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Dejan Brkić, Pavel Praks

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Dejan Brkić, PhD, Researcher, IT4Innovations, VŠB - Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic and Research and Development Center "Alfatec", Bulevar Nikole Tesle, 18000 Niš, Serbia, dejanbrkic0611@gmail.com - ORCID id: 0000-0002-2502-0601

Pavel Praks, PhD, Researcher, IT4Innovations, VŠB - Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic, pavel.praks@vsb.cz - ORCID id: 0000-0002-3913-7800

The author of the discussed paper (Vatankhah and Kouchakzadeh 2008; 2009, Vatankhah 2014) has made significant contributions related to the Colebrook equation for flow friction (Colebrook 1939; Colebrook and White 1937); Eq. (1):

$$\frac{1}{\sqrt{f}} = -2 \cdot \log_{10}\left(\frac{2.51}{R} \cdot \frac{1}{\sqrt{f}} + \frac{\varepsilon}{3.71 \cdot D}\right) \qquad (1)$$

Where f is Darcy flow friction factor, R is the Reynolds number, and ε/D is the relative roughness of inner pipe surface (all three quantities are dimensionless).

Approximations based on genetic programming

It is underlined that it is important to develop not only accurate (Gregory and Fogarasi 1985, Brkić 2011a; 2012, Winning and Coole 2013) but also computational efficient approximations to the Colebrook equation (Clamond 2009, Giustolisi et al. 2011, Ćojbašić and Brkić 2013, Winning and Coole 2013, Brkić and Ćojbašić 2017). To be computational efficient the approximations need to contain the least possible number of logarithmic and non-integer power expressions (Brkić and Praks 2019). Knowing that appropriately trained artificial neural networks (Özger and Yıldırım 2009, Brkić and Ćojbašić 2016) can simulate the Colebrook equation accurately not knowing the structure of the equation but only knowing the input parameters which are the Reynolds number, R and the relative roughness, ε/D and the

corresponding output parameter which is the flow friction factor, f; {R, ε/D}→f, we tried to use that fact to extract the accurate and computational accurate explicit approximations using artificial intelligent techniques (Smidt and Lipson 2009, Dubčáková 2011), Eq. (2). We used Eureqa, artificial intelligence software tool based on genetic programming (Praks and Brkić 2018a). The symbolic regression approach adopted herein is based upon genetic programming wherein a population of functions is allowed to breed and mutate with the genetic propagation into subsequent generations based upon a survival-of-the-fittest criteria.

$$\frac{1}{\sqrt{f_0}} \approx 1.15 \cdot \alpha + 0.569 \cdot \beta + 0.292 \cdot \alpha \cdot \beta + 0.478 \cdot \sin(0.939 \cdot \alpha - \beta) + 0.122 \cdot \sin^2(0.939 \cdot \alpha - \beta) - 1.284 - 0.12 \cdot \alpha^2 - 0.162 \cdot \beta^2 \qquad (2)$$

Where $\alpha = \log_{10}(R)$; $\beta = -\log_{10}(\varepsilon/D)$.

It is revealed that the input parameters the Reynolds number, R and the relative roughness ε/D need to be normalized because their raw form where R>1000 and ε/D<<1 can cause a problem for the genetic programming tool to recognize the pattern and to estimate flow friction f accurately and we use $\alpha=\log_{10}(R)$ and $\beta=-\log_{10}(\varepsilon/D)$. The practical domain of the Reynolds number, R is between around 4000 to 10^8 which means it is expressed in relatively large numbers, while the relative roughness, ε/D is up to 0.05 which means that it is expressed in relatively small numbers. As explained, the different scale is a problem for the artificial intelligence to recognize pattern (Özger and Yıldırım 2009, Brkić and Ćojbašić 2016) where we discovered that the normalization in the form $\alpha=\log_{10}(R)$ and $\beta=-\log_{10}(\varepsilon/D)$ can overwhelm the problem. This means that Eq. (2) practically contain only two logarithmic expressions used only for normalization $\{\alpha=\log_{10}(R); \beta=-\log_{10}(\varepsilon/D)\}\rightarrow f$, and not a single non-integer power expression. In that way the estimated relative error of f_0 calculated through Eq. (2) compared with the accurate friction factor f; $(|f-f_0|/f)\cdot 100\%$ is less than 2%. On the other hand the approximation presented with Eq. (2) contains sinus trigonometric function which can introduce higher computational cost.

Introducing one additional logarithmic form, i.e. one more fixed-point iterative step; Eq. (3); Brkić (2017a), the error in this case can be reduced ten times, and it is up to 0.2%. The strategy with acceleration using one additional internal iterative step; two in total including initial starting iteration (Chen 1979, Shorle et al. 1980, Brkić 2011b), is widely used.

$$\frac{1}{\sqrt{f_{i+1}}} \approx -2 \cdot \log_{10}\left(\frac{2.51}{R} \cdot \frac{1}{\sqrt{f_i}} + \frac{\varepsilon}{3.71 \cdot D}\right) \qquad (3)$$

On the other hand, using the raw parameters without normalization; {R, ε/D}→f and with acceleration through Eq. (3) the maximal error of no more than 1.55% is introduced using only one logarithmic form; Eq. (4):

$$\frac{1}{\sqrt{f_1}} \approx -2 \cdot \log_{10}\left(\frac{2.51}{R} \cdot \left(5.05 - 30.73 \cdot \frac{\varepsilon}{\mathrm{D}} + \frac{3.4 \cdot R + \frac{R^2}{469647.7}}{46137.9 + \mathrm{R} + \frac{\mathrm{R}^2}{3250657.6} + \frac{\frac{\varepsilon}{\mathrm{D}} \cdot \mathrm{R}^2}{515.25}}\right) + \frac{\varepsilon}{3.71 \cdot D}\right) \qquad (4)$$

The accelerative procedure through Eq. (3) is very fast where for Eq. (4) the relative error decreases as follow: $f_0 \rightarrow 42.7\%$, $f_1 \rightarrow 1.55\%$, $f_2 \rightarrow 0.22\%$, $f_3 \rightarrow 0.036\%$, etc. The expression for f_0 is polynomial and hence very simple, but with the relative error that can reach up to 42.7% not sufficiently accurate. On the other hand, with two logarithmic expressions, the relative error f_2 introduced with Eq. (4) is about 0.22% which is almost the same as for f_1 introduced with Eq. (2) that requires three logarithmic expressions in total (two for normalization and one for acceleration).

Three-point methods

Approximations with virtually three iterative steps are also available (Zigrang and Sylvester 1982, Serghides 1984). Using only function evaluations and one evaluation of the first derivative (Sharma and Arora 2016), the Colebrook equation can be solved practically with the neglected error within one iteration step; Eq. (5), (Džunić et al. 2011). Through the three-point iterative procedure the friction factor is evaluated in points x_0, y_0 and z_0 where the optimal starting point x_0 for the whole domain of the

Colebrook equation is x_0=7.273124147 which is chosen after numerous tests where $x_0 = \frac{1}{\sqrt{f_0}}$. The final solution is at the point x_1; Eq. (5) where $f_1 = \frac{1}{\sqrt{x_1}}$; (Praks and Brkić 2018bc):

$$\left.\begin{array}{c} \frac{1}{\sqrt{f_0}} = x_0 = 7.273124147 \\ F(x_0) = x_0 + 2 \cdot \log_{10}\left(\frac{2.51 \cdot x_0}{R} + \frac{\varepsilon}{3.71 \cdot D}\right) \\ F'(x_0) = \frac{186242}{\ln(10) \cdot \left(93121 \cdot x_0 + 10000 \cdot \frac{\varepsilon}{D} \cdot R\right)} + 1 = \frac{251}{50 \cdot R \cdot \ln(10) \cdot \left(\frac{100}{371} \cdot \frac{\varepsilon}{D} + \frac{2.51 \cdot x_0}{R}\right)} + 1 \\ y_0 = x_0 - \frac{F(x_0)}{F'(x_0)} \\ F(y_0) = y_0 + 2 \cdot \log_{10}\left(\frac{2.51 \cdot y_0}{R} + \frac{\varepsilon}{3.71 \cdot D}\right) \\ z_0 = y_0 - \frac{F(x_0)}{F(x_0) - 2 \cdot F(y_0)} \cdot \frac{F(y_0)}{F'(x_0)} \\ F(z_0) = z_0 + 2 \cdot \log_{10}\left(\frac{2.51 \cdot z_0}{R} + \frac{\varepsilon}{3.71 \cdot D}\right) \\ x_1 = z_0 - \frac{F(z_0)}{F'(x_0) \cdot \left[1 - 2 \cdot \frac{F(y_0)}{F(x_0)} - \left(\frac{F(y_0)}{F(x_0)}\right)^2\right] \cdot \left[1 - \frac{F(z_0)}{F(y_0)}\right] \cdot \left[1 - 2 \cdot \frac{F(z_0)}{F(x_0)}\right]} \\ x_1 = \frac{1}{\sqrt{f_1}} \rightarrow f \end{array}\right\} \quad (5)$$

In Eq. (5) is used the first derivative F'(x_0) of the Colebrook function in point x_0 in respect to parameter x, where the Colebrook equation is expressed in the suitable form to start the Newton-Raphson iterative procedure; Eq. (6):

$$F = \frac{1}{\sqrt{f}} + 2 \cdot \log_{10}\left(\frac{2.51}{R} \cdot \frac{1}{\sqrt{f}} + \frac{\varepsilon}{3.71 \cdot D}\right) = 0 \quad (6)$$

The corresponding VPA (Variable precision arithmetic) approximations of the first derivative F'(x) with 16 digits precision expression is =2.180158299154324/(R·(0.2695417789757412·ε/D+(2.51·x)/R))+1.0= 80883.87289862543/(93121.0·x+10000.0·R·ε/D)+1.0 where $x = \frac{1}{\sqrt{f}}$

Optimization problem

As noted in the discussed paper, the first derivative of the Colebrook equation expressed in the form suitable for the Newton-Raphson procedure in respect to the Reynolds number, R is used for the

optimization problem in pipe network simulations (Brkić 2016). For the purpose of this discussion we generated in MATLAB the appropriate first derivative F'(R); Eq. (7):

$$F'(R) = -\frac{251\cdot\frac{1}{\sqrt{f}}}{50\cdot R^2\cdot ln(10)\cdot\left(\frac{100}{371}\cdot\frac{\varepsilon}{D}+\frac{2.51}{R}\cdot\frac{1}{\sqrt{f}}\right)} = -\frac{186242\cdot\frac{1}{\sqrt{f}}}{R\cdot ln(10)\cdot\left(93121\cdot\frac{1}{\sqrt{f}}+10000\cdot R\cdot\frac{\varepsilon}{D}\right)} \quad (7)$$

The corresponding VPA (Variable precision arithmetic) approximations of the first derivative F'(R) with 16 digits precision is =-(186242.0·x)/(23025.85092994046·R^2·ε/D +214419.0264446985·x·R)= -(80883.87289862543·x)/(R·(93121.0·x+10000.0·R·ε/D)) where $x = \frac{1}{\sqrt{f}}$

Padé polynomials

In the discussed paper, the improved versions of the approximations proposed by Sonnad and Goudar (2006), Mikata and Walczak (2016), Brkić (2017b), Vatankhah and Kouchakzadeh (2008, 2009), and Vatankhah (2014) are shown. In addition, the second logarithmic form in Eq. (12) of the discussed paper can be simplified using Padé approximation (Baker and Graves-Morris 1996); ln(1-a·δ/d)=ln(1-θ)≈p/q where p and q are defined by Eq. (8):

$$\left.\begin{aligned} p &\approx \theta\cdot(-0.18333\cdot\theta^2+\theta-1) \\ q &\approx -0.05\cdot\theta^3 + 0.6\cdot\theta^2 - 1.5\cdot\theta + 1 \end{aligned}\right\} \quad (8)$$

For the domain of applicability of the Colebrook equation that is 0<ε/D<0.1 and 4000<R<10^8; θ=a·δ/d is always between 0 and 0.22; 0<θ<0.22, the relative percentage error of the Padé approximation p/q compared with the accurate ln(1-a·δ/d) is between $3.5\cdot10^{-8}$% and $2\cdot10^{-5}$%. In this case the Padé approximation is very accurate, and also computationally cheap compared with logarithmic function: it is a division of two polynomials of degree of 3; we can call Padé approximation (3/3) because of two polynomials of degree of 3 used (Praks and Brkić 2018d).

Further to start Newton-Raphson procedure $\delta_{i+1}=\delta_i-\zeta(\delta_i)/\zeta'(\delta_i)$ starting from i=0, in order to evaluate δ, Eq. (12) of the discussed paper need to be rearranged where ζ(δ)=0, and where ζ is functional symbol; Eq. (9):

$$\zeta(\delta) = \ln(d) + \ln\left(1 - \frac{a \cdot \delta}{d}\right) - \delta \approx \ln(d) + \frac{\theta \cdot (-0.18333 \cdot \theta^2 + \theta - 1)}{-0.05 \cdot \theta^3 + 0.6 \cdot \theta^2 - 1.5 \cdot \theta + 1} - \delta = 0 \quad (9)$$

δ is the unknown parameter to be solved, a is constant a=2/ln(10)=0.8686 and d is parameter d=c/b-a·ln(b); where b=2.51/R and c=(ε/D)/3.71.

Also, the Newton-Raphson procedure requires the first derivative ζ'(δ)=-a/(d-a·δ)-1, which is computationally cheap. Because it is equally cheap but slightly more accurate, we use the first derivative of ζ(δ)=ln(d)+ln(1-a·δ/d)-δ=0 and not the derivative of its Padé version. Knowing that δ ranges between 1.61and 13.91, the starting point δ_0 for the Newton-Raphson procedure should be selected from that domain; $\delta_{i+1}=\delta_i-\zeta(\delta_i)/\zeta'(\delta_i)$ starting from i=0. The Newton-Raphson is faster than the simple fixed-point iterative procedure (Brkić 2017a), so hopefully Eq. (9) should be evaluated only few times; where the stopping criteria is $\delta_{i+1}-\delta_i \approx 0$.

Disclaimer: The views expressed are those of the authors and may not in any circumstances be regarded as stating an official position of the Alfatec or of the IT4Innovations, VŠB - Technical University Ostrava. Both authors contributed equally to this discussion.

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Notation

The following symbols are used in this discussion:

a = constant, a=2/ln(10)=0.8686;

b = parameter related to Reynolds number (dimensionless);

c = parameter related to ε and D (dimensionless);

D = inner diameter of pipe (length; the same unit as for D);

d = parameter related to a, b and c (dimensionless);

f = Darcy friction factor implicitly related to R, ε and D (dimensionless);

R = Reynolds number (dimensionless);

p and q = Padé polynomials related to θ;

x = parameter related to friction factor f (dimensionless);

y = parameter related to x (dimensionless);

z = parameter related to y (dimensionless);

α= parameter related to the Reynolds number R (dimensionless);

β= parameter related to ε and D (dimensionless);

δ = unknown parameter implicitly related to a and d;

ε = average height of protrusion of inner pipe surface (length; the same unit as for D);

θ = parameter related to a, δ and d

functional symbols:

F – functional symbol related to the Colebrook equation

ln - Napierian natural logarithm

log – Briggsian logarithm

sin – sine trigonometric function

‘ – represents first derivative of function

ζ - functional symbol related to parameter δ

Indexes:

i – counter

0 – refers to first iteration

1 – refers to second iteration

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