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Solution Operator for Non-Autonomous Perturbation of Gibbs Semigroup

To the memory of Hagen Neidhardt

Valentin A. Zagrebnov

Aix-Marseille Université, CNRS, Centrale Marseille, I2M

Institut de Mathématiques de Marseille (UMR 7373)

CMI - Technopôle Château-Gombert

39 rue F. Joliot Curie, 13453 Marseille, France

Abstract

The paper is devoted to a linear dynamics for non-autonomous perturbation of the Gibbs semigroup on a separable Hilbert space. It is shown that evolution family $\{U(t, s)\}_{0 \leq s \leq t}$ solving the non-autonomous Cauchy problem can be approximated in the *trace-norm* topology by product formulae. The rate of convergence of product formulae approximants $\{U_n(t, s)\}_{\{0 \leq s \leq t, n \geq 1\}}$ to the solution operator $\{U(t, s)\}_{\{0 \leq s \leq t\}}$ is also established.

1 Introduction and main result

The aim of the paper is two-fold. Firstly, we study a linear dynamics, which is a non-autonomous perturbation of Gibbs semigroup. Secondly, we prove *product formulae* approximations of the corresponding to this dynamics *solution operator* $\{U(t, s)\}_{\{0 \leq s \leq t\}}$, known also as *evolution family*, *fundamental solution*, or *propagator*, see [1] Ch. VI, Sec.9.

To this end we consider on separable Hilbert space \mathfrak{H} a linear non-autonomous dynamics given by evolution equation of the type:

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= -C(t)u(t), \quad u(s) = u_s, \quad s \in [0, T) \subset \mathbb{R}_0^+, \quad t \in \mathcal{I} := [0, T], \quad (1.1) \\ C(t) &:= A + B(t), \quad u_s \in \mathfrak{H}, \end{aligned}$$

where $\mathbb{R}_0^+ = \{0\} \cup \mathbb{R}^+$ and linear operator A is generator of a Gibbs semigroup. Note that for the autonomous Cauchy problem (ACP) (1.1), when $B(t) = B$, the outlined

programme corresponds to the Trotter product formula approximation of the Gibbs semigroup generated by a closure of operator $A + B$, [20] Ch.5.

The main result of the present paper concerns the non-autonomous Cauchy problem (nACP) (1.1) under the following

Assumptions:

(A1) The operator $A \geq \mathbb{1}$ in a separable Hilbert space \mathfrak{H} is self-adjoint. The family $\{B(t)\}_{t \in \mathcal{I}}$ of non-negative self-adjoint operators in \mathfrak{H} is such that the bounded operator-valued function $(\mathbb{1} + B(\cdot))^{-1} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is strongly measurable.

(A2) There exists $\alpha \in [0, 1)$ such that inclusion: $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$, holds for a.e. $t \in \mathcal{I}$. Moreover, the function $B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is strongly measurable and essentially bounded in the operator norm:

$$C_\alpha := \text{ess sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha}\| < \infty. \quad (1.2)$$

(A3) The map $A^{-\alpha}B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is Hölder continuous in the operator norm: for some $\beta \in (0, 1]$ there is a constant $L_{\alpha, \beta} > 0$ such that one has estimate

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq L_{\alpha, \beta} |t - s|^\beta, \quad (t, s) \in \mathcal{I} \times \mathcal{I}. \quad (1.3)$$

(A4) The operator A is generator of the Gibbs semigroup $\{G(t) = e^{-tA}\}_{t \geq 0}$, that is, a strongly continuous semigroup such that $G(t)|_{t > 0} \in \mathcal{C}_1(\mathfrak{H})$. Here $\mathcal{C}_1(\mathfrak{H})$ denotes the *-ideal of trace-class operators in C^* -algebra $\mathcal{L}(\mathfrak{H})$ of bounded operators on \mathfrak{H} .

Remark 1.1 Assumptions (A1)-(A3) are introduced in [4] to prove the operator-norm convergence of product formula approximants $\{U_n(t, s)\}_{0 \leq s \leq t}$ to solution operator $\{U(t, s)\}_{0 \leq s \leq t}$. Then they were widely used for product formula approximations in [10]-[15] in the context of the evolution semigroup approach to the nACP, see [6]-[9].

Remark 1.2 The following main facts were established (see, e.g., [4, 7, 18, 19]) about the nACP for perturbed evolution equation of the type (1.1):

(a) By assumptions (A1)-(A2) the operators $\{C(t) = A + B(t)\}_{t \in \mathcal{I}}$ have a common $\text{dom}(C(t)) = \text{dom}(A)$ and they are generators of contraction holomorphic semigroups. Hence, the nACP (1.1) is of parabolic type [5, 16].

(b) Since domains $\text{dom}(C(t)) = \text{dom}(A)$, $t \geq 0$, are dense, the nACP is well-posed with time-independent regularity subspace $\text{dom}(A)$.

(c) Assumptions (A1)-(A3) provide the existence of evolution family solving nACP (1.1) which we call the solution operator. It is a strongly continuous, uniformly bounded family of operators $\{U(t, s)\}_{(t, s) \in \Delta}$, $\Delta := \{(t, s) \in \mathcal{I} \times \mathcal{I} : 0 \leq s \leq t \leq T\}$, such that the conditions

$$\begin{aligned} U(t, t) &= \mathbb{1} \quad \text{for } t \in \mathcal{I}, \\ U(t, r)U(r, s) &= U(t, s), \quad \text{for } t, r, s \in \mathcal{I} \quad \text{for } s \leq r \leq t, \end{aligned} \quad (1.4)$$

are satisfied and $u(t) = U(t, s)u_s$ for any $u_s \in \mathfrak{H}_s$ is in a certain sense (e.g., classical, strict, mild) solution of the nACP (1.1).

(d) Here $\mathfrak{H}_s \subseteq \mathfrak{H}$ is an appropriate regularity subspace of initial data. Assumptions (A1)-(A3) provide $\mathfrak{H}_s = \text{dom}(A)$ and $U(t, s)\mathfrak{H} \subseteq \text{dom}(A)$ for $t > s$.

In the present paper we essentially focus on convergence of the product *approximants* $\{U_n(t, s)\}_{(t, s) \in \Delta, n \geq 1}$ to solution operator $\{U(t, s)\}_{(t, s) \in \Delta}$. Let

$$s = t_1 < t_2 < \dots < t_{n-1} < t_n < t, \quad t_k := s + (k-1)\frac{t-s}{n}, \quad (1.5)$$

for $k \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, be partition of the interval $[s, t]$. Then corresponding approximants may be defined as follows:

$$\begin{aligned} W_k^{(n)}(t, s) &:= e^{-\frac{t-s}{n}A} e^{-\frac{t-s}{n}B(t_k)}, \quad k = 1, 2, \dots, n, \\ U_n(t, s) &:= W_n^{(n)}(t, s)W_{n-1}^{(n)}(t, s) \times \dots \times W_2^{(n)}(t, s)W_1^{(n)}(t, s). \end{aligned} \quad (1.6)$$

It turns out that if the assumptions (A1)-(A3), *adapted* to a Banach space \mathfrak{X} , are satisfied for $\alpha \in (0, 1)$, $\beta \in (0, 1)$ and in addition the condition $\alpha < \beta$ holds, then solution operator $\{U(t, s)\}_{(t, s) \in \Delta}$ admits the operator-norm approximation

$$\text{ess sup}_{(t, s) \in \Delta} \|U_n(t, s) - U(t, s)\| \leq \frac{R_{\beta, \alpha}}{n^{\beta - \alpha}}, \quad n \in \mathbb{N}, \quad (1.7)$$

for some constant $R_{\beta, \alpha} > 0$. This result shows that convergence of the approximants $\{U_n(t, s)\}_{(t, s) \in \Delta, n \geq 1}$ is determined by the smoothness of the perturbation $B(\cdot)$ in (A3) and by the parameter of inclusion in (A2), see [13].

The Lipschitz case $\beta = 1$ was considered for \mathfrak{X} in [10]. There it was shown that if $\alpha \in (1/2, 1)$, then one gets estimate

$$\text{ess sup}_{t \in \mathcal{I}} \|U_n(t, s) - U(t, s)\| \leq \frac{R_{1, \alpha}}{n^{1 - \alpha}}, \quad n = 2, 3, \dots \quad (1.8)$$

For the Lipschitz case in a Hilbert space \mathfrak{H} the assumptions (A1)-(A3) yield a stronger result [4]:

$$\text{ess sup}_{(t, s) \in \Delta} \|U_n(t, s) - U(t, s)\| \leq R \frac{\log(n)}{n}, \quad n = 2, 3, \dots \quad (1.9)$$

Note that actually it is the best of known estimates for operator-norm rates of convergence under conditions (A1)-(A3).

The estimate (1.7) was improved in [12] for $\alpha \in (1/2, 1)$ in a Hilbert space using the *evolution semigroup* approach [2, 3, 9]. This approach is quite different from technique used for (1.9) in [4], but it is the same as that employed in [10].

Proposition 1.3 [12] *Let assumptions (A1)-(A3) be satisfied for $\beta \in (0, 1)$. If $\beta > 2\alpha - 1 > 0$, then estimate*

$$\text{ess sup}_{(t, s) \in \Delta} \|U_n(t, s) - U(t, s)\| \leq \frac{R_{\beta}}{n^{\beta}}, \quad (1.10)$$

holds for $n \in \mathbb{N}$ and for some constant $R_\beta > 0$.

Note that the condition $\beta > 2\alpha - 1$ is weaker than $\beta > \alpha$ (1.7), but it does not cover the Lipschitz case (1.8) because of condition $\beta < 1$.

The main result of the present paper is the lifting of *any* known operator-norm bounds (1.7)-(1.10) (we denote them by $R_{\alpha,\beta}\varepsilon_{\alpha,\beta}(n)$) to estimate in the trace-norm topology $\|\cdot\|_1$. This is a subtle matter even for ACP, see [20] Ch.5.4.

- The first step is the construction for nACP (1.1) a *trace-norm* continuous solution operator $\{U(t,s)\}_{(t,s) \in \Delta}$, see Theorem 2.2 and Corollary 2.3.

- Then in Section 3 for assumptions (A1)-(A4) we prove (Theorem 1.4) the corresponding *trace-norm* estimate $R_{\alpha,\beta}(t,s)\varepsilon_{\alpha,\beta}(n)$ for difference $\|U_n(t,s) - U(t,s)\|_1$.

Theorem 1.4 *Let assumptions (A1)-(A4) be satisfied. Then the estimate*

$$\|U_n(t,s) - U(t,s)\|_1 \leq R_{\alpha,\beta}(t,s)\varepsilon_{\alpha,\beta}(n), \quad (1.11)$$

holds for $n \in \mathbb{N}$ and $0 \leq s < t \leq T$ for some $R_{\alpha,\beta}(t,s) > 0$.

2 Preliminaries

Besides Remark 1.2(a)-(d) we also recall the following assertion, see, e.g., [16], Theorem 1, [17], Theorem 5.2.1.

Proposition 2.1 *Let assumptions (A1)-(A3) be satisfied.*

(a) *Then solution operator $\{U(t,s)\}_{(t,s) \in \Delta}$ is strongly continuously differentiable for $0 \leq s < t \leq T$ and*

$$\partial_t U(t,s) = -(A + B(t))U(t,s). \quad (2.1)$$

(b) *Moreover, the unique function $t \mapsto u(t) = U(t,s)u_s$ is a classical solution of (1.1) for initial data $\mathfrak{H}_s = \text{dom}(A)$.*

Note that solution of (1.1) is called *classical* if $u(t) \in C([0,T], \mathfrak{H}) \cap C^1([0,T], \mathfrak{H})$, $u(t) \in \text{dom}(C(t))$, $u(s) = u_s$, and $C(t)u(t) \in C([0,T], \mathfrak{H})$ for all $t \geq s$, with convention that $(\partial_t u)(s)$ is the right-derivative, see, e.g., [16], Theorem 1, or [1], Ch.VI.9.

Since the involved into (A1), (A2) operators are non-negative and self-adjoint, equation (2.1) implies that the solution operator consists of *contractions*:

$$\partial_t \|U(t,s)u\|^2 = -2(C(t)U(t,s)u, U(t,s)u) \leq 0, \quad \text{for } u \in \mathfrak{H}. \quad (2.2)$$

By (A1) $G(t) = e^{-tA} : \mathfrak{H} \rightarrow \text{dom}(A)$. Applying to (2.1) the *variation of constants* argument we obtain for $U(t,s)$ the integral equation :

$$U(t,s) = G(t-s) - \int_s^t d\tau G(t-\tau)B(\tau)U(\tau,s), \quad U(s,s) = \mathbb{1}. \quad (2.3)$$

Hence evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$, which is defined by equation (2.3), can be considered as a *mild* solution of nACP (2.1) for $0 \leq s \leq t \leq T$ in the Banach space $\mathcal{L}(\mathfrak{H})$ of bounded operators, cf. [1], Ch.VI.7.

Note that assumptions (A1)-(A4) yield for $0 \leq s < t \leq T$, $\tau \in (s,t)$ and for the closure $\overline{A^{-\alpha}B(\tau)}$:

$$\|G(t-s)A^\alpha\|_1 \leq \frac{M_\alpha}{(t-\tau)^\alpha} \quad \text{and} \quad \|\overline{A^{-\alpha}B(\tau)}\| \leq C_\alpha. \quad (2.4)$$

Then (2.2), (2.4) give the trace-norm estimate

$$\left\| \int_s^t d\tau G(t-\tau)B(\tau)U(\tau,s) \right\|_1 \leq \frac{M_\alpha C_\alpha}{1-\alpha} (t-s)^{1-\alpha}, \quad (2.5)$$

and by (2.3) we ascertain that $\{U(t,s)\}_{(t,s)\in\Delta} \in \mathcal{C}_1(\mathfrak{H})$ for $t > s$.

Therefore, we can construct solution operator $\{U(t,s)\}_{(t,s)\in\Delta}$ as a *trace-norm* convergent Dyson-Phillips series $\sum_{n=0}^\infty S_n(t,s)$ by iteration of the integral formula (2.3) for $t > s$. To this aim we define the recurrence relation

$$\begin{aligned} S_0(t,s) &= U_A(t-s), \\ S_n(t,s) &= - \int_s^t ds G(t-\tau)B(\tau)S_{n-1}(\tau,s), \quad n \geq 1. \end{aligned} \quad (2.6)$$

Since in (2.6) the operators $S_{n \geq 1}(t,s)$ are the n -fold trace-norm convergent Bochner integrals

$$\begin{aligned} S_n(t,s) &= \int_s^t d\tau_1 \int_s^{\tau_1} d\tau_2 \dots \int_s^{\tau_{n-1}} d\tau_n \\ &G(t-\tau_1)(-B(\tau_1))G(\tau_1-\tau_2) \dots G(\tau_{n-1}-\tau_n)(-B(\tau_n))G(\tau_n-s), \end{aligned} \quad (2.7)$$

by contraction property (2.2) and by estimate (2.5) there exist $0 \leq s \leq t$ such that $M_\alpha C_\alpha (t-s)^{1-\alpha} / (1-\alpha) =: \xi < 1$ and

$$\|S_n(t,s)\|_1 \leq \xi^n, \quad n \geq 1. \quad (2.8)$$

Consequently $\sum_{n=0}^\infty S_n(t,s)$ converges for $t > s$ in the trace-norm and satisfies the integral equation (2.3). Thus we get for solution operator of nACP the representation

$$U(t,s) = \sum_{n=0}^\infty S_n(t,s). \quad (2.9)$$

This result can be extended to any $0 \leq s < t \leq T$ using (1.4).

We note that for $s \leq t$ the above arguments yield the proof of assertions in the next Theorem 2.2 and Corollary 2.3, but only in the *strong* ([19] Proposition 3.1, main Theorem in [18]) and in the *operator-norm* topology, [4] Lemma 2.1. While for $t > s$ these arguments prove a generalisation of Theorem 2.2 and Corollary 2.3 to the *trace-norm* topology in Banach space $\mathcal{C}_1(\mathfrak{H})$:

Theorem 2.2 *Let assumptions (A1)-(A4) be satisfied. Then evolution family $\{U(t,s)\}_{(t,s) \in \Delta}$ (2.9) gives for $t > s$ a mild trace-norm continuous solution of nACP (2.1) in Banach space $\mathcal{C}_1(\mathfrak{H})$.*

Corollary 2.3 *For $t > s$ the evolution family $\{U(t,s)\}_{(t,s) \in \Delta}$ (2.9) is a strict solution of the nACP :*

$$\begin{aligned} \partial_t U(t,s) &= -C(t)U(t,s), \quad t \in (s,T) \quad \text{and} \quad U(s,s) = \mathbb{1}, \\ C(t) &:= A + B(t), \end{aligned} \quad (s,T) \subset [0,T], \quad (2.10)$$

in Banach space $\mathcal{C}_1(\mathfrak{H})$.

Proof. Since by Remark 1.2(c),(d) the function $t \mapsto U(t,s)$ for $t \geq s$ is strongly continuous and since $U(t,s) \in \mathcal{C}_1(\mathfrak{H})$ for $t > s$, the product $U(t+\delta,t)U(t,s)$ is continuous in the trace-norm topology for $|\delta| < t-s$. Moreover, since $\{u(t)\}_{s \leq t \leq T}$ is a classical solution of nACP (1.1), equation (2.1) implies that $U(t,s)$ has strong derivative for any $t > s$. Then again by Remark 1.2(d) the trace-norm continuity of $\delta \mapsto U(t+\delta,t)U(t,s)$ and by inclusion of ranges: $\text{ran}(U(t,s)) \subseteq \text{dom}(A)$ for $t > s$, the trace-norm derivative $\partial_t U(t,s)$ at $t(>s)$ exists and belongs to $\mathcal{C}_1(\mathfrak{H})$.

Therefore, $U(t,s) \in C((s,T], \mathcal{C}_1(\mathfrak{H})) \cap C^1((s,T], \mathcal{C}_1(\mathfrak{H}))$ with $U(s,s) = \mathbb{1}$ and $U(t,s) \in \mathcal{C}_1(\mathfrak{H})$, $C(t)U(t,s) \in \mathcal{C}_1(\mathfrak{H})$ for $t > s$, which means that solution $U(t,s)$ of (2.10) is *strict*, cf. [19] Definition 1.1. \square

We note that these results for ACP in Banach space $\mathcal{C}_1(\mathfrak{H})$ are well-known for Gibbs semigroups, see [20], Chapter 4.

Now, to proceed with the proof of Theorem 1.4 about trace-norm convergence of the solution operator approximants (1.6) we need the following preparatory lemma.

Lemma 2.4 *Let self-adjoint positive operator A be such that $e^{-tA} \in \mathcal{C}_1(\mathfrak{H})$ for $t > 0$, and let V_1, V_2, \dots, V_n be bounded operators $\mathcal{L}(\mathfrak{H})$. Then*

$$\left\| \prod_{j=1}^n V_j e^{-t_j A} \right\|_1 \leq \prod_{j=1}^n \|V_j\| \|e^{-(t_1+t_2+\dots+t_n)A/4}\|_1, \quad (2.11)$$

for any set $\{t_1, t_2, \dots, t_n\}$ of positive numbers.

Proof. At first we prove this assertion for *compact* operators: $V_j \in \mathcal{C}_\infty(\mathfrak{H})$, $j = 1, 2, \dots, n$. Let $t_m := \min\{t_j\}_{j=1}^n > 0$ and $T := \sum_{j=1}^n t_j > 0$. For any $1 \leq j \leq n$, we define an integer $\ell_j \in \mathbb{N}$ by

$$2^{\ell_j} t_m \leq t_j \leq 2^{\ell_j+1} t_m.$$

Then we get $\sum_{j=1}^n 2^{\ell_j} t_m > T/2$ and

$$\prod_{j=1}^n V_j e^{-t_j A} = \prod_{j=1}^n V_j e^{-(t_j - 2^{\ell_j} t_m)A} (e^{-2^{\ell_j} t_m A})^{\ell_j}. \quad (2.12)$$

By the definition of the $\|\cdot\|_1$ -norm and by inequalities for singular values $\{s_k(\cdot)\}_{k \geq 1}$ of compact operators

$$\begin{aligned} \left\| \prod_{j=1}^n V_j e^{-t_j A} \right\|_1 &= \sum_{k=1}^{\infty} s_k \left(\prod_{j=1}^n V_j e^{-(t_j - 2^{\ell_j} t_m) A} (e^{-t_m A})^{2^{\ell_j}} \right) \\ &\leq \sum_{k=1}^{\infty} \prod_{j=1}^n s_k \left(e^{-(t_j - 2^{\ell_j} t_m) A} \right) [s_k(e^{-t_m A})]^{2^{\ell_j}} s_k(V_j) \\ &\leq \sum_{k=1}^{\infty} s_k(e^{-t_m A})^{\sum_{j=1}^n 2^{\ell_j}} \prod_{j=1}^n \|V_j\|. \end{aligned} \quad (2.13)$$

Here we used that $s_k(e^{-(t_j - 2^{\ell_j} t_m) A}) \leq \|e^{-(t_j - 2^{\ell_j} t_m) A}\| \leq 1$ and that $s_k(V_j) \leq \|V_j\|$. Let $N := \sum_{j=1}^n 2^{\ell_j}$ and $T_m := N t_m > T/2$. Since

$$\sum_{k=1}^{\infty} s_k(e^{-tA/q})^q = (\|e^{-tA/q}\|_q)^q,$$

the inequality (2.13) yields for $q = N$:

$$\left\| \prod_{j=1}^n V_j e^{-t_j A} \right\|_1 \leq \left(\|e^{-T_m A/N}\|_N \right)^N \prod_{j=1}^n \|V_j\|. \quad (2.14)$$

Now we consider an integer $p \in \mathbb{N}$ such that $2^p \leq N < 2^{p+1}$. It then follows that $T/4 < T_m/2 < 2^p T_m/N$, and hence we obtain

$$\begin{aligned} \left(\|e^{-T_m A/N}\|_{q=N} \right)^N &= \sum_{k=1}^{\infty} s_k^N(e^{-T_m A/N}) \\ &\leq \sum_{k=1}^{\infty} s_k^{2^p}(e^{-2^p T_m A/2^p N}) \leq \sum_{k=1}^{\infty} s_k^{2^p}(e^{-TA/2^{p+2}}) = \|e^{-TA/2^2}\|_1, \end{aligned} \quad (2.15)$$

where we used that $s_k(e^{-T_m A/N}) = s_k(e^{-2^p T_m A/2^p N}) \leq \|e^{-T_m A/N}\| \leq 1$, and that $s_k(e^{-(t+\tau)A}) \leq \|e^{-tA}\| s_k(e^{-\tau A}) \leq s_k(e^{-\tau A})$ for any $t, \tau > 0$. Therefore, the estimates (2.14), (2.15) give the bound (2.11).

Now, let $V_j \in \mathcal{L}(\mathfrak{H})$, $j = 1, 2, \dots, n$, and set $\tilde{V}_j := V_j e^{-\varepsilon A}$ for $0 < \varepsilon < t_m$. Hence, $\tilde{V}_j \in \mathcal{C}_1(\mathfrak{H}) \subset \mathcal{C}_{\infty}(\mathfrak{H})$ and $s_k(\tilde{V}_j) \leq \|\tilde{V}_j\| \leq \|V_j\|$. If we set $\tilde{t}_j := t_j - \varepsilon$, then

$$\left\| \prod_{j=1}^n V_j e^{-t_j A} \right\|_1 \leq \prod_{j=1}^n \|V_j\| \|e^{-(\tilde{t}_1 + \tilde{t}_2 + \dots + \tilde{t}_n) A/4}\|_1. \quad (2.16)$$

Since the semigroup $\{e^{-tA}\}_{t \geq 0}$ is $\|\cdot\|_1$ -continuous for $t > 0$, we can take in (2.16) the limit $\varepsilon \downarrow 0$. This gives the result (2.11) in general case. \square

3 Proof of Theorem 1.4

We follow the line of reasoning of the *lifting* lemma developed in [20], Ch.5.4.1.

1. By virtue of (1.4) and (1.6) we obtain for difference in (1.11) formula:

$$U_n(t, s) - U(t, s) = \prod_{k=n}^1 W_k^{(n)}(t, s) - \prod_{l=n}^1 U(t_{l+1}, t_l). \quad (3.1)$$

Let integer $k_n \in (1, n)$. Then (3.1) yields the representation:

$$\begin{aligned} U_n(t, s) - U(t, s) &= \left(\prod_{k=n}^{k_n+1} W_k^{(n)}(t, s) - \prod_{l=n}^{k_n+1} U(t_{l+1}, t_l) \right) \prod_{k=k_n}^1 W_k^{(n)}(t, s) \\ &+ \prod_{l=n}^{k_n+1} U(t_{l+1}, t_l) \left(\prod_{k=k_n}^1 W_k^{(n)}(t, s) - \prod_{l=k_n}^1 U(t_{l+1}, t_l) \right), \end{aligned}$$

which implies the trace-norm estimate

$$\begin{aligned} \|U_n(t, s) - U(t, s)\|_1 &\leq \left\| \prod_{k=n}^{k_n+1} W_k^{(n)}(t, s) - \prod_{l=n}^{k_n+1} U(t_{l+1}, t_l) \right\|_1 \left\| \prod_{k=k_n}^1 W_k^{(n)}(t, s) \right\|_1 \\ &+ \left\| \prod_{l=n}^{k_n+1} U(t_{l+1}, t_l) \right\|_1 \left\| \prod_{k=k_n}^1 W_k^{(n)}(t, s) - \prod_{l=k_n}^1 U(t_{l+1}, t_l) \right\|_1. \end{aligned} \quad (3.2)$$

2. Now we assume that $\lim_{n \rightarrow \infty} k_n/n = 1/2$. Then (1.5) yields $\lim_{n \rightarrow \infty} t_{k_n} = (t + s)/2$, $\lim_{n \rightarrow \infty} t_n = t$ and uniform estimates (1.7)-(1.10) with the bound $R_{\alpha, \beta} \varepsilon_{\alpha, \beta}(n)$ imply

$$\operatorname{ess\,sup}_{(t, s) \in \Delta} \left\| \prod_{k=n}^{k_n+1} W_k^{(n)}(t, s) - U(t, (t+s)/2) \right\| \leq R_{\alpha, \beta}^{(1)} \varepsilon_{\alpha, \beta}(n), \quad (3.3)$$

$$\operatorname{ess\,sup}_{(t, s) \in \Delta} \left\| \prod_{k=k_n}^1 W_k^{(n)}(t, s) - U((t+s)/2, s) \right\| \leq R_{\alpha, \beta}^{(2)} \varepsilon_{\alpha, \beta}(n), \quad (3.4)$$

for $n \in \mathbb{N}$ and for some constants $R_{\alpha, \beta}^{(1,2)} > 0$.

3. Since $\lim_{n \rightarrow \infty} k_n/n = 1/2$ and $t > s$, by definition (1.6) and by Lemma 2.4 for contractions $\{V_k = e^{-\frac{t-s}{n}B(t_k)}\}_{k=1}^n$ there exists $a_1 > 0$ such that

$$\left\| \prod_{k=k_n}^1 W_k^{(n)}(t, s) \right\|_1 = \left\| \prod_{k=k_n}^1 e^{-\frac{t-s}{n}A} e^{-\frac{t-s}{n}B(t_k)} \right\|_1 \leq a_1 \|e^{-\frac{t-s}{2}A}\|_1. \quad (3.5)$$

Similarly there is $a_2 > 0$ such that

$$\left\| \prod_{l=n}^{k_n+1} U(t_{l+1}, t_l) \right\|_1 \leq a_2 \|e^{-\frac{t-s}{2}A}\|_1. \quad (3.6)$$

4. Since for $t > s$ the trace-norm $c(t-s) := \|e^{-\frac{t-s}{2}A}\|_1 < \infty$, by (3.2)-(3.6) we obtain the proof of the estimate (1.11) for

$$R_{\alpha,\beta}(t,s) := (a_1 R_{\alpha,\beta}^{(1)} + a_2 R_{\alpha,\beta}^{(2)})c(t-s), \quad (3.7)$$

and $0 \leq s < t \leq T$. \square

Corollary 3.1 *By virtue of Lemma 2.4 the proof of Theorem 1.4 can be carried over almost verbatim for approximants $\{\widehat{U}_n(t,s)\}_{(t,s) \in \Delta, n \geq 1}$:*

$$\begin{aligned} \widehat{W}_k^{(n)}(t,s) &:= e^{-\frac{t-s}{n}B(t_k)} e^{-\frac{t-s}{n}A}, \quad k = 1, 2, \dots, n, \\ \widehat{U}_n(t,s) &:= \widehat{W}_n^{(n)}(t,s) \widehat{W}_{n-1}^{(n)}(t,s) \times \dots \times \widehat{W}_2^{(n)}(t,s) \widehat{W}_1^{(n)}(t,s), \end{aligned} \quad (3.8)$$

as well as for self-adjoint approximants $\{\widetilde{U}_n(t,s)\}_{(t,s) \in \Delta, n \geq 1}$:

$$\begin{aligned} \widetilde{W}_k^{(n)}(t,s) &:= e^{-\frac{t-s}{n}A/2} e^{-\frac{t-s}{n}B(t_k)} e^{-\frac{t-s}{n}A/2}, \quad k = 1, 2, \dots, n, \\ \widetilde{U}_n(t,s) &:= \widetilde{W}_n^{(n)}(t,s) \widetilde{W}_{n-1}^{(n)}(t,s) \times \dots \times \widetilde{W}_2^{(n)}(t,s) \widetilde{W}_1^{(n)}(t,s). \end{aligned} \quad (3.9)$$

For the both case the rate of convergence $\varepsilon_{\alpha,\beta}(n)$ for approximants (3.8),(3.9) is the same as in (1.11).

Note that extension of Theorem 1.4 to Gibbs semigroups generated by a family of non-negative self-adjoint operators $\{A(t)\}_{t \in \mathcal{I}}$ can be done along the arguments outlined in Section 2 of [18]. To this end one needs to add more conditions to (A1)-(A4) that allow to control the family $\{A(t)\}_{t \in \mathcal{I}}$.

Here we also comment that a general scheme of the *lifting* due to Lemma 2.4 and Theorem 1.4 can be applied to any symmetrically-normed ideal $\mathcal{C}_\phi(\mathfrak{H})$ of compact operators $\mathcal{C}_\infty(\mathfrak{H})$, [20] Ch.6. We return to this point elsewhere.

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