1 - c = \kappa.$$

– and the method of Birkhoff [5,6], relying on both the method of characteristics (see [17, §2.1]) and elliptic regularity, which handles (1.1) in its whole generality.

$$f^{\pm}(x) = \frac{c}{2}\int_0^{\infty} \exp(-r)\rho(x \mp r)\mathrm{d}r, \qquad \rho(x) = f^{+}(x) + f^{-}(x), \tag{A.1}$$

is the expression of a 1D stationary kinetic solution by characteristics (see e.g. [9, §3.3, eqn (7)]). The diffusion approximation suggests that a stationary macroscopic solution satisfies a damped diffusion equation,

$$-\partial_{xx}\rho(x) + \lambda^2\rho = 0, \qquad \rho(x) = A\exp(\lambda x) + B\exp(-\lambda x).$$

This exact expression of ρ may be used in order to deduce $f(x,v)$ through (A.1); however, for ρ solution of the elliptic equation, [6, Theorem A] gives

$$\forall (x,r) \in \mathbb{R}\times\mathbb{R}^+, \qquad \rho(x) = \frac{\rho(x+r) + \rho(x-r)}{2M(r)},$$

with the weight M in the mean-value being a modified Bessel function,

$$M(r) = \Gamma(\tfrac{1}{2})\sqrt{\frac{\lambda r}{2}}\mathscr{I}_{-\frac{1}{2}}(\lambda r) = \Gamma(\tfrac{1}{2})\sqrt{\frac{\lambda r}{2}}\sqrt{\frac{2}{\pi\lambda r}}\cosh(\lambda r),$$

and since $\Gamma(1/2) = \sqrt{\pi}$, $M(r) = \cosh(\lambda r)$. By integrating (A.1) in v,

$$\begin{aligned}
\rho(x) &= \frac{c}{2}\int_0^{\infty} \exp(-r)\Big(\rho(x-r) + \rho(x+r)\Big)\mathrm{d}r \\
&= c\int_0^{\infty} \exp(-r)\cosh(\lambda r)\rho(x)\mathrm{d}r \\
&= \rho(x)c\mathscr{L}[\cosh(\lambda r)](p=1), \qquad \text{(Laplace transform)} \\
\text{so,} \qquad 1 &= \left.\frac{cp}{p^2-\lambda^2}\right|_{p=1}, \qquad \lambda^2 = 1 - c = \kappa.
\end{aligned}$$

Accordingly, stationary kinetic solutions are given by (A.1),

$$\begin{aligned}
f^{\pm}(x) &= \frac{c}{2}\int_0^{\infty} \exp(-r)\Big(A\exp((x-r)\sqrt{\kappa}) + B\exp(-(x+r)\sqrt{\kappa})\Big)\mathrm{d}r \\
&= A\exp(x\sqrt{\kappa})\frac{c}{2}\int_0^{\infty} \exp(-r(1+\sqrt{\kappa}))\mathrm{d}r \\
&\quad + B\exp(-x\sqrt{\kappa})\frac{c}{2}\int_0^{\infty} \exp(-r(1-\sqrt{\kappa}))\mathrm{d}r \\
&= \tilde{A}\frac{\exp(x\sqrt{\kappa})}{1+\sqrt{\kappa}} + \tilde{B}\frac{\exp(-x\sqrt{\kappa})}{1-\sqrt{\kappa}},
\end{aligned}$$

which are identical to "damped modes" previously obtained by both Caseology and the direct approach. In all these computations, a constant opacity parameter $\sigma > 0$ can be handled very simply through a scaling of the space variable, $x \to \sigma x$.

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