Optimal Bound on the Combinatorial Complexity of Approximating Polytopes

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Abstract

Convex bodies play a fundamental role in geometric computation, and approximating such bodies is often a key ingredient in the design of efficient algorithms. We consider the question of how to succinctly approximate a multidimensional convex body by a polytope. We are given a convex body K of unit diameter in Euclidean d-dimensional space (where d is a constant) along with an error parameter $\varepsilon > 0$. The objective is to determine a polytope of low combinatorial complexity whose Hausdorff distance from K is at most ε. By *combinatorial complexity* we mean the total number of faces of all dimensions of the polytope. In the mid-1970's, a result by Dudley showed that $O(1/\varepsilon^{(d-1)/2})$ facets suffice, and Bronshteyn and Ivanov presented a similar bound on the number of vertices. While both results match known worst-case lower bounds, obtaining a similar upper bound on the total combinatorial complexity has been open for over 40 years. Recently, we made a first step forward towards this objective, obtaining a suboptimal bound. In this paper, we settle this problem with an asymptotically optimal bound of $O(1/\varepsilon^{(d-1)/2})$.

Our result is based on a new relationship between ε-width caps of a convex body and its polar. Using this relationship, we are able to obtain a volume-sensitive bound on the number of approximating caps that are "essentially different." We achieve our result by combining this with a variant of the witness-collector method and a novel variable-width layered construction.

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1 Introduction

Convex objects are of central importance in numerous areas of geometric computation. Efficiently approximating a multi-dimensional convex body by a convex polytope is a natural and fundamental problem. Given a closed, convex set K of unit diameter in Euclidean d-dimensional space and an error parameter $\varepsilon > 0$, the objective is to produce a convex polytope of low combinatorial complexity whose Hausdorff distance[1] from K is at most ε. The *combinatorial complexity* of a polytope is the total number of faces of all dimensions. Throughout, we assume that the dimension d is a constant.

Dudley showed that, for $\varepsilon \leq 1$, any convex body K of unit diameter can be ε-approximated by a convex polytope P with $O(1/\varepsilon^{(d-1)/2})$ facets [20]. This bound is known to be tight in the worst case and is achieved when K is a Euclidean ball [16]. Alternatively, Bronshteyn and Ivanov showed the same bound holds for the number of vertices, which is also the best possible [15]. Similar bounds are widely used in algorithms based on ε-kernels to approximate the diameter, width, minimum enclosing cylinder, and bichromatic closest pair, among others (see [1, 5, 17]). Unfortunately, no construction is known that matches both bounds simultaneously. This issue has been noted by Clarkson [18], where he cites communications with Jeff Erickson showing that both bounds can be attained but at the cost of sacrificing convexity.

McMullen's *Upper-bound Theorem* [27] implies that a polytope with n facets (resp., vertices) has $O(n^{\lfloor d/2 \rfloor})$ vertices (resp., facets), and this bound is attained by cyclic polytopes. Applying this to Dudley's or Bronshteyn and Ivanov's constructions yields a very weak upper bound of roughly $O\big(1/\varepsilon^{(d^2-d)/4}\big)$ on the combinatorial complexity of ε-approximating polytopes. (Alternative constructions are known that yield a complexity of roughly $O(1/\varepsilon^{d-2})$ [2, 12], but this is nearly quadratic in the lower bound.)

Because it is often useful to convert between vertex-based and facet-based representations of convex polytopes (as the convex hull of points and the intersection of halfspaces, respectively), this blowup has been a major impediment to the application of fundamental polytopal structures such as convex hulls, Delaunay triangulations, and Voronoi diagrams in dimensions $d > 3$. Efficiently representing the combinatorial structure of a polytope's faces is of great practical importance to several algorithms [13]. However, the high combinatorial complexity of existing polytope approximations severely limits the efficiency of such data structures.

In this paper we achieve a major breakthrough by resolving this decades-old problem. We present a construction for approximating a convex body that not only simultaneously achieves the bound of $O(1/\varepsilon^{(d-1)/2})$ on the number of vertices and facets, but in fact establishes this bound on the total combinatorial complexity (sum of faces of all dimensions).

Theorem 1.1. *Let $K \subset \mathbb{R}^d$ be a convex body of unit diameter, where d is a fixed constant. For all sufficiently small positive ε (independent of K) there exists an ε-approximating convex polytope P to K of combinatorial complexity $O(1/\varepsilon^{(d-1)/2})$.*

We considered this problem earlier [6], obtaining a result that was suboptimal. That paper introduced two useful techniques: a width-based variant of Bárány's [11] economical cap cover based on Macbeath regions and a multi-layered approach to the witness-collector method [19]. That result fell short, however, due in part to a weak understanding of the distribution of Macbeath region

[1] The *Hausdorff distance* between any two sets is the maximum Euclidean distance between any point in one set and its closest point in the other set. While there are other metrics for polytope similarity (see, e.g. [16]), Hausdorff is the measure most often used in computational geometry. Approximations sensitive to the diameter and the directional width can be obtained by applying an affine transformation to K.

volumes in the cap cover. In this paper, we introduce an important new result, a volume-sensitive bound on the number of Macbeath regions (in Theorem 4.2). This new bound comes about by establishing a correspondence between Macbeath regions in the original convex body and its polar body, and demonstrating a reciprocal relationship in the volumes of corresponding regions. A classical result in the theory of convex bodies states that the volume of a convex body and its polar dual have a reciprocal relationship. The dimensionless product of these two quantities is called the *Mahler volume* [26]. Our correspondence can be viewed as a "local" extension of the Mahler-volume concept, and it allows this reciprocal relationship to be applied in the context of approximation. This enables a more sophisticated application of the witness-collector method. We believe that this primal-polar approach is an important new technique, which will be useful in other optimization problems involving convex approximation.

Overview of Methods

Before delving into technical matters, let us survey the broader context behind our work and give a high-level view of our approach. Convex approximation by polytopes is in essence a covering problem. Given a convex body K of unit diameter, an *ε-width cap* is the intersection of K with a halfspace that cuts off a slice of width ε from K. Clearly, any collection of ε-width caps that covers all of K's boundary yields an ε-approximation of K having (at most) as many facets. Clarkson [18] observed that, as ε tends to zero, computing such a cover involves sampling the boundary of K according to a metric that is sensitive to K's shape, with proportionately more samples in areas of higher curvature. Intuitively, such a metric should capture the notion of the "local feature size" at any point of K.

In recent works, we have demonstrated that shape-sensitive sampling can be achieved through the use of Macbeath regions. Given a point $x \in K$, the *Macbeath region* $M(x)$ is a maximal centrally symmetric convex shape centered at x and contained within K. (Formal definitions and properties are provided in Section 2.1.) Macbeath regions enjoy many useful properties. They can be computed efficiently, they have nice packing and covering properties, and up to constant scaling factors, $M(x)$ approximates the minimum volume cap centered at x as well as the unit balls centered at x in both the Hilbert and Blaschke geometries induced by K [11, 29, 30]. Macbeath regions have been introduced to computational geometry as a tool to prove lower bounds for range searching [10, 14]. Later on, Macbeath regions have been used to prove existential results [6, 7, 21, 28]. More recently, the explicit computation of such regions has been used to obtain the fastest algorithms known for several approximation problems such as ε-kernel, diameter, and width [5, 8].

In spite of their obvious relevance to convex approximation, there is still much that is not known about Macbeath regions. In the context of convex approximation, the number of disjoint (shrunken) Macbeath regions that can be placed within distance ε of K's boundary is closely related to the complexity of approximating K. In earlier work [6], we showed that for any convex body of unit diameter, such a set has size $O(1/\varepsilon^{(d-1)/2})$. The volume distribution of such a set is a question of key importance. To see why this is nontrivial, consider the unit hypercube. Parallel to each facet is an ε-width cap (and Macbeath region) of very large volume $\Theta(\varepsilon)$. Since the volume of the portion of K lying within distance ε of the boundary is $\Theta(\varepsilon)$, a packing argument implies that there cannot be more than a constant number of disjoint Macbeath regions associated with such large volume caps. On the other hand, ε-width caps (and Macbeath regions) that are orthogonal to the main diagonals, that is, close to the vertices of the hypercube, have very small volume of $\Theta(\varepsilon^d)$. A packing argument provides no useful bound on their number. Nonetheless, there cannot be many of them. To see why, observe that a small volume ε-width cap can only exist where K's boundary has high curvature, and the total curvature of any convex body is bounded. In this paper, we formalize and generalize this intuition. In particular, we show (in Theorem 4.2) that the number of disjoint ε-width Macbeath regions of volume v is $O(\min(\varepsilon/v, v/\varepsilon^d))$. Note that in both of the above extremes, this yields a tight bound of $O(1)$ on the number of Macbeath regions.

To prove this volume-sensitive bound we explore the nature of the Macbeath regions and caps in the polar body K^* of K. We show that for any cap C of width ε in the primal body K, there is a corresponding cap C' of width $\Theta(\varepsilon)$ in the polar body K^* such that the base of C' is a constant factor approximation of the polar of the base of C. Supplementing this new fundamental result with Mahler-volume upper and lower bounds (which imply a reciprocal relationship between the volume of a convex body and its polar), we conclude that the product of the volume of C and the volume of C' is roughly the same for every cap C. The volume-sensitive bound follows by applying packing in either the original body or the polar, depending on volume.

In order to apply this new volume-sensitive bound to convex approximation, we recall the witness-collector approach to bound the combinatorial complexity of the convex hull of a set of points [6, 19]. Let $S \subset \mathbb{R}^d$ be a set of points. We define a set $\mathscr{W}$ of regions called *witnesses* and a set $\mathscr{C}$ of regions called *collectors*, which satisfy the following properties:

(1) Each witness of $\mathscr{W}$ contains a point of S in its interior.

(2) Any halfspace H either contains a witness $W \in \mathscr{W}$ or $H \cap S$ is contained in a collector $C \in \mathscr{C}$.

(3) Each collector $C \in \mathscr{C}$ contains a constant number of points of S.

Devillers *et al.* [19] showed that given a set of witnesses $\mathscr{W}$ and collectors $\mathscr{C}$ satisfying the above properties, the combinatorial complexity of the polytope K defined as the convex hull of S is $O(|\mathscr{C}|)$. Hence, to prove Theorem 1.1, it suffices to provide a set S whose convex hull ε-approximates K and a corresponding set of witnesses and collectors of cardinality $O(1/\varepsilon^{(d-1)/2})$.

The key tool for our construction is the aforementioned collection of disjoint Macbeath regions lying within distance ε of K's boundary. This set of Macbeath regions has the desired size, and the corresponding caps satisfy properties (1) and (2) above, where the Macbeath regions represent the witnesses and the caps represent the collectors. Since Macbeath regions approximate caps, if we define a set S by picking one arbitrary point inside each Macbeath region, then we are guaranteed to obtain an ε-approximation of K.

However, the construction may fail to satisfy property (3). While a cap may only intersect a constant number of Macbeath regions of larger or similar volume, any given cap may intersect a large number of Macbeath regions of smaller volume. We dealt with this in [6] by arranging the Macbeath regions into $O(\log \frac{1}{\varepsilon})$ *layers* of thickness ε, moving low-volume Macbeath regions into the innermost layers. In this way, we prevented the caps from intersecting Macbeath regions of smaller volume, and assuring property (3). However, the approximation error grew from ε to $O(\varepsilon \log \frac{1}{\varepsilon})$ because of the increased total thickness of the layers. An ε-approximation was achieved through a compensatory scaling of ε, causing the complexity to grow to $O((\log \frac{1}{\varepsilon}/\varepsilon)^{(d-1)/2})$.

In this paper we show how to exploit our volume-sensitive bound to obtain an ε-approximation of optimal combinatorial complexity. As in [6], we place Macbeath regions in different layers of thickness according to their volumes, the outermost layers corresponding to Macbeath regions of larger volume and the innermost layers corresponding to smaller volumes. However, we no longer use layers of constant thickness. Instead, the middle layer (numbered layer 0), which corresponds

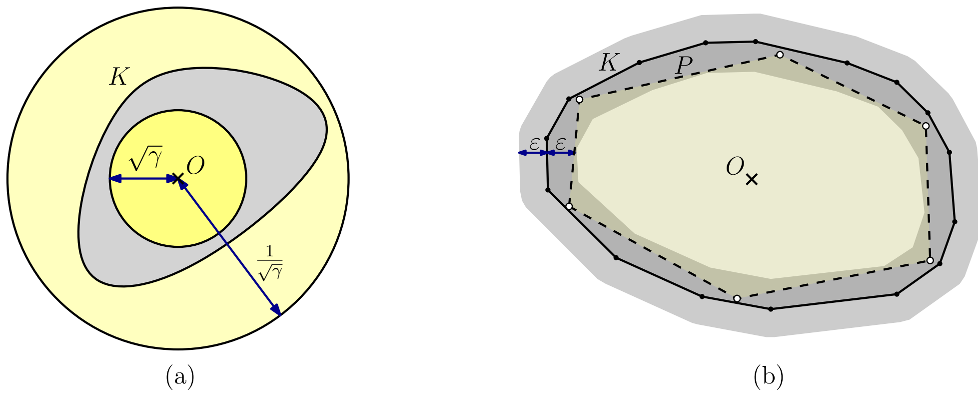

Figure 1: (a) A convex body K in γ-canonical form and (b) an inner ε-approximation P.

to Macbeath regions of volume $v = \varepsilon^{(d+1)/2}$, has the maximum thickness ε, and a layer of number i (that may be positive or negative) has thickness ε/i^2. The sum of the thicknesses of all layers is then given by $\sum_i \varepsilon/i^2 = O(\varepsilon)$, thus eliminating the wasteful log factor. The Macbeath regions inside each layer have width proportional to the thickness of the layer and, thanks to the volume-sensitive bound, the total number of Macbeath regions remains $O(1/\varepsilon^{(d-1)/2})$, yielding the optimal bound for the combinatorial complexity.

The remainder of the paper is organized as follows. In Section 2, we introduce the various geometric preliminaries upon which our construction relies, and summarize the salient properties of Macbeath regions, which are central to our construction. In Section 3, we investigate the relationship between ε-width caps in the primal body K and its polar K^*. In Section 4, we show that the number of disjoint Macbeath regions of width ε and volume $\Theta(v)$ is $O(\min(\varepsilon/v, v/\varepsilon^d))$. Finally, we prove Theorem 1.1 in Section 5.

2 Geometric Preliminaries

Much of the material in this section has been presented in [5–7] and can be skimmed on first reading. We include it here for the sake of completeness. The proofs of all the lemmas in this subsection that are omitted can be found in these papers or are straightforward adaptations of the proofs given therein.

Consider a convex body K in d-dimensional space $\mathbb{R}^d$. Let ∂K denote the boundary of K. Let O denote the origin of $\mathbb{R}^d$. Given a parameter $0 < \gamma \leq 1$, we say that K is *γ-fat* if there exist concentric Euclidean balls B and B', such that $B \subseteq K \subseteq B'$, and $\text{radius}(B)/\text{radius}(B') \geq \gamma$. We say that K is *fat* if it is γ-fat for a constant γ (possibly depending on d, but not on ε).

Let B_0 denote the ball of unit radius centered at the origin and for $\alpha > 0$, let αB_0 denote the ball of radius α centered at the origin. We say that a convex body K is in *γ-canonical form*[2] if it is nested between $\sqrt{\gamma}B_0$ and $B_0/\sqrt{\gamma}$ (see Figure 1(a)). A body in γ-canonical form is γ-fat and, for constant γ, it is fat and has $\Theta(1)$ diameter.

[2]This definition differs from our earlier papers but has the elegant feature that a convex body K is in γ-canonical form if and only if its polar K^* (see Section 2.2) is as well.

We say that a convex body P is an *ε-approximation* (see Figure 1(b)) to another convex body K if they are within Hausdorff error ε of each other. Further, we say that P is an *inner* (resp., *outer*) approximation if $P \subseteq K$ (resp., $P \supseteq K$). The next lemma shows that, up to constant factors, the problem of approximating a convex body can be reduced to the problem of approximating a convex body in canonical form. The proof is an easy consequence of John's Theorem [24]. (Also, see Lemma 2.1 of [6].)

Lemma 2.1. *Let $K \subset \mathbb{R}^d$ be a convex body. There exists a non-singular affine transformation T such that $T(K)$ is in $(1/d)$-canonical form and if P is any $(2\varepsilon/\sqrt{d})$-approximation to $T(K)$, then $T^{-1}(P)$ is an ε-approximating polytope to K.*

In light of this result, we may assume that K is presented in γ-canonical form, for any constant γ (depending on dimension), and that ε has been appropriately scaled. (This scaling will only affect the constant factors hidden in our asymptotic bounds. The transformation also preserves directionally sensitive notions of approximation [5].) Henceforth, we will focus on the problem of ε-approximating a convex body K in canonical form.

Finally, we define two useful notions of distance from the boundary of a convex body. Let K be a convex body and let x be a point. Define $\delta(x)$ to be the minimum distance from x to any point on ∂K. We define a ray-based notion of distance of a point x as well. Consider the intersection point p of ∂K and the ray emanating from O and passing through x. Define x's *ray-distance*, denoted $\mathrm{ray}(x)$, to be $\|xp\|$ (see Figure 2(a)). We have the following lemma, which shows that for points inside a convex body in γ-canonical form for constant γ, these two distance measures are the same to within a constant factor.

Lemma 2.2. *Let K be a convex body in γ-canonical form. For any point $x \in K$, $\delta(x) \leq \mathrm{ray}(x) \leq \delta(x)/\gamma$.*

2.1 Caps and Macbeath Regions

Given a convex body K, a *cap* C is defined to be the nonempty intersection of the convex body K with a halfspace (see Figure 2(b)). Let h denote the hyperplane bounding this halfspace. We define the *base* of C to be $h \cap K$. The *apex* of C is any point in the cap such that the supporting hyperplane of K at this point is parallel to h. The *width* of C, denoted $\mathrm{width}(C)$, is the distance between h and this supporting hyperplane. Given any unit vector u and any sufficiently small width w, there is a unique cap of width w whose base is orthogonal to u and lies on the same side of the origin as indicated by u. We refer to this as the cap that is *orthogonal* to u. Given any cap C of width w and a real $\lambda \geq 0$, we define its *λ-expansion*, denoted C^λ, to be the cap of K cut by a hyperplane parallel to and at distance λw from this supporting hyperplane. (Note that $C^\lambda = K$, if λw exceeds the width of K along the defining direction.)

We begin with some simple geometric facts about caps. An easy consequence of convexity is that, for $\lambda \geq 1$, C^λ is a subset of the region obtained by scaling C by a factor of λ about its apex. This implies the following lemma.

Lemma 2.3. *Let $K \subset \mathbb{R}^d$ be a convex body and $\lambda \geq 1$. For any cap C of K, $\mathrm{vol}(C^\lambda) \leq \lambda^d \cdot \mathrm{vol}(C)$.*

Another consequence of convexity is that containment of caps is preserved if the halfspaces defining both caps are consistently scaled about a point that is common to both caps. This is stated in the following lemma.

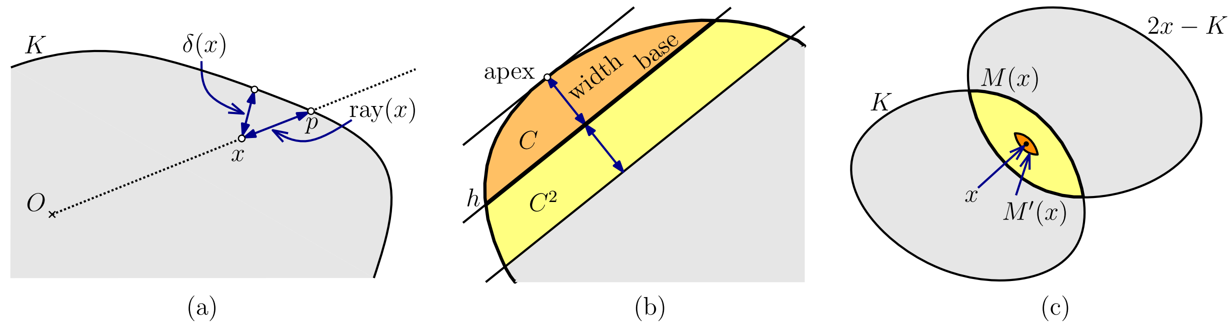

Figure 2: (a) Notions of distance, (b) cap concepts, and (c) Macbeath regions.

Lemma 2.4. *Let K be a convex body, and let $C_1 \subseteq C_2$ be two caps of K. Let H_1 and H_2 be their respective defining halfspaces, and let H_1^λ and H_2^λ be the respective halfspaces obtained by scaling by $\lambda \geq 1$ about any point $p \in C_1$. Then $K \cap H_1^\lambda \subseteq K \cap H_2^\lambda$.*

The next two lemmas apply to bodies in γ-canonical form. The first shows that for a point p in a cap near the boundary, the angle between Op and the normal to the base of the cap is bounded away from $\pi/2$. The second gives upper and lower bounds on the volume of a cap of width α.

Lemma 2.5. *Let $K \subset \mathbb{R}^d$ be a convex body in γ-canonical form for constant γ, and let Δ_0 be a sufficiently small constant (depending on d and γ). Let C be a cap of width at most Δ_0 and let p be any point inside C. Then the cosine of the angle between Op and the normal to the base of C is at least $\gamma/2$.*

Lemma 2.6. *Let $K \subset \mathbb{R}^d$ be a convex body in γ-canonical form for constant γ, and let $0 < \alpha < 1$ be a positive real. Then there exist constants c and c' (depending on d and γ) such that for any cap C of width α, $c\alpha^d \leq \mathrm{vol}(C) \leq c'\alpha$.*

Given a point $x \in K$ and a real parameter $\lambda \geq 0$, the *Macbeath region* $M^\lambda(x)$ (also called an *M-region*) is defined as:

$$M^\lambda(x) \;=\; x + \lambda((K - x) \cap (x - K)).$$

It is easy to see that $M^1(x)$ is the intersection of K and the reflection of K around x (see Figure 2(c)), and so $M^1(x)$ is centrally symmetric about x. $M^\lambda(x)$ is a scaled copy of $M^1(x)$ by the factor λ about x. We refer to x as the *center* of $M^\lambda(x)$ and to λ as its *scaling factor*. As a convenience, we define $M(x) = M^1(x)$ and $M'(x) = M^{1/5}(x)$. We refer to the latter as the *shrunken* Macbeath region.

Macbeath regions have found numerous uses in the theory of convex sets and the geometry of numbers (see Bárány [11] for an excellent survey). They have also been applied to a growing number of results in the field of computational geometry, particularly to construct lower bounds for range searching [9, 10, 14] and to bound the complexity of an ε-approximating polytope [3, 6].

Given any point $x \in K$, we define a *minimal cap* $C(x)$ to be the cap with minimum volume that contains x. There generally may be multiple minimum-volume caps containing a given point, and if so, $C(x)$ denotes any such cap. Clearly, the base of $C(x)$ must pass through x. In fact, a standard variational argument implies x is the centroid of the base (for otherwise, we could decrease the cap volume by an infinitesimal rotation of the base about x [23]). Indeed, our use of minimal caps is primarily due to the fact that they are well centered around their defining point, and not volume properties. We also let $C^\lambda(x)$ refer to the λ-expansion of $C(x)$, that is, $C^\lambda(x) = (C(x))^\lambda$.

We now present lemmas that encapsulate key properties of Macbeath regions, which will be useful in our analysis. The first lemma shows that if two shrunken Macbeath regions have a nonempty intersection, then a constant factor expansion of one contains the other [14, 23].

Lemma 2.7. *Let K be a convex body, and let $\lambda \leq 1/5$ be any real. If $x, y \in K$ such that $M^\lambda(x) \cap M^\lambda(y) \neq \emptyset$, then $M^\lambda(y) \subseteq M^{4\lambda}(x)$.*

The next two lemmas are useful in situations when we know that a Macbeath region partially overlaps a cap of K, and allow us to conclude that a constant factor expansion of the cap will fully contain the Macbeath region. The first applies to shrunken Macbeath regions and the second to Macbeath regions with scaling factor one.

Lemma 2.8. *Let K be a convex body. Let C be a cap of K and x be a point in K such that $C \cap M'(x) \neq \emptyset$. Then $M'(x) \subseteq C^2$.*

Lemma 2.9. *Let K be a convex body. If x is a point in a cap C of K, then $M(x) \subseteq C^2$.*

The next lemma shows that any sufficiently small cap is contained within a suitable constant factor expansion of the Macbeath region centered at the centroid of its base [14, 23]. In particular, it implies that the minimal cap associated with a point is contained within a suitable constant factor expansion of the Macbeath region centered at that point.

Lemma 2.10. *Let $K \subset \mathbb{R}^d$ be a convex body in γ-canonical form for constant γ, and let Δ_0 be a sufficiently small constant (depending on d and γ). Let C be a cap of K of width at most Δ_0 and let x denote the centroid of the base of this cap. Then $C \subseteq M^{3d}(x)$.*

The following four lemmas are easy consequences of standard properties of Macbeath regions. The first gives lower and upper bounds on the width of the minimal cap. The second generalizes Lemma 2.10 to caps formed by expanding the minimal cap. The third states that if the shrunken Macbeath regions associated with two caps overlap, then a constant factor expansion of any one cap is contained in a suitable constant factor expansion of the other. The last states that all the points in a shrunken Macbeath region have similar distances from the boundary of K.

Lemma 2.11. *Let K, C and x be as defined in Lemma 2.10. Then $\delta(x) \leq \mathrm{width}(C) \leq c\delta(x)$ for a suitable constant c (depending on d and γ).*

Lemma 2.12. *Let $\lambda \geq 1$ and let K, C, and x be as defined in Lemma 2.10. Then $C^\lambda \subseteq M^{3d(2\lambda-1)}(x)$.*

Lemma 2.13. *Let $K \subset \mathbb{R}^d$ be a convex body in γ-canonical form, where γ is a constant. Let Δ_0 be the constant of Lemma 2.10 and let $\lambda \geq 1$ be any real. There exists a constant $\beta \geq 2$ such that the following holds. Let C_1 and C_2 be any two caps of K of width at most Δ_0. Let x_1 and x_2 denote the centroids of the bases of the caps C_1 and C_2, respectively. If $M'(x_1) \cap M'(x_2) \neq \emptyset$, then $C_1^\lambda \subseteq C_2^{\beta\lambda}$.*

Lemma 2.14. *Let K be a convex body. If $x \in K$ and $x' \in M'(x)$, then $4\delta(x)/5 \leq \delta(x') \leq 4\delta(x)/3$.*

2.2 Polarity and the Mahler Volume

Some of our analysis will involve the well known concept of *polarity*. Let us recall some general facts (see, e.g., Eggleston [22]). Given vectors $u, v \in \mathbb{R}^d$, let $\langle u, v \rangle$ denote their dot product, and let $\|v\| = \sqrt{\langle v, v \rangle}$ denote v's Euclidean length. (Throughout, we use the terms *point* and *vector* interchangeably.) Given a bounded convex body $K \in \mathbb{R}^d$ that contains the origin in its interior, define its *polar*, denoted K^*, to be the convex set

$$K^* \;=\; \{u : \langle u, v \rangle \leq 1, \text{ for all } v \in K\}.$$

The polar enjoys many useful properties. For example, it is well known that K^* is bounded and $(K^*)^* = K$. Further, if K_1 and K_2 are two convex bodies such that $K_1 \subseteq K_2$ then $K_2^* \subseteq K_1^*$.

It will be convenient to also define the *polar* of a point. Given a point $v \in \mathbb{R}^d$ (not the origin), we define v^* to be the hyperplane that is orthogonal to v and at distance $1/\|v\|$ from the origin, on the same side of the origin as v. The polar of a hyperplane is defined as the inverse of this mapping. We may equivalently define K^* as the intersection of the closed halfspaces that contain the origin, bounded by the hyperplanes v^*, for all $v \in K$.

For any centrally symmetric convex body K and real $\lambda > 0$, we use λK to denote the body obtained by scaling K by a factor of λ about its center. Given any convex body K and real $\lambda \geq 1$, we say that a point $x \in K$ is λ-centered in K if there exists an ellipsoid E centered at x such that $E \subseteq K \subseteq \lambda E$. It is well known that every convex body K contains a d-centered point. This follows from the fact that the ellipsoid E of largest volume contained inside a convex body K, called the *John Ellipsoid*, satisfies $E \subseteq K \subseteq dE$. The following lemma shows that the centroid of K is $O(1)$-centered (the proof is presented in [25]).

Lemma 2.15. *For any convex body K, its centroid is d-centered.*

An important concept related to polarity is the *Mahler volume*, which is defined to be the product of the volumes of a convex body and its polar. There is a large literature on the Mahler volume (see, e.g., Kuperberg [26]). It is well known that the Mahler volume is affine invariant. The following lemma establishes a constant upper and lower bound on the Mahler volume of a convex body when the polar is computed with respect to an $O(1)$-centered point. Note that the convex body need not be centrally symmetric.

Lemma 2.16. *Let K be a convex body, let $x \in K$ be a λ-centered point for constant λ, and let K^* denote the polar of K with respect to x. Then* $\mathrm{vol}(K) \cdot \mathrm{vol}(K^*) = \Theta(1)$.

Proof. By definition, there is an ellipsoid E centered at x such that $E \subseteq K \subseteq \lambda E$. By standard properties of the polar transformation, we have $(\lambda E)^* \subseteq K^* \subseteq E^*$. Since $(\lambda E)^* = (1/\lambda)E^*$, it follows that $(1/\lambda)E^* \subseteq K^* \subseteq E^*$. Thus

$$\begin{aligned} \frac{1}{\lambda^d} \cdot \mathrm{vol}(E) \cdot \mathrm{vol}(E^*) \;&\leq\; \mathrm{vol}(K) \cdot \mathrm{vol}(K^*) \\ &\leq\; \lambda^d \cdot \mathrm{vol}(E) \cdot \mathrm{vol}(E^*). \end{aligned}$$

Since the Mahler volume is invariant under linear invertible transformation, $\mathrm{vol}(E) \cdot \mathrm{vol}(E^*)$ equals the square of the volume of a unit ball, which is $\Theta(1)$. The desired result follows immediately. $\square$

The following lemma is now an immediate consequence of Lemmas 2.15 and 2.16.

Lemma 2.17. *Let K be a convex body and let K^* denote the polar of K with respect to its centroid. Then* $\mathrm{vol}(K) \cdot \mathrm{vol}(K^*) = \Theta(1)$.

3 Caps of the Polar

Let K denote a convex body. The goal of this section is to establish certain fundamental and novel relationships between ε-width caps in K and its polar K^*. Assuming K satisfies certain fatness assumptions, we show that the base of a cap in K and the base of a corresponding cap in K^*, treated as $(d-1)$-dimensional bodies, are roughly related as polars of each other. We establish this in Lemmas 3.1 and 3.2.

It will be useful to consider the notion of a cap in a dual setting (see, e.g., [3,4]). Given a convex body K and a point z that is exterior to K, we define the *dual cap* of K with respect to z to be the set of $(d-1)$-dimensional hyperplanes that pass through z and do not intersect K's interior (see Figure 3). If K is full dimensional and contains the origin, it follows that a hyperplane h lies in the dual cap if and only if the point h^* lies on the base of the cap of K^* defined by the hyperplane z^*. We can define the polar of a dual cap as the set of points that results by taking the polar of each hyperplane of the dual cap.

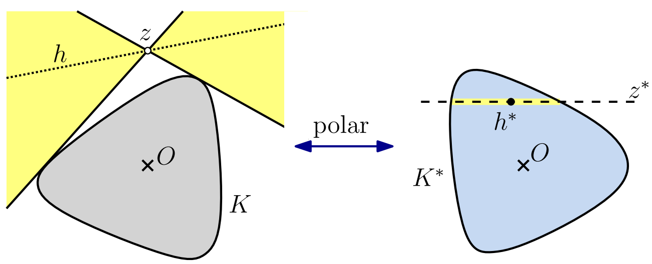

Figure 3: Definition of a dual cap and its polar.

The following lemma will be useful in proving Lemma 3.2, wherein we establish a polar relationship between the base of a cap in a convex body and the base of a corresponding cap in its polar. For the sake of concreteness, we state our results in terms of an arbitrary direction, which we call *vertical*, and any hyperplane orthogonal to this direction is called *horizontal*. Since the direction is arbitrary, there is no loss of generality.

Lemma 3.1. *Let $z \in \mathbb{R}^d$ be a point that lies on a vertical ray from the origin O, and let K be a $(d-1)$-dimensional convex body whose interior intersects the segment Oz at some point x (see Figure 4). Let G be the polar of the dual cap of K with respect to z, let $\overline{K}$ be the vertical projection of K, and let h be the hyperplane parallel to K passing through z. Then $G - h^* = \alpha\overline{K}^*$, where $\alpha = \|xz\|/\|Oz\|$.*

Note that G is a $(d-1)$-dimensional convex body that lies on the horizontal hyperplane z^*. Since h passes through z and does not intersect K, h^* is a point lying within G, and therefore the translate $G - h^*$ is horizontal and contains the origin. Since $\overline{K}$ is also horizontal and contains the origin, so does $\overline{K}^*$ (see Figure 4).

Proof. Let u be the unit vector orthogonal to K, and let v be the unit vector in the vertical direction. Let g be any hyperplane as described above. Letting $w = g^*$, it follows that g is the set of points p satisfying $\langle w, p\rangle = 1$. In order to find the polar of $\overline{K}$, our approach is to find the equation of the hyperplane g' whose normal has no vertical component and which passes through the vertical projection of the intersection of g with the hyperplane containing K. Clearly, by taking

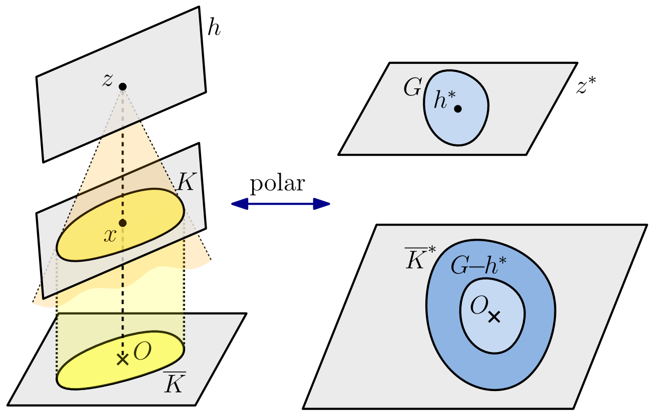

Figure 4: Statement of Lemma 3.1.

the polar of this hyperplane, we will obtain a point in $\overline{K}^*$ and, moreover, all points in $\overline{K}^*$ can be generated in this manner.

We claim that the hyperplane g' is defined by the equation $\langle w', p\rangle = \alpha$, where $w' = w - u/\langle u, z\rangle$. Note that w' has no vertical component since $\langle w', z\rangle = 0$. To show that g' passes through the vertical projection of the intersection of g with the hyperplane containing K, let y be any point on the intersection of g with the hyperplane that contains K. We may express y as $y = x + a$, where $\langle a, u\rangle = 0$. The vertical projection of y, denoted y', equals $y - \langle y, v\rangle\, v$. Using the facts that $\langle u, a\rangle = 0$ and $\langle w, y\rangle = \langle w, z\rangle = 1$, we obtain

$$\begin{aligned}
\langle w', y'\rangle &= \left\langle w - \frac{1}{\langle u, z\rangle} u, y - \langle y, v\rangle v \right\rangle \\
&= \langle w, y\rangle - \frac{\langle u, y\rangle}{\langle u, z\rangle} - \langle y, v\rangle \cdot \langle w, v\rangle + \frac{\langle y, v\rangle}{\langle u, z\rangle} \cdot \langle u, v\rangle \\
&= 1 - \frac{\langle u, y\rangle}{\langle u, z\rangle} - \langle y, v\rangle \frac{\langle w, z\rangle}{\|z\|} + \frac{\langle y, v\rangle}{\|z\|} \\
&= 1 - \frac{\langle u, x + a\rangle}{\left\langle u, \frac{1}{1-\alpha} x \right\rangle} \;=\; 1 - (1-\alpha)\frac{\langle u, x\rangle}{\langle u, x\rangle} \;=\; \alpha,
\end{aligned}$$

which proves the claim.

Taking the polar of g', we obtain the point $w'/\alpha = (w - u/\langle u, z\rangle)/\alpha$, which lies in $\overline{K}^*$. Note that this point is a scaled and translated version of w, the polar of g. Since each hyperplane g generates a point in both $\overline{K}^*$ and G related by this transformation, and every point in both these bodies is generated by some hyperplane g, we see that $G - u/\langle u, z\rangle = \alpha\overline{K}^*$. Observe that $u/\langle u, z\rangle$ is the vector orthogonal to h that lies on G, so it must be h^*. Thus, $G - h^* = \alpha\overline{K}^*$, completing the proof. □

For the rest of this section, we focus on the case where K is a convex body in γ-canonical form for constant γ. In order to relate approximating elements for K with approximating elements for K^*, we introduce a mapping of ε-width caps of K to $\Theta(\varepsilon)$-width caps of K^*. Let C be an ε-width cap of K. For concreteness, assume that space has been rotated so that C's base is horizontal with C lying above the origin. For a suitably large constant c (to be specified later), shoot a ray vertically upwards from O, and let $x \in K^*$ be a point on this ray such that $\delta(x) = \varepsilon/c$. Define $\pi(C)$ to be a minimum volume cap of K^* that contains x (see Figure 5). Our next lemma shows that there is a polar relationship between the bases of C and $\pi(C)$.

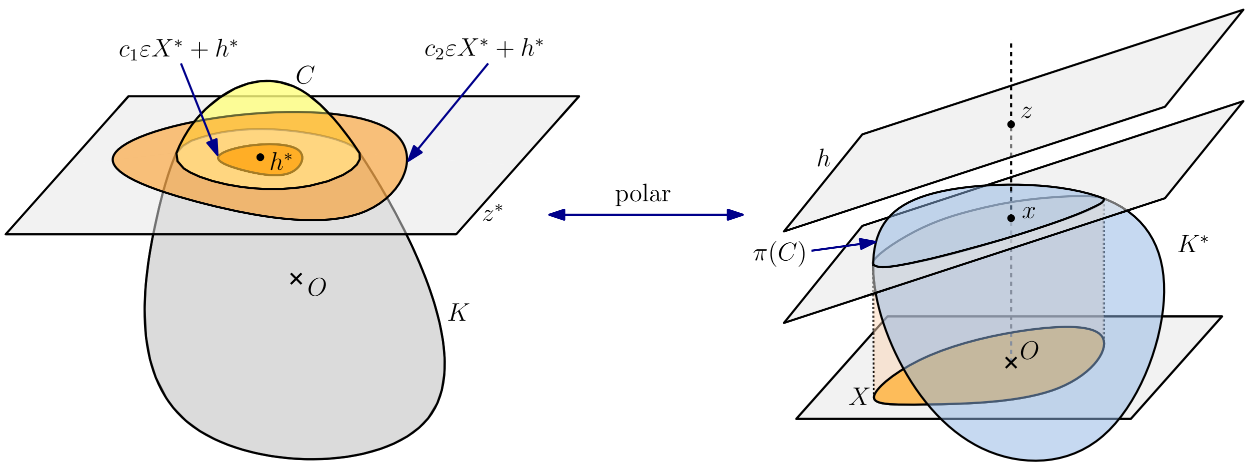

Figure 5: Statement of Lemma 3.2.

Lemma 3.2. *Given a convex body K in γ-canonical form for a constant γ, there exist constants c_1, c_2 such that the following holds. Let C be a horizontal ε-width cap of K lying above the origin. Let X be the vertical projection of the base of $\pi(C)$. Let z be the point that is the polar of the hyperplane passing through* $\text{base}(C)$, *and let h be the hyperplane parallel to* $\text{base}(\pi(C))$ *passing through z. Then $c_1\varepsilon X^* \subseteq \text{base}(C) - h^* \subseteq c_2\varepsilon X^*$.*

Proof. Let z' be the point of intersection of the vertical ray Ox with ∂K^* (see Figure 6). By the definition of the polar transformation, z lies on the ray Ox (outside K^*) such that $c_1'\varepsilon \le \|z'z\| \le c_2'\varepsilon$ for suitable constants c_1' and c_2'. Let $B_1 = \text{base}(\pi(C))$. Let D and D_1 be the dual caps of K^* and B_1, respectively, with respect to z. Recall that the polar of D is $\text{base}(C)$. Let G_1 be the polar of D_1. Since $B_1 \subseteq K^*$, we have $D \subseteq D_1$. It follows that $\text{base}(C) \subseteq G_1$. Applying Lemma 3.1, we obtain $G_1 - h^* = (\|xz\|/\|Oz\|)X^*$. Since $\|xz\| = \Theta(\varepsilon)$ and $\|Oz\| = \Theta(1)$, it follows that $\text{base}(C) - h^* \subseteq c_2\varepsilon X^*$, for a suitable constant c_2. This establishes one part of the desired inequality.

We establish the other part of the inequality using a similar approach. The key insight is that a scaled and translated version of B_1 has the property that its dual cap with respect to z is a subset of D, the dual cap of K^* with respect to z. Towards this end, let ψ_1 be the (infinite) generalized cone defined by rays shot from z' to B_1. Note that ψ_1 will enclose $K^* \setminus \pi(C)$ due to convexity. Consider the generalized cone Γ_1 with apex O and base B_1. Scale this cone about O by a suitable factor f_1 to produce a cone Γ_2 whose base (call it B_2) is tangent to K^*. Since $\delta(x) = \varepsilon/c$ and, by Lemma 2.11, $\text{width}(\pi(C)) = O(\delta(x))$, it is easy to see that $f_1 \le 1 + c_3'(\varepsilon/c)$ for a suitable constant c_3'. Note that Γ_2 encloses the entirety of $\pi(C)$. We now expand ψ_1 by this same factor f_1 about O to produce ψ_2. Clearly, ψ_2 encloses both $K^* \setminus \pi(C)$ and $\pi(C)$ itself, so it encloses all of K^*. For sufficiently large constant c, we can show that the apex of ψ_2 lies below z. This is because the distance between z' and the apex of ψ_2 is $(f_1 - 1)\|Oz'\| \le c_3'(\varepsilon/c)\|Oz'\| \le c_3'(\varepsilon/c)(1/\sqrt{\gamma})$. Recall that z is at distance at least $c_1'\varepsilon$ from z'. Thus, by choosing c larger than $c_3'/(c_1'\sqrt{\gamma})$, we can ensure that the apex of ψ_2 lies below z. We scale ψ_1 about O such that the apex of the resulting cone ψ_3

is z. The scaling factor f_2 is $\|Oz\|/\|Oz'\| = 1 + \Theta(\varepsilon)$. We scale B_1 by the same factor f_2 about O to obtain B_3. Clearly ψ_3 encloses the entirety of K^*. Further, B_3 is at vertical distance $\Theta(\varepsilon)$ below z. To see this, let x' be the point of intersection of Oz with B_3. Note that $\|x'z\| = f_2\|xz'\| = \Theta(\varepsilon)$.

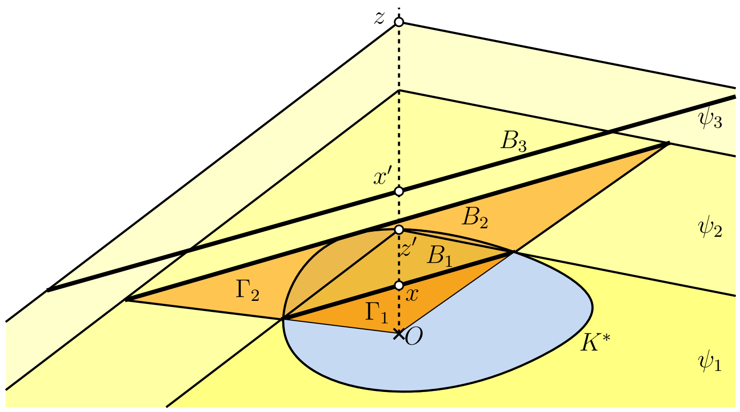

Figure 6: Definitions in the proof of Lemma 3.2.

Let D_3 be the dual cap of B_3 with respect to z, and let G_3 be the polar of D_3. Since $D_3 \subseteq D$, it follows that $G_3 \subseteq \text{base}(C)$. Applying Lemma 3.1, we obtain $G_3 - h^* = (\|x'z\|/\|Oz\|)(X')^*$, where X' is the vertical projection of B_3 (and so is a constant-factor expansion of X). Since $\|x'z\| = \Theta(\varepsilon)$ and $\|Oz\| = \Theta(1)$, it follows that $G_3 - h^* = c_1\varepsilon X^*$, for some constant c_1. Thus, we have $c_1\varepsilon X^* \subseteq \text{base}(C) - h^*$, which is the other part of our desired inequality. □

Given this polar-like relationship between C and $\pi(C)$, we can apply the Mahler volume to bound the product of their volumes.

Lemma 3.3. *Let C be as defined in Lemma 3.2. Then* $\text{vol}(C) \cdot \text{vol}(\pi(C)) = \Theta(\varepsilon^{d+1})$.

Proof. Recall that K is in γ-canonical form for constant γ. Thus K^* is also in γ-canonical form for constant γ and, by Lemma 2.5, $\text{area}(\text{base}(\pi(C))) = \Theta(\text{area}(X))$. By Lemma 3.2, base$(C)$ is sandwiched between two scaled copies of X^*, where the scaling factor is $\Theta(\varepsilon)$. Thus $\text{area}(\text{base}(C)) = \Theta(\varepsilon^{d-1}\,\text{area}(X^*))$. We have

$$\begin{aligned}&\text{area}(\text{base}(C)) \cdot \text{area}(\text{base}(\pi(C)))\\ &= \Theta(\varepsilon^{d-1}\,\text{area}(X^*) \cdot \text{area}(X)) = \Theta(\varepsilon^{d-1}).\end{aligned}$$

In the last step we have used the fact that for any minimal volume cap containing a point x, x is the centroid of the base of the cap. Thus, it follows from the definition of $\pi(C)$ and X that X^* is the polar of X with the centroid of X as origin, and so $\text{area}(X^*) \cdot \text{area}(X) = \Theta(1)$ by Lemma 2.17. Finally, the lemma follows by noting that the caps C and $\pi(C)$ each have $\Theta(\varepsilon)$ width and thus their volumes are $\Theta(\varepsilon)$ times the areas of their respective bases. □

Next, we show that the bound on the product of volumes holds within the neighborhood of the ray, and specifically to any shrunken Macbeath region that intersects the ray. This will be applied in Section 4 to establish our volume-sensitive bounds on the number of Macbeath regions.

Lemma 3.4. *Let C be a horizontal ε-width cap of K lying above the origin. Let $y \in K^*$ be a point at distance ε/c from ∂K^*, where c is a sufficiently large constant. Suppose that the ray* *from the origin shot vertically upwards intersects the Macbeath region $R = M'(y)$ of K^*. Then $\mathrm{vol}(C) \cdot \mathrm{vol}(R) = \Theta(\varepsilon^{d+1})$.*

Proof. Let x be a point in the intersection of $M'(y)$ with the ray shot vertically upwards from O, and let E denote the minimum volume cap containing x. By Lemma 2.14, $4\delta(y)/5 \leq \delta(x) \leq 4\delta(y)/3$. It follows from Lemma 3.3 that $\mathrm{vol}(C) \cdot \mathrm{vol}(E) = \Theta(\varepsilon^{d+1})$. By Lemmas 2.8 and 2.10, we have $\mathrm{vol}(M'(x)) = \Theta(\mathrm{vol}(E))$. As $M'(x)$ intersects R, by Lemma 2.7, a constant factor expansion of R encloses $M'(x)$ and vice versa, so $\mathrm{vol}(R) = \Theta(\mathrm{vol}(M'(x)))$. Thus $\mathrm{vol}(C) \cdot \mathrm{vol}(R) = \Theta(\varepsilon^{d+1})$, as desired. ☐

Intuitively, the points inside a scaled Macbeath region $M'(x)$ are similar in many respects. Two points within a shrunken Macbeath region in the polar body can be thought of as generating similar caps in the original body in the sense of satisfying a "sandwiching" property. This property will be used in Section 4 to allow us to focus on a discrete set of caps as defined by a finite collection of Macbeath regions. This will be proved in Lemma 3.6 below. The following technical lemma will be useful for proving it.

Lemma 3.5. *Consider a Macbeath region $R = M'(y)$ for K. Consider two rays r and r' shot from the origin through R (see Figure 7). Let $z \notin K$ be a point on r. Let h be a hyperplane passing through z that does not intersect K. Let z' be the point of intersection of r' with h. Then, $\mathrm{ray}(z') = O(\mathrm{ray}(z) + \delta(y))$.*

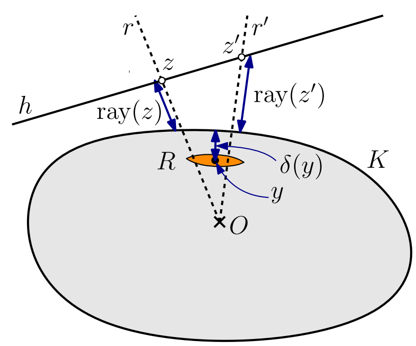

Figure 7: Statement of Lemma 3.5.

Proof. Consider a hyperplane h' that is parallel to h and passes through a point $p \in r \cap M$ (see Figure 8). We claim that the distance between h and h' is $O(\mathrm{ray}(z)+\delta(y))$. To see this, let t be the point of intersection of r with ∂K. By Lemma 2.14, $\delta(p) = O(\delta(y))$. Applying Lemma 2.2, we have $\|pt\| = \mathrm{ray}(p) = O(\delta(p))$. It follows that $\|pz\| = \|pt\| + \|tz\| = \mathrm{ray}(p) + \mathrm{ray}(z) = O(\mathrm{ray}(z) + \delta(y))$, which proves the claim.

Let C be the cap induced by h'. Since C intersects R, by Lemma 2.8 the cap C^2 encloses R. Let h'' denote the hyperplane passing through the base of C^2. Observe that the distance between h'' and h is no more than twice the distance between h' and h, and is thus $O(\mathrm{ray}(z) + \delta(y))$.

Let p' be any point in $r' \cap M$. It follows easily from Lemma 2.5 that $\|p'z'\|$ is at most a constant times the distance between h'' and h. Thus, $\|p'z'\| = O(\mathrm{ray}(z) + \delta(y))$. It follows that $\mathrm{ray}(z') = O(\mathrm{ray}(z) + \delta(y))$, as desired. ☐

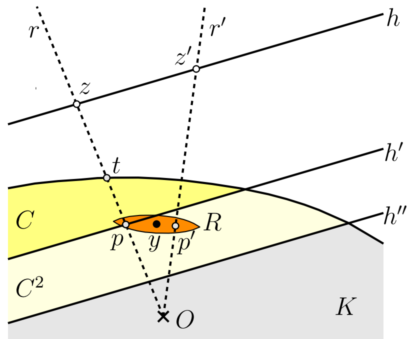

Figure 8: Proof of Lemma 3.5.

Lemma 3.6. *For any choice of constants c_1 and c_2, there exist constants c_0, σ, such that the following holds. Let $R = M'(y)$ be any Macbeath region of K^*, where $\delta(y) = \varepsilon/c_0$ (see Figure 9). There is a point $z \notin K^*$ on the ray Oy such that the following holds. Consider any ray r from the origin that intersects R. Let C be a cap of K whose base is orthogonal to r such that $c_1\varepsilon \leq \mathrm{width}(C) \leq c_2\varepsilon$. Then the cap E of K induced by the hyperplane z^* satisfies $E^{1/\sigma} \subseteq C \subseteq E$.*

Proof. Since K is in γ-canonical form, it is easy to see that the polar of any hyperplane that induces a cap of K of width between $c_1\varepsilon$ and $c_2\varepsilon$ is a point outside K^*, whose ray-distance with respect to K^* is at least $c_1'\varepsilon$ and at most $c_2'\varepsilon$ for suitable constants c_1' and c_2' (depending only on c_1, c_2, d, and γ). Let β be a sufficiently large constant. We assume that the constant c_0 in the statement of this lemma is β/c_1'. In other words, $\delta(y) = c_1'\varepsilon/\beta$.

Let $z, z' \notin K^*$ be points on the ray Oy such that $\mathrm{ray}(z) = c_2'\beta\varepsilon$ and $\mathrm{ray}(z') = c_1'\varepsilon/\beta$. Let E and E' be caps of K induced by hyperplanes z^* and $(z')^*$, respectively. Let r be the ray given in the statement of the lemma (drawn vertically in Figure 9), and let C be as described in the lemma. Let x be the polar of the hyperplane passing through the base of C. Recall that $c_1'\varepsilon \leq \mathrm{ray}(x) \leq c_2'\varepsilon$.

We claim that $E' \subseteq C \subseteq E$. Since $\mathrm{ray}(z') = c_1'\varepsilon/\beta$, $\delta(y) = c_1'\varepsilon/\beta$ and $\mathrm{ray}(x) \geq c_1'\varepsilon$, it follows that $\mathrm{ray}(x) \geq (\beta/2)(\mathrm{ray}(z') + \delta(y))$. Since β was chosen to be a sufficiently large constant, by Lemma 3.5, any hyperplane passing through z' that does not intersect K^* separates K^* from x. Thus, taking the polar, it follows that $E' \subseteq C$. Similarly, since $\mathrm{ray}(x) \leq c_2'\varepsilon$, $\delta(y) = c_1'\varepsilon/\beta$, and $\mathrm{ray}(z) = c_2'\beta\varepsilon$, it follows that $\mathrm{ray}(z) \geq (\beta/2)(\mathrm{ray}(x) + \delta(y))$. Consequently, by Lemma 3.5, any

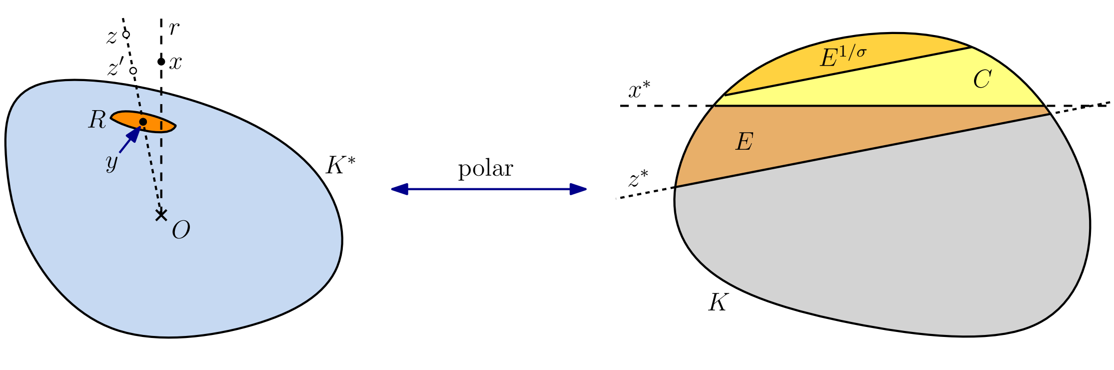

Figure 9: Statement and proof of Lemma 3.6.

hyperplane passing through x that does not intersect K^* separates K^* from z. Thus, taking the polar, $C \subseteq E$. Putting it together, we obtain $E' \subseteq C \subseteq E$.

Let w denote the distance from O to ∂K^* along the ray Oy. By basic properties of the polar transformation, the widths of the caps E and E' are

$$\left(\frac{1}{w} - \frac{1}{w + c'_2 \beta\varepsilon}\right) \text{ and } \left(\frac{1}{w} - \frac{1}{w + c'_1 \varepsilon/\beta}\right),$$

respectively. It is now easy to verify that $\text{width}(E)/\,\text{width}(E') = O(\beta^2)$, which is bounded above by some constant σ. Thus $E^{1/\sigma} \subseteq E'$. Since $E' \subseteq C \subseteq E$, it follows that $E^{1/\sigma} \subseteq C \subseteq E$. ☐

4 Volume of Macbeath Regions

In this section, we present a bound on the number of disjoint shrunken Macbeath regions associated with ε-width caps, that is sensitive to the volume. In turn, this volume-sensitive bound leads to a more elegant proof of the bound on the total number of disjoint Macbeath regions associated with ε-width caps. Our proofs are based on the relationship we developed in the last section between a cap in a convex body and the corresponding cap in its polar.

Lemma 4.1. *Let K be a convex body in γ-canonical form for constant γ. There exists a set $\mathscr{R}$ of Macbeath regions $M'(x)$ of K, where $\delta(x) = \varepsilon$, such that the following properties hold:*

(i) For any $v > 0$, the number of Macbeath regions in $\mathscr{R}$ of volume $\Theta(v)$ is $O(\varepsilon/v)$.

(ii) Any ray emanating from the origin of K intersects some Macbeath region of $\mathscr{R}$.

Proof. Let $\mathscr{R}'$ be a maximal set of disjoint Macbeath regions $M^\lambda(x)$, where $\lambda = 1/20$ and $\delta(x) = \varepsilon$. We scale each Macbeath region of $\mathscr{R}'$ by a factor of 4 about its center to obtain the set $\mathscr{R}$.

Observe that all the Macbeath regions of $\mathscr{R}'$ lie within distance $O(\varepsilon)$ of ∂K, and so must lie in a region of volume $O(\varepsilon)$. For any v, it follows from disjointness that $\mathscr{R}'$ contains $O(\varepsilon/v)$ Macbeath regions of volume $\Theta(v)$. Since the Macbeath regions of $\mathscr{R}$ are obtained by scaling the Macbeath regions of $\mathscr{R}'$ by a constant factor, this proves property (i).

To prove (ii), consider any ray emanating from the origin of K. Let y be a point on this ray such that $\delta(x) = \varepsilon$. Consider the Macbeath region $R_1 = M^\lambda(y)$. Observe that, since $\mathscr{R}'$ is maximal, R_1 must intersect a Macbeath region $R_2 \in \mathscr{R}'$. Thus, by Lemma 2.7, the 4-factor expansion of R_2 must fully enclose R_1. By construction, the 4-factor expansion of R_2 is an element $R \in \mathscr{R}$. As R encloses R_1, R encloses y as well. Thus, any ray emanating from the origin must intersect some element of $\mathscr{R}$. ☐

We are now ready to prove the volume-sensitive bound on the number of disjoint Macbeath regions.

Theorem 4.2. *Let $K \subset \mathbb{R}^d$ be a convex body in γ-canonical form for constant γ. Let $\mathscr{C}$ be a set of caps of K of width $\Theta(\varepsilon)$ and volume $\Theta(v)$, such that the Macbeath regions $M'(x)$ centered at the centroids x of the bases of these caps are disjoint. Then*

$$|\mathscr{C}| = O\left(\min\left(\frac{\varepsilon}{v}, \frac{v}{\varepsilon^d}\right)\right).$$

Proof. Let $\mathscr{R}$ be the set of Macbeath regions described in the lemma. We will establish the bound given in the lemma for $\mathscr{R}$. As $|\mathscr{C}| = |\mathscr{R}|$, this will prove the lemma.

As in the proof of Lemma 4.1, the $O(\varepsilon/v)$ bound on $|\mathscr{R}|$ follows easily from disjointness and the fact that the Macbeath regions of $\mathscr{R}$ are all fully contained within distance $O(\varepsilon)$ of ∂K. In the rest of the proof, we will show that $|\mathscr{R}| = O(v/\varepsilon^d)$.

Construct the set of Macbeath regions as described in Lemma 4.1, with the convex body referred to in the lemma being the polar body K^*, and with ε in the lemma set to ε/c for sufficiently large constant c. Call this set $\mathscr{R}'$. With each Macbeath region $R' \in \mathscr{R}'$, we associate a canonical cap E of K as described in the statement of Lemma 3.6. We will show that, for every Macbeath region $R \in \mathscr{R}$, there is a canonical cap E which satisfies the following properties: (i) $R \subseteq E^2$, (ii) $\mathrm{vol}(E^2) = O(v)$, and (iii) it is associated with a Macbeath region $R' \in \mathscr{R}'$ whose volume is $\Theta(V)$, where $V = \varepsilon^{d+1}/v$.

Consider a Macbeath region $R \in \mathscr{R}$ and let C be the associated cap of $\mathscr{C}$. Let h be the base of C. Shoot a ray orthogonal to h from the origin. From Lemma 4.1(ii), we know that this ray will intersect some Macbeath region $R' \in \mathscr{R}'$. We will show that the canonical cap E associated with R' satisfies the above properties (i)–(iii).

By Lemma 3.6, we have $E^{1/\sigma} \subseteq C \subseteq E$. By Lemma 2.8, we have $R \subseteq C^2$ and, by Lemma 2.10, we have $C \subseteq R^{15d}$. Thus $\mathrm{vol}(R) = \Theta(\mathrm{vol}(C)) = \Theta(v)$. To prove (i), we apply Lemma 2.4. Since $C \subseteq E$, the lemma implies that $C^2 \subseteq E^2$. Since $R \subseteq C^2$, it follows that $R \subseteq E^2$. To prove (ii), note that by Lemma 2.3, $\mathrm{vol}(E^2) = O(\mathrm{vol}(E^{1/\sigma}))$. Since $E^{1/\sigma} \subseteq C$, we have $\mathrm{vol}(E^2) = O(\mathrm{vol}(C)) = O(v)$. To prove (iii), we use the fact that $\mathrm{vol}(C) = \Theta(v)$ and apply Lemma 3.4. It follows that the volume of R' is $\Theta(V)$, where $V = \varepsilon^{d+1}/v$.

To bound $|\mathscr{R}|$, recall that the Macbeath regions of $\mathscr{R}$ are disjoint and have volume $\Theta(v)$. By properties (i) and (ii), each Macbeath region of $\mathscr{R}$ is contained in some cap E^2 of volume $O(v)$. By a straightforward packing argument, the number of Macbeath regions that can be contained in such a cap is $O(1)$. Further, by property (iii), these caps are associated with Macbeath regions $R' \in \mathscr{R}'$ of volume $\Theta(V)$. It follows that the number of Macbeath regions of $\mathscr{R}$ is asymptotically bounded by the number of Macbeath regions $R' \in \mathscr{R}'$ of volume $\Theta(V)$. By Lemma 4.1(i), the number of such Macbeath regions R' is $O(\varepsilon/V) = O(v/\varepsilon^d)$. This completes the proof. $\square$

We note that using the previous theorem, the bound on the total number of disjoint Macbeath regions from [6] follows easily.

Corollary 4.3. *Let $K \subset \mathbb{R}^d$ be a convex body in γ-canonical form for constant γ. Let $\mathscr{C}$ be a set of caps of K of width $\Theta(\varepsilon)$, such that the Macbeath regions $M'(x)$ centered at the centroids x of the bases of these caps are disjoint. Then $|\mathscr{C}| = O(1/\varepsilon^{(d-1)/2})$.*

Proof. Partition $\mathscr{C}$ into disjoint sets containing caps of $\mathscr{C}$ whose volumes differ by a factor of at most 2. By Lemma 2.6, the volume of any cap in $\mathscr{C}$ is $\Omega(\varepsilon^d)$ and $O(\varepsilon)$, and so there are are $O(\log \frac{1}{\varepsilon})$ such sets. Consider a set S in the partition containing caps of volume between v and $2v$, where $v < \varepsilon^{(d+1)/2}$. By Theorem 4.2, $|S| = O\left(\min\left(\frac{\varepsilon}{v}, \frac{v}{\varepsilon^d}\right)\right) = O\left(\frac{v}{\varepsilon^d}\right)$. Summing up the cardinalities of all those sets S corresponding to volumes $v < \varepsilon^{(d+1)/2}$, we obtain a geometric progression that sums to $O(1/\varepsilon^{(d-1)/2})$. Next consider a set S' in the partition containing caps of volume between v and $2v$, where $v \geq \varepsilon^{(d+1)/2}$. By Theorem 4.2, $|S'| = O\left(\min\left(\frac{\varepsilon}{v}, \frac{v}{\varepsilon^d}\right)\right) = O\left(\frac{\varepsilon}{v}\right)$. Summing up the cardinalities of all those sets S' corresponding to volumes $v \geq \varepsilon^{(d+1)/2}$, we obtain a geometric progression that also sums to $O(1/\varepsilon^{(d-1)/2})$. Putting together the two cases, which together address the entirety of $\mathscr{C}$, we see that $|\mathscr{C}| = O(1/\varepsilon^{(d-1)/2})$. $\square$

5 Combinatorial Complexity

The bound on the total combinatorial complexity of approximating polytopes established in [6] is sub-optimal by a factor that is polynomial in $\log \frac{1}{\varepsilon}$. In this section, we show how to apply the volume-sensitive bounds developed in the previous sections to eliminate this overhead and obtain an optimal bound. The key idea is to reduce the width of Macbeath regions that have either very large or very small volumes. The reason we can do so is that our volume-sensitive bounds show that their numbers are low, so we can afford the increase in their number that may come about through the use of thinner Macbeath regions. Overall, we can adjust parameters to maintain the asymptotic bound of $O(1/\varepsilon^{(d-1)/2})$ on the total number of Macbeath regions. By using thinner Macbeath regions, we can house them in thinner layers during the stratification process. As the approximation quality of this method is determined by the total width of all the layers, our new strategy allows us to improve the approximation quality. To be precise, we can reduce the total width of all the layers from $O(\varepsilon \log \frac{1}{\varepsilon})$ in the construction of [6] to $O(\varepsilon)$ which, in turn, allows us to eliminate the polylog$(1/\varepsilon)$ overhead and obtain an optimal bound for the combinatorial complexity.

5.1 Cap Covering

In this subsection, we present a variant of the cap covering lemma proved in [6]. We will employ this lemma later in establishing an optimal bound on the combinatorial complexity of an approximating polytope. A novel feature of this variant is that the canonical caps in the cover have different widths.

Before presenting the lemma, we need some definitions. Throughout this section, we assume that the convex body K is in γ-canonical form for constant γ. For any integer j, define $v_j = \varepsilon^{(d+1)/2} \cdot 2^j$, $a_j = \max(j^2, 1)$, and $w_j = \varepsilon / a_j$. We say that a cap C of K is of type j if $v_j \leq \mathrm{vol}(C) < 2v_j$. By Lemma 2.6, the volume of any ε-width cap of K is at least $\Omega(\varepsilon^d)$ and at most $O(\varepsilon)$. It follows that an ε-width cap is of type j, where j takes integral values ranging from $-O(\log \frac{1}{\varepsilon})$ to $+O(\log \frac{1}{\varepsilon})$. We say that a set of caps $\mathscr{C}$ is *ε-balanced* if there exist constants b_1 and b_2 such that, for all j, any type-j cap $C \in \mathscr{C}$ satisfies $b_1 w_j \leq \mathrm{width}(C) \leq b_2 w_j$. More precisely, we say that $\mathscr{C}$ is (ε, b_1, b_2)-balanced. Note that balanced caps of type 0 have volume $\Theta(\varepsilon^{(d+1)/2})$ and width $\Theta(\varepsilon)$. Roughly speaking, balanced caps whose volumes are larger or smaller than type 0 caps by a factor of 2^j have widths that are smaller by a factor of j^2. We will prove each property of the following lemma separately.

Lemma 5.1. *Let K be a convex body in γ-canonical form for constant γ. There exists a set $\mathscr{R}$ of disjoint centrally symmetric convex bodies $R_1, \ldots, R_k$, and a collection $\mathscr{C}$ of associated ε-balanced caps $C_1, ..., C_k$ such that the following hold for some constant σ (depending only on d and γ):*

(1) $k = O(1/\varepsilon^{(d-1)/2})$.

(2) *For each i, $R_i \subseteq C_i \subseteq R_i^{\sigma}$.*

(3) *For any direction u, there is a cap C whose base is orthogonal to u, and which satisfies $R_i \subseteq C$ and $C_i^{1/\sigma} \subseteq C \subseteq C_i$, for some i.*

Let $\mathscr{A} = \{A_1, \ldots, A_k\}$ be a maximal set of (ε, b_1, b_2)-balanced caps, such that the Macbeath regions $M'(x_i)$ centered at the centroids x_i of the bases of the caps $A_i^{1/\beta}$ are disjoint. Here β is the constant of Lemma 2.13, and b_1 and b_2 are constants that we will specify later in the proof of Lemma 5.4. We let $R_i = M'(x_i)$ and $C_i = A_i^{\beta}$. Let $\mathscr{R}, \mathscr{C}$, and $\mathscr{A}'$ be the sets consisting of R_i, C_i, and $A_i^{1/\beta}$, respectively, for $1 \leq i \leq k$. It follows from Lemma 2.3 that constant factor expansions of ε-balanced caps are ε-balanced, so the sets $\mathscr{C}$ and $\mathscr{A}'$ are also ε-balanced (for different constants).

In Lemma 5.2, we establish Property 1 by showing a bound on $|\mathscr{A}'|$. In Lemma 5.3, we establish Property 2, and in Lemmas 5.4 and 5.5, we establish Property 3.

Lemma 5.2. *The number of ε-balanced caps such that the Macbeath regions $M'(x)$ centered at the centroids x of the bases of these caps are disjoint is $O(1/\varepsilon^{(d-1)/2})$.*

Proof. First we bound the number of such caps of type j where $j \geq 0$. As these caps are ε-balanced, they have width $\Theta(w_j)$. By Theorem 4.2, their number n_j is $O(w_j/v_j)$. Recalling that $v_j = \varepsilon^{(d+1)/2} \cdot 2^j$ and $w_j = \varepsilon/\max(j^2,1)$, we have

$$n_j = O\left(\frac{\varepsilon/\max(j^2,1)}{\varepsilon^{\frac{d+1}{2}} \cdot 2^j}\right) = O\left(\frac{1}{\varepsilon^{\frac{d-1}{2}}} \cdot \frac{1}{2^j \max(j^2,1)}\right).$$

Summing n_j over all $j \geq 0$, we obtain a total of $N_+ = O(1/\varepsilon^{(d-1)/2})$ caps.

Next we bound the number of such caps of type j where $j < 0$. By Theorem 4.2, their number n_j is $O(v_j/w_j^d)$. Thus

$$n_j = O\left(\frac{\varepsilon^{(d+1)/2} \cdot 2^j}{(\varepsilon/j^2)^d}\right) = O\left(\frac{1}{\varepsilon^{(d-1)/2}} \cdot 2^j j^{2d}\right).$$

Summing n_j over all $j < 0$, we obtain a total of $N_- = O(1/\varepsilon^{(d-1)/2})$ caps. Therefore, the total number of caps, $N_+ + N_- = O(1/\varepsilon^{(d-1)/2})$, as desired. ☐

Lemma 5.3. *There exists a constant σ such that for each i, $R_i \subseteq C_i \subseteq R_i^\sigma$.*

Proof. Recall that $R_i = M'(x_i)$. By Lemma 2.8, the expansion $A_i^{2/\beta}$ will fully contain R_i. Since $A_i^{2/\beta} \subseteq A_i^\beta = C_i$, we obtain $R_i \subseteq C_i$. By Lemma 2.12,

$$C_i = (A_i^{1/\beta})^{\beta^2} \subseteq M^{3d(2\beta^2-1)}(x_i) = R_i^{15d(2\beta^2-1)} \subseteq R_i^\sigma$$

for any constant $\sigma \geq 15d(2\beta^2-1)$. Thus, $R_i \subseteq C_i \subseteq R_i^\sigma$, as desired. ☐

Lemma 5.4. *Given any ε-width cap C of type j, the cap C^{1/a_j} is a (ε, b_1, b_2)-balanced cap for suitable constants b_1 and b_2.*

Proof. Since C is of type j, we have $\varepsilon^{(d+1)/2}2^j \leq \mathrm{vol}(C) < \varepsilon^{(d+1)/2}2^{j+1}$. Let $C' = C^{1/a_j}$. Clearly, $\mathrm{vol}(C') \leq \mathrm{vol}(C)$ and, by Lemma 2.3, $\mathrm{vol}(C) \leq a_j^d \cdot \mathrm{vol}(C')$. Thus

$$\varepsilon^{\frac{d+1}{2}}2^{j+1} \;>\; \mathrm{vol}(C') \;\geq\; \frac{\varepsilon^{\frac{d+1}{2}}2^j}{a_j^d} \;=\; \varepsilon^{\frac{d+1}{2}}2^{j-d\log(a_j)}.$$

Letting k denote the type of C', we have $\varepsilon^{(d+1)/2}2^k \leq \mathrm{vol}(C') < \varepsilon^{(d+1)/2}2^{k+1}$. These inequalities readily imply that

$$j+1 \;>\; k \;>\; j - d\log(a_j) - 1.$$

It is easy to see that $a_j = \Theta(a_k)$. As the width of C' is ε/a_j, which is $\Theta(\varepsilon/a_k)$, it follows that we can choose b_1 and b_2 such that C' is (ε, b_1, b_2)-balanced. ☐

Lemma 5.5. *There exists a constant σ such that the following holds. For any direction u, there is a cap C whose base is orthogonal to u, and which satisfies $R_i \subseteq C$ and $C_i^{1/\sigma} \subseteq C \subseteq C_i$, for some i.*

Proof. Let F be an ε-width cap whose base is orthogonal to direction u. Suppose that F is of type j. We will show that the cap $C = F^{1/a_j}$ satisfies the properties given in the statement of the lemma.

By Lemma 5.4, C is (ε, b_1, b_2)-balanced. Let $R = M'(x)$ be the Macbeath region centered at the centroid x of the base of the cap $C^{1/\beta}$. By our construction, there must exist a Macbeath region R_i which intersects R. Recall that $C_i = A_i^{\beta}$ and $R_i = M'(x_i)$, where x_i is the centroid of the base of the cap $A_i^{1/\beta}$. Since $M'(x_i) \cap M'(x) \neq \emptyset$, by Lemma 2.7, $M'(x_i) \subseteq M(x)$. Also, by Lemma 2.9, $M(x) \subseteq C^{2/\beta}$. Clearly $C^{2/\beta} \subseteq C$. Putting it together, we have $R_i = M'(x_i) \subseteq M(x) \subseteq C^{2/\beta} \subseteq C$, which proves the first part of the lemma.

It remains to show that $C_i^{1/\sigma} \subseteq C \subseteq C_i$. Since $M'(x_i) \cap M'(x) \neq \emptyset$, we can apply Lemma 2.13 to caps $A_i^{1/\beta}$ and $C^{1/\beta}$ (for $\lambda = 1$) to obtain $A_i^{1/\beta} \subseteq C$. Applying Lemma 2.13 again to caps $C^{1/\beta}$ and $A_i^{1/\beta}$ (for $\lambda = \beta$), we obtain $C \subseteq A_i^{\beta}$. Recalling that $C_i = A_i^{\beta}$, we have $C_i^{1/\beta^2} \subseteq C \subseteq C_i$. The claim now follows for any positive constant $\sigma \geq \beta^2$. $\square$

5.2 Witness-Collector Technique

In this section, we will provide a quick overview of the witness-collector approach [6], which is central to our construction. Recall that K is a convex body in γ-canonical form for some constant γ. The general strategy is as follows. First, we build a set $\mathscr{R}$ of disjoint centrally symmetric convex bodies lying within K and close to its boundary. These bodies will possess certain key properties to be specified later. For each $R \in \mathscr{R}$, we select a point arbitrarily from this body, and let S denote this set of points. The approximation P is defined as the convex hull of S. In Lemma 5.14, we will prove that P is an ε-approximation of K and, in Lemma 5.15, we will apply a deterministic variant of the witness-collector approach [19] to show that P has low combinatorial complexity.

Let $\mathscr{H}$ denote the set of all halfspaces in $\mathbb{R}^d$. We define a set $\mathscr{W}$ of regions called *witnesses* and a set $\mathscr{C}$ of regions called *collectors*, which satisfy the following properties:

(1) Each witness of $\mathscr{W}$ contains a point of S in its interior.

(2) Any halfspace $H \in \mathscr{H}$ either contains a witness $W \in \mathscr{W}$ or $H \cap S$ is contained in a collector $C \in \mathscr{C}$.

(3) Each collector $C \in \mathscr{C}$ contains a constant number of points of S.

The key idea of the witness-collector method is encapsulated in the following lemma, which was proved in [6].

Lemma 5.6. *Given a set of witnesses and collectors satisfying the above properties, the combinatorial complexity of the convex hull P of S is $O(|\mathscr{C}|)$.*

5.3 Stratification

A natural choice for the witnesses and collectors would be the convex bodies R_i and the caps C_i, respectively, from Lemma 5.1. As shown in [6], these bodies do not work for our purposes. The main difficulty is that Property (3) could fail, since a cap C_i could intersect a non-constant number of bodies of $\mathscr{R}$, and hence contain a non-constant number of points of S. Following the general approach of that earlier paper, we show that it is possible to construct a set of witnesses and collectors that satisfy all the requirements by scaling and translating the convex bodies from Lemma 5.1 into a stratified placement according to their volumes. The properties we obtain are specified below in Lemma 5.7.

Our choice of witnesses and collectors will be based on the following lemma. Specifically, the convex bodies $R_1, \ldots, R_k$, will play the role of the witnesses and the regions $C_1, \ldots, C_k$, will play the role of the collectors. The lemma strengthens Lemma 5.1, achieving the critical property that any collector C_i intersects only a constant number of convex bodies of $\mathscr{R}$. As each witness set R_i will contain one point, this ensures that a collector contains only a constant number of input points (Property (3) of the witness-collector system). This strengthening is achieved while maintaining the same number of collectors asymptotically, as in Lemma 5.1. Also, the collectors are no longer simple caps, but have a more complex shape as described in the proof (this, however, has no adverse effect in our application).

Lemma 5.7. *Let $\varepsilon > 0$ be a sufficiently small parameter. Let $K \subset \mathbb{R}^d$ be a convex body in canonical form. There exists a collection $\mathscr{R}$ of $k = O(1/\varepsilon^{(d-1)/2})$ disjoint centrally symmetric convex bodies $R_1, \ldots, R_k$ and associated regions $C_1, \ldots, C_k$ such that the following hold:*

1. *Let C be any cap of width ε. Then there is an i such that $R_i \subseteq C$.*

2. *Let C be any cap. Then there is an i such that either (i) $R_i \subseteq C$ or (ii) $C \subseteq C_i$.*

3. *For each i, the region C_i intersects at most a constant number of bodies of $\mathscr{R}$.*

As mentioned earlier, our proof of this lemma is based on a stratified placement of the convex bodies from Lemma 5.1, which are distributed among $O(\log \frac{1}{\varepsilon})$ layers that lie close to the boundary of K. Let $\alpha = c_0\,\varepsilon$, where c_0 is a suitable constant to be specified later. We begin by applying Lemma 5.1 to K using $\varepsilon = \alpha$. This yields a collection $\mathscr{R}'$ of $k = O(1/\alpha^{(d-1)/2})$ disjoint centrally symmetric convex bodies $\{R'_1, \ldots, R'_k\}$ and associated caps $\mathscr{C}' = \{C'_1, \ldots, C'_k\}$. Our definition of the convex bodies R_i and regions C_i required in Lemma 5.7 will be based on R'_i and C'_i, respectively. In particular, the convex body R_i will be obtained by translating a scaled copy of R'_i into an appropriate layer, based on the type of the cap C'_i.

Recall that the caps of $\mathscr{C}'$ are (α, b_1, b_2)-balanced for some constants b_1 and b_2, and have integral types ranging from $-t$ to $+t$, where $t = O(\log \frac{1}{\alpha})$. By definition, the width of any cap of type j in $\mathscr{C}'$ is at most $b_2 w_j$, where $w_j = \alpha/\max(j^2, 1) = c_0\varepsilon/\max(j^2, 1)$.

We partition the set $\mathscr{R}'$ of convex bodies into $2t+1$ groups, based on the type of the associated cap C'. More precisely, for $-t \le j \le t$, group j consists of bodies $R' \in \mathscr{R}'$, such that the associated cap C' is of type j.

Next we describe how the layers are constructed. We will construct $2t+1$ layers corresponding to the $2t+1$ groups of $\mathscr{R}'$. Our construction uses a constant parameter c_1. For $-t-1 \le j \le t-1$, let T_j denote the linear transformation that represents a uniform scaling by a factor of $\prod_{i=j+1}^{t}(1-c_1 w_i)$ about the origin, and let T_t denote the identity transformation. Further, define $K_j = T_j(K)$ (see Figure 10(a)) and let $K_t = K$. For $-t \le j \le t$, define layer j, denoted L_j, to be the difference $K_j \setminus K_{j-1}$. Whenever we refer parallel supporting hyperplanes for two bodies K_i and K_j, we assume that both hyperplanes lie on the same side of the origin.

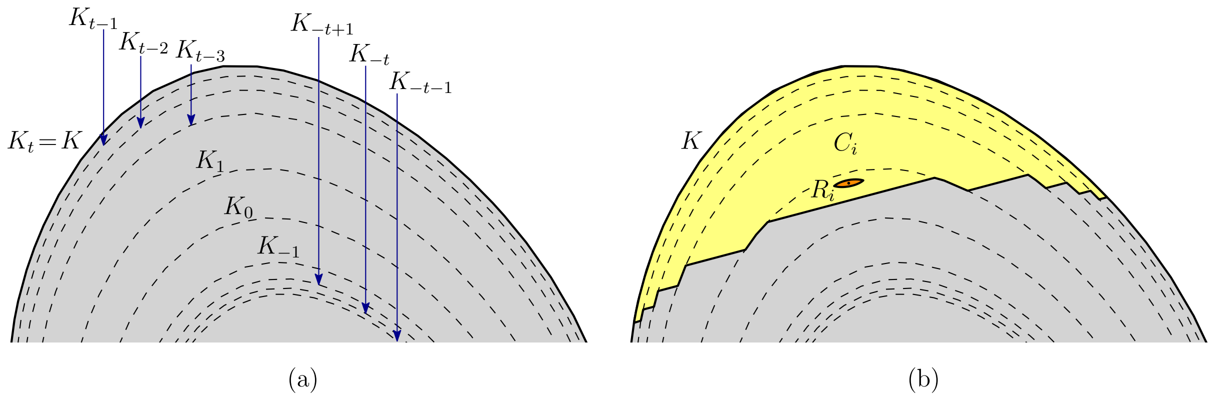

Figure 10: (a) Stratified placement of the bodies R_i and (b) the region C_i corresponding to a body R_i. (Figure not to scale.)

The following lemma describes some straightforward properties of these layers and the scaling transformations. In particular, the lemma shows that the t layers lie close to the boundary of K (within distance ε) and layer j has a "thickness" of $\Theta(w_j)$.

Lemma 5.8. *Let $\varepsilon > 0$ be a sufficiently small parameter and c_1 be any constant. For sufficiently small constant c_0 in the definition of α (depending on c_1), the layered decomposition and the scaling transformations described above satisfy the following properties:*

(a) For $-t \leq j \leq t$, the distance between parallel supporting hyperplanes of K_{j-1} and K_j is at most $c_1 w_j / \sqrt{\gamma}$.

(b) For $-t \leq j \leq t$, the distance between parallel supporting hyperplanes of K_{j-1} and K_j is at least $\sqrt{\gamma} c_1 w_j / 2$.

(c) The distance between parallel supporting hyperplanes of K and K_{-t-1} is at most ε.

(d) For $-t-1 \leq j \leq t$, the scaling factor for T_j is at least 1/2 and at most 1.

(e) For $-t-1 \leq j \leq t$, T_j preserves volumes up to a constant factor.

Proof. To prove (a), let h_1, h_2 denote parallel supporting hyperplanes of K_j, K_{j-1}, respectively. Since K is in γ-canonical form, and the scaling factor of the transformation T_j is at most 1, it follows that h_1 is at distance at most $1/\sqrt{\gamma}$ from the origin. Since h_2 is obtained by scaling h_1 by a factor of $1 - c_1 w_j$ about the origin, it follows that the distance between h_1 and h_2 is at most $c_1 w_j / \sqrt{\gamma}$.

To prove (c), let h_1, h_2 denote parallel supporting hyperplanes of K, K_{-t-1}, respectively. The upper bound of (a) implies that the distance between h_1 and h_2 is at most $\sum_{j=-t}^{t} c_1 w_j / \sqrt{\gamma}$. Recall that $w_j = \alpha / \max(j^2, 1)$, where $\alpha = c_0 \varepsilon$. By choosing a sufficiently small constant c_0 in the definition of α (depending on c_1 and γ), we can ensure that the distance between h_1 and h_2 is at most ε.

In the rest of this proof, we will assume that c_0 in the definition of α is sufficiently small, so (c) holds. To prove (b), let h_1, h_2 denote parallel supporting hyperplanes of K_j, K_{j-1}, respectively. Since K is in γ-canonical form, it follows from (c) that h_1 is at distance at least $\sqrt{\gamma} - \varepsilon$ from the origin. Since h_2 is obtained by scaling h_1 by a factor of $1 - c_1 w_j$ about the origin, it follows that the distance between h_1 and h_2 is at least $c_1(\sqrt{\gamma} - \varepsilon)w_j$, which is at least $\sqrt{\gamma} c_1 w_j/2$ for $\varepsilon \leq \sqrt{\gamma}/2$.

To prove (d), note that we only need to show the lower bound on the scaling factor of T_j, since the upper bound is obvious. Again, let h_1, h_2 denote parallel supporting hyperplanes of K, K_{-t-1}, respectively. Since K is in γ-canonical form, h_1 is at distance at least $\sqrt{\gamma}$ from the origin. Recall that T_{-t-1} maps h_1 to h_2 and, as shown above, the distance between h_1 and h_2 is at most ε. It follows that the scaling factor of T_{-t} is at least $1 - \varepsilon/\sqrt{\gamma}$. By choosing $\varepsilon \leq \sqrt{\gamma}/2$, we can ensure that the scaling factor of T_{-t-1} is at least $1/2$. Clearly, this lower bound on the scaling factor also applies to any transformation T_j, $-t-1 \leq j \leq t$. This proves (d). Note that (e) is an immediate consequence. □

Recall that the width of a cap of type j in $\mathscr{C}'$ is at most $b_2 w_j$ for some constant b_2. In order to ensure that layer j can accommodate caps of type j, we construct the layered decomposition of Lemma 5.8 for a constant $c_1 = 2b_2/\sqrt{\gamma}$. This choice ensures that the distance between parallel supporting hyperplanes of K_{j-1} and K_j, respectively, is at most $(2b_2/\gamma)w_j$ and at least $b_2 w_j$ (properties (a) and (b) in Lemma 5.8).

Let H' be a halfspace and let $C' = K \cap H'$ be a type-j cap in $\mathscr{C}'$. It will be convenient to associate a set of caps with C' that occur frequently in our construction and analysis. For $j \leq r \leq t$, define $E_r = K_r \cap T_j(H')$ and define $F_r = T_r(C')$. Both E_r and F_r are caps of K_r. This is obviously true for E_r. To see that F_r is a cap of K_r, note that $F_r = T_r(C') = T_r(K \cap H') = T_r(K) \cap T_r(H') = K_r \cap T_r(H')$. Also, note that $E_j = F_j$.

We are now ready to define the sets $\mathscr{R}$ and $\mathscr{C}$ required in Lemma 5.7. Let $R' \in \mathscr{R}'$ be a body in group j and let the cap in $\mathscr{C}'$ associated with it be $C' = K \cap H'$. We define a body $R \in \mathscr{R}$ and an associated region $C \in \mathscr{C}$, based on R' and C' as follows. We define $R = T_j(R')$, and define

$$C \;=\; \bigcup_{r=j}^{t} E_r^{\sigma} \cap L_r,$$

where σ is the constant of Lemma 5.1. (See Figure 10(b).)

In Lemma 5.9, we show that the regions R are contained in layer j if R' is in group j. In Lemmas 5.10 and 5.11, we establish Properties 1 and 2 of Lemma 5.7. Finally, in Lemmas 5.12 and 5.13, we establish Property 3 of Lemma 5.7.

Lemma 5.9. *If R' is in group j, then $R \subseteq F_j \subseteq L_j$.*

Proof. By Property 2 of Lemma 5.1, $R' \subseteq C'$. Applying the transformation T_j to these two sets yields $R \subseteq F_j$. Next we show that $F_j \subseteq L_j$. Since C' is a cap of type j, its width is at most $b_2 w_j$. By Lemma 5.8(d), the scaling factor for T_j is at most 1. Thus, the width of F_j is at most $b_2 w_j$. By Lemma 5.8(b) and our remarks following Lemma 5.8, the distance between any parallel supporting hyperplanes of K_{j-1} and K_j, respectively, is at least $b_2 w_j$. It follows that $F_j \subseteq K_j \setminus K_{j-1} = L_j$. This completes the proof. □

Lemma 5.10. *For any direction u, there is a cap A whose base is orthogonal to u, and which satisfies $R \subseteq A \subseteq C$, for some $R \in \mathscr{R}$ and $C \in \mathscr{C}$. Further, the width of the cap A is at most ε.*

Proof. By Property 3 of Lemma 5.1, there exists a cap $\hat{A} = K \cap \hat{H}$ whose base is orthogonal to u, and which satisfies $R' \subseteq \hat{A}$ and $(C')^{1/\sigma} \subseteq \hat{A} \subseteq C'$ for some $R' \in \mathscr{R}'$ and $C' \in \mathscr{C}'$. Let C' be a type-j

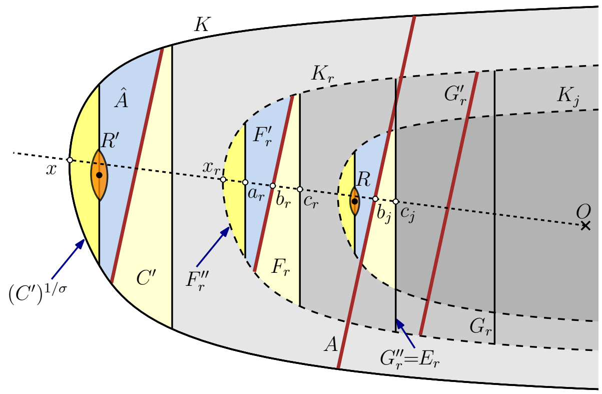

Figure 11: Proof of Lemma 5.10.

cap. Define $H = T_j(\hat{H})$ and $A = K \cap H$. We will show that the cap A possesses the properties given in the statement of the lemma. See Figure 11 for a representation of the definitions.

Since $R' \subseteq \hat{A} = K \cap \hat{H}$, we can apply the transformation T_j to these sets to obtain $R \subseteq K_j \cap H \subseteq K \cap H = A$.

Next we show that the width of the cap A is at most ε. Recall that $\hat{A} \subseteq C'$. Applying the transformation T_j to these sets, we obtain $K_j \cap H \subseteq F_j$. By Lemma 5.9, $F_j \subseteq L_j$. Thus $K_j \cap H \subseteq L_j$. Also, by Lemma 5.8(c), the distance between any parallel supporting hyperplanes of K and K_{-t-1} is at most ε. Since $K_j \cap H \subseteq L_j$, it follows that the width of the cap $A = K \cap H$ is at most ε.

It remains to show that $A \subseteq C$. By the definition of A and C, it suffices to show that for $j \leq r \leq t$, $K_r \cap H \ \subseteq \ E_r^{\sigma}$. Note that for $r = j$, applying T_j to both sides of $\hat{A} \subseteq C'$, we obtain $K_j \cap H \ \subseteq \ F_j = E_j \subseteq E_j^{\sigma}$ (for any $\sigma \geq 1$).

However, the proof is more involved when $r > j$. In this case, we will need to exploit the fact that $\hat{A}$ is sandwiched between two caps with parallel bases, that is, $(C')^{1/\sigma} \subseteq \hat{A} \subseteq C'$. Recall that F_r and F_j are the caps of K_r and K_j, respectively, defined as $F_r = T_r(C')$ and $F_j = T_j(C')$. Define $F'_r = T_r(\hat{A})$, $F'_j = T_j(\hat{A})$, and $F''_r = T_r((C')^{1/\sigma})$. We have $F''_r \subseteq F'_r \subseteq F_r$ and $F'_j \subseteq F_j$.

Let x denote the apex of C' and x_r denote the point $T_r(x)$. Let a_r, b_r, and c_r denote the points of intersection of the bases of the caps F''_r, F'_r, and F_r, respectively, with the line segment Ox. Similarly, let b_j and c_j denote the points of intersection of the bases of the caps F'_j and F_j, respectively, with the line segment Ox. Consider scaling caps F''_r, F'_r and F_r as described in Lemma 2.4, about the point x_r with scaling factor $\|c_j x_r\|/\|a_r x_r\|$. Let G''_r, G'_r, and G_r denote the caps of K_r obtained from F''_r, F'_r, and F_r, respectively, through this transformation. By Lemma 2.4, $G''_r \subseteq G'_r \subseteq G_r$. Note that F''_r and E_r are caps of K_r with parallel bases and the base of E_r passes through the point c_j (since the hyperplanes passing through the bases of the caps E_r and F_j are the same). Also, $K_r \cap H$ and G'_r are caps of K_r with parallel bases. Our choice of the scaling factor thus implies that $G''_r = E_r$ and $K_r \cap H \subseteq G'_r$.

Putting these facts together, we have $K_r \cap H \subseteq G_r$. Note that G_r and E_r are caps of K_r with parallel bases. Thus, to prove that $K_r \cap H \subseteq E_r^{\sigma}$, it suffices to show that $\mathrm{width}(G_r)/\,\mathrm{width}(E_r) \leq \sigma$. Note that

$$\frac{\mathrm{width}(G_r)}{\mathrm{width}(E_r)} \;=\; \frac{\mathrm{width}(G_r)}{\mathrm{width}(G''_r)} \;=\; \frac{\mathrm{width}(F_r)}{\mathrm{width}(F''_r)} \;=\; \frac{\mathrm{width}(C')}{\mathrm{width}((C')^{1/\sigma})} \;=\; \sigma,$$

which completes the proof. $\square$

Lemma 5.11. *Let A be any cap of K. Then either (i) there is a body $R \in \mathscr{R}$ such that $R \subseteq A$ or (ii) there is a region $C \in \mathscr{C}$ such that $A \subseteq C$. Furthermore, if the width of A is ε, then (i) holds.*

Proof. Taking u to be the unit vector orthogonal to the base of the cap A and applying Lemma 5.10, it follows that there exists a cap A' whose base is parallel to the base of A and which satisfies $R \subseteq A' \subseteq C$, for some $R \in \mathscr{R}$ and $C \in \mathscr{C}$. Further, the width of the cap A' is at most ε.

We consider two cases, depending on whether $A' \subseteq A$ or $A \subseteq A'$. In the first case, we have $R \subseteq A' \subseteq A$ and, in the second case, we have $A \subseteq A' \subseteq C$. Thus, either $R \subseteq A$ or $A \subseteq C$.

Further, if the width of A is ε, then $A' \subseteq A$ because the width of A' is at most ε. Thus the first case holds implying that $R \subseteq A$. $\square$

In Lemma 5.13, we bound the number of bodies of $\mathscr{R}$ that overlap any region $C \in \mathscr{C}$ (Property 3 of Lemma 5.7). Recall that C corresponds to a cap $C' \in \mathscr{C}'$. Let C' be of type j. We first establish a constant bound on the number of bodies of $\mathscr{R}$ that overlap $E_j^\sigma \cap L_j$. Then we bound the number of bodies of $\mathscr{R}$ that overlap $E_r^\sigma \cap L_r$ for $r > j$. Our analysis exploits the fact that the volume of E_r exceeds the volume of E_j by a factor that is at most polynomial in $r - j$ (i.e., the number of layers between K_r and K_j), while the volume of the bodies of $\mathscr{R}$ in layer r exceeds the volume of the bodies of $\mathscr{R}$ in layer j by a factor that is exponential in $r - j$. This allows us to show that the number of bodies of $\mathscr{R}$ that overlap C is bounded by a constant.

Before presenting Lemma 5.13, we establish a polynomial bound on the growth rate of the volume of the caps E_r in the following lemma.

Lemma 5.12. *Let C' be a type-j cap. For $j+1 \le r \le t$, $\mathrm{vol}(E_r) = O((r-j)^{3d}) \cdot \mathrm{vol}(E_j)$.*

Proof. Recall that $F_r = T_r(C')$ and $E_j = F_j = T_j(C')$. By Lemma 5.8(e), T_j and T_r preserve volumes up to constant factors, and so $\mathrm{vol}(F_r) = \Theta(\mathrm{vol}(F_j))$. Thus, to prove the lemma, it suffices to show that $\mathrm{vol}(E_r)/\,\mathrm{vol}(F_r) = O((r-j)^{3d})$. In turn, in light of Lemma 2.3, it suffices to prove that $\mathrm{width}(E_r)/\,\mathrm{width}(F_r) = O((r-j)^3)$.

Towards this end, recall that the width of E_r is upper bounded by the distance between parallel supporting hyperplanes of K_r and K_{j-1} which, by Lemma 5.8(a), is at most $O(\sum_{i=j}^r w_i)$. Further, by Lemma 5.8(d), the width of F_r is at least half the width of C'. As C' is of type j, by definition its width is $\Theta(w_j)$. It follows that the width of F_r is $\Omega(w_j)$. Thus, we have shown that

$$\frac{\mathrm{width}(E_r)}{\mathrm{width}(F_r)} = O\left(\frac{\sum_{i=j}^r w_i}{w_j}\right).$$

Clearly,

$$\frac{\sum_{i=j}^r w_i}{w_j} \le (r-j+1) \cdot \left(\frac{\max_{i=j}^r w_i}{w_j}\right).$$

To complete the proof, we will show that $(\max_{i=j}^r w_i)/w_j = O((r-j)^2)$. Recall that for any i, $w_i = \alpha/\max(i^2, 1)$. We consider three cases: (1) $r \ge j \ge 0$, (2) $0 > r \ge j$, and (3) $r \ge 0 > j$. In Case 1, we have $\max_{i=j}^r w_i = w_j$, and so the quantity of interest is 1. In Case 2, $\max_{i=j}^r w_i = w_r$. Thus

$$\frac{\max_{i=j}^r w_i}{w_j} \;=\; \frac{w_r}{w_j} = \frac{1/r^2}{1/j^2} = \frac{j^2}{r^2} \;=\; \left(\frac{(r-j)+|r|}{|r|}\right)^2 \;=\; \left(\frac{r-j}{|r|}+1\right)^2 \;\le\; (r-j+1)^2.$$

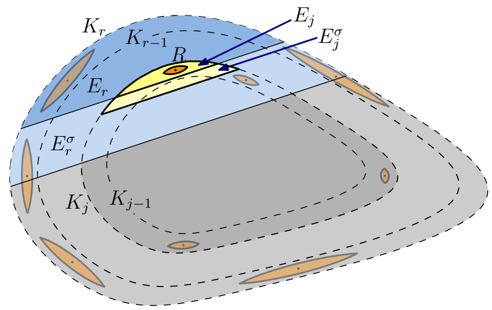

Figure 12: Proof of Lemma 5.13.

In Case 3, $\max_{i=j}^r w_i = w_0$. Thus

$$\frac{\max_{i=j}^r w_i}{w_j} = \frac{1}{1/j^2} = j^2 \leq (r-j)^2.$$

In all three cases, we have shown that $(\max_{i=j}^r w_i)/w_j = O((r-j)^2)$, as desired. $\square$

Lemma 5.13. *Any region $C \in \mathscr{C}$ intersects $O(1)$ bodies of $\mathscr{R}$.*

Proof. Suppose that R' is in group j. Recall that $R = T_j(R')$, $C' = K \cap H'$ and $C = \bigcup_{r=j}^t (E_r^\sigma \cap L_r)$. We begin by bounding the number of bodies of $\mathscr{R}$ that overlap $E_j^\sigma \cap L_j$. (See Figure 12.) We assert that all the bodies of $\mathscr{R}$ in layer j have volumes $\Omega(\mathrm{vol}(E_j))$. To prove this, recall that the type-j caps of $\mathscr{C}'$ have the same volume as C' to within a factor of 2. Also, recall that the body of $\mathscr{R}'$ associated with a cap of $\mathscr{C}'$ has the same volume as the cap to within a constant factor (immediate consequence of Property 2 of Lemma 5.1). It follows that all the bodies of $\mathscr{R}'$ in group j have volumes $\Omega(\mathrm{vol}(C'))$. By Lemma 5.8(e), the scaling transformations used in our construction preserve volumes to within a constant factor. Also, recall that the bodies of $\mathscr{R}$ in layer j are scaled copies of the bodies of $\mathscr{R}'$ in group j. It follows that the bodies of $\mathscr{R}$ in layer j all have volumes $\Omega(\mathrm{vol}(E_j))$.

Next, we assert that any body of $\mathscr{R}$ that overlaps $E_j^\sigma \cap L_j$ is contained within the cap $E_j^{2\sigma}$. To prove this, recall from the proof of Lemma 5.1 that the bodies of $\mathscr{R}'$ are $(1/5)$-scaled disjoint Macbeath regions with respect to K. It follows that the bodies of $\mathscr{R}$ in layer j are $(1/5)$-scaled disjoint Macbeath regions with respect to K_j. By Lemma 2.8, it now follows that any body of $\mathscr{R}$ that overlaps $E_j^\sigma \cap L_j$ is contained within the cap $E_j^{2\sigma}$. By Lemma 2.3, $\mathrm{vol}(E_j^{2\sigma}) = O(\mathrm{vol}(E_j))$. Since the bodies of $\mathscr{R}$ in layer j have volumes $\Omega(\mathrm{vol}(E_j))$, it follows by a simple packing argument that at most a constant number of bodies of $\mathscr{R}$ are contained within $E_j^{2\sigma} \cap L_j$. Hence, the number of bodies of $\mathscr{R}$ that overlap $E_j^\sigma \cap L_j$ is $O(1)$.

Next we bound the number of bodies of $\mathscr{R}$ that overlap $E_r^\sigma \cap L_r$, where $j+1 \leq r \leq t$. (See Figure 12.) By Lemma 5.12, we have $\mathrm{vol}(E_r) = O((r-j)^{3d}) \cdot \mathrm{vol}(E_j)$. Recall that the volume of the bodies of $\mathscr{R}'$ in group r exceeds the volume of the bodies of $\mathscr{R}'$ in group j by a factor of $\Omega(2^{r-j})$. It follows from Lemma 5.8(e) and our construction that the volume of the bodies of $\mathscr{R}$ in layer r exceeds the volume of the bodies of $\mathscr{R}$ in layer j by a factor of $\Omega(2^{r-j})$. For the same reasons as discussed above, any body of $\mathscr{R}$ that overlaps $E_r^\sigma \cap L_r$ is contained within $E_r^{2\sigma}$, and $\mathrm{vol}(E_r^{2\sigma}) = O(\mathrm{vol}(E_r))$. Putting this together with the upper bound on $\mathrm{vol}(E_r)$ given above, we have $\mathrm{vol}(E_r^{2\sigma}) = O((r-j)^{3d}) \cdot \mathrm{vol}(E_j)$. By a simple packing argument, it follows that the number of bodies of $\mathscr{R}$ that are contained within $E_r^{2\sigma} \cap L_r$ is $O((r-j)^{3d}/2^{r-j})$. This bounds the number of bodies of $\mathscr{R}$ that overlap $E_r^{\sigma} \cap L_r$. It follows that the number of bodies of $\mathscr{R}$ that overlap $C = \bigcup_{r=j+1}^{t}(E_r^{\sigma} \cap L_r)$ is on the order of $\sum_{j+1 \le r \le t}(r-j)^{3d}/2^{r-j} = O(1)$, which completes the proof. $\square$

5.4 Polytope Approximation

Finally, we can assemble all the pieces to obtain the desired approximation. Let S be a set of points containing one point inside each body of $\mathscr{R}$ defined in Lemma 5.7 and no other point.

Lemma 5.14. *The polytope $P = \mathrm{conv}(S)$ is an inner ε-approximation of K.*

Proof. A set of points S is said to *stab* a cap if the cap contains at least one point of S. It is well known that if a set of points $S \subset K$ stabs all caps of width ε of K, then $\mathrm{conv}(S)$ is an inner ε-approximation of K [15]. Let C be a cap of width ε. By Lemma 5.7, Property 1, there is a convex body $R_i \subseteq C$. Since S contains a point that is in R_i, we have that the cap C is stabbed. $\square$

To bound the combinatorial complexity of $\mathrm{conv}(S)$, and hence conclude the proof of Theorem 1.1, we use the witness-collector approach [19]. The proof is the analogue to the one in [6] and is included for completeness.

Lemma 5.15. *The number of faces of $P = \mathrm{conv}(S)$ is $O(1/\varepsilon^{(d-1)/2})$.*

Proof. Define the witness set $\mathscr{W} = R_1, \ldots, R_k$ and the collector set $\mathscr{C} = C_1, \ldots, C_k$, where the R_i's and C_i's are as defined in Lemma 5.7. As there is a point of S in each body R_i, Property (1) of the witness-collector method is satisfied. To prove Property (2), let H be any halfspace. If H does not intersect K, then Property (2) of the witness-collector method holds trivially. Otherwise let $C = K \cap H$. By Property 2 of Lemma 5.7, there is an i such that either $R_i \subseteq C$ or $C \subseteq C_i$. It follows that H contains witness R_i or $H \cap S$ is contained in collector C_i. Thus Property (2) of the witness-collector method is satisfied. Finally, Property 3 of Lemma 5.7 implies Property (3) of the witness-collector method. Thus, we can apply Lemma 5.6 to conclude that the number of faces of P is $O(|\mathscr{C}|) = O(k)$, which proves the lemma. $\square$

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