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A Complexity Theory for Hard Enumeration Problems

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Abstract

Complexity theory provides a wealth of complexity classes for analyzing the complexity of decision and counting problems. Despite the practical relevance of enumeration problems, the tools provided by complexity theory for this important class of problems are very limited. In particular, complexity classes analogous to the polynomial hierarchy and an appropriate notion of problem reduction are missing. In this work, we lay the foundations for a complexity theory of hard enumeration problems by proposing a hierarchy of complexity classes and by investigating notions of reductions for enumeration problems.

1. Introduction

While decision problems often ask for the existence of a solution to some problem instance, enumeration problems aim at outputting all solutions. In many domains, enumeration problems are thus the most natural kind of problems. Just take the database area (usually the user is interested in all answer tuples and not just in a yes/no answer) or diagnosis (where the user wants to retrieve possible explanations, and not only whether one exists) as two examples. Nevertheless, the complexity of enumeration problems is far less studied than the complexity of decision problems.

It should be noted that even simple enumeration problems may produce big output. To capture the intuition of easy to enumerate problems – despite a possibly exponential number of output values – various notions of tractable enumeration classes have been proposed in [1]. The class DelayP ("polynomial delay") contains all enumeration problems where, for given instance $x$, (1) the time to compute the first solution, (2) the time between outputting any two consecutive solutions, and (3) the time to detect that no further solution exists, are all polynomially bounded in the size of $x$. The class IncP ("incremental polynomial time") contains those enumeration problems where, for given instance $x$, the time to compute the next solution and for detecting that no further solution exists is polynomially bounded in the size of both $x$ and of the already computed solutions. Obviously, the relationship DelayP $\subseteq$ IncP holds. In [2], the proper inclusion DelayP $\subsetneq$ IncP is mentioned. For these tractable enumeration classes, a variety of membership results exist, a few examples are given in [3, 4, 5, 6, 7, 8].

There has also been work on intractable enumeration problems. Intractability of enumeration is typically proved by showing intractability of a related decision problem rather than directly proving lower bounds by relating one enumeration
problem to the other. Tools for a more fine-grained analysis of intractable enumeration problems are missing to date. For instance, up to now we are not able to make a differentiated analysis of the complexity of the following typical enumeration problems:

\[
\begin{align*}
\text{E-}\Pi_k\text{SAT} &/ \text{E-}\Sigma_k\text{SAT} \\
\text{Instance:} & \quad \psi = \forall \vec{y}_1 \exists \vec{y}_2 \ldots Q_k \vec{y}_k \phi(\vec{x}, \vec{y}) / \psi = \exists \vec{y}_1 \forall \vec{y}_2 \ldots Q_k \vec{y}_k \phi(\vec{x}, \vec{y}) \\
\text{Output:} & \quad \text{All assignments for } \vec{x} \text{ such that } \psi \text{ is true.}
\end{align*}
\]

This is in sharp contrast to decision problems, where the polynomial hierarchy is crucial for a detailed complexity analysis. As a matter of fact, it makes a big difference, if an NP-hard problem is in NP or not. Indeed, NP-complete problems have an efficient transformation into SAT and can therefore be solved by making use of powerful SAT-solvers. Similarly, problems in \(\Sigma^P_2\) can be solved by using ASP-solvers. Finally, also for problems on higher levels of the polynomial hierarchy, the number of quantifier alternations in the QBF-encoding matters when using QBF-solvers. For counting problems, an analogue of the polynomial hierarchy has been defined in form of the \(#\cdot C\)-classes with \(C \in \{P, coNP, \Pi^P_2, \ldots\}\) \cite{9, 10}. For enumeration problems, no such analogue has been studied.

**Goal and Results.** The goal of this work is to lay the foundations for a complexity theory of hard enumeration problems by defining appropriate complexity classes for intractable enumeration and a suitable notion of problem reductions. We propose to extend tractable enumeration classes by oracles. We will thus get a hierarchy of classes \(\text{Del}C, \text{Inc}C\), where various complexity classes \(C\) are used as oracles. As far as the definition of an appropriate notion of reductions is concerned, we follow the usual philosophy of reductions: if some enumeration problem can be reduced to another one, then we can use this reduction together with an enumeration algorithm for the latter problem to solve the first one. We observe that two principal kinds of reductions are used for decision problems, namely many-one reductions and Turing reductions. Similarly, we shall define a more declarative-style and a more procedural-style notion of reduction for enumeration problems. Our results are summarized below.

- **Enumeration complexity classes.** In Section 3 we introduce a hierarchy of complexity classes of intractable enumeration via oracles and prove that it is strict unless the polynomial hierarchy collapses.

- **Declarative-style reductions.** In Section 4 we introduce a declarative-style notion of reductions. While they enjoy some desirable properties, we do not succeed in exhibiting complete problems under this type of reductions.

- **Procedural-style reductions.** In Section 5 we introduce a procedural-style notion of reductions, show that they have the desirable properties any reduction should possess, and show that the enumeration problems associated with the typical quantified Boolean satisfiability problems are complete in our hierarchy of enumeration problems.

- **Completeness results for natural problems.** Starting with these completeness results, we establish a chain of reductions among several natural enumeration problems from areas such as generalized satisfiability, circumscription, model-based diagnosis, abduction, and repairs of inconsistent databases in Section 6, thus proving completeness of these problems in classes of our hierarchy.

This work is an extension of the conference article \cite{11}.
2. Preliminaries

In the following, \( \Sigma \) denotes a finite alphabet and \( R \) denotes a polynomially bounded, binary relation \( R \subseteq \Sigma^\ast \times \Sigma^\ast \), i.e., there is a polynomial \( p \) such that for all \((x, y) \in R\), \(|y| \leq p(|x|)\) holds. For every string \( x \), \( R(x) = \{ y \in \Sigma^\ast \mid (x, y) \in R \} \). A string \( y \in R(x) \) is called a solution for \( x \). With such a relation \( R \), we can associate several natural computational problems:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exist_(R)</td>
<td>Instance: ( x \in \Sigma^\ast )  Question: Is ( R(x) \neq \emptyset ) ?</td>
</tr>
<tr>
<td>Exist-AnotherSol_(R)</td>
<td>Instance: ( x \in \Sigma^\ast ), ( Y \subseteq R(x) )  Output: Is ( (R(x) \setminus Y) \neq \emptyset ) ?</td>
</tr>
<tr>
<td>AnotherSol_(R)</td>
<td>Instance: ( x \in \Sigma^\ast ), ( Y \subseteq R(x) )  Output: ( y \in R(x) \setminus Y ) or declare that no such ( y ) exists.</td>
</tr>
<tr>
<td>Check_(R)</td>
<td>Instance: ( (x, y) \in \Sigma^\ast \times \Sigma^\ast )  Question: Is ( (x, y) \in R ) ?</td>
</tr>
<tr>
<td>ExtSol_(R)</td>
<td>Instance: ( (x, y) \in \Sigma^\ast \times \Sigma^\ast )  Question: Is there some ( y' \in \Sigma^\ast ) such that ( (x, yy') \in R ) ?</td>
</tr>
</tbody>
</table>

In counting complexity, the following computational problem arises:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count_(R)</td>
<td>Instance: ( x \in \Sigma^\ast )  Output: (</td>
</tr>
</tbody>
</table>

A binary relation \( R \) also gives rise to an enumeration problem, which aims at outputting the set \( R(x) \) of solutions for \( x \).

<table>
<thead>
<tr>
<th>Problem</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Enum_(R)</td>
<td>Instance: ( x \in \Sigma^\ast )  Output: ( R(x) = { y \in \Sigma^\ast \mid (x, y) \in R } ).</td>
</tr>
</tbody>
</table>

We assume the reader to be familiar with the polynomial hierarchy – the complexity classes \( \text{P} \), \( \text{NP} \), \( \text{coNP} \) and, more generally, \( \Delta^P_k \), \( \Sigma^P_k \), and \( \Pi^P_k \) for \( k \in \{0, 1, \ldots \} \).

As we usually deal with exponential runtime of algorithms in the course of an enumeration process, we will also make use of the weak \( \text{EXP} \) hierarchy (see [12]) in the following. It is an exponential time analogue of the polynomial hierarchy defined as follows:

**Definition 1 (Exponential Hierarchy).**

\[
\begin{align*}
\Delta^\text{EXP}_0 &= \Sigma^\text{EXP}_0 = \Pi^\text{EXP}_0 = \text{EXP} \\
\Delta^\text{EXP}_k &= \text{EXP}^{\Sigma^\text{EXP}_{k-1}}, \text{ for } k \geq 1 \\
\Sigma^\text{EXP}_k &= \text{NEXP}^{\Sigma^\text{EXP}_{k-1}}, \text{ for } k \geq 1 \\
\Pi^\text{EXP}_k &= \text{coNEXP}^{\Sigma^\text{EXP}_{k-1}}, \text{ for } k \geq 1
\end{align*}
\]
A hierarchy of counting complexity classes similar to the polynomial hierarchy was defined [9, 10]. Let $C$ be a complexity class of decision problems. Then $\# \cdot C$ denotes the class of all counting problems $P$, such that there exists a relation $R$ with \text{Check}_R \in C$ and $P = \text{Count}_R$. For $\#P$ – the best known counting complexity class – we have $\#P \subseteq \# \cdot \text{P} \subseteq \# \cdot \text{NP} \subseteq \# \cdot \text{coNP} \subseteq \# \cdot \Sigma^P_2 \subseteq \# \cdot \Pi^P_2 \subseteq \ldots$ hold.

A complexity class $C$ is closed under a reduction $\leq_r$ if, for any two binary relations $R_1$ and $R_2$ we have that $R_2 \in C$ and $R_1 \leq_r R_2$ imply $R_1 \in C$. Furthermore, a reduction $\leq_r$ is transitive if for any three binary relations $R_1, R_2, R_3$, it is the case that $R_1 \leq_r R_2$ and $R_2 \leq_r R_3$ implies $R_1 \leq_r R_3$.

3. Complexity Classes for Enumeration

In Section 1, we have already recalled two important tractable enumeration complexity classes, $\text{DelayP}$ and $\text{IncP}$ from [1]. Note that in [2, 13], these classes are defined slightly differently by restricting $\text{DelayP}$ and $\text{IncP}$ to those problems $\text{Enum}_R$ such that the corresponding problem $\text{Check}_R$ is in $\text{P}$ — in addition to the constraints on the allowed delays. We adhere to the definition of tractable enumeration classes from [1].

In contrast to counting complexity, defining a hierarchy of enumeration problems via the $\text{Check}_R$ problem of binary relations $R$ does not seem appropriate. Note that while the same counting problem $P$ can be defined by different relations, i.e., $P = \text{Count}_{R_1}$ and $P = \text{Count}_{R_2}$ for $R_1 \neq R_2$, the relation associated with an enumeration problem is fixed.

Thus, we need an alternative approach for defining meaningful enumeration complexity classes. To this end, we first fix our computation model: Observe that there is a subtle difference between how Turing Machines and random access machines (RAMs) can access data in their “memory”. Due to the linear nature of the tapes of Turing Machines, accessing an exponential size data structure requires exponential time, even if just a small portion of its data is actually read. This is usually not a problem when studying decision or search problems, since writing an exponentially sized data structure onto the tape already requires exponential time, and as a result the additional exponential time to read the data structure has no more effect on the overall runtime.

This situation, however, changes for enumeration problems and polynomial delay, since it is now possible that, while computing an exponential number of solutions, an exponential size data structure accumulates. Thus computing another solution with only polynomial delay is not possible with Turing Machines if this requires some information from such a data structure. However, by maintaining suitable index structures, it might be possible to gather the necessary information in polynomial time from such a data structure on a random access machine (RAM) that allows to access its memory directly and not first needs to move the read/write head over the correct tape position. In fact, due to this property that allows to access (polynomially sized) parts of exponential size data in polynomial time, it is common to use the RAM model as a computational model for the study of enumeration problems (cf. [2]). We restrict ourselves here to polynomially bounded RAMs, i.e., throughout the computation of such a machine, the size of the content of each register is polynomially bounded in the size of the input.

For enumeration, we will also make use of RAMs with an output-instruction, as defined in [2]. This model can be extended further by introducing decision oracles. The input to the oracle is stored in special registers and the oracle takes consecutive non-empty registers as input. Moreover, following [14], we use a computational model that does not delete the input of an oracle call once such a call is made. For
3 COMPLEXITY CLASSES FOR ENUMERATION

a detailed definition, refer to [2]. It is important to note that due to the exponential runtime of an enumeration algorithm and the fact that the input to an oracle is not deleted when the oracle is executed, the input to an oracle call may eventually become exponential as well. Clearly, this can only happen if exponentially many consecutive special registers are non-empty, since we assume also each special register to be polynomially bounded.

Before introducing enumeration complexity classes with oracles, we recall the definitions of precomputation and postcomputation time, see [15]. Given a binary relation, an instance \( x \) and a RAM \( M \) that enumerates \( R(x) \), the precomputation time is the time between the start of the RAM \( M \) and the first output of \( R(x) \), and the postcomputation time is the time between the last output of \( R(x) \) and the termination of \( M \). It is convenient to think of these time bounds in terms of additional solutions of \( R(x) \): As a first step in computing \( R(x) \), \( M \) outputs a 0-th solution, and after the termination of \( M \), a \((|R(x)| + 1)\)-th solution is output. Thus the pre- and postcomputation time correspond to the delay between outputting the \( 0-th \) and first solution, respectively outputting the \(|R(x)|\)-th and \((|R(x)| + 1)\)-th solution.

**Definition 2 (enumeration complexity classes).** Let \( R \) be a binary relation, and \( C \) a decision complexity class. Then we say that:

- **Enum** \( R \in \text{Del}^C \) if there is a RAM \( M \) with oracle \( L \) in \( C \) and a polynomial \( p \), such that for any instance \( x \), \( M \) enumerates \( R(x) \) with delay \( p(|x|) \) between the \( i \)-th and \( (i + 1) \)-th solution for every \( 0 \leq i \leq |R(x)| \).

- **Enum** \( R \in \text{Inc}^C \) if there is a RAM \( M \) with oracle \( L \) in \( C \) and a polynomial \( p \), such that for any instance \( x \), \( M \) enumerates \( R(x) \) with delay \( p(|x| + i) \) between the \( i \)-th and \( (i + 1) \)-th solution for every \( 0 \leq i \leq |R(x)| \).

- **Enum** \( R \in \text{Del}C \) if there is a RAM \( M \) with oracle \( L \) in \( C \) and a polynomial \( p \), such that for any instance \( x \), \( M \) enumerates \( R(x) \) with delay \( p(|x|) \) between the \( i \)-th and \( (i + 1) \)-th solution for every \( 0 \leq i \leq |R(x)| \). Moreover, the size of every oracle call is bound by \( p(|x|) \).

- **Enum** \( R \in \text{Inc}C \) if there is a RAM \( M \) with oracle \( L \) in \( C \) and a polynomial \( p \), such that for any instance \( x \), \( M \) enumerates \( R(x) \) with delay \( p(|x| + i) \) between the \( i \)-th and \( (i + 1) \)-th solution for every \( 0 \leq i \leq |R(x)| \). Moreover, the size of every oracle call made between outputting the \( i \)-th and \( (i + 1) \)-th solution is bound by \( p(|x| + i) \) for every \( 0 \leq i \leq |R(x)| \).

It is easy to see that by taking \( C = \text{P} \), the definitions of \( \text{Del}C \) and \( \text{Inc}C \) are equivalent to the definitions of the classes \( \text{DelayP} \) respectively \( \text{IncP} \). Note that the restriction of the oracle inputs to polynomial size makes a crucial difference when it comes to \( \text{Del}C \), where we have a discrepancy between the polynomial restriction (w.r.t. the input \( x \)) on the time between two consecutive solutions are output and the possibly exponential size (w.r.t. the input \( x \)) of oracle calls. No such discrepancy exists for \( \text{Inc}C \), where the same polynomial upper bound w.r.t. the already computed solutions (resp. all solutions) applies both to the allowed time and to the size of the oracle calls:

**Proposition 1.** Let \( C \) be a decision complexity class. Then \( \text{Inc}C = \text{Inc}^C \)

*Proof.* The inclusion \( \text{Inc}C \subseteq \text{Inc}^C \) is given by definition, thus we need to show that \( \text{Inc}^C \subseteq \text{Inc}C \). Let \( R \) be a binary relation with \( \text{Enum}_R \in \text{Inc}^C \), \( x \) an instance to the enumeration problem and let \( M \) be a RAM with oracle \( C \) that outputs \( R(x) \) with incremental polynomial delay \( \mathcal{O}((|x| + i)^k) \) between the \( i \)-th and \( (i + 1) \)-th
solution for every $0 \leq i \leq |R(x)|$. The input to the $C$-oracle is of size at most $O(|x|)$ between the $i$-th and $(i+1)$-th solution and thus also polynomially bounded. \qed

In fact, the lower computational power of Del$C$ compared with Inc$C$ can be compensated by equipping the lower class with a slightly more powerful oracle.

**Theorem 2.** Let $k \geq 0$. Then Del$^+ \Delta^P_{k+1} = \text{Inc} \Sigma^P_k$.

**Proof.** Let $k \geq 0$. We start the proof by showing that Del$^+ \Delta^P_{k+1} = \text{Inc} \Sigma^P_k$. So let $\text{Enum}_R \in \text{Inc} \Sigma^P_k$ with corresponding binary relation $R$ and let $x \in \Sigma^*$. Fix an incremental delay algorithm $A$ which uses a $\Sigma^P_k$-oracle witnessing the membership $\text{Enum}_R \in \text{Inc} \Sigma^P_k$ and let $<^*$ be an order on $R(x)$ induced by algorithm $A$, i.e. the $i$-th output of $A$ on input $x$ is the $i$-th element in $<^*$. We define the following decision problem:

<table>
<thead>
<tr>
<th>ANOTHERSOLEXT$^A_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Instance:</strong> $y_1, \ldots, y_n, y', x \in \Sigma^*$</td>
</tr>
<tr>
<td><strong>Question:</strong> Is $y'$ a prefix of $y_{n+1}$, where $y_{n+1}$ is the $(n+1)$-th element in $R(x)$ w.r.t. $&lt;^*$?</td>
</tr>
</tbody>
</table>

We first note that ANOTHERSOLEXT$^A_R \in \Delta^P_{k+1}$. Indeed, assume that we have given an instance $y_1, \ldots, y_n, y', x \in \Sigma^*$. Then we can use $A$ to enumerate the first $n+1$ elements of $R(x)$ in time $O(\text{poly}(|x|, |n+1|)) = O(\text{poly}(|y_1|, \ldots, |y_n|, |y'|, |x|))$ and then check whether $y'$ is a prefix of $y_{n+1}$. As $A$ uses a $\Sigma^P_k$-oracle, this decision can be made within $P^{\Sigma^P_k} = \Delta^P_{k+1}$. The membership $\text{Enum}_R \in \text{Del}^+ \Delta^P_{k+1}$ follows immediately, as we can construct a polynomial delay algorithm with an ANOTHERSOLEXT$^A_R$-oracle that enumerates $\text{Enum}_R$ in a similar way to commonly used enumeration algorithms, see [2,5] or Proposition 6. This enumeration algorithm starts by computing some $a \in \Sigma$ such that the oracle on this input returns ‘yes’. Then $y_1$ (and also every other $y_i$) can be computed by repeatedly extending the previous input to the oracle by the unique element in $\Sigma$ such that the decision oracle ANOTHERSOLEXT$^A_R$ returns ‘yes’.

Next we need to show that Del$^+ \Delta^P_{k+1} \subseteq \text{Inc} \Sigma^P_k$, so let $\text{Enum}_R \in \text{Del}^+ \Delta^P_{k+1}$, and let $A$ be an algorithm witnessing this membership. Moreover, let $L \in P^{\Sigma^P_k}$ be the language used for the $\Delta^P_{k+1}$-oracle in $A$, with a polynomial $q$ and a language $L' \in \Sigma^P_k$ such that $L \in \text{DTIME}^{L'}(q(n))$. Let $p$ be the polynomial for the delay of $A$. We describe an algorithm $B$ that has an incremental delay of $p(X) \cdot Y \cdot q(p(X) \cdot Y)$ (here the indeterminate $X$ stands for the size of the input and the indeterminate $Y$ for the number of previously output solutions) that uses a $\Sigma^P_k$-oracle, such that $B$ enumerates $\text{Enum}_R$. For this, let $x \in \Sigma^*$ and assume that we want to enumerate $R(x)$. The algorithm $B$ works as follows:

- Let $y_1$ be the first element output by algorithm $A$ on input $x$. As this output can be computed by $A$ in time $p(|x|)$, at most $p(|x|)$ many calls to the $L$-oracle have been made with an input of size at most $p(|x|)$. Thus the answer of every oracle call can be computed in time $q(p(|x|))$ using an $L'$-oracle. Therefore $y_1$ can be output by $B$ in time $p(|x|) \cdot q(p(|x|))$ by running $A$ until the first output, and simulating the oracle calls accordingly.

- For $n \geq 2$, let $y_n$ be the $n$-th element output by $A$. As with $y_1$, we can make the $n$-th output of $B$ by running $A$ until $y_n$ is output and simulating the oracle calls accordingly. Indeed, $A$ takes $p(|x|) \cdot n$ steps to output $y_n$, with at most $p(|x|) \cdot n$ oracle calls to $L$ with an input of size bounded by $p(|x|) \cdot n$. Thus $y_n$ can be computed in time $p(|x|) \cdot n \cdot q(p(|x|) \cdot n)$ using an $L'$-oracle.
In fact, given that the classes $\Delta^P_k$ describe problems that can be solved in polynomial time by a deterministic machine with a $\Sigma^P_{k+1}$-oracle, it is not very surprising that for enumeration algorithms, $\Delta^P_k$-oracles can be simulated by $\Sigma^P_{k+1}$-oracles, as long as one accounts for the potentially accumulating input to the $\Delta^P_k$-oracles.

This property is not restricted to the case shown in Theorem 2, but applies to all $\Delta^P_k$-oracles. As a result, in the remainder of this work, we omit any result regarding the classes Del$\Delta^P_k$, Del$^+\Delta^P_k$ or Inc$\Delta^P_k$, as they are given implicitly by a result on Del$\Sigma^P_k$ or Inc$\Sigma^P_k$ respectively. Indeed, Theorem 2 shows the equality of Del$^+\Delta^P_k$ and Inc$\Sigma^P_{k-1}$. Concerning the remaining classes, observe that Del$^P_C = $ Del$C$ for any decision complexity class C; in particular we obtain Del$\Delta^P_k = $ Del$\Sigma^P_k$. A similar results holds for the Inc$C$ classes even without the bound on the input size to the decision oracle. This is due to the fact that the size of any input $a$ to the decision oracle is at most polynomial in the combined size of the instance and number of outputs. Thus an incremental delay gives us enough time to make poly($|a|$) many oracles calls, meaning that Inc$\Delta^P_k = $ Inc$\Sigma^P_{k-1}$.

3.1. A Hierarchy of Enumeration Complexity Classes

We now prove that our classes provide strict hierarchies under the assumption that the polynomial hierarchy is strict.

Theorem 3. Let $k \geq 0$. Then, unless the polynomial hierarchy collapses to the $(k+1)$-st level,

$$\text{Del} \Sigma^P_k \subseteq \text{Del} \Sigma^P_{k+1}, \quad \text{Del}^+ \Sigma^P_k \subseteq \text{Del}^+ \Sigma^P_{k+1}, \quad \text{Inc} \Sigma^P_k \subseteq \text{Inc} \Sigma^P_{k+1}$$

and Inc$\Sigma^P_k \subseteq$ Inc$\Sigma^P_{k+1}$.

Proof. Let $k \geq 0$. By Theorem 2 and the definition of our classes we have that

$$\text{Del}^+ \Sigma^P_k \subseteq \text{Del}^+ \Delta^P_{k+1} = \text{Inc} \Sigma^P_k \subseteq \text{Del}^+ \Sigma^P_{k+1} \subseteq \text{Inc} \Sigma^P_{k+1},$$

so we only need to show that Del$\Sigma^P_k \subseteq$ Del$\Sigma^P_{k+1}$ and inc$\Sigma^P_k \subseteq$ inc$\Sigma^P_{k+1}$. Let $L$ be a $\Sigma^P_{k+1}$-complete problem. Define a relation $R_L = \{(x, 1) \mid x \in L\}$. It is clear that Check$\_R_L$ is $\Sigma^P_{k+1}$-complete. Moreover, the enumeration problem $\text{Enum}\_R_L$ is in Del$\Sigma^P_{k+1}$ (thus also in Del$^+\Sigma^P_{k+1}$). Assume that $\text{Enum}\_R_L \in$ Del$\Sigma^P_k$ (or $\text{Enum}\_R_L \in$ Inc$\Sigma^P_k$). Then, as there is only one output to the enumeration problem, Check$\_R_L$ can be decided in polynomial time using a $\Sigma^P_{k+1}$-oracle, meaning that Check$\_R_L \in \Delta^P_{k+1}$ and thus the polynomial hierarchy collapses to the $(k+1)$-st level.

The statement of the following proposition is twofold: First, it shows that the complexity classes based on Del$\Sigma^P_k$ with a polynomial bound on the decision oracle and the complexity classes based on Del$\Sigma^P_k$ without such a bound, respectively, are very likely to be distinct. Second, the gap between Del$\Sigma^P_k$ and Del$^+\Sigma^P_k$ cannot be overcome by equipping the oracle-bounded enumeration complexity class with a slightly more powerful oracle, in contrast to the result of Theorem 2. Note that the proposition refers to the weak EXP hierarchy as defined in Definition 1.

Proposition 4. Let $k \geq 1$. If EXP $\not\subseteq \Delta^\text{EXP}_{k+1}$, then

$$\text{Del} \Sigma^P_k \not\subseteq \text{Del}^+ \Sigma^P_k \not\subseteq \text{Del} \Sigma^P_{k+1}.$$
3 COMPLEXITY CLASSES FOR ENUMERATION

![Diagram showing hierarchy among enumeration complexity classes for all \( k \geq 1 \). The lines without ‘?’ represent strict inclusions under some reasonable complexity theoretic assumption. For the line with ‘?’; inclusion holds, but it is not clear whether it is strict.]

**Lemma 5.** Let \( R \) be a binary relation and \( k \geq 0 \). If \( \text{Enum}_R \in \text{Del}\Sigma^P_k \), then \( \text{Check}_R \in \text{EXP} \).

**Proof.** Let \((x, y)\) be an instance of \( \text{Check}_R \) and let \( k \geq 0 \). Further let \( A \) be an enumeration algorithm witnessing the membership \( \text{Enum}_R \in \text{Del}\Sigma^P_k \). In order to decide whether \((x, y) \in R\), we simply enumerate all of \( R(x) \) and check whether \( y \in R(x) \). Let \( q, r \) be polynomials such that the decision of the \( \Sigma^P_k \)-oracle can be computed in \( O(2^{q(|x|)}) \), and the size of the input to any such decision oracle while executing \( A \) is bounded by \( r \). By the definition of polynomial delay and the fact that \( R \) is a polynomial relation, there is some polynomial \( h \) such that \( R(x) \) can be enumerated in time \( O(2^{q(r(|x|)) + h(|x|)}) \), i.e., in exponential time.

**of Proposition 4** Assume that \( \text{EXP} \subseteq \Delta^P_{k+1} \). Then there exists some polynomial \( q \) and a language \( L \) such that \( L \in \Delta^P_{k+1} \setminus \text{EXP} \) and \( L \) can be decided in time \( O(2^{q(|x|)}) \) using a \( \Sigma^P_k \)-oracle. Define the following enumeration problem:

**Enumeration Problem**

- **Instance:** \( x \in \Sigma^* \).
- **Output:** All \( \{0, 1\} \)-words of length \( q(|x|) \), and 2 if \( x \in L \).

First note that \( \text{Enum}_{D_0} \in \text{Del}^+\Sigma^P_k \) by an algorithm \( A \) that enumerates all \( 2^{q(|x|)} \) words in \( \{0, 1\}^{|x|} \) in \( O(2^{q(|x|)}) \). While enumerating the trivial part of the output, \( A \) also has enough time to compute whether \( x \in L \), and then makes the last output (’2’ or nothing) accordingly. Next assume that \( \text{Enum}_{D_0} \in \text{Del}\Sigma^P_k \) (or \( \text{Del}\Sigma^P_{k+1} \)). Then, by Lemma 5 \( \text{Check}_{D_0} \in \text{EXP} \). Therefore we can check for all \( x \in \Sigma^* \) whether \((x, 2) \in D_0 \), which is equivalent to \( x \in L \). Thus we can decide \( L \) in exponential time, a contradiction. This proves also the second claim.

By generalizing the statement of Lemma 5, we even have that \( \text{Del}C \) and \( \text{Del}^+\Sigma^P_k \) are incomparable for any \( k \geq 1 \) and any \( C \) within \( \text{EXP} \) under the assumption that the weak \( \text{EXP} \) hierarchy does not collapse. This is due to the fact that \( \text{Check}_R \) for an enumeration problem \( \text{Enum}_R \in \text{Del}C \) is within \( \text{EXP} \), whereas \( \text{Del}^+\Sigma^P_k \) contains an enumeration problem with corresponding check problem in \( \Delta^P_{k+1} \setminus \text{EXP} \).

The relations among the enumeration complexity classes introduced in this chapter are summarized in Figure 1.
3.2. Connections to Decision Complexity

For a lot of problems, the set of all solutions can be enumerated by repeatedly solving differing instances of a corresponding decision problem and using the information gained from to construct the solutions.

For the class DelC, it turns out that the decision problem ExtSol\(R\) is most relevant for achieving the aforementioned goal. Indeed, the standard enumeration algorithm that constructs solutions bit by bit and tests whether the current partial candidate solution can still be extended to a solution \([2,5]\), and which outputs the solutions in lexicographical order, gives the following relationship.

**Proposition 6.** Let \(R\) be a binary relation. Further let \(k \geq 0\) and \(C \in \{\Delta_k^P, \Sigma_k^P\}\). If \(\text{ExtSol}_R \in C\) then \(\text{Enum}_R \in \text{DelC}\).

**Proof.** Let \(k \geq 1\). Further let \(q\) be a polynomial such that for all \((x, y) \in R\), \(|y| \leq q(|x|)\) holds. As \(\text{Del}\Delta_{k+1}^P = \text{Del}\Sigma_k^P\), we show that \(\text{Enum}_R \in \text{DelC}\) by giving an enumeration algorithm \(A\) with access to \(\text{ExtSol}_R\) as an oracle. Assume that we have given \(x\) as an input for \(\text{Enum}_R\). For every \(y_1 \in \Sigma\), we make an oracle call with \((x, y_1)\) as input. If the oracle returns ‘yes’, \(y_1\) can be extended to some \(y \in R(x)\), so for every \(y_2 \in \Sigma\), we query \((x, y_1 y_2)\) to the oracle. Repeating this \(q(|x|)\) times gives a first answer \(y \in R(x)\). With the use of backtracking, we can enumerate all of \(R(x)\) with a polynomial delay and access to the oracle \(\text{ExtSol}_R\), where every input to an oracle call is polynomially bounded by \(q\). \(\square\)

When considering decision problems, typically not the problem \(\text{ExtSol}_R\) is studied, but instead the problem \(\text{Exist}_R\) that just asks, given some instance, whether there exists any solution at all. While in general unrelated, for a big class of problems, the so-called self-reducible problems, the problem \(\text{ExtSol}_R\) can be solved via \(\text{Exist}_R\).

**Definition 3 (self-reducibility).** Let \(\leq_T\) denote Turing reductions. We say that a binary relation \(R\) is self-reducible, if \(\text{ExtSol}_R \leq_T \text{Exist}_R\).

Self-reducibility thus allows to reduce enumeration to the typical decision problem associated with \(R\). More formally, for self-reducible problems, Proposition 6 can be refined as follows.

**Proposition 7.** Let \(R\) be a binary relation, which is self-reducible, and \(k \geq 0\). Then \(\text{Exist}_R \in \Delta_k^P\) if and only if \(\text{Enum}_R \in \text{Del}\Sigma_k^P\).

The above proposition gives a characterization of the class \(\text{Del}\Sigma_k^P\) in terms of the complexity of decision problems in the case of self-reducible relations. Analogously, the notion of “enumeration self-reducibility” introduced by Kimelfeld and Kolaitis, that directly relates the search problem \(\text{AnotherSol}_R\) to its corresponding decision problem \([4]\), allows a characterization of the class \(\text{Inc}\Sigma_k^P\).

**Definition 4 ([4], enumeration self-reducibility).** A binary relation \(R\) is enumeration self-reducible if \(\text{AnotherSol}_R \leq_T \text{Exist-AnotherSol}_R\).

The following proposition generalizes two results to the higher levels of the enumeration hierarchy, by following analogous arguments as for the basic level \(\text{IncP}\). The first result is given by Strozecki in \([2, \text{Prop. 2.14}]\), and the second result is given by Kimelfeld and Kolaitis \([3, \text{Prop. 2.2}]\).

**Proposition 8.** Let \(R\) be a binary relation and let \(k \geq 0\).

- The functional problem \(\text{AnotherSol}_R\) can be solved in polynomial time with access to a \(\Delta_k^P\) oracle if and only if \(\text{Enum}_R \in \text{Inc}\Sigma_k^P\).
- Let \(R\) be enumeration self-reducible. Then \(\text{Exist-AnotherSol}_R \in \Delta_k^P\) if and only if \(\text{Enum}_R \in \text{Inc}\Sigma_k^P\).
4 DECLARATIVE-STYLE REDUCTIONS

As far as we know, only a few kinds of reductions between enumeration problems have been investigated so far. One such reduction is implicitly described in [16]. It establishes a bijection between sets of solutions. Different approaches introduced in [17] and in [18] relax this condition and allow for non-bijective reduction relations. We go further in that direction in proposing a declarative style reduction relaxing the isomorphism requirement while closing the relevant enumeration classes.

Definition 5 (e-reduction). Let $R_1, R_2 \subseteq \Sigma^*$ be binary relations. Then $\text{Enum}_x R_1$ reduces to $\text{Enum}_x R_2$ via an e-reduction, $\text{Enum}_x R_1 \preceq_e \text{Enum}_x R_2$, if there exist a function $\sigma : \Sigma^* \rightarrow \Sigma^*$ computable in polynomial time and a relation $\tau \subseteq \Sigma^* \times \Sigma^* \times \Sigma^*$, s.t. for all $x \in \Sigma^*$ the following holds. For $y \in \Sigma^*$, let $\tau(x, y, -) := \{z \in \Sigma^* \mid (x, y, z) \in \tau\}$ and for $z \in \Sigma^*$, let $\tau(x, -, z) := \{y \in \Sigma^* \mid (x, y, z) \in \tau\}$. Then:

1. $R_1(x) = \bigcup_{y \in R_2(\sigma(x))} \tau(x, y, -)$;
2. For all $y \in R_2(\sigma(x))$, either $\tau(x, y, -) = \emptyset$, or $\emptyset \subset \tau(x, y, -) \subseteq R_1(x)$ and $\tau(x, y, -)$ can be enumerated with polynomial delay in $|x|$; moreover $\tau(x, y, -) = \emptyset$ can only hold for a number of $y$’s which is polynomially bounded in $|x|$;
3. For all $z \in R_1(x)$, we have $\tau(x, -, z) \subseteq R_2(\sigma(x))$ and the size of $\tau(x, -, z)$ is polynomially bounded in $|x|$.

Intuitively, $\tau$ establishes a relationship between instances $x$, solutions $y \in R_2(\sigma(x))$ and solutions $z \in R_1(x)$. We can thus use $\tau$ to design an enumeration algorithm for $R_1(x)$ via an enumeration algorithm for $R_2(\sigma(x))$. The conditions imposed on $\tau$ have the following meaning: By condition 1, the solutions $z \in R_1(x)$ can be computed by iterating through the solutions $y \in R_2(\sigma(x))$ and computing $\tau(x, y, -) \subseteq R_1(x)$. Conditions 2 and 3 make sure that the delay of enumerating $R_1(x)$ only differs by a polynomial from the delay of enumerating $R_2(\sigma(x))$: Condition 2 ensures that, for every $y$, the set $\tau(x, y, -)$ can be enumerated with polynomial delay and that we do not encounter more than a polynomial number of “useless” $y$ (i.e., a solution $y \in R_2(\sigma(x))$ which is associated with no solution $z \in R_1(x)$).

In principle, we may thus get duplicates $z$ associated with different values of $y$. However, Condition 3 ensures that each $z$ can be associated with at most polynomially many values $y$. Using a priority queue storing all $z$ that are output, we can avoid duplicates, cf. the proof of Proposition [9] or [2]. Figure 2 illustrates the idea of $\tau$, and a concrete example of an e-reduction is provided next.

Example 1. The idea of the relation $\tau$ can also be nicely demonstrated on an e-reduction from E-3-COLOURABILITY to E-4-COLOURABILITY (enumerating all
valid 3- respectively 4-colourings of a graph). We intentionally choose this reduction since there is no bijection between the solutions of the two problems.

Recall the classical many-one reduction between these problems, which takes a graph \( G \) and defines a new graph \( G' \) by adding an auxiliary vertex \( v \) and connecting it to all the other ones. This reduction can be extended to an \( e \)-reduction with the following relation \( \tau \): With every graph \( G \) in the first component of \( \tau \), we associate all valid 4-colourings (using 0, 1, 2, and 3) of \( G' \) in the third component of \( \tau \). With each of those we associate the corresponding 3-colouring of \( G \) in the second component. They are obtained from the 4-colourings by first making sure that \( v \) is coloured with 3 (by “switching” the colour of \( v \) with 3) and then by simply reading off the colouring of the remaining vertices.

The \( e \)-reductions have two desirable and important properties, as stated next.

**Proposition 9.**

1. **Reducibility via \( e \)-reductions is a transitive relation.**

2. Let \( \mathcal{C} \in \{\Sigma^P_k, \Delta^P_k \mid k \geq 0\} \). The classes \( \text{Del}^C \), \( \text{Del}^C \), and \( \text{Inc}^C \) are closed under \( e \)-reductions.

**Proof.** We first show that the enumeration classes \( \text{Del}^C \), \( \text{Del}^C \), and \( \text{Inc}^C \) are closed under \( e \)-reductions for \( \mathcal{C} \in \{\Sigma^P_k, \Delta^P_k \mid k \geq 0\} \). Let \( R_1, R_2 \) be binary relations with \( \text{Enum}_{R_1} \leq_e \text{Enum}_{R_2} \). Further let \( \sigma \) and \( \tau \) be relations corresponding to the reduction \( \text{Enum}_{R_1} \leq_e \text{Enum}_{R_2} \) and assume that \( \text{Enum}_{R_2} \in \text{Del}^C \) (the cases where \( \text{Enum}_{R_2} \in \text{Del}^C \) or \( \text{Enum}_{R_2} \in \text{Inc}^C \) work along the same lines). Let \( \mathcal{A} \) denote the enumeration algorithm for \( \text{Enum}_{R_2} \) with a polynomial delay and decision oracle \( \mathcal{C} \), and let \( \mathcal{B} \) be the polynomial delay algorithm enumerating \( \tau(x, y, -) \) for all \( x, y \in \Sigma^* \). Moreover, let \( p \) be a polynomial such that for all \( z \in R_1(x) \), we have \( |\tau(x, y, z)| \leq p(|x|) \). The idea for an enumeration algorithm for \( \text{Enum}_{R_1} \) is to enumerate (without output) \( R_2(\sigma(x)) \) via \( \mathcal{A} \), and for every element \( y \) that would be output by \( \mathcal{A} \), repeatedly add \( p(|x|) \) elements of \( \tau(x, y, -) \) to a priority queue. Then, whenever those elements are added to the queue, an element of the queue is output. This way, one can ensure polynomial (respectively incremental) delay with corresponding priority queue.

To give a detailed explanation of the enumeration algorithm for \( \text{Enum}_{R_1} \), fix some \( x \in \Sigma^* \). Denote by \( \text{Newoutput}_A(\sigma(x)) \) a new output made by the enumeration algorithm \( \mathcal{A} \) when enumerating \( R_2(\sigma(x)) \), and similarly by \( \text{Newoutput}_B(y) \) a new output made by the enumeration algorithm \( \mathcal{B} \) when enumerating \( \tau(x, y, -) \). Algorithm 1 gives the algorithm for enumerating \( R_1(x) \). It is easy to see that this algorithm indeed works with a polynomial (respectively incremental) delay. When adding some element \( z \) to the priority queue (line 10 of Algorithm 1), if \( z \) is already in the queue, then \( \text{output\_queue} \) remains unchanged, otherwise \( z \) is added as the last element. Moreover, note that we do not delete elements in the priority queue in order to avoid the duplicates in the output.

To show that reducibility via \( e \)-reductions is a transitive relation, let \( R_1, R_2, R_3 \subseteq \Sigma^* \) be binary relations. Further let \( \text{Enum}_{R_1} \leq_e \text{Enum}_{R_2} \) with corresponding polynomial \( p_1 \) and relations \( \tau_1 \) and \( \sigma_1 \) and \( \text{Enum}_{R_2} \leq_e \text{Enum}_{R_3} \) with corresponding polynomial \( p_2 \) and relations \( \tau_2 \) and \( \sigma_2 \). Define relations \( \sigma_3 \) as \( \sigma_3 := \sigma_2 \circ \sigma_1 \) and \( \tau_3 \) as

\[
\tau_3 := \{(x, y, z) \in \Sigma^* \times \Sigma^* \times \Sigma^* \mid \exists \zeta \in \Sigma^* \text{ with } (x, \zeta, z) \in \tau_1 \text{ and } (\sigma_1(x), y, \zeta) \in \tau_2}\}
\]

Let \( x, y \in \Sigma^* \). To show that \( \tau_3(x, y, -) \) can be enumerated with a polynomial delay in \(|x|\), we use the same idea for a polynomial delay enumeration as we did to show that the reduction closes the enumeration classes: First a single \( y' \in R_2(\sigma_1(x)) \) is computed by \( \tau_2(\sigma_1(x), y, -) \) (with a polynomial delay in \( |x| \)) and thus polynomial delay in \(|x|\), and then at most polynomially many \( y'' \) from \( \tau_1(x, y', -) \) are
Algorithm 1 Enumerate $R_1(x)$

1: $i = 1$
2: output_queue = $\emptyset$
3: while $A$ has not output all of $R_2(\sigma(x))$ do
4: \hspace{1em} $y = \text{Newoutput}_A(\sigma(x))$
5: \hspace{1em} output_delay = 1
6: \hspace{1em} while output_delay < $p(|x|)$ do
7: \hspace{2em} $z = \text{Newoutput}_B(y)$
8: \hspace{2em} if $z \neq \emptyset$ then
9: \hspace{3em} Add $z$ to output_queue
10: \hspace{3em} output_delay = output_delay + 1
11: \hspace{2em} else
12: \hspace{3em} $y = \text{Newoutput}_A(\sigma(x))$
13: \hspace{2em} end if
14: \hspace{1em} end while
15: \hspace{1em} end while
16: Output (without deleting) the elements of output_queue starting from the $i$-th element
17: \hspace{1em} output_delay = $p(|x|)$
18: \hspace{1em} end if
19: end while
20: if $y \neq \emptyset$ then
21: \hspace{1em} Output (without deleting) the $i$-th element of output_queue
22: \hspace{1em} end if
23: $i = i + 1$
24: end while

added to some priority queue. An element of the queue is output, and again polynomially many elements are added by the queue (possibly by first computing some new $y' \in R_2(\sigma_1(x))$). This way, all of $\tau_3(x, y, -)$ can be enumerated with a polynomial delay in $|x|$. Moreover, it is easy to see that for all $x, z \in \Sigma^*$ we have $|\tau_3(x, - , z)| \leq p_1(p_2(|x|))$ and that $R_2(x) = \bigcup_{y \in R_3(\sigma_3(x))} \tau_3(x, y, -)$. It follows that indeed $\text{Enum}_R_1 \leq_e \text{Enum}_R_3$.

Note that, as an additional benefit, $\text{Enum}_R_1 \leq_e \text{Enum}_R_2$ does not imply $\text{Exist}_R_1 \leq_m \text{Exist}_R_2$. An example for this behavior can be found in Theorem 15 below.

Thus, besides being a natural extension of existing reductions, Proposition 9 also shows that $e$-reductions have important desirable properties. As a result, they are very well suited for reasoning about enumeration algorithms and complexities. For example, they allow to compare different enumeration problems, thus supporting to make statements on one problem “relative” to some other problem, and can be used to derive enumeration algorithms – hence complexity upper bounds.

In fact, they have already been implicitly used previously for these purposes. For example, an important result from [19] compares the complexity of two well-known problems by implicitly using an $e$-reduction:

**Example 2.** We denote by $\text{Dom-Enum}$ the problem of enumerating all subset-minimal dominating sets of a given graph, and by $\text{Trans-Enum}$ the problem of enumerating all minimal hitting sets of a hypergraph. In [19], it was shown that $\text{Trans-Enum}$ can be enumerated via an enumeration algorithm for $\text{Dom-Enum}$, by mapping a hypergraph to an instance of $\text{Dom-Enum}$, and giving a (non-bijective) map between the minimal dominating sets and the minimal hitting sets. In fact this defines the following $e$-reduction $\text{Trans-Enum} \leq_e \text{Dom-Enum}$:
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- Given a hypergraph \( \mathcal{H} \), \( \sigma(\mathcal{H}) \) is defined as the co-bipartite incidence graph of \( \mathcal{H} \), with

\[
V(\sigma(\mathcal{H})) = V(\mathcal{H}) \cup \{y_e \mid e \in E(\mathcal{H})\} \cup \{v\},
\]
\[
E(\sigma(\mathcal{H})) = \{\{x, y_e\} \mid x \in V(\mathcal{H}), x \in e\} \cup \{\{x, y\} \mid x, y \in V(\mathcal{H})\}
\]
\[
\quad \cup \{\{e, f\} \mid e, f \in E(\mathcal{H})\}.
\]

- Given \( \sigma(\mathcal{H}) \) and some minimal dominating set \( D \subseteq \sigma(\mathcal{H}) \), we have that

\[
\tau(\sigma(\mathcal{H}), D, -) = \begin{cases} 
\emptyset & : D = \{x, y_e\} \text{ for some } x \in V(\mathcal{H}), e \in E(\mathcal{H}), \\
D & : \text{otherwise}.
\end{cases}
\]

Note that \( \text{Dom-Enum} \leq_e \text{Trans-Enum} \) holds as well. The original question studied in [19] was whether any of these problems can be enumerated in a time that is polynomially in both, the size of the input and the size of the output. For the two problems \( \text{Dom-Enum} \) and \( \text{Trans-Enum} \), this is a longstanding open problem. The conclusion in [19] was that if such an enumeration algorithm exists for one of the problems, then this is also possible for the other one. It can be shown that the existence of the \( e \)-reductions between \( \text{Dom-Enum} \) and \( \text{Trans-Enum} \) also induces the same result. Moreover, they are also equivalent for all of the classes that are closed under \( e \)-reductions as given in Proposition 9, i.e., if one of the two problems is in any of these classes, then the other problem is as well.

Another example that implicitly uses \( e \)-reductions is [20]. In later sections of this paper, we will use them to retrieve reductions among enumeration problems as well. And even though we do not show any completeness results under \( e \)-reduction, we will actually use them to show hardness under alternative reductions that will be introduced in the next section.

5. Procedural-style Reductions

Although Turing reductions are too strong to show completeness results for classes in the polynomial hierarchy, because the classes \( \Sigma_k^p \) and \( \Delta_k^{p+1} \) have the same closure, Turing style reductions turn out to be meaningful in our case. In this section we introduce two types of reductions that are motivated by Turing reductions. Both of them are able to reduce between enumeration problems for which the \( e \)-reductions seem to be too weak.

Towards this goal, we first have to define the concept of RAMs with an oracle for enumeration problems (such oracles were already introduced in [2]). The intuition behind the definition of such enumeration oracle machines is the following: For algorithms (i.e., Turing machines or RAMs in the case of enumeration) using a decision oracle for the language \( L \), we usually have a special instruction that given an input decides in one step whether \( x \in L \), and then executes the next step of the algorithm accordingly. For an algorithm \( \mathcal{A} \) using an enumeration oracle, an input \( x \) to some \( \text{Enum}_R \)-oracle returns in a single step (using the instruction \text{NOO}, see the definition below) a single element of \( R(x) \), and then \( \mathcal{A} \) can proceed according to this output.

**Definition 6 (Enumeration Oracle Machines).** Let \( \text{Enum}_R \) be an enumeration problem. An **Enumeration Oracle Machine with an enumeration oracle** \( \text{Enum}_R \) (OM\(_R\)) is a RAM that has a sequence of new registers \( A_e, O^e(0), O^e(1), \ldots \) and a new instruction \text{NOO} (next Oracle output). An \( \text{OM}_R \) is **oracle-bounded** if the size of all inputs to the oracle is at most polynomial in the size of the input to the \( \text{OM}_R \).
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When executing NOO, the machine writes – in one step – some \( y_k \in R(x) \) to register \( A_e \), where \( x \) is the word stored in \( O^e(0), O^e(1), \ldots \) and \( y_k \) is defined as follows:

**Definition 7 (Next Oracle Output).** Let \( R \) be a binary relation, \( \pi_1, \pi_2, \ldots \) be the run of an EOM \( R \) and assume that the \( k^{\text{th}} \) instruction is NOO, i.e., \( \pi_k = \text{NOO} \). Denote with \( x_i \) the word stored in \( O^e(0), O^e(1), \ldots \) at step \( i \). Let \( K = \{ \pi_i \in \{\pi_1, \ldots, \pi_{k-1}\} \mid \pi_i = \text{NOO} \text{ and } x_i = x_k \} \). Then the oracle output \( y_k \) in \( \pi_k \) is defined as an arbitrary \( y_k \in R(x_k) \) s.t. \( y_k \) has not been the oracle output in any \( \pi_i \in K \). If no such \( y_k \) exists, then the oracle output in \( \pi_k \) is undefined.

When executing NOO in step \( \pi_k \), if the oracle output \( y_k \) is undefined, then the register \( A_e \) contains some special symbol in step \( \pi_{k+1} \). Otherwise in step \( \pi_{k+1} \) the register \( A_e \) contains \( y_k \).

Observe that since an EOM \( M^e \) is a polynomially bounded RAM and the complete oracle output is stored in the register \( A_e \), only such oracle calls are allowed where the size of each oracle output is guaranteed to be polynomially bounded in the size of the input of \( M^e \).

Using EOMs, we can now define another type of reductions among enumeration problems, reminiscent of classical Turing reductions. I.e., we say that one problem Enum\( R_1 \) reduces to another problem Enum\( R_2 \), if Enum\( R_1 \) can be solved by an EOM using Enum\( R_2 \) as an enumeration oracle.

**Definition 8 (D-reductions, I-reductions).** Let \( R_1 \) and \( R_2 \) be binary relations.

- We say that Enum\( R_1 \) reduces to Enum\( R_2 \) via D-reductions, Enum\( R_1 \leq_D \) Enum\( R_2 \), if there is an oracle-bounded EOM\( R_2 \) that enumerates \( R_1 \) in DelP and is independent of the order in which the Enum\( R_2 \)-oracle enumerates its answers.

- We say that Enum\( R_1 \) reduces to Enum\( R_2 \) via I-reductions, Enum\( R_1 \leq_I \) Enum\( R_2 \), if there is an EOM\( R_2 \) that enumerates \( R_1 \) in IncP and is independent of the order in which the Enum\( R_2 \)-oracle enumerates its answers.

The following proposition shows that these reductions have desirable properties:

**Proposition 10.** Reducibility via D-reductions as well as reducibility via I-reductions are transitive relations.

**Proof.** This can be proven along the same lines as Proposition 11 by substituting occurrences of enumeration RAMs with decision oracles by enumeration RAMs.

For D-reductions, we require the EOM\( R_2 \) to be oracle-bounded. We would like to point out that this restriction is essential: if we drop it, then the classes DelC are not closed under the resulting reduction. Indeed, suppose that the \( D_u \)-reduction is defined similar to the D-reduction but without the restriction to an oracle-bounded EOM. Then, abusing the decision problem \( \text{ANOTHERSOL} \) from the proof of Theorem 2 as an enumeration problem \( \text{Enum}\( R_A \) (with output 'yes' or 'no'), we can show that for any \( k \geq 1 \) and any Enum\( R \in \text{Inc}\( P \) we have that Enum\( R \leq_D \) Enum\( R_A \) but Enum\( R_A \notin \text{Del}\( P \).

**Proposition 11.** Let \( C \in \Sigma_k^P \), \( k \geq 0 \). The class Del\( C \) is closed under D-reductions and the class Inc\( C \) is closed under I-reductions.

**Proof.** Let \( M \) be an oracle-bounded enumeration oracle machine with an enumeration oracle Enum\( R_2 \) witnessing that Enum\( R \leq_D \) Enum\( R_2 \). Let \( A \) be the polynomial delay algorithm with access to a polynomially bounded C-oracle.
construct a RAM $M'$ that enumerates $\text{ENUM}_{R_1}$ with a polynomial delay using a polynomially bounded decision oracle $C$, by modifying the RAM $M$ as follows: Every time $M$ makes a call to an $\text{ENUM}_{R_2}$-oracle, we use the algorithm $\mathcal{A}$ to retrieve what should be written to the register $A_e$. Assume that $x$ is the input to an oracle call of the RAM $M$. Then the new RAM $N$ assigns two fixed addresses $a^0_x$ and $a^1_x$ to $x$. Then $N$ can simulate the algorithm $\mathcal{A}$ on the registers $R(2^a_x), \ldots, R(2^b_x)$ until $\mathcal{A}$ would output some $y \in \Sigma^*$. The RAM $N$ writes $y$ to $A_e$, and a simulation of a single oracle call is completed. Whenever $x$ is the input for a NOO-call, $N$ continues to simulate $\mathcal{A}$ on those registers; this way, the enumeration of $R_2(x)$ does not need to start from the beginning every time $x$ is the input of an oracle call. The proof of the closure of $\text{IncC}$ under $I$-reductions can be done along the same lines. 

Observe that the $e$-reductions introduced in the previous section are a particular case of $D$- and $I$-reductions. We will also use $e$-reductions later when we establish completeness results for specific problems.

**Proposition 12.** Let $R_1$ and $R_2$ be binary relations.

- If $\text{ENUM}_{R_1} \leq_e \text{ENUM}_{R_2}$, then $\text{ENUM}_{R_1} \leq_D \text{ENUM}_{R_2}$.
- If $\text{ENUM}_{R_1} \leq_D \text{ENUM}_{R_2}$, then $\text{ENUM}_{R_1} \leq_I \text{ENUM}_{R_2}$.

**Proof.** The proof of the second claim follows immediately from the definitions. Now, if $\text{ENUM}_{R_1} \leq_e \text{ENUM}_{R_2}$, then Algorithm developed in the proof of Proposition provides an oracle bounded $\text{EOM}_{R_2}$ that enumerates $R_1$ in $\text{DelP}$. For enumerating $R_1(x)$ the calls to the enumeration oracle $\text{ENUM}_{R_2}$ are only made on the input $\sigma(x)$, where $\sigma$ is the function used in the $e$-reduction.

Now, unlike for $e$-reductions, the next theorem shows that the $D$- and $I$-reduction induce complete problems for the enumeration complexity classes introduced in Section 3.

**Theorem 13.** Let $R$ be a binary relation and $k \geq 1$ such that $\text{EXIST}_R$ is $\Sigma^P_k$-hard.

- $\text{ENUM}_R$ is $\text{Del}\Sigma^P_k$-hard via $D$-reductions.
- $\text{ENUM}_R$ is $\text{Inc}\Sigma^P_k$-hard via $I$-reductions.
- If additionally $\text{EXIST}_R$ is in $\Sigma^P_k$ and $R$ is self-reducible, then $\text{ENUM}_R$ is $\text{Del}\Sigma^P_k$-complete via $D$-reductions and $\text{Inc}\Sigma^P_k$-complete via $I$-reductions.

**Proof.** Let $R$ be a relation such that $\text{EXIST}_R$ is $\Sigma^P_k$-complete for some $k \geq 1$.

- We have to prove that for any $\text{ENUM}_R' \in \text{Del}\Sigma^P_k$, $\text{ENUM}_R' \leq_D \text{ENUM}_R$ (Inc$\Sigma^P_k$-hardness via $I$-reductions can be shown along the same lines). So let $R'$ be a binary relation such that $\text{ENUM}_R' \in \text{Del}\Sigma^P_k$. By definition there is some $L \in \Sigma^P_k$ such that $\text{ENUM}_R' \in \text{Del}L$. Moreover let $\mathcal{A}$ be an algorithm witnessing this membership. As $\text{EXIST}_R$ is $\Sigma^P_k$-complete, we have that $L \leq_m \text{EXIST}_R$, so any input $x$ to an $L$-decision oracle when enumerating $\text{ENUM}_R$ can be transformed to an instance $x' \in \text{EXIST}_R$ such that $x \in L$ iff $x' \in \text{EXIST}_R$, and this transformation can be done in polynomial time in the size of $x$. Moreover, since the size of the oracle input is polynomial, this reduction can be computed within the time bounds of a polynomial delay, i.e. whenever a polynomial delay algorithm with an $L$-oracle makes an oracle call with an input $x$, the same algorithm can also perform a transformation to some $x'$ before that oracle call, without violating the polynomial delay restriction. Therefore we can enumerate $\text{ENUM}_R'$ with an oracle bounded enumeration
oracle machine with \( \text{ENUM}_R \) as follows: Whenever \( A \) would make a decision oracle call to \( L \) with input \( x \), instead the machine transforms this to some \( x' \in \Sigma^* \), and then makes a \( \text{NOO} \)-instruction with input \( x' \) to the \( \text{ENUM}_R \)-oracle. The \( \text{NOO} \)-instruction writes a nonempty string to the register \( A_r \) if and only if \( x' \in \text{EXIST}_R \) and thus if and only if \( x \in L \). It follows that we can simulate the decision oracle call with an enumeration oracle call.

- Membership of \( \text{ENUM}_R \) in \( \text{Del}\Sigma_k^P \) in this case follows immediately from Proposition 6. Since \( \text{Del}\Sigma_k^P \subseteq \text{Inc}\Sigma_k^P \), this also shows membership in \( \text{Inc}\Sigma_k^P \).

As a consequence, the enumeration problems \( \text{E-}\Sigma_k\text{SAT} \) and also \( \text{E-}\Pi_k\text{SAT} \) are natural complete problems for our enumeration complexity classes:

**Corollary 14.** Let \( k \geq 1 \). Then

1. \( \text{E-}\Sigma_k\text{SAT} \) is complete for \( \text{Del}\Sigma_k^P \) via \( D \)-reductions.
2. \( \text{E-}\Pi_k\text{SAT} \) and \( \text{E-}\Sigma_{k+1}\text{SAT} \) are complete for \( \text{Inc}\Sigma_{k+1}^P \) via \( I \)-reductions. In particular \( \text{E-SAT} \) the enumeration variant of the traditional \( \text{SAT} \) problem is \( \text{IncNP} \)-complete via \( I \)-reductions.

**Proof.** The results for \( \text{E-}\Sigma_k\text{SAT} \) follow immediately from Theorem 13. It only remains to prove that \( \text{E-}\Pi_k\text{SAT} \) and \( \text{E-}\Sigma_{k+1}\text{SAT} \) are equivalent under \( I \)-reductions. Note that \( \text{E-}\Pi_k\text{SAT} \leq_I \text{E-}\Sigma_{k+1}\text{SAT} \) follows immediately from the fact that \( \text{E-}\Pi_k\text{SAT} \) is a special case of \( \text{E-}\Sigma_{k+1}\text{SAT} \), so it suffices to show that \( \text{E-}\Sigma_{k+1}\text{SAT} \leq_I \text{E-}\Pi_k\text{SAT} \). Thus consider an instance \( \psi \) of \( \text{E-}\Sigma_{k+1}\text{SAT} \) given as \( \psi(x) := \exists y_0 \forall y_1 \ldots Q_k y_k \phi(x, y_0, \ldots, y_k) \). We can enumerate all solutions to \( \psi \) as follows: The first input to a \( \text{E-}\Pi_k\text{SAT} \)-oracle is \( \psi_0(x, y_0) := \forall y_1 \ldots Q_k y_k \phi(x, y_0, \ldots, y_k) \), with free variables \( x \) and \( y_0 \). A single \( \text{NOO} \) instruction thus gives a solution \( x_0, y_0' \) for \( \psi_0 \), and \( x_0 \) can be output as a solution to \( \psi \). The next solution can be found by calling a \( \text{NOO} \) instruction with the input \( \psi_1(x, y_0) = \forall y_1 \ldots Q_k y_k (\phi(x, y_0, \ldots, y_k) \land (x_0 \neq x)) \). We only need to add the clauses of \( (x_0 \neq x) \) to the input registers of the oracle tape, and we can choose an encoding such that this does not alter the previous input, but extends it. The output \( x_1, y_0'' \) of the second oracle call gives the second output \( x_1 \) for the \( \text{E-}\Sigma_{k+1}\text{SAT} \) problem. By repeating this method until an oracle call gives back the empty solution, we can enumerate the solutions of \( \psi \).

Observe that, via different reductions, \( \text{E-}\Sigma_k\text{SAT} \) is complete for both, \( \text{Inc}\Sigma_k^P \) and for the presumably smaller class \( \text{Del}\Sigma_k^P \). This provides additional evidence that the two reductions nicely capture \( \text{Inc}\Sigma_k^P \) and \( \text{Del}\Sigma_k^P \), respectively.

Roughly speaking Theorem 13 says that any enumeration problem whose corresponding decision problem is hard, is hard as well. An interesting question is whether there exist easy decision problems for which the corresponding enumeration problem is hard. We answer positively to this question in the next section.

6. Completeness Results

In this section, we prove completeness results under procedural-style reductions for several problems. We start by considering generalized satisfiability in Schaefer’s framework, and then continue by looking into minimal models of Boolean formulæ. Next, we study enumeration within the framework of abduction, and finally turn to the problem of enumerating repairs of an inconsistent database.
6 COMPLETENESS RESULTS

6.1. Generalized Satisfiability

In this subsection, we revisit a classification theorem obtained for the enumeration of generalized satisfiability [20] in our framework. It is convenient to first introduce some notation.

A logical relation of arity \( k \) is a relation \( R \subseteq \{0, 1\}^k \). A constraint, \( C \), is a formula \( C = R(x_1, \ldots, x_k) \), where \( R \) is a logical relation of arity \( k \) and the \( x_i \)'s are variables. An assignment \( m \) of truth values to the variables satisfies the constraint \( C \) if \( (m(x_1), \ldots, m(x_k)) \in R \). A constraint language \( \Gamma \) is a finite set of nontrivial logical relations. A \( \Gamma \)-formula \( \phi \) is a conjunction of constraints using only logical relations from \( \Gamma \). A \( \Gamma \)-formula \( \phi \) is satisfied by an assignment \( m : \text{var}(\phi) \to \{0, 1\} \) if \( m \) satisfies all constraints in \( \phi \).

Throughout the text we refer to different types of Boolean relations following Schaefer’s terminology, see [21, 20]. We say that a constraint language is Schaefer if every relation in \( \Gamma \) is either Horn, dualHorn, bijunctive, or affine.

\[
\begin{array}{|c|c|}
\hline
\text{E-SAT}(\Gamma) \\
\text{Instance:} & \phi \text{ a } \Gamma\text{-formula} \\
\text{Output:} & \text{All satisfying assignments of } \phi. \\
\hline
\end{array}
\]

The following theorem gives the complexity of this problem according to \( \Gamma \).

**Theorem 15.** Let \( \Gamma \) be a finite constraint language. If \( \Gamma \) is Schaefer, then \( \text{E-SAT}(\Gamma) \) is in DelP, otherwise it is DelNP-complete via \( D \)-reductions and IncNP-complete via \( I \)-reductions.

**Proof.** The polynomial cases were studied in [20]. Let us now consider the case where \( \Gamma \) is not Schaefer. Membership of \( \text{E-SAT}(\Gamma) \) in DelNP is clear. For the hardness, let us introduce \( T \) and \( F \) as the two unary constant relations \( T = \{1\} \) and \( F = \{0\} \). According to Schaefer’s dichotomy theorem [21], deciding whether a \( \Gamma \cup \{F, T\} \)-formula is satisfiable is NP-complete. Since this problem is self-reducible, according to Theorem 13, \( \text{E-SAT}(\Gamma \cup \{F, T\}) \) is DelNP-complete via \( D \)-reductions.

Let \( \text{Imp} \) denote the binary implication relation \( \text{Imp}(x, y) \equiv (x \to y) \). Given a \( \Gamma \cup \{F, T\} \)-formula \( \varphi \), one can construct a \( \Gamma \cup \{\text{Imp}\} \) formula \( \varphi' \) as follows. Let \( V \) denote the set of variables in \( \varphi \), \( V_0 \) the set of variables \( x \) on which lies the constraint \( F(x) \), and \( V_1 \) the set of variables \( x \) on which lies the constraint \( T(x) \). Let \( f \) and \( t \) be two fresh variables. Given a set of variables \( W \) and a variable \( z \), \( \varphi[W/z] \) denotes the formula obtained from \( \varphi \) in replacing each occurrence of a variable from \( W \) by the variable \( z \). Then we set \( \varphi' = \varphi[V_0/f] \land \varphi[V_1/t] \land \bigwedge_{x \in V} \text{Imp}(x, f) \land \bigwedge_{x \in V} \text{Imp}(f, x) \). Clearly models of \( \varphi \) coincide with models of \( \varphi' \) in which \( f \) is set to false and \( t \) to true, and \( \varphi' \) might have two additional models, the all-zero one and the all-one one.

In [20] it is essentially proven that if \( \Gamma \) is not Schaefer then the implication relation \( \text{Imp} \) can be expressed by a \( \Gamma \)-formula. Therefore, in this case the construction described above shows that \( \text{E-SAT}(\Gamma \cup \{F, T\}) \leq_c \text{E-SAT}(\Gamma) \), thus proving the DelNP-completeness via \( D \)-reductions and IncNP-completeness via \( I \)-reductions of the latter problem, according to Proposition 12.

To come back to the above discussion, we point out that there exist constraint languages \( \Gamma \) such that the decision problem \( \text{SAT}(\Gamma) \) is in \( P \), while the enumeration problem \( \text{E-SAT}(\Gamma) \) is DelNP-complete, namely 0-valid or 1-valid constraint languages that are not Schaefer.

6.2. Enumeration of Minimal Models

Let us now turn to the complexity of enumerating minimal models of a Boolean formula. Such an enumeration is ubiquitous in practical settings, ranging from verification, to databases and to knowledge representation among many others. One can
consider subset-minimality as well as cardinal-minimality. Therefore we consider the
two following enumeration problems: \( E\text{-CIRCUMSCRIPTION} \) denotes the problem of
enumerating all subset-minimal models of a boolean formula and \( E\text{-CARDMINSAT} \) denotes the problem of enumerating all cardinality-minimal models of a boolean
formula. We pinpoint the complexity of these two enumeration problems.

Observe that contrary to the satisfiability problems we discussed above, enumeration of minimal models is not obviously self-reducible. In particular, the problem \( E\text{-CIRCUMSCRIPTION} \) is very unlikely to be self-reducible: The problem of decid-
ing if a partial truth assignment can be extended to a subset minimal model is
\( \Sigma_2^P \)-complete [22], while deciding the existence of a minimal model is clearly \( \text{NP} \)-complete. Thus \( E\text{-CIRCUMSCRIPTION} \) is not self-reducible unless the polynomial
hierarchy collapses to the first level.

**Theorem 16.** \( E\text{-CIRCUMSCRIPTION} \) is \( \text{IncNP} \)-complete via \( I \)-reductions.

**Proof.** Hardness follows immediately from Theorem[13] since deciding the existence of a (minimal) model of a Boolean formula is \( \text{NP} \)-complete.

To obtain membership, we show that \( E\text{-CIRCUMSCRIPTION} \) can be even enumerated in \( \text{Del}^+\text{NP} \). Consider a boolean formula \( \phi(x_1, \ldots, x_n) \), and assume that we want to enumerate the minimal models of \( \phi \). We start by copying \( \phi \) to the oracle registers. A very first minimal model \( x' \) can be achieved by a greedy al-
grithm using an \( \text{NP} \) oracle. Next we extend \( \phi \) in the oracle registers to \( \phi' = \phi \land ((x_1 < x'_1) \lor \ldots \lor (x_n < x'_n)) \) and again get a minimal model \( x'' \) for \( \phi' \) us-
ing a greedy algorithm with an \( \text{NP} \)-oracle. This is also a minimal model for \( \phi \); repeatedely extending \( \phi \) and then computing a minimal model via a greedy al-
grithm achieves the membership \( E\text{-CIRCUMSCRIPTION} \in \text{Del}^+\text{NP} \) and thus also \( E\text{-CIRCUMSCRIPTION} \in \text{IncNP} \).

What makes the completeness result obtained for \( E\text{-CIRCUMSCRIPTION} \) surpris-
ing is the discrepancy from the behaviour of the counting variant of the problem:
The counting variant of \( E\text{-CIRCUMSCRIPTION} \) is a prototypical \# \cdot \text{coNP}-complete
problem [23], and thus of the same hardness as the counting variant of \( E\text{-\Pi}_1\text{SAT} \). However, for enumeration we have that \( E\text{-CIRCUMSCRIPTION} \) shows the same com-
plexity as \( E\text{-\Sigma}_1\text{SAT} \), which is considered to be lower than that of \( E\text{-\Pi}_1\text{SAT} \).

**Theorem 17.** \( E\text{-CARDMINSAT} \) is \( \text{DelNP} \)-complete via \( D \)-reductions and \( \text{IncNP} \)-
complete via \( I \)-reductions.

**Proof.** Hardness follows again immediately from Theorem[13] since deciding the existence of a (minimal) model of a Boolean formula is \( \text{NP} \)-complete.

Membership of \( E\text{-CARDMINSAT} \) in \( \text{DelNP} \) (and a fortiori in \( \text{IncNP} \)) follows from the fact that polynomial time with access to an \( \text{NP} \)-oracle gives us the possi-
bility to compute first the minimal cardinality of models. Having this information
we can enumerate all cardinality-minimal models by the standard binary search tree
with an \( \text{NP} \)-oracle that enumerates them in lexicographical order.

### 6.3. Model-based Diagnosis

The problem of model-based diagnosis [24] is ubiquitous in practical settings. A
usual formulation of this problem is as follows: given a system description composed
of some components, and an observation inconsistent with the system description,
the goal is to identify a cardinality-minimal set of components which, if declared
faulty and thus put out of the system description, restores the consistency be-
tween the model and the observation. In recent years the practical enumeration
of minimum-size diagnoses has gained interest through translations to MaxSAT
instances and the use of efficient MaxSAT solvers (see e.g. [25]).

Formally a system description can be represented by a consistent set of proposi-
tional formulæ, each encoding the normal behavior of a component. The observation
is represented by an additional formula that is inconsistent with the former set of formulæ. Thus we get the following enumeration problem.

<table>
<thead>
<tr>
<th>E-MODEL-BASED DIAGNOSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Instance:</strong> B = {( \phi_1, \ldots, \phi_n )} a consistent set of formulæ, ( \mu ) a satisfiable formula inconsistent with B</td>
</tr>
<tr>
<td><strong>Output:</strong> All cardinality-maximal subsets ( B' \subseteq B ) such that ( \bigwedge_{\phi \in B'} \phi \land \mu ) is satisfiable.</td>
</tr>
</tbody>
</table>

Observe that for this enumeration problem, the corresponding decision problem \( \text{EXIST}_R \) is trivial because there always is a solution \( B' \). Nevertheless we prove hardness by an \( e \)-reduction from a previously identified hard problem. So contrary to all results above, we obtain here a hardness result for an enumeration problem which does not rely on hardness of any related decision problem.

The following theorem settles the complexity of \( \text{E-MODEL-BASED DIAGNOSIS} \) in our hierarchy.

**Theorem 18.** \( \text{E-MODEL-BASED DIAGNOSIS} \) is \( \text{DelNP} \)-complete via \( D \)-reductions and \( \text{IncNP} \)-complete via \( I \)-reductions.

**Proof.** Membership of \( \text{E-MODEL-BASED DIAGNOSIS} \) in \( \text{DelNP} \) (and a fortiori in \( \text{IncNP} \)) follows from the fact that polynomial time with access to an \( \text{NP} \)-oracle gives us the possibility to compute first the maximal cardinality of subsets of \( B \) that are consistent with \( \mu \). Having this information we can enumerate all such cardinality-maximal subsets by the standard binary search tree with an \( \text{NP} \)-oracle that enumerates them in lexicographical order.

For hardness we show that \( \text{E-CARDMINSAT} \leq_e \text{E-MODEL-BASED DIAGNOSIS} \), thus proving that \( \text{E-MODEL-BASED DIAGNOSIS} \) is \( \text{DelNP} \)-hard via \( D \)-reductions and \( \text{IncNP} \)-hard via \( I \)-reductions, according to Proposition \[12\] and Proposition \[16\].

Let \( \phi \) be a formula given as an instance of \( \text{E-CARDMINSAT} \), and assume \( \text{var}(\phi) = \{x_1, \ldots, x_n\} \). Without loss of generality suppose in addition that \( \phi \) is not 0-valid, i.e., is not satisfied by the all-zero assignment. We define a corresponding instance \( \sigma(\phi) \) of \( \text{E-MODEL-BASED DIAGNOSIS} \) as follows: Let \( B \) consist of the \( n \) formulæ \( \phi'_i = (\phi \lor x_0) \land (\neg x_i \lor x_0) \) for \( i = 1, \ldots, n \), and let \( \mu = \neg x_0 \), where \( x_0 \) is a fresh variable. Observe that on the one hand the set \( B \) is consistent (set \( x_0 \) to true), and on the other hand it is inconsistent with \( \mu \) (since by assumption \( \phi \) is not 0-valid). It is easy to see that there is a bijection \( \pi \) mapping the cardinality-minimal models \( x' \) of \( \phi \) to cardinality-maximal subsets of \( B \) that are consistent with \( \mu \). Thus we can define a relation \( \tau \) with \( \tau(\sigma(\phi), x', -) = \pi(x') \), showing that \( \text{E-CARDMINSAT} \leq_e \text{E-MODEL-BASED DIAGNOSIS} \). \( \square \)

**6.4. Abduction**

We now provide an example of an enumeration problem hard in a higher level of the hierarchy, namely abduction. Abduction is a well-studied form of non-monotonic reasoning. Given a certain consistent knowledge, abductive reasoning is used to generate explanations for observed manifestations. We consider here propositional abduction, in which the knowledge base as well as the manifestations are represented by propositional formulæ \[22\].

The enumeration problem can be formalized as follows:
E-ABDUCTION
Instance: \( \Gamma \) a set of formulæ (knowledge),
\( H \) a set of literals (hypotheses),
\( q \) a variable that does not occur in \( H \) (manifestation).
Output: All sets \( E \subseteq H \) such that \( \Gamma \land E \) is satisfiable and
\( \Gamma \land E \models q \) (equivalently, \( \Gamma \land E \land \neg q \) is unsatisfiable).

The decision problem ABDUCTION corresponding to this variant of abduction
is known to be \( \Sigma^p_2 \)-complete, see, e.g., [26]. Based on this we obtain hardness of
the enumeration problem. Though this is maybe not very surprising, we mention
this problem here as a first example of a problem whose enumeration is even harder
than those we studied before, at least under the assumption that the polynomial
hierarchy does not collapse.

**Theorem 19.** E-ABDUCTION is \( \text{Del}\Sigma^p_2 \)-complete via \( D \)-reductions and \( \text{Inc}\Sigma^p_2 \)-complete
via \( I \)-reductions.

**Proof.** As we have mentioned above, the associated decision problem is \( \Sigma^p_2 \)-complete. Also, note that the problem is self-reducible in the sense of Definition 3. Given an
instance \((\Gamma, H, q)\) of E-ABDUCTION and a partial solution \( E \subseteq H \),
\( E \) can be ex-
tended to an explanation iff \((\Gamma \land E, H, q) \in \text{Abduction}\). Therefore, according to
the last statement of Theorem 13, the theorem follows. \( \Box \)

6.5. Database Repairs

Finally, we provide an example of an enumeration problem from the database
domain, more specifically the problem of enumerating all “repairs” of an inconsistent
database. In this problem, we are given a database instance \( D \) (i.e., simply a finite
structure given by a domain and a collection of relations; for our purposes, it is
most convenient to represent \( D \) as a set of ground atoms) and a set \( C \) of integrity
constraints (i.e., in general, a set of formulæ from some fragment of first-order logic).
Now suppose that \( D \) is inconsistent, i.e., \( D \not\models C \) holds. Then we are interested in
enumerating the possible repairs. A repair is a database instance \( I \) which satisfies \( C \)
and which differs minimally from \( D \). The term “differs minimally” can be defined
in several ways. The most common approach is the one introduced by Arenas et al. [27]:
here, repairs are obtained from the original database by inserting or deleting
ground atoms to or from \( D \), respectively. Then “minimal difference” means that the
symmetric set difference \( \Delta(D, I) \) is minimal w.r.t. subset inclusion. More formally,
let \( \Delta(D, I) = (D \setminus I) \cup (I \setminus D) \). Then \( I \) is a repair of \( D \) w.r.t. \( C \) if \( I \models C \) (i.e., \( I \)
is consistent w.r.t. \( C \)) and there does not exist an instance \( I' \) that satisfies \( C \) and
\( \Delta(D, I') \subsetneq \Delta(D, I) \).

Many different fragments of first-order logic have been studied for the definition
of the integrity constraints \( C \). We only look at one such fragment here, namely
equality generating dependencies (EGDs, for short). EGDs are a classical formal-
ism for defining database constraints [28]. They generalize functional dependencies
(FDs), which in turn generalize key dependencies (KDs). Formally, they are defined
as formulæ of the form
\[ \forall \bar{x} (\phi(\bar{x}) \rightarrow x_i = x_j) \]
where \( \phi \) is a conjunction of atoms over the predicate symbols occurring in \( D \), \( \bar{x} \)
denotes the variables occurring in these atoms, and \( x_i, x_j \) are two variables from \( \bar{x} \).
The meaning of such an EGD is that, for every variable binding \( \lambda \) on the variables \( \bar{x} \),
such that applying \( \lambda \) to the atoms in \( \phi(\bar{x}) \) sends all atoms of \( \phi(\bar{x}) \) into the database
instance \( D \), it must be the case that \( \lambda(x_i) = \lambda(x_j) \) holds. Following the usual
notational convention in the database literature, we will omit the quantifiers and
denote EGDs of the above form simply as \( \phi(\vec{x}) \rightarrow x_i = x_j \) with the understanding that all variables occurring in an EGD are universally quantified.

The enumeration problem considered here is defined as follows:

E-Repair
Instance: Database instance \( D \), set \( C \) of EGDs
Output: \( D \) if \( D \models C \) and All repairs \( I \) of \( D \) w.r.t. \( C \) if \( D \not\models C \).

As in Section 6.3, we observe that for this enumeration problem, the corresponding decision problem \( \text{Exist}_R \) is trivial because for every inconsistent database instance, the existence of a repair is guaranteed. Moreover, it is worth mentioning that a different variant of repair enumeration has been recently studied in [29], where a different type of integrity constraints was considered and prereferences were taken into account.

Several decision problems have been studied in the context of repairs. The most intensively studied one is the Consistent Query Answering problem, i.e., given a database instance \( D \), a set \( C \) of integrity constraints and a boolean query \( Q \), is the query \( Q \) true in every repair \( I \) of \( D \) w.r.t. \( C \). For EGDs as constraint language and conjunctive queries as query language, this problem has been shown to be \( \Pi^P_2 \)-complete [30]. Another decision problem is the Repair Checking problem, i.e., given database instances \( D \) and \( I \), and a set \( C \) of integrity constraints, is \( I \) a repair of \( D \) w.r.t. \( C \). In other words, this is the Check problem corresponding to the enumeration problem E-Repair. For EGDs, this decision problem is \( \mathsf{NP} \)-complete [30]. Definitely a bit more surprising than in case of E-Abduction, we get \( \mathsf{Inc}\Sigma^P_2 \)-completeness also for the E-Repair problem. In particular, the hardness-proof is more involved than for the previously studied problems.

**Theorem 20.** E-Repair is \( \mathsf{Inc}\Sigma^P_2 \)-complete via \( I \)-reductions.

**Proof.** For the membership proof, it is important to note that repairs of \( D \) w.r.t. a set of EGDs can only be obtained by deletions, i.e., if a constraint is violated, it does not help to add further atoms. In other words, EGDs have a monotonic behaviour in the sense that if a database instance \( I \) violates an EGD, then also every superset of \( I \) does. This allows us to redefine E-Repair as Enum\_R, where the relation \( R \) consists of pairs \( ((D,C),V) \) such that

- \( D \) is a database instance containing \( N \geq 0 \) atoms, which are numbered as \( A_1, \ldots, A_N \),
- \( C \) is a set of EGDs and
- \( V \) is a word of length \( N \) over \( \{0,1\} \), where \( V(i) = 0 \) means that we delete \( A_i \) from \( D \) and \( V(i) = 1 \) means that we retain \( A_i \) in \( D \).

We observe that \( \text{ExtSol}_R \in \Sigma^P_2 \): Given some word \( V_1 \in \{0,1\}^{N-k} \), we need a non-deterministic guess for the instantiation of some \( V_2 \in \{0,1\}^k \) and then we need to call a \( \mathsf{NP} \)-oracle (or, equivalently, two \( \mathsf{NP} \)-oracles) to check if the word \( V_1V_2 \) represents a repair. Thus \( \text{Enum}_R \in \mathsf{Del}\Sigma^P_2 \) by Proposition 6, meaning that E-Repair \( \in \mathsf{Inc}\Sigma^P_2 \).

For the hardness proof, we first note that the \( \mathsf{Inc}\Sigma^P_2 \)-completeness of E-\( \Pi_1 \)SAT via \( I \)-reductions from Corollary 4 can be strengthened in the sense that we may restrict the instances to 3-DNF (i.e., the matrix of the formulæ under investigation is in DNF such that each implicant consists of 3 literals). Then our \( \mathsf{Inc}\Sigma^P_2 \)-completeness proof is by an \( e \)-reduction from E-\( \Pi_1 \)SAT, where we assume w.l.o.g., that the instances of E-\( \Pi_1 \)SAT are in 3-DNF.
Consider an arbitrary instance of E-$\Pi_1$SAT, i.e., a quantified boolean formula $\psi(\vec{x}) = \forall \vec{y} \phi(\vec{x}, \vec{y})$. Let $\vec{x} = x_1, \ldots, x_k$ and $\vec{y} = y_1, \ldots, y_\ell$. By the restriction to 3-DNF, $\phi(\vec{x}, \vec{y})$ is of the form $\phi(\vec{x}, \vec{y}) = \bigvee_{m=1}^k (l_{11} \land l_{12} \land l_{13})$, where $m$ denotes the number of implicants in $\phi$ and $l_{ij}$ with $j \in \{1, 2, 3\}$ is the $j$-th literal in the $i$-th implicant, i.e., each $l_{ij}$ is of the form $x_\alpha, \neg x_\alpha, y_\beta, \neg y_\beta$ with $\alpha \in \{1, \ldots, k\}$ and $\beta \in \{1, \ldots, \ell\}$.

From this we define an instance $\sigma(\psi(\vec{x})) = (D, C)$ of E-REPAIR. The relation symbols used in this instance are $\{a, q\} \cup \{p_1, \ldots, p_k\} \cup \{b_0, b_1\}$ It is convenient to define the following subformulae:

$$\chi = b_0(v_0) \land b_1(v_1) \land q(v_0, v_1) \land a(v_0, v_0) \land a(v_0, v_0, v_1) \land a(v_0, v_1, v_0) \land a(v_1, v_0, v_0) \land a(v_1, v_0, v_1) \land a(v_1, v_1, v_0) \land a(v_1, v_1, v_1)$$

$$\pi = \bigwedge_{i=1}^k p_i(w_1, w'_i)$$

$$\psi^* = \bigwedge_{i=1}^m a(l^*_i, l^*_{i2}, l^*_{i3})$$

where we define $l^*_i = z$ if $l_{ij}$ is the positive literal $z$ and $l^*_i = z'$ if $l_{ij}$ is the negative literal $\neg z$. For example, $\{(x \land \neg y \land z) \lor (\neg x \land y \land z)\}^* = a(x, y, z) = a(x, y', z)$.

Then the database $D$ of our instance of E-REPAIR looks as follows:

$$D = P \cup Q \cup A \cup E$$

with

$$P = \{p_1(0, 1), p_1(1, 0), p_2(0, 1), p_2(1, 0), \ldots, p_k(0, 1), p_k(1, 0)\}$$

$$Q = \{q(0, 1), q(1, 0)\}$$

$$A = \{a(0, 0, 0), a(0, 0, 1), a(0, 1, 0), a(0, 1, 1), a(1, 0, 0), a(1, 0, 1), a(1, 1, 0)\}$$

$$E = \{b_0(0), b_1(1)\}$$

Finally, we define the set $C$ of integrity constraints:

$$C = \bigcup_{1 \leq i \leq k} \{\chi \land \pi \land p_i(x, x') \land p_i(x', x) \rightarrow x = x'\} \quad \text{(C1)}$$

$$\cup \{\chi \land \pi \land \bigwedge_{i=1}^k p_i(x_i, x'_i) \land \bigwedge_{j=1}^\ell q(y_j, y'_j) \land \psi^* \rightarrow x_1 = x'_1\} \quad \text{(C2)}$$

The intuition of this reduction is as follows: all of the EGDs in line (C1) are clearly violated, i.e.: $\chi$ can be sent into the database $D$ by instantiating $v_0$ to 0 and $v_1$ to 1; likewise $\pi$ can be sent into the database $D$ by instantiating each pair $(w_i, w'_i)$ either to $(0, 1)$ or to $(1, 0)$. Finally, each of the atoms $p_i(x, x')$ and $p_i(x', x)$ can be sent into $D$ by instantiating $(x, x')$ to either $(0, 1)$ or $(1, 0)$. In either case, the left-hand side of the EGD is satisfied but the right-hand side is not. Hence, there are three possibilities to eliminate this EGD-violation:

(D1) we can delete one of the atoms in $Q \cup A \cup E$,

(D2) we can delete both atoms $p_i(0, 1)$ and $p_i(1, 0)$ for some $i \in \{1, \ldots, k\}$, or

(D3) we can delete one of the atoms $p_i(0, 1), p_i(1, 0)$ for every $i \in \{1, \ldots, k\}$. 
Both (D1) as well as (D2) lead to a repair, as the subformulae \( \chi \) respectively \( \pi \) are always violated, so we need to consider (D3). The effect of retaining \( p_i(0,1) \) with \( i \in \{1,\ldots,k\} \) is that the variables \( (x_i, x'_i) \) in line (C2) are instantiated to \((0,1)\); likewise, the effect of retaining \( p_i(1,0) \) is that the the variables \( (x_i, x'_i) \) in line (C2) are instantiated to \((1,0)\). Hence, there is a one-to-one correspondence between the choices of retained ground atom \( p_i(,.) \) for every \( i \in \{1,\ldots,k\} \) and truth assignments to the \( x_i \)-variables in \( \psi \), namely retaining \( p_i(0,1) \) (resp. \( p_i(1,0) \)) corresponds to setting the propositional variable \( x_i \) in \( \psi \) to false (resp. true) and to set its dual \( \neg x_i \), which is encoded by \( x'_i \) according to our definition of \( \psi^* \), to true (resp. false). For a repair \( I \) that only uses deletions of the form (D3), denote by \( \rho(I) \) the corresponding truth assignment on the propositional variables \( \{x_1,\ldots,x_k\} \). We then define the relation \( \tau \) for the \( e \)-reduction as follows:

\[
\tau(\sigma(\psi(x)), I, \neg) = \begin{cases} 
\emptyset & : I \text{ is the result of a deletion of the form (D1) or (D2)}, \\
\rho(I) & : \text{otherwise}.
\end{cases}
\]

To show that \( \tau \) and \( \sigma \) indeed define a valid \( e \)-reduction from E-HSAT, first note that \( \tau(\sigma(\psi(x)), I, \neg) \) maps to the empty set at most \( k+1 \) times. So it suffices to show that the repairs we can get if we apply a deletion of the form (D3) are indeed models of \( \psi \). We distinguish two cases: First suppose that the EGD in line (C2) is violated by the instance resulting from the deletion of exactly one of \( p_i(0,1), p_i(1,0) \) for every \( i \in \{1,\ldots,k\} \). Then we have to delete at least one more atom from \( D \). This means that we either delete one atom from \( Q \cup A \cup E \) to falsify \( \chi \) (and possibly also further atoms in \( q(y_i,y'_i) \land \psi^* \) or we delete the remaining atom with leading symbol \( p_i \) to falsify atom \( p_i(x_i,x'_i) \) on the left-hand side of the EGD. In either case, the resulting database instance is not a repair due to the minimality condition.

It remains to consider the case that the EGD in line (C2) is satisfied by the instance \( I \) resulting from the deletion of exactly one of \( p_i(0,1), p_i(1,0) \) for every \( i \in \{1,\ldots,k\} \). Clearly, the equality on the right-hand side is always violated, no matter how we instantiate \( (x_1,x'_1) \), since the only options are \((0,1)\) and \((1,0)\). This means that the only way to satisfy the EGD in line (C2) is that there is no way to send the left-hand side into the database instance \( I \). This means that no matter how we instantiate \( (y_j,y'_j) \) – i.e., either to \((0,1)\) or to \((1,0)\) – there is always one atom in \( \psi^* \) that is instantiated to an \( a \)-atom outside \( I \), namely \( a(1,1,1) \). By our encoding of \( \psi \) by \( \psi^* \) this has the following meaning for the quantified boolean formula \( \psi \) no matter how we extend the truth assignment \( \rho(I) \) to an assignment on \( \{y_1,\ldots,y_{\ell}\} \) (corresponding to the possible ways of instantiating \( (y_j,y'_j) \) for every \( j \in \{1,\ldots,\ell\} \)), there is at least one implicant in \( \psi \) for which all three literals evaluate to true. In other words, there is a one-to-one correspondence between repairs obtained by deleting exactly one of \( p_i(0,1), p_i(1,0) \) for every \( i \in \{1,\ldots,k\} \) and models of \( \psi \).

\[\Box\]

7. Conclusion

We introduced a hierarchy of enumeration complexity classes, extending the well-known tractable enumeration classes \( \text{DelayP} \) and \( \text{IncP} \), just as the \( \Delta^P_l \)-classes of the polynomial-time hierarchy extend the class \( P \). We show that under reasonable complexity assumptions these hierarchies are strict. We introduced a declarative type of reduction among enumeration problems which is transitive and under which the classes in our hierarchies are closed. Moreover, they turn out to be special cases of the procedural-style \( D \)- and \( I \)-reductions that we defined afterwards. These procedural style reductions allow one to get completeness results for the classes of our hierarchy. The \( e \)-reductions are conceptually simpler and easier to obtain; so they will be used to obtain further hardness results for enumeration problems. In
this way we obtain completeness results for diverse problems such as generalized satisfiability in the Schaefer framework, circumscription, model-based diagnosis, abduction, and repairs.

Prior to our work, lower bounds for enumeration problems were only of the form “\( \text{Enum}_R \) is not in \( \text{DelayP} \) (or \( \text{IncP} \)) unless \( P = NP \).” We have provided here a framework which allows one to pinpoint the complexity of such problems in a better way in terms of completeness in our enumeration hierarchy.

As far as future work is concerned, of course, we want to close the gap left in Figure 1, i.e., clarifying the precise relationship between the enumeration complexity classes \( \text{Del}^+ \Sigma^P_k \) and \( \text{Del}^+ \Delta^{P+1}_k \). For a wider field of research activities in the area of hard enumeration problems, recall that our hierarchies of enumeration complexity classes build upon the tractable classes \( \text{DelayP} \) and \( \text{IncP} \). Note that in [1], yet another notion of tractable enumeration is proposed, namely the class \( \text{OutputP} \) (also referred to as \( \text{TotalP} \) in [2]). A problem is in \( \text{OutputP} \) if the time needed to output all solutions to a given instance is bounded by a polynomial in the combined size of the input plus the output. It is easy to show that this class is closed w.r.t. \( e \)-reductions and only minor modifications of the \( D \)- and \( I \)-reductions are needed to close \( \text{OutputP} \) also under these reductions. However, while we have seen quite a close relationship between \( \text{DelayP} \) and \( \text{IncP} \) in this work, the class \( \text{OutputP} \) seems to behave very differently from the others. For instance, \( \text{OutputP} \) seems to be incomparable (under common complexity-theoretic assumptions) with all higher classes from our hierarchies built upon \( \text{DelayP} \) or \( \text{IncP} \). Sure, we may also define a hierarchy of classes by allowing, for instance, a \( \Sigma^P_k \) oracle for some \( k \geq 1 \) on top of \( \text{OutputP} \). However, it is unclear what complete problems for any of the resulting classes would look like. A major research effort is required to even get a basic understanding of such classes.

Finally, a natural next research target is to put the machinery developed here to work. In Section 6 we have already proved a few completeness results for our new complexity classes. Many more intractable enumeration problems (especially in the AI and database domains) wait for a precise complexity classification. We believe that such work – complementing the search for restrictions to make intractable enumeration problems tractable, cf. [31, 29] – is needed to get a better understanding of the true sources of complexity of hard enumeration problems.
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