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Flat inputs: theory and applications

Florentina Nicolau*, Witold Respondek †, Jean-Pierre Barbot ‡, and Achour Ouslimani §

Abstract

In this paper, we study the problem of constructing flat inputs for multi-output dynamical systems. The notion of flat inputs has been introduced by Waldherr and Zeitz in [30, 31] and can be seen as dual to that of flat outputs. In the single-output case, a flat input can be constructed if and only if the original dynamical system together with its output is observable. In the multi-output case, the observability is not necessary for the existence of flat inputs. The observable case has been treated in [31], where a system of linear algebraic equations has been proposed in order to determine the control vector fields associated to the flat inputs. The goal of this paper is to treat the unobservable case for multi-output dynamical systems. We start by discussing the case when the dynamical system together with the given output is observable and we present a generalization of the results of [31] by relating them with the notion of minimal differential weight. Then we give our main results. We consider the unobservable case for which locally, on an open and dense subset of $\mathbb{R}^n$, we construct control vector fields $g_1, \ldots, g_m$ such that the associated control system is flat. Finally, we explain how our results can be applied to private communication.

Keywords: Flat inputs, flatness, observed dynamical systems, constructing control vector fields, private communication.

1 Introduction

Consider the following nonlinear observed dynamics:

$$\Sigma : \dot{x} = f(x), \quad y = h(x), \tag{1}$$

where x is the state defined on $\mathbb{R}^n$ (or more generally, on an open subset of $\mathbb{R}^n$ or on an n-dimensional manifold) and $y \in \mathbb{R}^m$ are the measurements, supposed independent everywhere.

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The vector field f and the functions h_i, $1 \leq i \leq m$, are supposed smooth (the word smooth will always mean $\mathcal{C}^\infty$-smooth). The problem that we are studying in this paper is to find control vector fields $g_1, \ldots, g_m$ (or equivalently, to place the actuators or the inputs) such that the control-affine system $\Sigma_c : \dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$, associated to Σ, is flat with the original measurements $(h_1, \ldots, h_m)$ being a flat output.

The notion of flatness was introduced in control theory in the 1990's, by Fliess, Lévine, Martin and Rouchon [3, 4], see also [11, 12, 18, 25], and has attracted a lot of attention because of its multiple applications in the problems of constructive controllability and motion planning (see, e.g., [16, 19] and references therein). Flat systems form a class of control systems whose set of trajectories can be parametrized by m functions and their time-derivatives, m being the number of controls. More precisely, the control system $\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, is *flat* if we can find m functions $\varphi_i(x, u, \ldots, u^{(q)})$ such that

$$x = \gamma(\varphi, \ldots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(s)}), \tag{2}$$

for certain integers q and s, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ is called *a flat output*. Therefore the time-evolution of all state and control variables can be determined from that of flat outputs without integration and all trajectories of the system can be completely parameterized.

The construction of a flat output $h = \varphi$ can be seen as a problem of sensor placement in order to achieve flatness of the resulting input–state-output system. Dual to this, one can consider the problem of actuator placement, i.e., given (f, h), find control vector fields $g_1, \ldots, g_m$, in order to achieve the same property. This dual problem has been introduced by Waldherr and Zeitz [30, 31] who call inputs $u_1, \ldots, u_m$ multiplying, resp., $g_1, \ldots, g_m$, as *flat inputs* (which are objects dual to flat outputs).

In the single-output case, a flat input can be constructed if and only if the system Σ together with its output h is observable and the control vector field associated to the flat input can be computed from a system of linear algebraic equations, see [30]. In the multi-output case, the observability is not necessary for the existence of flat inputs. The observable case has been discussed in [31] and the control vector fields (there are as many as the number of outputs) associated to the flat inputs can be determined in a similar way as that for the SISO case. Another approach in the observable case, based on the notion of unimodularity, has been recently proposed in [5], together with an algorithm for constructing flat inputs and it is then shown that in the observable case, the integrability condition is always satisfied. The goal of this paper is to treat the unobservable multi-output case (the two-output case has been solved by the authors in [21]). It is crucial to distinguish the observability (or unobservability) of controlled systems from observability of uncontrolled ones (recall that for nonlinear systems, the observability property depends on the control [6, 8]). Here we deal with unobservable uncontrolled system that become at least locally weakly observable due to a suitable design of flat inputs. We give a complete solution for the local construction of flat inputs: we show that locally (and, as a consequence, on an open and dense subset of the state space) there always exist control vector fields $g_1, \ldots, g_m$ such that the control-affine system Σ_c is flat with h being a flat output. The link between the observed (via the given output h) subsystem and the unobserved one is made with the help of flat inputs and some linking terms.

First of all, it is important to highlight that this paper completely solves (under some suitable constant rank conditions) the problem of constructing flat inputs, that is, for any observed dynamical system (Σ, h), the proposed solution always allows to construct control vector fields $g_1, \ldots, g_m$ such that Σ_c is flat with h being a flat output. When comparing

this problem with that of verifying flatness for control systems, an interesting phenomenon can be noted: contrary to flat control systems that are very rare (the class of flat control systems is of codimension infinity among all control systems), any dynamical system (Σ, h) can be rendered flat by adding suitable control vector fields (or equivalently, suitable flat inputs). Our method constructs the control vector fields in the simplest possible way: indeed, $g_1, \ldots, g_m$ are the simplest possible nonlinear control vector fields: they are linear (if (Σ, h) is observable) and they are in the multi-chained form (if (Σ, h) is not observable).

The main results of the paper are based on several normal forms (for which h is a flat output). The construction of the first normal form NF1 is performed on an open and dense subset of $\mathbb{R}^n$ and is valid around any point x_0 of that set. In order to compute the control vector fields of the flat control system NF1, we do not need to solve any partial differential equation. The nature of the nominal point around which we work (equilibrium or not) plays an important role in our study. If x_0 is not an equilibrium point of the original system, i.e., $f(x_0) \neq 0$, then we can still simplify the solution of the problem: we render the system Σ flat, with h being a flat output, around any point x_0 (not just on an open and dense subset) provided that a suitable codistribution has constant rank around x_0. Furthermore, the new control system NF2 that we construct turns out to be not only flat, which is our primary goal, but the differential weight of h (which equals the the sum of minimal numbers of time-derivatives of h_i needed to express all states x and controls u) as a flat output of NF2 is the lowest (minimal possible) among all differential weights of h as a flat output of any Σ_c. Contrary to NF1, constructing NF2 requires suitable coordinates and in order to obtain them, we have to solve some partial differential equations. The normal forms NF1 and NF2 apply when the number of unobserved (with respect to the original output h) directions is at least $m-1$. Finally, a third normal form NF3 is proposed for the case when the number of unobserved states is smaller than $m-1$. NF3 is the analogue of NF1, that is, it is established on an open and dense subset and we do not need to solve any partial differential equation in order to construct its associated control vector fields; but NF3 is also reminiscent of NF2 since the differential weight of h as a flat output of NF3 is the lowest among all differential weights of h as a flat output of any Σ_c.

One of the motivations to construct a flat input for a given output is that with such an input, the tracking problem for that output can be solved with no need to calculate the zero dynamics (see, e.g., [7]), but constructing flat inputs may be useful for other problems as well: in this paper, we explain how it can be applied to private communication.

The paper is organized as follows. In Section 2, we recall the definition of flatness and the notion of differential weight of a flat system. We discuss (from the point of view of minimal differential weight) the case when the dynamical system Σ together with a given output h is observable and we present a generalization of the results of [31]. In Section 3, we give our main results. We completely describe the local case (and, as a consequence, we obtain solutions almost everywhere, that is, on an open and dense subset). We illustrate our results by several examples throughout the paper and we explain how it can be applied to private communication in Section 4. We provide proofs in Section 5.

2 Definitions and problem statement

Consider a nonlinear control systems of the form $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$, where x is the state defined on $\mathbb{R}^n$ (more generally, an open subset X of $\mathbb{R}^n$ or an n-dimensional manifold X) and u is the control taking values in $\mathbb{R}^m$. Fix an integer $l \geq -1$ and denote $\overline{u}^l = (u, \dot{u}, \ldots, u^{(l)})$.

For $l = -1$, the sequence $\overline{u}^{-1}$ is empty.

Definition 2.1. The system $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$ is *x-flat* at $(x_0, \bar{u}_0^l) \in X \times \mathbb{R}^{m(l+1)}$, for $l \geq -1$, if there exist a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and m smooth functions $\varphi_i = \varphi_i(x)$, $1 \leq i \leq m$, defined in a neighborhood of x_0, having the following property: there exist an integer $s > l$ and smooth functions γ_i, $1 \leq i \leq n$, and δ_j, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

for any C^{s-1}-control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

Remark 2.1. There exists a more general notion of flatness for which the functions φ_i may depend on the control and its successive time-derivatives up to a certain order q (i.e., $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(q)})$, as in the introduction of the paper), but we do not need it since, in our study, all functions φ_i depend on the state x only (hence only the notion of x-flatness is required).

The minimal number of derivatives of components of a flat output, needed to express x and u, will be called the differential weight of that flat output and is formalized as follows. By definition, for any flat output φ of Ξ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{array}{rcl} x & = & \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u & = & \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}). \end{array} \tag{3}$$

Moreover, we can choose $(s_1, \ldots, s_m)$, γ and δ such that (see [27]) if for any other m-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ and functions $\tilde{\gamma}$ and $\tilde{\delta}$, we have

$$\begin{array}{rcl} x & = & \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u & = & \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{array}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^{m}(s_i + 1) = m + \sum_{i=1}^{m} s_i$ the differential weight of φ. A flat output of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ. We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output. The differential weight of Ξ is at least $n + m$, since we have to express n states and m independent controls and in order to do that, we need at least $n + m$ derivatives (taking into account also those of order zero).

Flatness is a property of the state-space dynamics $\dot{x} = F(x, u)$ of a control system. It can also be described as a property of the input-output map for a dummy output y. In fact, x-flatness is equivalent to the existence of an $\mathbb{R}^m$-valued dummy output $y = \varphi(x)$ that renders the system $\dot{x} = F(x, u)$ observable [9, 14] and left-invertible [26]. Indeed, expressing the state as $x = \gamma(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)})$ and the control as $u = \delta(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$ corresponds, resp., to observability and left invertibility.

Let us now consider the dynamical system Σ together with the output $y = h(x) \in \mathbb{R}^m$, given by (1). In order to emphasize the fact that the system is observed[1] we will use the notation (Σ, h).

[1]When we say that a dynamical system Σ is observed, this does not mean that Σ is necessarily observable with respect to the output h.

Remark 2.2. The state x of the dynamical system Σ, given by (1), is defined on $\mathbb{R}^n$, but all results of the paper also hold when x is defined on an open subset of $\mathbb{R}^n$ or, more generally, on an n-dimensional manifold.

The problem that we are studying in this paper is the construction of control vector fields $g_1, \ldots, g_m$ (whose inputs $u_1, \ldots, u_m$ are called *flat inputs*) such that the control-affine system Σ_c, associated to Σ, and given by

$$\Sigma_c : \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i,$$

is x-flat with respect to the the original output $(h_1, \ldots, h_m)$. In that case, we will say that the pair (Σ_c, h) is x-flat.

As we have already noticed, flatness is closely related to observability. Thus for the problem of constructing flat inputs, it is natural to start by checking observability of Σ with respect to h. We denote by $\mathcal{H}(x)$ the observability codistribution

$$\mathcal{H}(x) = \text{span}\ \{\mathrm{d}L_f^j h_i(x), j \geq 0, 1 \leq i \leq m\} \tag{4}$$

associated with the output h. In our study, we will distinguish the observable and unobservable case.

Definition 2.2. The observed system (Σ, h) is said to have observability quasi-indices $(\rho_1, \ldots, \rho_m)$ at $x_0 \in \mathbb{R}^n$ if $\sum_{i=1}^m \rho_i = n$ and span $\{\mathrm{d}L_f^j h_i(x_0), 0 \leq j \leq \rho_i - 1, 1 \leq i \leq m\} = \mathbb{R}^n$. If, moreover, the quasi-indices satisfy $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_m$, then the smallest, in the lexicographic ordering, quasi-indices are called observability indices.

The existence of observability quasi-indices (and thus of observability indices as well, see [14] where the notion of observability indices has been analyzed) implies observability of (Σ, h). More precisely, we can distinguish points in a neighborhood of x_0 with the help of exactly $\rho_1 + \ldots + \rho_m = n$ derivatives of the output components (actually, using ρ_i time-derivatives of $y_i(t) = h_i(x(t))$). If (Σ, h) is locally observable everywhere on $\mathbb{R}^n$ (see [9], for various concepts of nonlinear observability), it possesses observability quasi-indices on an open and dense subset $X \in \mathbb{R}^n$ and thus, although it is locally observable, fails to have observability quasi-indices around singular points of $\mathbb{R}^n \setminus X$. Contrary to the observability indices (which are unique), the observability quasi-indices $(\rho_1, \ldots, \rho_m)$ are not unique and we do not suppose any order relation between them. Since observability quasi-indices may depend on a point, we say that quasi-indices $(\rho_1, \ldots, \rho_m)$ are uniform in a subset $\mathcal{X}$ of $\mathbb{R}^n$ if $(\rho_1, \ldots, \rho_m)$ form quasi-indices at any $x_0 \in \mathcal{X}$.

Example 2.1. Consider the following dynamics

$$\begin{array}{rclrcl} \dot{x}_1 & = & x_2 + x_3 & \dot{x}_3 & = & x_4 \\ \dot{x}_2 & = & 0 & \dot{x}_4 & = & 0 \end{array} \tag{5}$$

with the outputs $y_1 = x_1$ and $y_2 = x_3$. A straightforward calculation shows that both couples $(\rho_1, \rho_2) = (2, 2)$ and $(\rho_1, \rho_2) = (3, 1)$ are (uniform) observability quasi-indices (but only the first one corresponds to the observability indices).

The following theorem states that the observed system (Σ, h) can be made flat of differential weight $n+m$ (which is the minimal possible) if and only if it admits observability quasi-indices and, moreover, gives a system of algebraic equations, whose coefficients are calculated in terms of the Lie derivatives of the outputs, to be solved in order to construct flat inputs.

Theorem 2.1. *Consider the observed system (Σ, h) around $x_0 \in \mathbb{R}^n$. The following conditions are equivalent:*

($\mathcal{O}1$) There exist observability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0;

($\mathcal{O}2$) There exist vector fields $g_1, \ldots, g_m$ such that the system (Σ_c, h) is x-flat at x_0 of differential weight $n+m$, with h being a minimal flat output;

Moreover, under ($\mathcal{O}1$), the vector fields $g_1, \ldots, g_m$ are given by

(i) $< \mathrm{d}L_f^q h_i, g_j >= 0$ and $< \mathrm{d}L_f^{\rho_i - 1} h_i(x), g_j >= D_i^j(x)$, for $1 \leq i, j \leq m$, $0 \leq q \leq \rho_i - 2$, where $\left(D_i^j(x)\right)$ is any smooth invertible $m \times m$-matrix,

or, equivalently, by

(ii) The distribution $\mathcal{G} = \mathrm{span}\,\{g_1, \ldots, g_m\}$ satisfies

$$\mathcal{G}^{\perp} = \mathrm{span}\,\{\mathrm{d}L_f^j h_i, 0 \leq j \leq \rho_i - 2, 1 \leq i \leq m\}.$$

If $(\rho_1, \ldots, \rho_m)$ are uniform quasi-indices on $\mathcal{X}$ (in particular, on $\mathcal{X} = \mathbb{R}^n$), then the vector fields $g_1, \ldots, g_m$ exist globally on $\mathcal{X}$, on which are given by (i), and they yield the global system Σ_c on $\mathcal{X}$ that is locally flat around any $x \in \mathcal{X}$ with the help of the globally defined flat output $(\varphi_1, \ldots, \varphi_m) = (h_1, \ldots, h_m)$.

The following two results are a direct consequence of Theorem 2.1, their proofs as well as the proof of Theorem 2.1 are given in Section 5.

Corollary 2.1. *If (Σ, h) is not locally observable at x_0 and admits $g_1, \ldots, g_m$ such that the system (Σ_c, h) is x-flat at x_0 with $\varphi = h$ being a minimal flat output at x_0, then the control system Σ_c, independently of the choice of $g_1, \ldots, g_m$, is never static feedback linearizable around x_0.*

Corollary 2.2. *Consider (Σ, h) a linear system of the form $\Sigma : \dot{x} = Ax$, $y = h(x) = Cx$. There exist constant vector fields $g_1 = b_1, \ldots, g_m = b_m$ such that the system $\Sigma_c : \dot{x} = Ax + \sum_{i=1}^m b_i u_i$ is x-flat at x_0 with flat outputs $\varphi = y = Cx$ if and only if the pair (A, C) is observable.*

According to Corollary 2.2, if the original dynamical system is linear and if we want to obtain a flat control system that is also linear, then flat inputs can be constructed if and only if $(\Sigma, h) = (Ax, Cx)$ is observable. If we do not necessarily require to remain within the linear category, flat inputs can (generically) always be constructed following the procedure explained in Section 3 (but for the price of loosing the linearity). Notice also that there is no hypotheses on the differential weight of the flat linear control system (Σ_c, Cx): for any constant control vector fields $g_i = b_i$ that render the linear system Σ_c flat with $\varphi = Cx$ being a flat output (of any differential weight), the pair (A, C) is necessarily observable.

Remark 2.3. Theorem 2.1 generalizes the result of [31] according to which if a system has observability quasi-indices $(\rho_1, \ldots, \rho_m)$ and the vector fields $g_1, \ldots, g_m$ satisfy (i) with $D_i^j(x) = \delta_i^j$, then the control system (Σ_c, h) is flat. All other vector fields g_i corresponding to all other invertible matrices $\big(D_i^j(x)\big)$ render the system feedback linearizable and are given up to a transformation $g_i \mapsto \sum_{j=1}^m D_i^j(x) g_j$. In our result, we prove additionally that such g_i's (and only such) lead to a flat system Σ_c of differential weight $n+m$, which is minimal and implies that Σ_c is static feedback linearizable (see [10, 13] and also [20], where this last property is related to the minimal differential weight $n+m$).

The conditions of Theorem 2.1 are local and valid around a nominal point x_0 that can be an equilibrium or not. If quasi-indices are uniform on $\mathbb{R}^n$, then the control vector fields $g_1, \ldots, g_m$ exist globally and the control system is x-flat on $\mathbb{R}^n$ with a flat output being globally defined as $\varphi_i = h_i$, $1 \leq i \leq m$, nevertheless representation (3) is, in general, local only. In particular, the map $(x, u) \to (h_1, \dot{h}_1, \ldots, h_1^{(s_1)}, \ldots, h_m, \dot{h}_m, \ldots, h_m^{(s_m)})$ need not be globally invertible, as shows the following example.

Example 2.2. Consider the following dynamics on $\mathbb{R}^2$

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2) \\ \dot{x}_2 & = & f_2(x_1, x_2), \end{array} \tag{6}$$

with the outputs $y_1 = \mathrm{e}^{x_1} \cos x_2$ and $y_2 = \mathrm{e}^{x_1} \sin x_2$. The above system is locally everywhere observable, with $(\rho_1, \rho_2) = (1, 1)$ uniform observability indices on $\mathbb{R}^2$. Hence, by Theorem 2.1, the control vector fields g_1 and g_2 exist globally on $\mathbb{R}^2$ and can be defined by $L_{g_j} h_i(x) = D_i^j(x)$, $1 \leq i, j \leq 2$, where $\big(D_i^j(x)\big)$ is any smooth invertible 2×2-matrix. They yield, on $\mathbb{R}^2$, the global control system $\Sigma_c : \dot{x} = f(x) + u_1 g_1 + u_2 g_2$. The flat output $\varphi(x) = h(x) = (\mathrm{e}^{x_1} \cos x_2, \mathrm{e}^{x_1} \sin x_2)$ is globally defined on $\mathbb{R}^2$ but the map $(x, u) \to (h_1, \dot{h}_1)$ is not globally invertible (because the map $x \to h(x) = (\mathrm{e}^{x_1} \cos x_2, \mathrm{e}^{x_1} \sin x_2)$ is not). Notice, however, that Σ_c is globally flat with another flat output $\tilde{\varphi} = (x_1, x_2)$.

Notice that there can be many observability quasi-indices, and different observability quasi-indices $(\rho_1, \ldots, \rho_m)$ lead to different control vector fields $g_1, \ldots, g_m$, not equivalent via $g_i \mapsto \sum_{j=1}^m \beta_i^j g_j$, and thus to flat control systems (Σ_c, h) that are not static feedback equivalent, see Example 2.3 below. In particular, to any choice of observability quasi-indices $(\rho_1, \ldots, \rho_m)$, there correspond flat inputs $u_1, \ldots, u_m$ giving (3) with $s_i = \rho_i$.

Example 2.3. Consider again system (5). For the first couple of quasi-indices $(\rho_1, \rho_2) = (2, 2)$, we get $\mathcal{G}^\perp = \text{span}\ \{\mathrm{d}x_1, \mathrm{d}x_3\}$. Thus $\mathcal{G} = \text{span}\ \{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4}\}$ and the associated flat control system is (or any system equivalent to it via an invertible static feedback $u = \beta(x)\tilde{u}$)

$$\begin{array}{rclrcl} \dot{x}_1 & = & x_2 + x_3 & \dot{x}_3 & = & x_4 \\ \dot{x}_2 & = & u_1 & \dot{x}_4 & = & u_2 \end{array} \tag{7}$$

with $h = (x_1, x_3)$ a minimal flat output of differential weight $n + 2 = 6$.

The second couple of quasi-indices $(\rho_1, \rho_2) = (3, 1)$ leads to $\mathcal{G}^\perp = \text{span}\ \{\mathrm{d}x_1, \mathrm{d}x_2 + \mathrm{d}x_3\}$. Thus $\mathcal{G} = \text{span}\ \{\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\}$ and the associated flat control system is (or any system equivalent to it via an invertible static feedback $u = \beta(x)\tilde{u}$)

$$\begin{array}{rclrcl} \dot{x}_1 & = & x_2 + x_3 & \dot{x}_3 & = & x_4 - u_1 \\ \dot{x}_2 & = & u_1 & \dot{x}_4 & = & u_2 \end{array} \tag{8}$$

with $h = (x_1, x_3)$ a minimal flat output of differential weight $n + 2 = 6$.

In some practical applications, it may be interesting to use more derivatives of a particular output component to decrease the number of derivatives of another (more sensitive) component. For instance, in the above example, if the output $y_1 = x_1$ contains less measurement noise than $y_2 = x_3$ or if the system were nonlinear, with fewer nonlinear terms in successive time-derivatives of $y_1 = x_1$ than of $y_2 = x_3$, then it is natural to consider the quasi-indices $(\rho_1, \rho_2) = (3, 1)$. Notice also that both resulting control systems (7) and (8) are static feedback linearizable with (x_1, x_3) playing the role of the linearizing outputs, but they are not static feedback equivalent to each other. This is so because their quasi-indices coincide with their controllability indices. The former being different implies that the latter are different as well and thus the systems cannot be static feedback equivalent.

It is well known that systems linearizable via invertible static feedback are flat and their description (3) uses the minimal possible, which is $n + m$, number of time-derivatives of the components of the flat output (see [20] where that property is discussed). So, according to Theorem 2.1, if the pair (Σ, h) is observable, then there always exist $g_1, \ldots, g_m$ such that the associated control system Σ_c is flat with h being a flat output and, moreover, $g_1, \ldots, g_m$ can be chosen such that Σ_c is static feedback linearizable and can be calculated via a system of algebraic equations. The goal of this paper is thus to solve the problem of finding flat inputs for the unobservable multi-output case and to generalize the results of [21] treating the two-output case.

Similarly to the definition of observability quasi-indices, we introduce the notion of unobservability quasi-indices. Recall that to the output h, we associated the observability codistribution $\mathcal{H}$ given by (4).

Definition 2.3. The observed system (Σ, h) is said to have unobservability quasi-indices $(\rho_1, \ldots, \rho_m)$ around x_0 if dim span $\{\mathrm{d}L_f^j h_i(x), 0 \leq j \leq \rho_i - 1, 1 \leq i \leq m\} = \dim \mathcal{H}(x) = \sum_{i=1}^m \rho_i = k < n$ in a neighborhood of x_0.

According to the above definition, if the systems (Σ, h) has unobservability quasi-indices around x_0, then it is unobservable at x_0 and, in addition, the associated codistribution $\mathcal{H}(x)$ is of constant rank, equal to k, in a neighborhood of x_0. This means that only k directions can be observed (and this is valid around any point of that neighborhood). The above definition is more restrictive than the lack of observability at a point: we require the system to be nowhere observable on a whole neighborhood of x_0 and, moreover, its observability defect to be constant on that neighborhood. As for the observability quasi-indices, there can be many unobservability quasi-indices and different unobservability quasi-indices $(\rho_1, \ldots, \rho_m)$ will lead to different control vector fields $g_1, \ldots, g_m$ and thus to different flat control systems (Σ_c, h), as explained in Section 3.

3 Main results

Throughout, we suppose that (Σ, h) is nowhere observable. More precisely, we assume that around any x_0 in an open and dense subset X of $\mathbb{R}^n$, we have $\dim \mathcal{H}(x) = k$ and there exist unobservability quasi-indices $(\rho_1, \ldots, \rho_m)$ of (Σ, h) which, by definition, satisfy $\rho_1 + \cdots + \rho_m = k$ (in the $\mathcal{C}^\infty$-case the values of k and $(\rho_1, \ldots, \rho_m)$ can be different on various connected components of X but this does not affect our constructions below, that are local, so given on an arbitrary but fixed connected component). Locally, around $x_0 \in X$, introduce coordinates

$$w_i^j = L_f^{j-1} h_i, \quad 1 \le j \le \rho_i,\ 1 \le i \le m, \tag{9}$$

in which the system Σ can be locally transformed into the following observed-unobserved form

$$\begin{array}{rcll} \dot{w}_i^j & = & w_i^{j+1}, & 1 \le j \le \rho_i - 1, \\ \dot{w}_i^{\rho_i} & = & a_i(w), & 1 \le i \le m, \end{array} \tag{10}$$

$$\dot{z} = b(w, z), \tag{11}$$

with $\dim w = k < n$ and $\dim w + \dim z = n$, where z consists of any coordinates completing w to a coordinate system, and $(h_1, \ldots, h_m) = (w_1^1, \ldots, w_m^1)$.

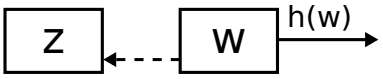


Figure 1: Observed-unobserved variables.

The w-coordinates of the above form are the states observed with the help of the output h and its successive time-derivatives, and there are k of them (recall that $k = \dim \mathcal{H}(x)$, where $\mathcal{H}(x) = \text{span}\ \{\mathrm{d}L_f^j h_i(x), j \ge 0, 1 \le i \le m\}$). The z-coordinates correspond to the unobserved directions, there are $n - k$ (which is the observability defect) of them and they complete w to a coordinate system. Figure 1 summarizes the above remark. We denote by a (resp., by b) the drift associated to the observed (resp., unobserved) subsystem.

The main results of the paper are given by Theorems 3.1, 3.2, and 3.3 below according to which, even in the unobservable case, we can always, at least locally on an open and dense subset, construct control vector fields $g_1, \ldots, g_m$ such that the associated control system (Σ_c, h) is x-flat with flat output $\varphi = h$. We first discuss Theorems 3.1 and 3.2 in Section 3.1, which apply to the case $n - k \ge m - 1$, and then Theorem 3.3 in Section 3.2, which deals with $n - k < m - 1$.

The nature of the nominal point around which we work plays an important role in our study. Theorems 3.1, 3.2, and 3.3 are based on normal forms given in terms of (w, z)-coordinates. For the normal forms of Theorems 3.1 and 3.2, in the z_i^j-variables, the lower index i is numbered between 2 and m since for the flat control system (Σ_c, h), the first output $h_1 = w_1^1$ (and the corresponding flat input u_1) has a special role and we distinguish it from the other outputs.

3.1 Construction of the control vector fields: the case $n - k \ge m - 1$

Theorem 3.1 presents a normal form NF1 whose construction is performed on a an open and dense subset $\mathcal{X} \subset \mathbb{R}^n$. Normal form NF1 contains vector fields $g_1, \ldots, g_m$ that render the system (Σ, h) locally flat, with h being a flat output. The idea is to complete the original output h to a new output $h^\ell = (h, \psi^1, \ldots, \psi^\ell)$ such that (Σ, h^ℓ) is locally observable.

The integers $\mu_i^{\ell_i}$ appearing in NF1 (and defined in Section 3.1.2), are related with observability quasi-indices of (Σ, h^ℓ). For each z_i-chain, we distinguish ℓ_i (where either $\ell_i = \ell$ or $\ell_i = \ell - 1$) variables z_i^q, where $q = \mu_i^1, \ldots, \mu_i^{\ell_i}$. Each z_i-chain exhibits ℓ_i (nonlinear) functions b_i^q that actually depend on certain z-variables only and to indicate on which variables

they depend, for $q = \mu_i^s$, denote $\overline{z}^q = (z_2^1, \ldots, z_2^{\mu_2^s}, \ldots, z_m^1, \ldots, z_m^{\mu_m^s})$. Observe that only the upper-index s of $q = \mu_i^s$ is actually involved in the right-hand side term, that is, $\overline{z}^{\mu_i^s}$ is the same for all $2 \leq i \leq m$ and equals $\overline{z}^{\mu_i^s} = (z_2^1, \ldots, z_2^{\mu_2^s}, \ldots, z_m^1, \ldots, z_m^{\mu_m^s})$.

Theorem 3.1. *Consider the observed system (Σ, h) and assume that $n-k \geq m-1$. Then:*

(A1) There exists $\mathcal{X} \subset \mathbb{R}^n$, open and dense, such that locally, around any $x_0 \in \mathcal{X}$, we can always construct $g_1, \ldots, g_m$ such that (Σ_c, h) is x-flat at any (x_0, u_0) satisfying $x_0 \in \mathcal{X}$ and $u_{10} \neq 0$.

(A2) Around any $x_0 \in \mathcal{X}$, an open and dense subset of $\mathbb{R}^n$, there exist local coordinates (w, z), with w given by (9), bringing (Σ, h) into (10)-(11) with the unobserved subsystem (11) of the form:

$$\dot{z}_i^q = \begin{cases} z_i^{q+1}, & q \neq \mu_i^1, \ldots, \mu_i^{\ell_i}, \\ b_i^q(w, \overline{z}^q), & q = \mu_i^1, \ldots, \mu_i^{\ell_i}, \end{cases} \quad 2 \leq i \leq m, \tag{12}$$

where the integers $1 \leq \mu_i^1 < \mu_i^2 < \cdots < \mu_i^{\ell_i}$ satisfy $\sum_{i=2}^m \mu_i^{\ell_i} = n$, and for this form, we can always locally construct $g_1, \ldots, g_m$ such that (Σ_c, h) is x-flat at $(x_0, u_0) \in \mathcal{X} \times \mathbb{R}^m$, with $u_{10} \neq 0$, and is given by the following form:

$$NF1: \begin{cases} \dot{w}_1^j = w_1^{j+1} & \dot{w}_i^j = w_i^{j+1} \\ \dot{w}_1^{\rho_1} = a_1(w) + u_1 & \dot{w}_i^{\rho_i} = a_i(w) + z_i^1 u_1 \\ & \dot{z}_i^q = \begin{cases} z_i^{q+1}, & q \neq \mu_i^1, \ldots, \mu_i^{\ell_i}, \\ b_i^q(w, \overline{z}^q) + z_i^{q+1} u_1, & q = \mu_i^1, \ldots, \mu_i^{\ell_i - 1}, \\ b_i^q(w, z) + u_i, & q = \mu_i^{\ell_i}, \end{cases} \\ & 2 \leq i \leq m, \end{cases}$$

where $1 \leq j \leq \rho_i - 1$, and $h = (w_1^1, \ldots, w_m^1)$ is a flat output of differential weight $n + m + \max\limits_{2 \leq i \leq m} \mu_i^{\ell_i}$.

An algorithm constructing NF1 will be given in Section 3.1.1, and a proof of Theorem 3.1, based on that algorithm, in Section 5.4. Item (A2) of Theorem 3.1 is valid around any nominal point x_0, equilibrium or not. If x_0 is not an equilibrium point of the original system, i.e., $f(x_0) \neq 0$, then we can still simplify the solution of the problem in Theorem 3.2 below. Indeed, first, we render the system (Σ, h) flat around any point x_0 (not just on an open and dense subset) provided that (Σ, h) has unobservability quasi-indices at x_0. Second, the proposed normal form NF2 is obtained by adding to (Σ, h) a multi-input chained form for the whole z-space (with respect to z-variables that rectify the z-component of the drift f) and not only a multi-input chained form on a submanifold (with respect to suitable chosen z-variables that need to be identified through the algorithm). Third, the control system (Σ_c, h) that we construct (by adding a multi-input chained form) turns out to be not only flat, which is our primary goal, but the differential weight of h as a flat output of NF2 is the lowest (minimal possible) among all differential weights of h as a flat output of any Σ_c.

Recall that $\dim \mathcal{H}(x) = k$ and let

$$n - k = p(m-1) + r \quad \text{and} \quad p^* = \begin{cases} p, & \text{if } r = 0, \\ p+1, & \text{if } r > 0, \end{cases} \tag{13}$$

be the Euclidean division of $n-k$ (the observability defect) and $m-1$ (the number of output components being m). Define the integers

$$\begin{array}{rcll} \mu_i & = & p, & 2 \leq i \leq m-r, \\ \mu_i & = & p+1, & m-r+1 \leq i \leq m, \end{array} \tag{14}$$

that will correspond to the lengths of the z_i-chains of the form NF2.

Theorem 3.2. *Consider the observed system (Σ, h) and assume that $n-k \geq m-1$. Then:*

(B1) For any $x_0 \in \mathbb{R}^n$ satisfying $f(x_0) \neq 0$ and such that (Σ, h) admits unobservability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0, there exist local coordinates (w, z) in a neighborhood X_0 of x_0, with w given by (9), bringing (Σ, h) into (10)-(11) with the unobserved subsystem (11) of the form:

$$\begin{array}{rcll} \dot{z}_i^q & = & 0, & 1 \leq q \leq \mu_i - 1, \\ \dot{z}_i^{\mu_i} & = & b_i, & 2 \leq i \leq m, \end{array} \tag{15}$$

where the integers μ_i are given by (14), and $b_2 = 1$ and $b_i = 0$, for $3 \leq i \leq m-1$, and for this form, we can always construct vector fields $g_1, \ldots, g_m$ on X_0 such that (Σ_c, h) is x-flat at $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^m$, with $u_{10} \neq 0$, of differential weight $n+m+p^*$, and is given by the following form with $h = (w_1^1, \ldots, w_m^1)$:

$$NF2 \left\{ \begin{array}{ll} \dot{w}_1^j = w_1^{j+1} & \dot{w}_i^j = w_i^{j+1} \\ \dot{w}_1^{\rho_1} = a_1(w) + u_1 & \dot{w}_i^{\rho_i} = a_i(w) + z_i^1 u_1 \\ & \dot{z}_i^q = z_i^{q+1} u_1 \\ & \dot{z}_i^{\mu_i} = b_i + u_i, \\ & \qquad 2 \leq i \leq m, \end{array} \right.$$

where $1 \leq j \leq \rho_i - 1$, $1 \leq q \leq \mu_i - 1$.

(B2) Let $x_0 \in \mathbb{R}^n$ and assume that $\dim \mathcal{H}(x) = k$ and is constant around x_0. If there exist $g_1, \ldots, g_m$ such that (Σ_c, h) is x-flat at $(x_0, \bar{u}_0^l) \in \mathbb{R}^n \times \mathbb{R}^{m(l+1)}$, for a certain $l \geq -1$, then the differential weight of h as a flat output of Σ_c is at least $n+m+p^*$. In particular, the differential weight of h as a flat output of NF2 is the lowest (minimal possible) among all differential weights of h as a flat output of any Σ_c.

An algorithm constructing $g_1, \ldots, g_m$ of normal form NF2 is given in Section 3.1.2, and a proof of Theorem 3.2, based on that algorithm, in Section 5.5.

Since a flat system is observable (with respect to its flat output and independently of the applied input signal), we have to render the original system (Σ, h) observable. For observability we need a link between the w- and the z-subsystems (Figure 2) but for Σ

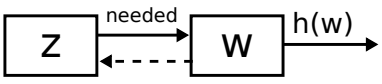


Figure 2: Link needed for observability.

there is no such a link, see the observed-unobserved form (10)-(11) illustrated by Figure 3. It follows that we have to create a link assuring observability with the help of the control vector fields (see Figure 4, where Π stands for products $z_i^{q+1} u_1$).

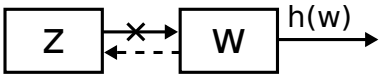


Figure 3: For Σ, no link between the w- and the z-subsystems.

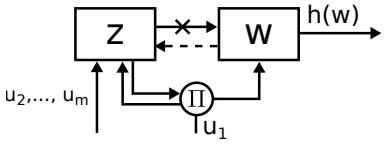


Figure 4: Observability obtained with the help of the g_i's.

3.1.1 Algorithm for constructing $g_1, \ldots, g_m$ of the normal form NF1

Consider the dynamical system (Σ, h) around any $x_0 \in \mathbb{R}^n$ such that (Σ, h) admits unobservability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0 (thus around any x_0 in X, an open and dense subset of $\mathbb{R}^n$) and transform it locally into the observed-unobserved form (10)-(11) and denote $x = (w, z)$. We first introduce coordinates z in which the subsystem $\dot{z} = b(w, z)$ of (10)-(11) takes form (12) and then for (12) we construct vector fields $g_1, \ldots, g_m$ yielding normal form NF1. Introduce the following notations: $X^0 = X$, $\mathcal{H}^0 = \mathcal{H}$, where $\mathcal{H}$ is the observability codistribution given by (4), and $k_0 = k = \dim \mathcal{H}(x)$.

Choose $m-1$ functions $\psi_i^1(w, z)$, for $2 \leq i \leq m$, satisfying $(\mathrm{d}\psi_2^1 \wedge \ldots \wedge \mathrm{d}\psi_m^1)(x_0) \neq 0 \bmod \mathcal{H}^0(x_0)$, where, in the (w, z)-coordinates, $\mathcal{H}^0(x) = \mathrm{span}\,\{\mathrm{d}w_i^j, 1 \leq i \leq m, 1 \leq j \leq \rho_i\}$. The functions ψ_i^1 always exist, since $\dim z = p(m-1)+r$, with $p \geq 1$, and can be, for instance, taken as $m-1$ well chosen original coordinates $z_{i_2}, \ldots, z_{i_m}$. Consider $\psi^1 = (\psi_2^1, \ldots, \psi_m^1)$ as dummy outputs that complete the original output h to a new output $h^1 = (h, \psi^1)$ and denote by $\mathcal{H}^1$ its associated observability codistribution $\mathcal{H}^1(x) = \mathcal{H}^0(x) + \mathrm{span}\,\{\mathrm{d}L_f^q \psi_i^1(x), 2 \leq i \leq m, q \geq 0\}$.

There exists an open and dense subset $X^1 \subset X^0$ (thus open and dense in $\mathbb{R}^n$ as well) such that for any $x_0 \in X^1$, we can find integers $\nu_2^1, \ldots, \nu_m^1$ and a neighborhood $X_{x_0}^1$ of x_0 satisfying $\mathcal{H}^1(x) = \mathcal{H}^0(x) + \mathrm{span}\,\{\mathrm{d}L_f^{q-1}\psi_i^1(x), 2 \leq i \leq m, 1 \leq q \leq \nu_i^1\}$ for $x \in X_{x_0}^1$, where $\dim \mathcal{H}^1(x) = k_0 + \sum_{i=2}^m \nu_i^1 = k_1$. Two cases are to be distinguished. Either $k_1 = n$, for $x \in X_{x_0}^1$, and then (f, h^1) is observable at x_0 and $(\rho_1, \ldots, \rho_m, \nu_2^1, \ldots, \nu_m^1)$ form observability quasi-indices at x_0. Or $k_1 < n$, for $x \in X_{x_0}^1$, and then (f, h^1) is not observable at x_0 and $(\rho_1, \ldots, \rho_m, \nu_2^1, \ldots, \nu_m^1)$ are unobservability quasi-indices at x_0 (of the output h^1). Notice that in both cases, observable and unobservable, the choice of $\nu_2^1, \ldots, \nu_m^1$ is, in general, not unique.

For $2 \leq i \leq m$, in both cases, introduce coordinates

$$z_i^q = L_f^{q-1}\psi_i^1, \quad 1 \leq q \leq \nu_i^1.$$

and define $\mu_i^1 = \nu_i^1$. In the observable case, we set $\ell = 1$, and the construction of the z-variables ends up, while in the unobservable case, we complete the output h^1 to a new output $h^2 = (h^1, \psi^2)$ by choosing $m-1$ functions $\psi_i^2(w, z)$, $2 \leq i \leq m$, such that $(\mathrm{d}\psi_2^2 \wedge \ldots \wedge \mathrm{d}\psi_m^2)(x_0) \neq 0 \bmod \mathcal{H}^1(x_0)$. Denote by $\mathcal{H}^2$ its associated observability codistribution $\mathcal{H}^2(x) = \mathcal{H}^1(x) + \mathrm{span}\,\{\mathrm{d}L_f^q \psi_i^2(x), 2 \leq i \leq m, q \geq 0\}$. There exists an open and dense subset $X^2 \subset X^1$ (thus open and dense in $\mathbb{R}^n$ as well) such that for any $x_0 \in X^2$, we can find integers $\nu_2^2, \ldots, \nu_m^2$ and a neighborhood $X_{x_0}^2$ of x_0 satisfying $\mathcal{H}^2(x) = \mathcal{H}^1(x) + \mathrm{span}\,\{\mathrm{d}L_f^{q-1}\psi_i^2(x), 2 \leq i \leq m, 1 \leq q \leq \nu_i^2\}$ for $x \in X_{x_0}^2$, where $\dim \mathcal{H}^2(x) = k_1 + \sum_{i=2}^m \nu_i^1 = k_2$. For $2 \leq i \leq m$,

introduce coordinates

$$z_i^{q+\mu_i^1} = L_f^{q-1}\psi_i^2, \text{ for } 1 \le q \le \nu_i^2,$$

and denote $\mu_i^2 = \mu_i^1 + \nu_i^2 = \nu_i^1 + \nu_i^2$. If $k_2 = n$, we set $\ell = 2$, and the construction of the z-variables stops, if $k_2 < n$, we continue.

Repeat this process until the system (f, h^ℓ), where $h^\ell = (h, \psi^1, \dots, \psi^\ell)$, obtained at the ℓ-th step, has observability quasi-indices $(\rho_1, \dots, \rho_m, \nu_i^j)$, where $2 \le i \le m$, $1 \le j \le \ell$, and $k + \sum_{i=2}^m \sum_{j=1}^\ell \nu_i^j = n$, on an open and dense subset $X^\ell \subset X^{\ell-1}$ (thus open and dense in $\mathbb{R}^n$). Notice that for any $1 \le j \le \ell - 1$, we have $\nu_i^j \ge 1$, for $2 \le i \le m$, but for $j = \ell$, we have either $\nu_i^\ell \ge 1$, for $2 \le i \le m$ (like in all preceding steps), if $n - k - \sum_{i=2}^m \mu_i^{\ell-1} \ge m - 1$ (first case), or $\nu_i^\ell = 1$, for $2 \le i \le d$, and $\nu_i^\ell = 0$, for $d + 1 \le i \le m$ (implying that the corresponding ψ_i^ℓ are absent), if $d = n - k - \sum_{i=2}^m \mu_i^{\ell-1} < m - 1$ (second case).

In the first case, we define $\mu_i^\ell = \mu_i^{\ell-1} + \nu_i^\ell$, for $2 \le i \le m$, and we set $\ell_i = \ell$ implying $\mu_i^{\ell_i} = \mu_i^\ell$. In the second case, for $1 \le i \le d$, we define $\mu_i^\ell = \mu_i^{\ell-1} + \nu_i^\ell$, and we set $\ell_i = \ell$ implying $\mu_i^{\ell_i} = \mu_i^\ell$, but for $d+1 \le i \le m$, we set $\ell_i = \ell - 1$ implying $\mu_i^{\ell_i} = \mu_i^{\ell-1}$ (since ψ_i^ℓ are absent for $d + 1 \le i \le m$).

Around any point $x_0 \in X^\ell$, we introduce $n-k$ coordinates $z_i^{q+\mu_i^{j-1}} = L_f^{q-1}\psi_i^j$, for $2 \le i \le m$, $1 \le q \le \nu_i^j$, $1 \le j \le \ell_i$, that are independent modulo $\mathcal{H} = \text{span}\,\{\mathrm{d}w_i^j, 1 \le i \le m, 1 \le j \le \rho_i\}$, and bring the unobserved subsystem (11) into form (12). For that form, set $Z_i^0 = z_i^1$ and $(Z_i^1, Z_i^2, Z_i^3, Z_i^4, \dots, Z_i^{2\ell_i-3}, Z_i^{2\ell_i-2}, Z_i^{2\ell_i-1}) = (z_i^{\mu_i^1}, z_i^{\mu_i^1+1}, z_i^{\mu_i^2}, z_i^{\mu_i^2+1}, \dots, z_i^{\mu_i^{\ell_i-1}}, z_i^{\mu_i^{\ell_i-1}+1}, z_i^{\mu_i^{\ell_i}})$, and define the control vector fields

$$g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + \sum_{i=2}^m \left(Z_i^0 \frac{\partial}{\partial w_i^{\rho_i}} + \sum_{s=1}^{\ell_i-1} Z_i^{2s} \frac{\partial}{\partial Z_i^{2s-1}} \right) \quad \text{and} \quad g_i = \frac{\partial}{\partial Z_i^{2\ell_i-1}},\ 2 \le i \le m,$$

which are in the multi-chained form on the submanifold $\{w_i^j = 0, \text{ for } 1 \le j \le \rho_i - 1 \text{ and } 1 \le i \le m, \text{ and } z_i^q = 0, \text{ for } q \ne \mu_i^1, \dots, \mu_i^{\ell_i} \text{ and } 2 \le i \le m\}$, where either $\ell_i = \ell - 1$ or $\ell_i = \ell$, and obtain the normal form NF1, with $(h_1, \dots, h_m) = (w_1^1, \dots, w_m^1)$, which is clearly x-flat at (x_0, u_0), where $u_{10} \ne 0$, with $(h_1, \dots, h_m)$ being a flat output of differential weight $n + m + \max_{2 \le i \le m} \mu_i^{\ell_i}$.

◁

The above algorithm provides an explicit construction, that uses differentiation and algebraic operations only, of g_i's.

Corollary 3.1. *Consider the dynamical system (Σ, h) around any $x_0 \in X^\ell$ and denote $\Psi_i^{q+\mu_i^{j-1}} = L_f^{q-1}\psi_i^j$, for $2 \le i \le m$, $1 \le q \le \nu_i^j$, $1 \le j \le \ell_i$, where the functions ψ_i^j and the set X^ℓ are given by the above algorithm. Denote $\varphi_i^q = L_f^{q-1}h_i$, for $1 \le q \le \rho_i$. Vector fields $g_1, \dots, g_m$ such that (Σ_c, h) is x-flat with h being a flat output can be constructed by*

$$\begin{cases} < \mathrm{d}\varphi_i^q, g_1 > = 0,\ 1 \le q \le \rho_i - 1, & < \mathrm{d}\varphi_i^{\rho_i}, g_1 > = \Psi_i^1, \\ < \mathrm{d}\varphi_1^{\rho_1}, g_1 > = 1, & < \mathrm{d}\Psi_i^q, g_1 > = \Psi_i^{q+1},\ q = \mu_i^1, \dots, \mu_i^{\ell_i-1}, \\ \qquad 1 \le i \le m, & < \mathrm{d}\Psi_i^q, g_1 > = 0,\ q \ne \mu_i^1, \dots, \mu_i^{\ell_i-1}, \\ & \qquad 2 \le i \le m, \end{cases} \tag{16}$$

for g_1, and

$$\begin{cases} < \mathrm{d}\varphi_i^q, g_j > = 0,\ 1 \le q \le \rho_i, & < \mathrm{d}\Psi_i^q, g_j > = 0,\ q \ne \mu_i^{\ell_i}, \\ \qquad 1 \le i \le m, & < \mathrm{d}\Psi_i^q, g_j > = \delta_{ij},\ q = \mu_i^{\ell_i}, \\ & \qquad 2 \le i \le m, \end{cases} \tag{17}$$

where $2 \le j \le m$, and $\delta_{ij} = 1$, if $i = j$, and 0, otherwise, for $g_2, \dots, g_m$. If all functions φ_i^q and Ψ_i^q are global, then the vector fields g_i are defined globally and assure local flatness around any $x_0 \in \mathbb{R}^n$.

Proof. The proof follows directly from the algorithm for constructing $g_1, \dots, g_m$ of NF1. □

Observe that if we regroup all n functions φ_i^q and Ψ_i^q of Corollary 3.1 into $\Phi = (\varphi_i^q, \Psi_i^q)$, systems (16) and (17) can be rewritten as

$$\mathrm{d}\Phi(x) \cdot g_i = d_i(x),\ 1 \le i \le m,$$

where the terms d_i denote the right hand-side of (16) and (17) and both, $\Phi(x)$ and $d_i(x)$ are known.

3.1.2 Algorithm for constructing $g_1, \dots, g_m$ of the normal form NF2

Consider the system (Σ, h) around any $x_0 \in \mathbb{R}^n$ satisfying $f(x_0) \neq 0$ and such that (Σ, h) admits unobservability quasi-indices $(\rho_1, \dots, \rho_m)$ at x_0. Bring Σ into the observed-unobserved form (10)-(11). By $f(x_0) \neq 0$, the first order partial differential equation $L_f\psi(x) = 0$ admits $n-1$ solutions independent at x_0. We choose among them any $n-k-1$ solutions independent modulo $\mathcal{H} = \text{span}\ \{\mathrm{d}w_i^j, 1 \le j \le \rho_i, 1 \le i \le m\}$ at x_0 and we label them as ψ_i^q, $2 \le i \le m$, where $1 \le q \le p$ for $2 \le i \le m-r$ and $1 \le q \le p+1$ for $m-r+1 \le i \le m$, and $(i,q) \neq (2,p)$ (recall that $n-k = p(m-1)+r$). Choose a solution of $L_f\psi(x) = 1$, and denote it ψ_2^p, such that the functions ψ_2^p, w_i^j, and the just constructed ψ_i^q, are independent at x_0. Introduce, around x_0, the new z-coordinates $z_i^q = \psi_i^q$, for $2 \le i \le m$, $1 \le q \le \mu_i$, where $\mu_i = p$, if $2 \le i \le m-r$, and $\mu_i = p+1$, if $m-r+1 \le i \le m$.

For the system (Σ, h), whose w-subsystem is given by (10) and the z-subsystem is represented by (15) in just constructed z_i^q-coordinates, define

$$g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + \sum_{i=2}^{m} \left(z_i^1 \frac{\partial}{\partial w_i^{\rho_i}} + \sum_{q=1}^{\mu_i - 1} z_i^{q+1} \frac{\partial}{\partial z_i^q} \right) \quad \text{and} \quad g_i = \frac{\partial}{\partial z_i^{\mu_i}}, \quad 2 \le i \le m. \tag{18}$$

In (w, z_i^q)-coordinates, the corresponding control system is in the form NF2, with $(h_1, \dots, h_m) = (w_1^1, \dots, w_m^1)$, which is clearly x-flat at (x_0, u_0), with $u_{10} \neq 0$, of differential weight $n + m + p^*$, see (13) for the definition of p^*, and with $(h_1, \dots, h_m)$ being a minimal flat output of NF2.

◁

The above algorithm provides an explicit construction of g_i's (that, once the solutions ψ_i^q of $L_f\psi(x) = 0$ and $L_f\psi(x) = 1$ are computed, uses differentiation and algebraic operations only).

Corollary 3.2. *Consider the dynamical system (Σ, h) around any $x_0 \in \mathbb{R}^n$ satisfying $f(x_0) \neq 0$ and such that (Σ, h) admits unobservability quasi-indices $(\rho_1, \dots, \rho_m)$ at x_0. Denote $\varphi_i^q = L_f^{q-1} h_i$, for $1 \le q \le \rho_i$, and compute the functions ψ_i^q, for $1 \le q \le \mu_i$, $2 \le i \le m$, of the above algorithm. Vector fields $g_1, \dots, g_m$ such that (Σ_c, h) is x-flat with h being a flat output can be constructed by*

$$\begin{cases} < \mathrm{d}\varphi_i^q, g_1 > = 0,\ 1 \le q \le \rho_i - 1, & < \mathrm{d}\varphi_i^{\rho_i}, g_1 > = \psi_i^1, \\ < \mathrm{d}\varphi_1^{\rho_1}, g_1 > = 1, & < \mathrm{d}\psi_i^q, g_1 > = \psi_i^{q+1},\ 1 \le q \le \mu_i - 1, \\ \qquad\qquad 1 \le i \le m, & < \mathrm{d}\psi_i^{\mu_i}, g_1 > = 0, \\ & \qquad\qquad 2 \le i \le m, \end{cases} \tag{19}$$

for g_1, and

$$\left\{\begin{array}{ll} <\mathrm{d}\varphi_i^q, g_j>=0,\ 1\le q\le \rho_i, & <\mathrm{d}\psi_i^q, g_j>=0,\ 1\le q\le \mu_i-1, \\ \qquad\qquad 1\le i\le m, & <\mathrm{d}\psi_i^{\mu_i}, g_j>=\delta_{ij}, \\ & \qquad\qquad 2\le i\le m, \end{array}\right. \tag{20}$$

where $2\le j\le m$, and $\delta_{ij}=1$, if $i=j$, and 0, otherwise, for $g_2,\dots,g_m$. If all functions φ_i^q and ψ_i^q are global, then the vector fields g_i are defined globally and assure local flatness around any $x_0\in\mathbb{R}^n$.

Proof. The proof follows directly from the algorithm for constructing $g_1,\dots,g_m$ of NF2. □

Observe that if we regroup all n functions φ_i^q and ψ_i^q of Corollary 3.2 into $\Phi=(\varphi_i^q,\psi_i^q)$, systems (19) and (20) can be rewritten as

$$\mathrm{d}\Phi(x)\cdot g_i = d_i(x),\ 1\le i\le m,$$

where the terms d_i denote the right hand-side of (19) and (20). In order to get the expressions of $\Phi(x)$ and $d_i(x)$, we first have to compute the solutions ψ_i^q of $L_f\psi(x)=0$ and $L_f\psi(x)=1$.

3.1.3 Discussion of the normal form NF1

The two extreme cases of the z-part of NF1 are $\ell=1$ and $\ell=p$, if $n-k=p(m-1)$, or $\ell=p+1$, if $n-k=p(m-1)+r$, where $1\le r\le m-2$. In the first case, we have

$$\begin{array}{rcl} \dot z_i^q &=& z_i^{q+1},\quad 1\le q\le \mu_i^1-1, \\ \dot z_i^{\mu_i^1} &=& b_i^{\mu_i^1}(w,z)+u_i,\quad 2\le i\le m, \end{array}$$

the z-subsystem is formed by $m-1$ almost linear chains (the nonlinearities as well as the inputs appear at the bottom level of each z_i-chain only). In the second case, we have $\mu_i^s=s$, $1\le s\le \ell_i$, with $\ell_i=p$, for $2\le i\le m-r$, and $\ell_i=p+1$, for $m-r+1\le i\le m$, and

$$\begin{array}{rcl} \dot z_i^q &=& b_i^q(w,\overline{z}^q)+z_i^{q+1}u_1, \\ & & \qquad 1\le q\le p-1, \\ \dot z_i^p &=& b_i^p(w,z)+u_i, \\ & & \qquad 2\le i\le m-r, \end{array} \qquad \begin{array}{rcl} \dot z_i^q &=& b_i^q(w,\overline{z}^q)+z_i^{q+1}u_1, \\ & & \qquad 1\le q\le p, \\ \dot z_i^{p+1} &=& b_i^{p+1}(w,z)+u_i, \\ & & \qquad m-r+1\le i\le m. \end{array}$$

This case reminds very much the form NF2; indeed the control vector fields g_i are in the multi-chained form (18), with μ_i replaced by ℓ_i (equal p or $p+1$), and it is because of g_1 only (the drift does not play any role) that for each q, the successive derivatives $L_{g_1}z_i^q=z_i^{q+1}$ provide $m-1$ new states z_i^{q+1} (as many as possible).

If x_0 is an equilibrium point of f, contrary to the case of the normal form NF2, then the drift cannot be rectified and we need to identify its triangular structure and to respect it. Any system can always be transformed into NF1 and this normal form actually works around any x_0, equilibrium or not, and in order to compute the z-coordinates we do not have to compute any partial differential equation, see the algorithm for the construction of NF1 and Corollary 3.1 in Section 3.1.1. For NF1, the linking terms, that is, the nonzero components of g_1, are the functions ψ_i^q defining the variables z_i^q, for $q=\mu_i^0,\mu_i^1,\dots,\mu_i^{\ell_i-1}$ (where we put $\mu_i^0=1$). Notice that the choice of linking terms ψ_i^q is far from being unique and may be important. The components of the vector field g_1 depend on z only, the link between the

observed subsystem and the (still) unobserved one is made (at each step s) with the help of $z_i^{\mu_i^s+1}u_1$ that involve the control u_1 only, and the remaining inputs (flatness implies that there are necessarily m inputs, which is the number of outputs) appear at the bottom levels of the z_i-chains.

In general, there is no reason for all z_i-chains to be (contrary to the normal form NF2) of length p or $p+1$, thus the differential weight of h may be greater than $n+m+p^*$, (which is the minimal possible and characterizes h as a flat output of NF2). In fact, the differential weight of h equals $n+m+\max\limits_{2\le i\le m}\mu_i^{\ell_i}$.

For fixed functions ψ_i^q, constructing the z-variables (and thus the form NF1) is performed on an open and dense subset $\mathcal{X}_\psi \subset \mathbb{R}^n$ that, indeed, depends on ψ. If it is important to work at a fixed point x_0 (around which unobservability quasi-indices exist) and if $x_0 \notin \mathcal{X}_\psi$, we can choose another family of dummy outputs $\tilde{\psi}$ because, in general, $\mathcal{X}_\psi \neq \mathcal{X}_{\tilde{\psi}}$. Choosing different ψ_i^q's will lead, in general, to different control vector fields and thus to different flat control systems with different differential weights.

3.1.4 Discussion of the normal form NF2

For the case $f(x_0) \neq 0$, corresponding to the normal form NF2, in order to connect the w- and the z-subsystems, one linking term for each w_i-chain is needed and the variables $(z_2^1, \dots, z_m^1)$ play indeed the role of those linking terms, as z_i^1 affects $\dot{w}_i^{\rho_i}$. Like for NF1, the link between the observed w-subsystem and the unobserved z-subsystem is made with the help of the control vector field g_1 only and the remaining $m-1$ inputs $u_2, \dots, u_m$ appear at the bottom level of each z_i-chain. It follows that u_1 is the only input that needs to be differentiated in order to express all states and controls with the help of flat outputs $h_1 = w_1^1, \dots, h_m = w_m^1$ and their derivatives. If the observability defect verifies $n-k = p(m-1)$, i.e., $n-k$ is a multiple of $m-1$, then each z_i-chain is of length p, and the control u_1 needs to be differentiated p times, so the differential weight is $n+m+p$. If $n-k = p(m-1)+r$, with $1 \le r \le m-2$, then the r first z_i-chains are of length $p+1$, the input u_1 needs to be differentiated $p+1$ times and, hence the differential weight is $n+m+p+1$.

Notice that there is a significant freedom in constructing g_i's, see the algorithm describing NF2 and Corollary 3.2 in Section 3.1.2. First we can choose any $n-k-1$ solutions (independent modulo $\mathcal{H}$) of $L_f\psi(x) = 0$. Second, there is a full arbitrariness of labelling them as ψ_i^q. In particular μ_i can be taken not as p or $p+1$ but (almost) arbitrary respecting $\sum_{i=2}^m \mu_i = n-k$. Our choice is simple (the form of g_i's is the simplest possible) and natural (chosen μ_i give the minimal differential weight) but other choices may be also interesting (in applications, for instance). Third, it is clear that we may replace the proposed $g_1, \dots, g_m$ by a more general triangular form

$$\bar{g}_1 = \frac{\partial}{\partial w_1^{\rho_1}} + \sum_{i=2}^m g_i^0(w, \overline{z}^1)\frac{\partial}{\partial w_i^{\rho_i}} + \sum_{i=2}^m \sum_{q=1}^{\mu_i-1} g_i^q(w, \overline{z}^{q+1})\frac{\partial}{\partial z_i^q},$$

and

$$\bar{g}_i = \frac{\partial}{\partial z_i^{\mu_i}}, \quad 2 \le i \le m,$$

where $\overline{z}^q = (z_2^1, \dots, z_2^q, \dots, z_m^1, \dots, z_m^q)$, and rk $\left(\frac{\partial g_i^q}{\partial z_j^{q+1}}\right)(w_0, z_0) = m-1$, $2 \le i, j \le m$, for any $0 \le q \le \mu_i - 1$. Then the new control system $\dot{x} = f(x) + \sum_{i=1}^m u_i \bar{g}_i(x)$, obtained with

the help of $\bar{g}_i$'s, is also flat with $h = (w_1^1, \dots, w_m^1)$ being a flat output of the same differential weight $n + m + p^*$ as previously.

If $f(x_0) \neq 0$, then the drift corresponding to the z-variables, that we denote by $b(w, z)$ in (11), can be rectified (even if $b(w_0, z_0) = 0$ and a non vanishing component of f is in the w-part, i.e., $a(w_0, z_0) \neq 0$) and, in well chosen z-coordinates, we have $b(w, z) = \frac{\partial}{\partial z_2^{\mu_2}}$ (which is of form (15)). It follows that, if x_0 is not an equilibrium point, the drift after rectification plays no role in choosing the linking terms $\psi_i^1 = z_i^1$ that can be any (linking terms meaning the w-components of g_1) and, moreover, it has no impact on the triangular structure of the z-part, which is completely determined by g_1. Notice that the components of the vector fields $g_1, \dots, g_m$ depend on the unobserved (with respect to the original output h) states only and they are designed to be in the multi-chained form. This particular form guarantees that the differential weight of h as a flat output of NF2 is the minimal possible (equal to $n+m+p^*$) among all differential weights of h as a flat output of any Σ_c. It is clear that NF2 is not linearizable via invertible static feedback, however, NF2 becomes static static feedback linearizable after pre-integrating p^* times the first control u_1, thus after the application of a dynamical precompensator of dimension p^*.

3.2 Construction of the control vector fields: the case $n - k < m - 1$

If the observability defect satisfies $n - k = r < m - 1$ (i.e., $p = 0$), then a result similar to Theorem 3.1 holds (but including it into Theorem 3.1 would significantly complicate the notations) and is given by Theorem 3.3 below.

Theorem 3.3. *Consider the observed system (Σ, h) and assume that $n - k = r < m - 1$. Then the set $\mathcal{X}$, consisting of points at which (Σ, h) admits unobservability quasi-indices, is open and dense in $\mathbb{R}^n$. Moreover:*

(A1)' Around any point x_0 of $\mathcal{X}$, we can always construct $g_1, \dots, g_m$ such that (Σ_c, h) is x-flat at any (x_0, u_0) satisfying $x_0 \in \mathcal{X}$ and $u_{10} \neq 0$. In particular, if $\mathcal{X} = \mathbb{R}^n$, then $g_1, \dots, g_m$ exist locally around any $x_0 \in \mathbb{R}^n$.

(A2)' Around any point x_0 of $\mathcal{X}$, independently of whether $f(x_0) = 0$ or $f(x_0) \neq 0$, choose any z-coordinates $(z_{m-r+1}^1, \dots, z_m^1)$ completing the w-coordinates, given by (9), to a local coordinate system and transform Σ into (10)-(11). For that form, the vector fields $g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + \sum_{i=m-r+1}^m z_i^1 \frac{\partial}{\partial w_i^{\rho_i}}$, $g_i = \frac{\partial}{\partial w_i^{\rho_i}}$, for $2 \leq i \leq m - r$, and $g_i = \frac{\partial}{\partial z_i^1}$, for $m - r + 1 \leq i \leq m$, yield the system (Σ_c, h) that is x-flat at $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^m$, satisfying $u_{10} \neq 0$, and is given by

$$NF3 \begin{cases} \dot{w}_i^j = w_i^{j+1} & \dot{w}_i^j = w_i^{j+1} \\ \dot{w}_i^{\rho_i} = a_i(w) + u_i, & \dot{w}_i^{\rho_i} = a_i(w) + z_i^1 u_1, \\ \quad 1 \leq i \leq m - r, & \dot{z}_i^1 = b_i^1(w, z) + u_i, \\ & \quad m - r + 1 \leq i \leq m, \end{cases}$$

where $b_i^1(w, z)$ are arbitrary functions of w and z, and $h = (w_1^1, \dots, w_m^1)$ is a minimal flat output of of differential weight $n + m + 1$.

(B2)' Let $x_0 \in \mathbb{R}^n$ and assume that $\dim \mathcal{H}(x) = k$ and is constant around x_0. If there exist $g_1, \dots, g_m$ such that (Σ_c, h) is x-flat at $(x_0, \bar{u}_0^l) \in \mathbb{R}^n \times \mathbb{R}^{m(l+1)}$, for a certain $l \geq -1$,

then the differential weight of h as a flat output of Σ_c is at least $n+m+1$. In particular, the differential weight of h as a flat output of NF3 is the lowest (minimal possible) among all differential weights of h as a flat output of any Σ_c.

3.2.1 Algorithm for constructing $g_1, \ldots, g_m$ of the normal form NF3

Consider the dynamical system (Σ, h) around any x_0 such that (Σ, h) admits unobservability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0 and transform it locally into the observed-unobserved form (10)-(11).

Since $\dim z = r < m - r$, it follows that at least $m - r$ controls, say $u_1, \ldots, u_{m-r}$, must affect the w-subsystem. So we can choose $g_i = \frac{\partial}{\partial w_i^{\rho_i}}$, for $2 \le i \le m - r$ (the control u_1 will play a special role), and thus modify the corresponding w-equations as $\dot{w}_i^{\rho_i} = a_i(w) + u_i$, for $2 \le i \le m - r$. The remaining equations for $\dot{w}_i^{\rho_i}$ have to be linked to the z-subsystem to render the whole system observable and we use all z-variables, where $z = (z^1_{m-r+1}, \ldots, z^1_m)$, as linking terms multiplying the control u_1, that is, we put

$$g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + \sum_{i=m-r+1}^{m} z_i^1 \frac{\partial}{\partial w_i^{\rho_i}} \quad \text{and} \quad g_i = \frac{\partial}{\partial z_i^1},\ m - r + 1 \le i \le m,$$

to get $\dot{w}_1^{\rho_1} = a_i(w) + u_1$, $\dot{w}_i^{\rho_i} = a_i(w) + z_i^1 u_1$ and $\dot{z}_i^1 = b_i^1(w, z) + u_i$, $m - r + 1 \le i \le m$. We thus have obtained the normal form NF3 which is clearly x-flat at (x_0, u_0), where $u_{10} \neq 0$, with $(h_1, \ldots, h_m) = (w_1^1, \ldots, w_m^1)$ being a minimal flat output of differential weight $n+m+1$.

$\triangleleft$

Like for the form NF1, the above algorithm provides an explicit construction, that uses differentiation and algebraic operations only, of g_i's. The algebraic system giving $g_1, \ldots, g_m$ for NF3 is very similar to that for NF1 and left to the reader.

3.2.2 Discussion of the normal form NF3

If the nominal point around which we work is not an equilibrium, i.e., $f(x_0) \neq 0$, the drift b of the unobserved subsystem (11) can be rectified, that is, in well chosen z-coordinates $z = (z^1_{m-r+1}, \ldots, z^1_m)$, we have $b(w, z) = \frac{\partial}{\partial z^1_{m-r+1}}$, and we obtain the normal form NF3 with $b^1_{m-r+1} = 1$ and $b^1_i = 0$, for $m - r + 2 \le i \le m$. This is the analogue of the normal form NF2 in the case $n - k < m - 1$.

The main difference between NF3 and the forms NF1 and NF2 is, as explained in the above algorithm, that the unobserved z-part of NF3 can be affected by at most r inputs, so the remaining controls necessarily appear in the w-equations. As for the case $n - k \ge m - 1$, the control vector fields are designed to be in the simplest possible form and only the vector field g_1 is used to link the observed and the unobserved subsystems although $g_2, \ldots, g_{m-r}$ affect the w-subsystem of NF3 as well (which is, in the case $n - k \ge m - 1$, affected by g_1 only). This, together with the fact that each z_i-chain is of length one, guarantee that the differential weight of h as a flat output of NF3 is the minimal possible, which is $n + m + 1$, compared to the differential weight of any Σ_c for which h is a flat output (recall that flatness of differential weight $n + m$ characterizes the observable case, see Section 2).

Notice also that the normal form NF3 of condition (A2)' is valid around any x_0 such that (Σ, h) admits unobservability quasi-indices at x_0. In particular, if quasi-indices exist around any $x_0 \in \mathbb{R}^n$, then NF3 is established around any $x_0 \in \mathbb{R}^n$ and not only on an open and dense subset like NF1.

Summing up, in all cases, we can always construct (locally, on an open and dense subset of $\mathbb{R}^n$), control vector fields $g_1, \ldots, g_m$ such that the associated control system (Σ_c, h) is flat and thus, generically, the problem of constructing flat inputs is completely solved.

3.3 Avoiding singularities in the control space

If the system (Σ, h) is observable at a given x_0, then for the corresponding flat control system (Σ_c, h), given by Theorem 2.1, even if $\varphi_i = h_i$ yield local flatness only, representing the states and the controls with the help of $\varphi_i = h_i$ and their derivatives is global with respect to u, so we never face singularities in the control space. This is no longer the case if (Σ, h) is unobservable and all normal forms NF1, NF2, and NF3 exhibit singularities in the control space. For all of them, the system ceases to be flat (with h as a flat output) at $u_1 = 0$ (which is a singular control for flatness). If we want to avoid singularities in the control space, we can construct another control system as follows: in all equations of NF1, NF2, and NF3, we replace u_1 by $\exp(u_1)$, except for the equation $\dot{w}_1^{\rho_1} = a_1(w) + u_1$ that we keep unchanged. For instance, for NF1, the equations involving u_1 become:

$$\begin{aligned} \dot{w}_1^{\rho_1} = a_1(w) + u_1 \quad \dot{w}_i^{\rho_i} &= a_i(w) + z_i^1 \exp(u_1) \\ \dot{z}_i^q &= b_i^q(w, \overline{z}^q) + z_i^{q+1} \exp(u_1), \quad q = \mu_i^1, \ldots, \mu_i^{\ell_i - 1}, \end{aligned}$$

where $2 \leq i \leq m$. We have thus constructed a new control system Σ_c that does no longer display singularities in the control space (representing the state z and the control u, with the help of components of $\varphi_i = h_i$ and their derivatives, is global), but the system is nonlinear with respect to u (which is the price for avoiding singularities).

4 Application to private communication

Since the trendsetting work of [24], it is known that the problem of private communication can be investigated with the help of a unidirectional synchronization of chaotic systems (see also [23] for an observer point of view). The goal of this section (which is based on the preliminary paper [22]) is to explain how constructing flat inputs for a given observed dynamic can be applied to private communication via both, using chaotic models and including the messages into the dynamics (see also [1], for a related approach using the design of nonlinear observers with unknown inputs, and [2] and [29], where invertibility and flatness of switched linear discrete-time systems is applied to private communication).

The general problem can be summarized as follow. Suppose that m confidential messages $u_1(t), \ldots, u_m(t)$ have to be sent to the receiver. To this aim, we use a transmitter (Σ, h) that consists of a dynamical system together with an output

$$\Sigma : \left\{ \begin{array}{rcll} \dot{x} & = & f(x), & x \in \mathbb{R}^n, \\ y & = & h(x), & y \in \mathbb{R}^m. \end{array} \right.$$

In order to transmit messages $(u_1(t), \ldots, u_m(t))$, we will add to Σ control vector fields $g_1, \ldots, g_m$ whose controls are, resp., $u_1, \ldots, u_m$ (that is, messages to be sent) in such a way that

$$\Sigma_c : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^m u_i g_i(x), \\ y & = & h(x), \end{array} \right.$$

is a flat system with $\varphi = (h_1, \ldots, h_m)$ being a flat output (see Definition 2.1). Both the sender and receiver know completely the equations of the flat control system $\Sigma_c : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$ and in order to transmit a message $u(t) = (u_1(t), \ldots, u_m(t))$, the sender calculates the corresponding output signal $y(t) = h(x(u(t), t))$ and transmits the masked information via the communication multiplexed channel. The receiver receives the signal $y(t)$ and knowing the system Σ_c calculates $u(t)$ (recall that Σ_c is flat). Notice that the flatness property and the fact that the original output forms a flat output are essential to express the confidential message $u(t)$ as function of the masked information $y(t)$ sent to the receiver (we thus study not only a synchronization problem, but also an input reconstruction problem). In order to compute $u(t)$ based on the knowledge of $y(t)$, left invertibility of Σ_c (which is a weaker property than flatness) would be sufficient but, in that case, the receiver should know not only $y(t)$ but also either the initial condition and the internal dynamics of the inverse (see [26]). So the advantage of using the flat system Σ_c is that the receiver uses the masked information $y(t)$ only. To assure that $y(t)$ is well masked (cannot be decrypted by a third person), we use as Σ a chaotic system.

To illustrate that general scheme, assume that two messages $u_1(t)$ and $u_2(t)$ have to be sent to the receiver. We use a transmitter, composed of two independent chaotic systems, a Lorenz circuit (x_ℓ, y_ℓ, z_ℓ), see, e.g., [17], and a Rössler circuit (x_r, y_r, z_r), see, e.g., [28], of the form (the notation (Ch) referring to the chaotic behavior):

$$(Ch) : \begin{cases} \dot{x}_\ell &= \sigma(y_\ell - x_\ell) \\ \dot{y}_\ell &= -k_d x_\ell z_\ell + r x_\ell - y_\ell \\ \dot{z}_\ell &= k_d x_\ell y_\ell - \beta z_\ell \\ \dot{x}_r &= -y_r - z_r \\ \dot{y}_r &= x_r + a y_r \\ \dot{z}_r &= b + z_r(x_r - c), \end{cases}$$

where the output $(h_1, h_2) = (x_\ell, z_\ell)$ is the masked information transmitted via the communication multiplexed channel, the parameters $a, b, c, \beta, \sigma, r$ and k_d are constant. In order to transmit messages $u_1(t)$ and $u_2(t)$, we add to (Ch) two control vector fields g_1 and g_2 whose controls are, resp., u_1 and u_2 (that is, messages to be sent):

$$(Ch_c) : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x), y_i = h_i(x), 1 \leq i \leq 2,$$

where g_1 and g_2 are chosen in such a way that (Ch_c) is flat with $\varphi = (h_1, h_2)$ being a flat output, and f is the right-hand side of (Ch). The chaotic behavior of (Ch) (depending on the values of the constant parameters) is crucial and has to be preserved by adding the modification $u_1 g_1(x) + u_2 g_2(x)$ to (Ch).

It is clear that with the given output (x_ℓ, z_ℓ) only the Lorenz variables can be observed and in the observed-unobserved form (10)-(11), using the new global w-coordinates $w_1^1 = x_\ell$, $w_1^2 = L_f x_\ell = \sigma(y_\ell - x_\ell)$, $w_2^1 = z_\ell$, the Lorenz circuit is equivalently given by: $\dot{w}_1^1 = w_1^2$, $\dot{w}_1^2 = -\sigma k_d w_1^1 w_2^1 + \sigma(r-1) w_1^1 - (\sigma+1) w_1^2 = a_1(w)$ and $\dot{w}_2^1 = k_d w_1^1 (\frac{1}{\sigma} w_1^2 + w_1^1) - \beta w_2^1 = a_2(w)$.

Notice that here, the unobserved subsystem, described by the Rössler circuit, is completely independent of the observed Lorenz circuit. Define the linking term $z_2^1 = \psi(w, x_r, y_r, z_r)$ by $\psi = y_r$ and compute its successive time-derivatives. We get $L_f \psi = x_r + a y_r$ and $L_f^2 \psi = -(y_r + z_r) + a(x_r + a y_r)$. It is clear that $\mathrm{d}\psi$, $\mathrm{d}L_f\psi$ and $\mathrm{d}L_f^2\psi$ are independent everywhere, and $z_2^j = L_f^{j-1}\psi$, $1 \leq j \leq 3$, (together with the w-coordinates) define a global system of coordinates on $\mathbb{R}^3 \times \mathbb{R}^3$. According to our results, we globally define $g_1 = \frac{\partial}{\partial w_1^2} + z_2^1 \frac{\partial}{\partial w_2^1}$ and

$g_2 = \frac{\partial}{\partial z_2^3}$ and obtain in (w, z)-coordinates a systems in NF1-form

$$(Ch_c) : \left\{ \begin{array}{rcl} \dot{w}_1^1 & = & w_1^2 \\ \dot{w}_1^2 & = & a_1(w) + u_1 \end{array} \quad \begin{array}{rcl} \dot{w}_2^1 & = & a_2(w) + z_2^1 u_1 \\ \dot{z}_2^1 & = & z_2^2 \\ \dot{z}_2^2 & = & z_2^3 \\ \dot{z}_2^3 & = & b_2^3(w, z) + u_2, \end{array} \right.$$

where $b_2^3(w, z) = -b - z_2^2 - az_2^3 - (z_2^3 - z_2^1 + az_2^2)(z_2^2 - az_2^1)$, which is flat with $(w_1^1, w_2^1) = (x_\ell, z_\ell)$ being a flat output. This system in the original coordinates is:

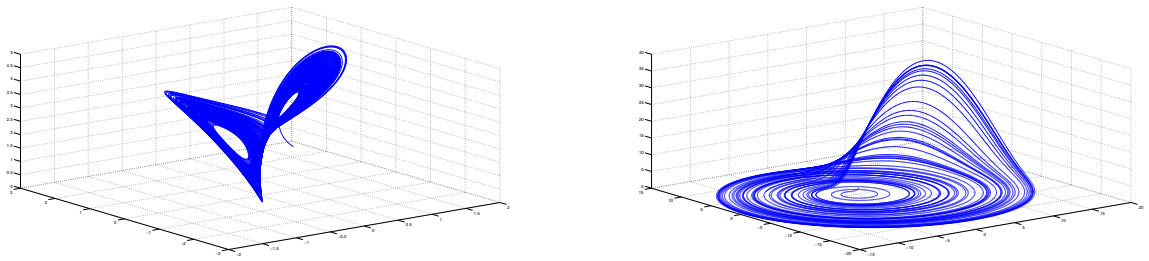
$$(\widetilde{Ch}_c) : \left\{ \begin{array}{rcl} \dot{x}_\ell & = & \sigma(y_\ell - x_\ell) \\ \dot{y}_\ell & = & -k_d x_\ell z_\ell + r x_\ell - y_\ell + \frac{1}{\sigma} u_1 \\ \dot{z}_\ell & = & k_d x_\ell y_\ell - \beta z_\ell + y_r u_1 \\ \dot{x}_r & = & -y_r - z_r \\ \dot{y}_r & = & x_r + a y_r \\ \dot{z}_r & = & b + z_r(x_r - c) - u_2. \end{array} \right.$$

The receiver knows completely the dynamics of the flat control system $(\widetilde{Ch}_c)$ as well as the transmitted output $(y_1(t), y_2(t)) = (x_\ell(t), z_\ell(t))$ and therefore using the flatness property the original messages $u = (u_1(t), u_2(t))$ can be recovered by the receiver as functions of the received output and its successive time-derivatives: $u_1(t) = \delta_1(y_1, \dot{y}_1, y_2)$ and $u_2(t) = \delta_1(y_1, \ldots, y_1^{(5)}, y_2, \ldots, y_2^{(4)})$.

Remark 4.1. An interesting question is why the unobserved part z is needed. The unobserved z-subsystem plays an important role for increasing the safety. Indeed, consider the system (Ch_c) in (w, z)-coordinates, the message u_1 has a degree of security significantly lower than that of u_2. The z-subsystem can be seen as a second level of security. This motivates even more the use of the quasi-indices: for a general two-output system, in order to decode u_1, we may need only the output $h_1 = w_1^1$ (if the function a_1 of equation $\dot{w}_1^{\rho_1} = a_1(w) + u_1$, allowing to decode u_1, depends on w_1^j-variables only) but we never need the linking term z_2^1, however, we always need it for decoding u_2. Notice that for the proposed transmitter (Ch_c), the function a_1 involves both outputs $h_1 = w_1^1$ and $h_2 = w_2^1$. Another issue (closely related to the above remark) is the motivation for using a two-output system for transmitting the two messages u_1 and u_2 (whose encodings are thus coupled), instead of using two independent systems, each one being used to encode a single message. Again it is clear that the proposed transmitter guarantees a higher security level for u_2: in order to decode it, we have to decode beforehand u_1.

4.1 Simulations

In this section we present a preliminary analysis that first shows, using numerical simulations, that the chaotic behavior of the transmitter (Ch) is preserved by adding control vector fields multiplied by suitable periodic messages into the dynamics, and second, that for suitable amplitudes of the encoded messages, the messages are recovered with a good accuracy by the receiver. The amplitudes should not be too small, otherwise the messages will be hidden into the chaos or will have the same order of magnitudes as numerical errors or perturbations and it will be difficult to distinguish between them; they should not be too large either, otherwise the chaotic behavior will be strongly modified and the messages will be easily detected by a hacker.



(a) Lorenz. (b) Rössler.

Figure 5: State variables phase plots without control.

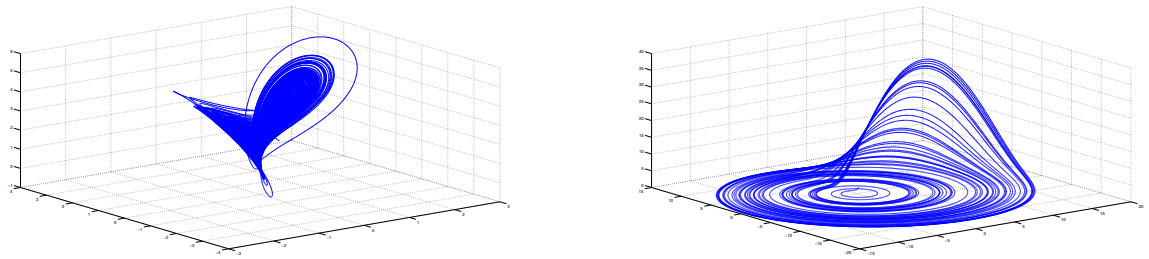
We will first give the phase plots for the uncontrolled dynamical transmitter composed by the Lorenz and Rössler circuits which are well know to be chaotic. Then we apply two sinusoidal input signals u_1 and u_2. We present three cases for different amplitudes of the inputs (the frequencies of u_1 and u_2 are the same for all cases). As expected, we observe that the larger the amplitudes of u_1 and u_2, the more the chaotic behavior is modified (but in all cases it seems to be preserved). The theoretical confirmation of this preliminary study will be the subject of a future work. Neither noise nor parameter uncertainties between the sender and the receiver nor the robustness with respect to different classical attacks have been considered for the simulations presented here. It is, however, obvious that in practice, those problems cannot be neglected and they will be discussed and taken into account in our future work.

Figures 5a and 5b gives the phase plots for the original Lorenz and Rössler subsystems (with no control), see the last paragraph of this section for the parameters values used in the simulations.

Figures 6a and 6b correspond to the solutions of the control system $(\widetilde{Ch}_c)$ and show that the chaotic behavior is preserved after modifying the original dynamical chaotic system (Ch) by applying the following inputs: $u_1 = 0.5 + 0.1\sin(2\pi t)$ and $u_2 = 0.5\sin(1.8\pi t)$ to the control system $(\widetilde{Ch}_c)$. For such (small) amplitudes, the first message is recovered with a good accuracy (Figure 7a), but the second one does not converge to the transmitted u_2 (Figure 7b). This is due to the fact that in order to decode u_2, we need the third derivative of y_r (the linking term between the two chaotic subsystems) and even if the decoded y_r is close to the transmitted one (Figure 8), it still contains some approximation errors (namely, the pick around 3.5 s in Figure 8) that are amplified at each new differentiation and lead to a decoded u_2 which is very far from the transmitted one.

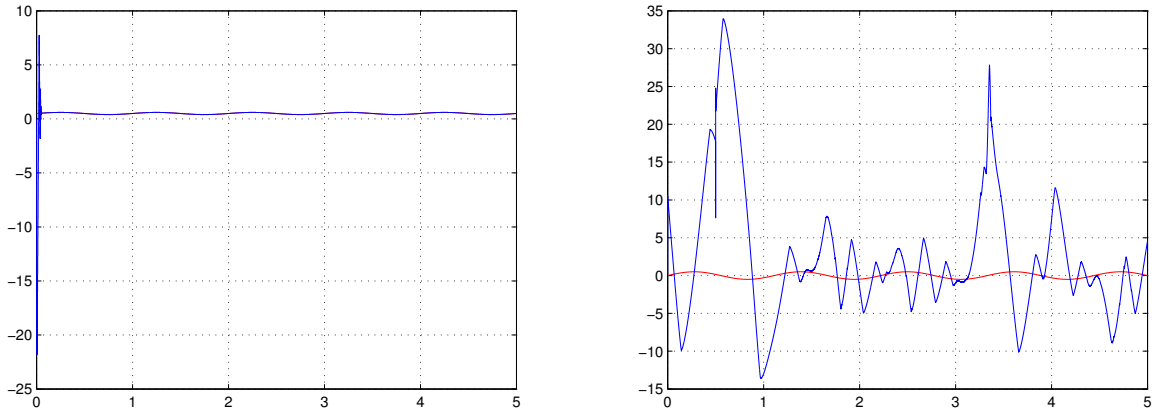
Figures 9a and 9b correspond to the solutions of the control system $(\widetilde{Ch}_c)$ obtained for $u_1 = 5 + \sin(2\pi t)$ and $u_2 = 5\sin(1.8\pi t)$. Notice that, the amplitude of the control values equals ten times that used in the previous example. For such inputs values, only the Lorenz strange attractor is slightly modified (a peak appears).

Figures 10a and 10b show that, for $u_1 = 5 + \sin(2\pi t)$ and $u_2 = 5\sin(1.8\pi t)$, the input u_1 (which corresponds to one of the confidential messages) can be exactly recovered after a very short time ($t > 0.1$ s) and u_2 can be recovered with a very good accuracy after 2 s. In order to obtain u_2, first we have to decode u_1, then y_r and, finally, to differentiate three times a filtered y_r. Figure 11 shows that, after 0.5 s, the real y_r of the transmitter and the decoded one coincide. The purpose of this paper is not to present the receiver (observer or/and differentiator), nevertheless it is important to note that the presented simulations



(a) Lorenz. (b) Rössler.

Figure 6: State variables phase plots for $u_1 = 0.5 + 0.1\sin(2\pi t)$, $u_2 = 0.5\sin(1.8\pi t)$.



(a) u_1 (b) u_2

Figure 7: To-be-sent messages in red and decoded messages in blue for $u_1 = 0.5+0.1\sin(2\pi t)$, $u_2 = 0.5\sin(1.8\pi t)$.

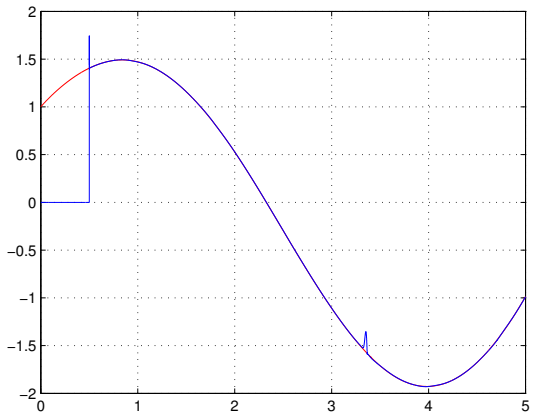
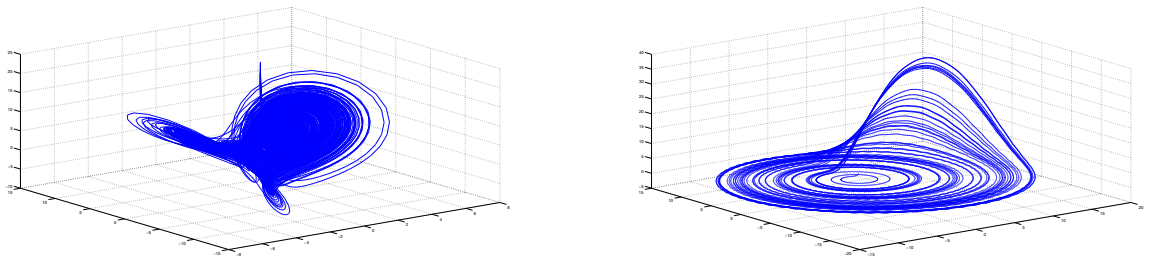
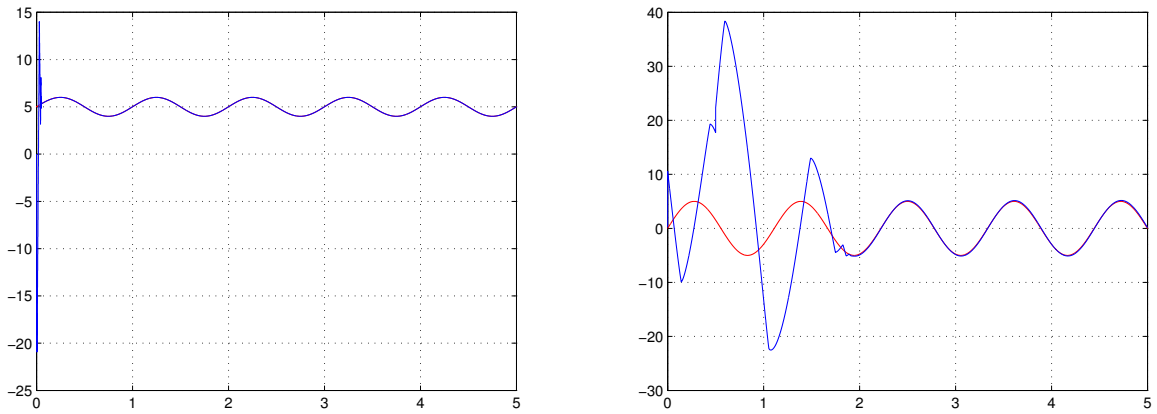


Figure 8: y_r of the transmitter in red and decoded y_r in blue for $u_1 = 0.5 + 0.1\sin(2\pi t)$, $u_2 = 0.5\sin(1.8\pi t)$.



(a) Lorenz. (b) Rössler.

Figure 9: State variables phase plots for $u_1 = 5 + \sin(2\pi t)$, $u_2 = 5\sin(1.8\pi t)$.



(a) u_1 (b) u_2

Figure 10: To-be-sent messages in red and decoded messages in blue for $u_1 = 5 + \sin(2\pi t)$, $u_2 = 5\sin(1.8\pi t)$.

have been obtained using two homogeneous differentiators [15] in order to compute u_1 and y_r. Then, after filtering, using a low pass filter, the obtained y_r, we use again a homogeneous differentiator to decode u_2. The above time constants (0.1 s, 2 s, etc.) are related to the differentiators gains.

From Figures 12a-12b, we see that for $u_1 = 10+2\sin(2\pi t)$, $u_2 = 10\sin(1.8\pi t)$, the chaotic behavior of the Lorenz subsystem and Rössler subsystem are modified. For such amplitudes, the messages are perfectly recovered by the receiver (as for the above case, after 0.1 s and 2 s, respectively) but the structure of the strange attractors are too strongly modified and a hacker would be able to detect the messages. We do not give the curves since they are very similar to those of the previous case.

The parameters for the simulations are given by $\beta = 8/3$, $\sigma = 10$, $r = 28$, $k_d = 10$, $a = 0.15$, $b = 0.20$, $c = 10$, with initial conditions $x_\ell(0) = y_\ell(0) = z_\ell(0) = 1$ and $x_r(0) = y_r(0) = z_r(0) = 1$.

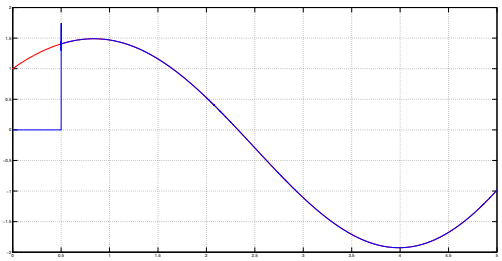
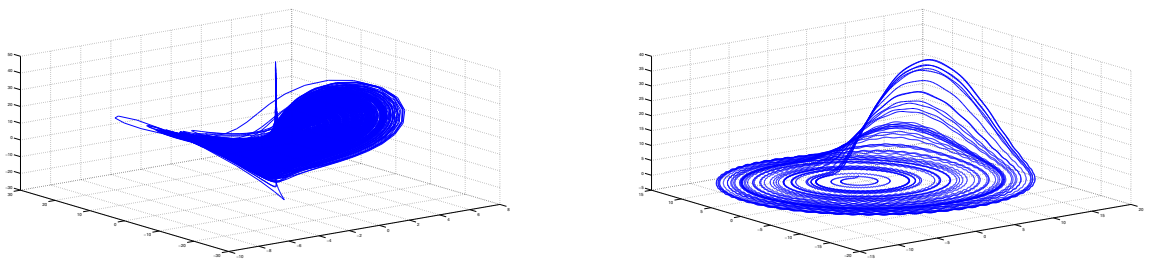


Figure 11: y_r of the transmitter in red and decoded y_r in blue for $u_1 = 5 + \sin(2\pi t)$, $u_2 = 5\sin(1.8\pi t)$.



(a) Lorenz. (b) Rössler.

Figure 12: State variables phase plots for $u_1 = 10 + 2\sin(2\pi t)$, $u_2 = 10\sin(1.8\pi t)$.

5 Proofs

5.1 Proof of Theorem 2.1

Proof of $(\mathcal{O}1) \Rightarrow (\mathcal{O}2)$. Consider the system Σ, around x_0, together with its output $h = (h_1(x), \ldots, h_m(x))$ and assume that there exist observability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0. Around x_0, the system Σ can be locally brought into

$$\begin{array}{rcll} \dot{w}_i^j & = & w_i^{j+1}, & 1 \leq j \leq \rho_i - 1, \\ \dot{w}_i^{\rho_i} & = & a_i(w), & 1 \leq i \leq m, \end{array}$$

with $w_i^j = L_f^{j-1} h_i$, for $1 \leq i \leq m$ and $1 \leq j \leq \rho_i$. For this form, define $g_i(w) = \frac{\partial}{\partial w_i^{\rho_i}}$, $1 \leq i \leq m$. It is clear that the corresponding control system

$$\Sigma_c : \left\{ \begin{array}{rcll} \dot{w}_i^j & = & w_i^{j+1}, & 1 \leq j \leq \rho_i - 1, \\ \dot{w}_i^{\rho_i} & = & a_i(w) + u_i, & 1 \leq i \leq m, \end{array} \right.$$

is static feedback linearizable around x_0 (with $h_i = w_i^1$, for $1 \leq i \leq m$, the linearizing outputs) and thus x-flat at x_0 of differential weight $n + m$, with h being a minimal flat output.

Proof of $(\mathcal{O}2) \Rightarrow (\mathcal{O}1)$. Suppose that there exist $g_1, \ldots, g_m$, around x_0, such that the associated control system $\Sigma_c : \dot{x} = f(x) + \sum_{i=1}^m g_i(x) u_i$ is x-flat at x_0 of differential weight $n + m$, with $h = (h_1(x), \ldots, h_m(x))$ being a minimal flat output. Let $\nu_1, \ldots, \nu_m$ denote the relative degrees of $h_1(x), \ldots, h_m(x)$, that is, $h_i^{(\nu_i)}$ is the lowest time-derivative of $h_i(x(t))$

depending explicitly on u. Since the flat outputs and all their successive derivatives are independent at x_0, we can introduce the new coordinates $\tilde{x}_i^j = L_f^{j-1} h_i$, for $1 \leq i \leq m$ and $1 \leq j \leq \nu_i$. We get

$$\begin{array}{rcl} \dot{\tilde{x}}_i^j & = & \tilde{x}_i^{j+1}, \quad 1 \leq j \leq \nu_i - 1, \\ \dot{\tilde{x}}_i^{\nu_i} & = & L_f^{\nu_i} h_i + \sum_{j=1}^m (L_{g_j} L_f^{\nu_i - 1} h_i) u_j, \quad 1 \leq i \leq m. \end{array} \tag{21}$$

Flatness of differential weight $n+m$ implies that we necessarily have $\sum_{i=1}^m \nu_i = n$ and that the decoupling matrix $D_i^j = L_{g_j} L_f^{\nu_i - 1} h_i$, for $1 \leq i, j \leq m$, is of full rank (otherwise the system would not be flat of differential weight $n+m$). It is clear that for the initial system Σ, the integers $\rho_i = \nu_i$, $1 \leq i \leq m$, are observability quasi-indices at x_0.

Proof of (i)-(ii). Suppose that (Σ, h) admits observability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0. It follows from the implication $(\mathcal{O}2) \Rightarrow (\mathcal{O}1)$ that desired vector fields $g_1, \ldots, g_m$ satisfy (i) for a suitable invertible matrix $\big(D_i^j(x)\big)$. Conversely, consider $g_1, \ldots, g_m$ satisfying (i). If $D(x) = I_m$, where I_m denotes the identity $m \times m$-matrix, then by the proof of the implication $(\mathcal{O}1) \Rightarrow (\mathcal{O}2)$, we get $g_i = \frac{\partial}{\partial w_i^{\rho_i}}$. If we take $D(x)$ instead of I_m, we have to replace $g(x) \mapsto g(x)\beta(x)$, where $\beta(x) = (D(x))^{-1}$ proving that for both choices of g_i's we get a feedback linearizable system for which h is a flat output of differential weight $n+m$. Independently of $D(x)$, all corresponding vector fields $g_1, \ldots, g_m$ define the same distribution $\mathcal{G}$ given by (ii).

5.2 Proof of Corollary 2.1

Consider the dynamical system (Σ, h) and assume that it is not locally observable at x_0, but that it admits control vector fields $g_1, \ldots, g_m$ such that the associated control system (Σ_c, h) is static feedback linearizable around x_0 (and thus of differential weight $n+m$, see [20]) with h being a minimal flat output. Then by Theorem 2.1, there exists observability quasi-indices $(\rho_1, \ldots, \rho_m)$ at x_0 implying that (Σ, h) is locally observable at x_0 and contradicting our assumption.

5.3 Proof of Corollary 2.2

Consider the linear dynamical system $\Sigma : \dot{x} = Ax$, $y = Cx$, and assume that the pair (A, C) is observable. Clearly, there exist observability quasi-indices for (A, C) and according to Theorem 2.1, there exist vector fields $g_1, \ldots, g_m$ such that the corresponding control system is x-flat of differential weight $n+m$, with $\varphi = Cx$ being a flat output. We construct the g_i's with the help of condition (i) of Theorem 2.1, where for $D_i^j(x)$ we take $D_i^j(x) = \delta_i^j$. It is clear that with that construction, we obtain constant vector fields $g_i = b_i$, for $1 \leq i \leq m$.

Let us now suppose that there exist constant vector fields $g_1 = b_1, \ldots, g_m = b_m$ such that the associated linear control system is flat at x_0. Since a flat control system is observable with respect to its flat output and since the observability of linear control systems does not depend on the control vector fields, it follows that the pair (A, C) is necessarily observable.

5.4 Proof of Theorem 3.1

Proof of (A2). The proof is given by the algorithm constructing the flat control system NF1 in Sections 3.1.1.

Proof of (A1). Condition (A1) is a direct consequence of (A2) which gives the construction of the flat control system NF1 around any point of $\mathcal{X}$, an open and dense subset of $\mathbb{R}^n$. Notice also that in (A1), flatness holds locally around any (x_0, u_0) satisfying $x_0 \in \mathcal{X}$ and $u_{10} \neq 0$, but the vector fields g_1 and g_2 are defined globally on all connected components of $\mathcal{X}$.

5.5 Proof of Theorem 3.2

Proof of (B1). The proof is given by the algorithm constructing the flat control system NF2 in Sections 3.1.2.

Proof of (B2). Consider the system Σ, around x_0, together with its output $h = (h_1(x), \ldots, h_m(x))$ and suppose that there exist $g_1, \ldots, g_m$ such that the associated control system $\Sigma_c : \dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i$ is x-flat at $(x_0, \bar{u}_0^l)$ with $\varphi = (h_1(x), \ldots, h_m(x))$ a minimal x-flat output, defined in a neighborhood of x_0. Suppose that the differential weight of h as a flat output of Σ_c is $n+m+s$. Recall that $n-k = p(m-1)+r$ is the Euclidean division of the observability defect $n-k$ and $m-1$. We will show that we necessarily have $s \geq p^*$ (recall that $p^* = p$,if $r = 0$ and $p^* = p+1$ otherwise). We denote by s_i the order of the highest derivative of φ_i, for $1 \leq i \leq m$, involved in the expression of x and u, i.e.,

$$x = \gamma(\bar{\varphi}_1^{s_1}, \ldots, \bar{\varphi}_m^{s_m}) \text{ and } u = \delta(\bar{\varphi}_1^{s_1}, \ldots, \bar{\varphi}_m^{s_m}), \tag{22}$$

where $\bar{\varphi}_i^j = (\varphi_i, \dot{\varphi}_i, \ldots, \varphi_i^{(j)})$ and $\sum_{i=1}^m s_i + m = n + m + s$.

Let ν_i, for $1 \leq i \leq m$, be the relative degree of φ_i, that is, the smallest integer such that the derivative $\varphi_i^{(\nu_i)}$ depends explicitly on the control u. We clearly have $\nu_i \geq 1$ (since all φ_i depend on x only), the functions $L_f^j \varphi_i = L_f^j h_i$, for $1 \leq i \leq m$ and $0 \leq j \leq \nu_i - 1$, are independent (as successive derivatives of flat outputs) and $\mathrm{d}L_f^{j-1} h_i \in \mathcal{H}$, so $\dim \mathcal{H}(x) = k$ implies $\sum_{i=1}^m \nu_i \leq k$ (and, in particular, $\sum_{i=1}^m \nu_i < n$, since $k < n$).

For $1 \leq i, j \leq m$, denote $\mathrm{rk}\ \left(L_{g_j} L_f^{\nu_i - 1} h_i\right)(x_0) = \mathrm{rk}\ \left(D_i^j\right)(x_0) = d > 0$. We clearly have $d \leq m-1$, otherwise the control system Σ_c would not be flat since $\sum_{i=1}^m \nu_i < n$. We first suppose that the rank d of the decoupling matrix $\left(D_i^j(x)\right)$ is constant around x_0 and show at the end of the proof of (B2) that the result still holds if the rank of $\left(D_i^j(x)\right)$ drops at x_0.

By the definition of the relative degree, we introduce new coordinates $\tilde{x}_i^j = L_f^{j-1}\varphi_i = L_f^{j-1} h_i$, for $1 \leq i \leq m$ and $1 \leq j \leq \nu_i$, and let ξ be complementary coordinates completing the $\tilde{x}$-variables to a coordinate system (in particular, $\dim \xi \geq n-k$). By applying a suitable feedback transformation (permute the components φ_i, if necessary), the system in the $(\tilde{x}, \xi)$-coordinates reads

$$\begin{array}{rclrcl} \dot{\tilde{x}}_i^j &=& \tilde{x}_i^{j+1},\ 1 \leq j \leq \nu_i - 1, & \dot{\tilde{x}}_i^j &=& \tilde{x}_i^{j+1},\ 1 \leq j \leq \nu_i - 1, \\ \dot{\tilde{x}}_i^{\nu_i} &=& v_i,\ 1 \leq i \leq d, & \dot{\tilde{x}}_i^{\nu_i} &=& a_i(\tilde{x}, \xi, v_1, \ldots, v_d),\ d+1 \leq i \leq m, \\ & & & \dot{\xi} &=& G(\tilde{x}, \xi, v), \end{array} \tag{23}$$

for a smooth map G and smooth functions a_i, all of them affine with respect to the controls. By hypothesis, the above system is flat with $\varphi = (\tilde{x}_1^1, \ldots, \tilde{x}_m^1)$ a minimal flat output. With the components $\varphi_1, \ldots, \varphi_d$ we can no longer produce any new state or control, therefore all ξ-variables as well as controls $v_{d+1}, \ldots, v_m$ have to expressed with the help of $\varphi_i^{(\nu_i + j)}$, for $d+1 \leq i \leq m$ and $j \geq 0$. For $d+1 \leq i \leq m$, each function $\varphi_i^{(\nu_i)} = a_i$ depends explicitly on (some of) $v_1, \ldots, v_d$, say on v_j with j depending on i. Then $\varphi_i^{(\nu_i+1)}$ depends on $\dot{v}_j$ and, finally,

$\varphi_i^{(s_i)}$ depends on $v_j^{(s_i-\nu_i)}$. It follows that $\mathrm{d}\varphi_i^{(\nu_i+1)}, \ldots, \mathrm{d}\varphi_i^{(s_i)}$ are independent modulo $\mathbb{X}+\mathbb{U}$, where $\mathbb{X} = \text{span}\,\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ and $\mathbb{U} = \text{span}\,\{\mathrm{d}u_1, \ldots, \mathrm{d}u_m\}$. Therefore the differential weight of $\varphi = h$ which is $n+m+s$ satisfies $s \geq \underset{d+1\leq i\leq m}{max}\ (s_i - \nu_i)$ yielding $s_i - \nu_i \leq s$, for $d+1 \leq i \leq m$. Flatness implies that using at most $(s+1)(m-d)$ functions $\varphi_i^{(\nu_i)} = a_i$, $\varphi_i^{(\nu_i+1)} = \dot{a}_i, \ldots, \varphi_i^{(\nu_i+s)} = a_i^{(s)}$, for $d+1 \leq i \leq m$, we should be able to express (using expressions that depend also on $\tilde{x}_i^j$ and $v_1, \ldots, v_d$) all state variables ξ (whose dimension is at least $n-k$) and $m-d$ control variables v_i, for $d+1 \leq i \leq m$. So we need

$$(s+1)(m-d) \geq n-k+m-d = (m-1)p + r + m - d,$$

yielding

$$s \geq \frac{m-1}{m-d}p + \frac{1}{m-d}r.$$

and we clearly have $s \geq p^*$.

If the rank of $\left(D_i^j(x)\right)$ is not constant around x_0, then the above proof holds on an open and dense subset $\mathcal{X}$ of $\mathbb{R}^n$. Thus on $\mathcal{X}$, the differential weight of h is at least $n+m+p^*$. Now suppose that $x_0 \notin \mathcal{X}$ (i.e., x_0 is such that the rank of $\left(D_i^j(x)\right)$ drops at x_0) and that the differential weight of h at x_0 is strictly less than $n+m+p^*$. It follows that h will have the same differential weight (as that at x_0) also on a neighborhood $\mathcal{X}_0$ of x_0. Since $\mathcal{X}$ is open and dense in $\mathbb{R}^n$, then the set $\mathcal{X} \cap \mathcal{X}_0$ is not empty and, according to what we have just proved for the constant rank case, on that intersection the differential weight of h should be at least $n+m+p^*$, contradicting the assumption that the differential weight at x_0 is strictly less than $n+m+p^*$.

5.6 Proof of Theorem 3.3

Proof of (A2)'. The proof is given by the (constructive) algorithm providing the construction of the flat control system NF3 associated to (Σ, h) in Section 3.2.1.

Proof of (A1)'. Statement (A1)' is a direct consequence of (A2)' which gives the construction of the flat control system NF3 around any point x_0 such that (Σ, h) admits unobservability quasi-indices at x_0 (thus around any point of an open and dense subset of $\mathbb{R}^n$). Moreover, if (Σ, h) admits unobservability quasi-indices at any $x_0 \in \mathbb{R}^n$, then locally, around any $x_0 \in \mathbb{R}^n$, we can always construct vector fields $g_1, \ldots, g_m$ with the desired properties.

Proof of (B2)'. Repeat the proof of condition (B2) of Theorem 3.1 for $p = 0$ and $r \neq 0$.

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