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► **To cite this version:**

Arnaud Guillin, Boris Nectoux. Low lying eigenvalues and convergence to the equilibrium of some Piecewise Deterministic Markov Processes generators in the small temperature regime. 2020. hal-02436593v1

**HAL Id: hal-02436593**

**<https://hal.science/hal-02436593v1>**

Preprint submitted on 13 Jan 2020 (v1), last revised 22 Jan 2022 (v3)

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# Low lying eigenvalues and convergence to the equilibrium of some Piecewise Deterministic Markov Processes generators in the small temperature regime

Arnaud Guillin\* and Boris Nectoux \*

## Abstract

In this work we study the number of small eigenvalues and the convergence to the equilibrium of the Bouncy Particle Sampler process and the Zig-Zag process generators in the small temperature regime. Such processes, which fall in the class of Piecewise Deterministic Markov Processes, are non diffusive and non reversible.

**Keywords.** Piecewise Deterministic Markov Processes, metastability, spectral theory, small temperature regime, semiclassical analysis.

**AMS classification.** 35P15, 35P20, 47F05, 35Q82, 35Q92.

## 1 Introduction and main results

### 1.1 Purpose and setting of this work

#### 1.1.1 Purpose

The quite recent growing interest for Piecewise Deterministic Markov Processes [11] (say PDMP hereafter) stems from their use within the Markov Chain Monte Carlo methodology. It aims at simulating a target probability distribution  $\pi$  by choosing a good Markov chain in the sense that it is ergodic and has stationary probability measure  $\pi$ . Let us be a little more precise concerning this probability measure  $\pi$ . Let  $\mathsf{M}$  be the *position* space and  $\pi$  be the Gibbs measure

$$\pi(dx) = \frac{e^{-\frac{2}{h}U}}{\int_{\mathsf{M}} e^{-\frac{2}{h}U}} dx, \quad (1)$$

associated with the potential function  $U : \mathsf{M} \rightarrow \mathbb{R}$  and the parameter  $h > 0$ ,  $dx$  being the Lebesgue measure on  $\mathsf{M}$ . The parameter  $h$  is proportional to the Boltzmann constant  $k_B$  through the relation  $h = k_B T$ ,  $T$  being the temperature of the underlying system. The Hastings-Metropolis algorithm [30] is surely the most well known method to create such a Markov chain by ensuring reversibility with respect to  $\pi$ . However, its performance may be questioned in terms of speed of convergence, computational cost and behavior with respect to the dimension of the problem.

PDMP may be shortly described as follows: between two jumps (eventually of only part of the coordinates) whose rates may of course depend of the position of the process, they have a deterministic behavior. PDMP may show remarkable feature [4, 17], as they are by essence non reversible and may thus exhibit faster rates of convergence towards equilibrium (see [3]). Of course there is still a lot of work to do to correctly assess the rate of convergence of such PDMP,

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see for example [1, 8, 12, 15], and the behavior with respect to dimension has still to be precisely understood (see however [6]). In practice, the second main advantage of these processes is that they can be used to sample the Gibbs measure (1) without sampling Brownian motions, as for example in Langevin type method such as MALA but only a countable collection of exponentially distributed random variables (using for example thinning procedure). Sampling from such distributions is a major aim of statistical simulations to compute macroscopic quantities or thermodynamic properties, see for instance [29].

However being non reversible, the Hasting-Metropolis trick is no more useful to guarantee that the PDMP has the correct invariant measure. The Bouncy Particle Sampler (BPS) process and the Zig-Zag (ZZ) process fall in the class of Piecewise Deterministic Markov Processes [11] designed to admit  $\pi \otimes \nu$  as invariant measure on the space  $\mathbf{E} = \mathbf{M} \times \mathbf{V}$ , where  $\nu$  is an instrumental measure on a space  $\mathbf{V}$  representing the *speed* of the process. It is then crucial to study their limiting properties as well as their speed of convergence to equilibrium, which has been the subject of quite a lot of recent and impressive works [2, 5, 8, 14, 32]. Let us also mention other contributions: [31] for a spectral analysis in dimension one with a constant jump rate on  $\mathbb{T} \times \{\pm 1\}$ , or [7] for an attempt of a spectral analysis in dimension 1 of the BPS and ZZ processes on the whole line, and [1, 13] for hypocoercive inequalities leading to the convergence of the semigroup to the equilibrium  $\mathbf{1}_{\mathbf{E}}$  in the weighted  $L^2$  space  $L^2(\mathbf{E}, \mathbb{C}; d\pi d\nu)$ . In this work, we provide, to the best of our knowledge, the first study of the spectral properties near the origin of the generators of the BPS process and the ZZ process (see Theorem 1) and its consequences on the rate of convergence to the equilibrium in the small temperature regime  $h \rightarrow 0$  (see Theorem 2). The main difficulty arising when studying these generators (see  $\mathbf{P}_h^{\text{BPS}}$  and  $\mathbf{P}_h^{\text{ZZ}}$  in the next section) is that they are not symmetric and not diffusive. The main results, namely Theorems 1 and 2, exhibit a metastable behavior of the processes associated with the operators  $\mathbf{P}_h^{\text{BPS}}$  and  $\mathbf{P}_h^{\text{ZZ}}$ : the convergence in  $L^2(\mathbf{E})$  to the equilibrium  $e^{-\frac{1}{h}U} \times \mathbf{1}_{\mathbf{V}}$  is very slow in the small temperature regime.

### 1.1.2 Some notations

In all this work,  $\mathbf{M} = \mathbb{T}^d$  where  $\mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$  and  $d \geq 1$ . Let also  $\mathbf{V}$  be either equal to  $\{\pm 1\}^d$  or to the  $d - 1$ -dimensional sphere  $S^d$  in  $\mathbb{R}^d$ . Denote by  $\nu$  the uniform probability measure on  $\mathbf{V}$ . In all this work,  $U : \mathbf{M} \rightarrow \mathbb{R}$  is a  $C^\infty$  function on  $\mathbf{M}$  and  $h > 0$  is a parameter which will be referred respectively as the potential function and the temperature of the underlying system. We denote by  $\mathbf{E}$  the space  $\mathbf{E} = \mathbf{M} \otimes \mathbf{V}$ . We define on  $L^2(\mathbf{E}, \mathbb{C}; dx d\nu)$ ,  $dx$  being the Lebesgue measure on  $\mathbf{M}$ , the operator  $\pi_v$  by:

$$\pi_v f(x) = \int_{\mathbf{V}} f(x, v) d\nu.$$

Notice that  $\pi_v$  is a bounded symmetric operator on  $L^2(\mathbf{E}, dx d\nu)$  and that  $\pi_v^2 = \pi_v$ . Thus  $\pi_v$  is an orthogonal projection on  $L^2(\mathbf{E}, \mathbb{C}; dx d\nu)$ .

Let  $\lambda_r : \mathbf{M} \rightarrow \mathbb{R}_+$  be a non negative bounded function such that:

$$r_0 := \inf_{\mathbf{M}} \lambda_r > 0. \tag{2}$$

In the following, for ease of notation, we denote by  $L^2(\mathbf{E})$  (resp.  $L^2(\mathbf{M})$ ) the space  $L^2(\mathbf{E}, \mathbb{C}; dx d\nu)$  (resp.  $L^2(\mathbf{M}, \mathbb{C}; dx)$ ). The Sobolev spaces of higher regularity will be denoted similarly. For instance,  $H^1(\mathbf{M})$  denotes the set of  $u \in L^2(\mathbf{M})$  such that  $\partial_x u \in L^2(\mathbf{M})$  (where  $\partial_x u := (\partial_{x_1} u, \dots, \partial_{x_d} u)^t$ ). Finally, we define the set  $\mathbf{C}_x(\mathbf{E})$  as the set of functions  $f \in L^2(\mathbf{E})$  such that the distribution  $v \cdot \partial_x f \in L^2(\mathbf{E})$  and for almost every  $v \in \mathbf{V}$ ,  $x \in \mathbf{M} \mapsto f(x, v)$  is  $C^\infty$ .

### 1.1.3 The Bouncy Particle Sampler process generator

In this section  $\mathbf{V} = \mathbb{S}^{d-1}$ . For  $f \in L^2(\mathbf{E})$ , the the jump operator  $\mathbf{B}$  is defined by:

$$\forall (x, v) \in \mathbf{M} \times \mathbf{V}, \mathbf{B}f(x, v) = f(x, v - 2v \cdot \mathbf{n}(x) \mathbf{n}(x)),$$

where for all  $x \in \mathbf{M}$ ,

$$\mathbf{n}(x) := \frac{\partial_x U(x)}{|\partial_x U(x)|} \text{ if } |\partial_x U(x)| \neq 0, \text{ else } \mathbf{n}(x) := 0.$$

For  $h > 0$ , let

$$\lambda_{h,J} := \frac{2}{h} (v \cdot \partial_x U)_+$$

be the jump rate, where the subscript  $J$  stands for jump. Here and in the following, for  $a \in \mathbb{R}$ ,  $(a)_+$  is defined by  $(a)_+ := \max(a, 0)$ . Notice that for all  $(x, v) \in \mathbf{M} \times \mathbf{V}$ , it holds:

$$\lambda_{h,J}(x, v) - \lambda_{h,J}(x, -v) = \frac{2}{h} v \cdot \partial_x U \quad \text{and} \quad \mathbf{B}\lambda_{h,J}(x, v) = \lambda_{h,J}(x, -v). \quad (3)$$

Let us consider the operator

$$\mathbf{P}_h^{\text{BPS}} = -v \cdot \mathbf{d}_{U,h} + \lambda_{1,J}(\mathbf{I} - \mathbf{B}) + \lambda_r(\mathbf{I} - \pi_v),$$

where  $\mathbf{I}$  is the identity operator,

$$\mathbf{d}_{U,h} = h \partial_x + \partial_x U = h e^{-\frac{1}{h}U} \partial_x e^{\frac{1}{h}U} \quad \text{and} \quad \lambda_{1,J} = 2(v \cdot \partial_x U)_+.$$

The formal adjoint of  $v \cdot \mathbf{d}_{U,h}$  in  $L^2(\mathbf{E})$  is the operator  $(v \cdot \mathbf{d}_{U,h})^* = v \cdot (-h \partial_x + \partial_x U) = -v \cdot h e^{\frac{1}{h}U} \partial_x e^{-\frac{1}{h}U}$ . The operator  $\mathbf{P}_h^{\text{BPS}}$  is linked to the the Bouncy Particle Sampler (BPS for short) generator  $-\mathbf{L}_h^{\text{BPS}}$  where  $\mathbf{L}_h^{\text{BPS}} = -v \cdot \partial_x + \lambda_{h,J}(\mathbf{I} - \mathbf{B}) + \frac{1}{h} \lambda_r(\mathbf{I} - \pi_v)$  introduced in [34] (see also [9]) through the relation

$$\mathbf{P}_h^{\text{BPS}} = h e^{-\frac{1}{h}U} \mathbf{L}_h^{\text{BPS}} e^{\frac{1}{h}U}.$$

Using (3), the formal adjoint operator of  $\mathbf{P}_h^{\text{BPS}}$  in  $L^2(\mathbf{E})$  is the differential operator

$$(\mathbf{P}_h^{\text{BPS}})^* = v \cdot \mathbf{d}_{U,h} + \lambda_{1,J}(\cdot, -\cdot)(\mathbf{I} - \mathbf{B}) + \lambda_r(\mathbf{I} - \pi_v).$$

### 1.1.4 The Zig Zag process generator

Let  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$  be the canonical basis of  $\mathbb{R}^d$ . When  $\mathbf{V} = \mathbb{S}^{d-1}$ , for  $f \in L^2(\mathbf{E})$ , one defines for  $k \in \{1, \dots, d\}$ , the jump operator  $\mathbf{B}^{(k)}$  as:

$$\forall (x, v) \in \mathbf{M} \times \mathbf{V}, \mathbf{B}^{(k)}f(x, v) = f(x, v - 2v \cdot \mathbf{n}^{(k)}(x) \mathbf{n}^{(k)}(x)),$$

where  $\mathbf{n}^{(k)}$  is defined by: for all  $x \in \mathbf{M}$ ,

$$\mathbf{n}^{(k)}(x) = \frac{\partial_{x_k} U(x)}{|\partial_{x_k} U(x)|} \mathbf{e}_k(x) \text{ if } |\partial_{x_k} U(x)| \neq 0, \text{ else } \mathbf{n}^{(k)}(x) = 0.$$

When  $\mathbf{V} = \{\pm 1\}^d$ , for all  $k \in \{1, \dots, d\}$ , the jump operator  $\mathbf{B}^{(k)}$  is defined for  $f \in L^2(\mathbf{E})$ , by:

$$\forall (x, v) \in \mathbf{M} \times \mathbf{V}, \mathbf{B}^{(k)}f(x, v) = f(x, v - 2v \cdot \mathbf{e}_k \mathbf{e}_k).$$

In this case,  $\mathbf{B}^{(k)}$  consists in negating the  $k$ -th component of  $v \in \{\pm 1\}^d$ . For  $h > 0$ , and  $k \in \{1, \dots, d\}$  let

$$\lambda_{h,J}^{(k)} = \frac{2}{h} (v \cdot \partial_{x_k} U \mathbf{e}_k)_+$$

be the  $k$ -th jump rate. Notice that for all  $\forall(x, v) \in \mathbb{M} \times \mathbb{V}$ , it holds for the two previous cases:

$$\lambda_{h,J}^{(k)}(x, v) - \lambda_{h,J}^{(k)}(x, -v) = \frac{2}{h} v \cdot \partial_{x_k} U \mathbf{e}_k \quad \text{and} \quad \mathbf{B}^{(k)} \lambda_{h,J}^{(k)}(x, v) = \lambda_{h,J}^{(k)}(x, -v). \quad (4)$$

Let us consider the operator

$$\mathbf{P}_h^{\text{ZZ}} = -v \cdot \mathbf{d}_{U,h} + \sum_{k=1}^d \lambda_{1,J}^{(k)} (\mathbf{I} - \mathbf{B}^{(k)}) + \lambda_r (\mathbf{I} - \pi_v),$$

where  $\lambda_{1,J}^{(k)} = 2(v \cdot \partial_{x_k} U \mathbf{e}_k)_+$ . The operator  $\mathbf{P}_h^{\text{ZZ}}$  is linked to the Zig-Zag (ZZ for short) process generator  $-\mathbf{L}_h^{\text{ZZ}}$  where  $\mathbf{L}_h^{\text{ZZ}} = -v \cdot \partial_x + \sum_{k=1}^d \lambda_{h,J}^{(k)} (\mathbf{I} - \mathbf{B}^{(k)}) + \frac{1}{h} \lambda_r (\mathbf{I} - \pi_v)$  through the relation

$$\mathbf{P}_h^{\text{ZZ}} = h e^{-\frac{1}{h}U} \mathbf{L}_h^{\text{ZZ}} e^{\frac{1}{h}U}.$$

We refer to [5] and references therein for more details on the ZZ process (see also [16]). Using (4), the formal adjoint operator of  $\mathbf{P}_h^{\text{ZZ}}$  in  $L^2(\mathbb{E})$  is the differential operator

$$(\mathbf{P}_h^{\text{ZZ}})^* = v \cdot \mathbf{d}_{U,h} + \sum_{k=1}^d \lambda_{1,J}^{(k)} (\cdot, -\cdot) (\mathbf{I} - \mathbf{B}) + \lambda_r (\mathbf{I} - \pi_v).$$

**Remark 1.** Notice that  $(\mathbf{P}_h^{\text{BPS}})^*(e^{-\frac{1}{h}U} \mathbf{1}_{\mathbb{V}}) = (\mathbf{P}_h^{\text{ZZ}})^*(e^{-\frac{1}{h}U} \mathbf{1}_{\mathbb{V}}) = 0$  and thus, the measure  $\pi \otimes \nu$  is invariant for both the BPS process and the ZZ process, where  $\pi$  is the Gibbs measure (1).

**Remark 2.** Let us explain the choice of scaling in  $h$  in the refreshment operator  $\mathbf{R}_v := \frac{1}{h} \lambda_r (\mathbf{I} - \pi_v)$ . This scaling is explained by the fact that in practice, a refreshment is added to balance the jump rate  $\lambda_{h,J}$  (or the  $\lambda_{h,J}^{(k)}$ 's when considering the ZZ process) in order to sample efficiently the measure  $\pi \otimes \nu$ . On the other hand, if the refreshment is too large compared to  $\lambda_{h,J}$ , the convergence rate towards  $\pi \otimes \nu$  becomes very poor. There is a trade-off between the added refreshment and  $\lambda_{h,J}$  (see for example [12, 15] for some very partial explanation). The relevant scaling when  $h \ll 1$  of  $\mathbf{R}_v$  is thus of the same order of  $\lambda_{h,J}$  which scales in  $h^{-1}$ . Other scalings for the refreshment operator are considered in Section 5.3 below.

## 1.2 Assumptions and main results

The following result will be needed.

**Lemma 3.** For all  $w \in \mathbf{C}_x(\mathbb{E})$ , it holds:

$$\text{Re} \langle \mathbf{P}_h^{\text{BPS}} u, u \rangle = \frac{1}{2} \int_{\mathbb{E}} \lambda_{1,J} |Bu - u|^2 + \int_{\mathbb{M}} \lambda_r \int_{\mathbb{V}} |(1 - \pi_v)u|^2, \quad (5)$$

and

$$\text{Re} \langle \mathbf{P}_h^{\text{ZZ}} u, u \rangle = \frac{1}{2} \sum_{k=1}^d \int_{\mathbb{E}} \lambda_{1,J}^{(k)} |\mathbf{B}^{(k)}u - u|^2 + \int_{\mathbb{M}} \lambda_r \int_{\mathbb{V}} |(1 - \pi_v)u|^2.$$

*Proof.* Let us prove (5). The other equality is proved similarly. Let us consider  $w \in \mathbf{C}_x(\mathbb{E})$ . Then, it holds:

$$\langle \mathbf{P}_h w, w \rangle = \int_{\mathbb{E}} (-v \cdot \mathbf{d}_{U,h} \bar{w}) w + \int_{\mathbb{E}} \lambda_1 (\mathbf{I} - \mathbf{B}) \bar{w} w + \int_{\mathbb{M}} \lambda_r \underbrace{\int_{\mathbb{V}} (1 - \pi_v) \bar{w} w \, d\nu}_{= \int_{\mathbb{V}} |(1 - \pi_v)w|^2 \, d\nu} d\pi.$$

Since for almost every  $v \in \mathbf{V}$ ,  $x \in \mathbf{M} \mapsto w(x, v)$  is smooth, it holds:  $\mathbf{d}_{U,h}\bar{w} = h\partial_x\bar{w} + \partial_x U \bar{w}$  and

$$\int_{\mathbf{E}} v \cdot \partial_x(|w|^2) = \int_{\mathbf{E}} v \cdot [(\partial_x\bar{w})w + (\partial_x w)\bar{w}]. \quad (6)$$

By the Stokes Theorem  $v \cdot \int_{\mathbf{M}} \partial_x(|w|^2) dx = 0$ , and therefore, one has:

$$\int_{\mathbf{E}} v \cdot \operatorname{Re}((\partial_x\bar{w})w) = 0,$$

and then,  $\operatorname{Re} \int_{\mathbf{E}} (-v \cdot \mathbf{d}_{U,h}\bar{w})w = \int_{\mathbf{E}} -v \cdot \partial_x U |w|^2$ . Let us recall that  $\mathbf{B}$  is symmetric on  $L^2(\mathbf{E})$ . Besides, one has:

$$\begin{aligned} \operatorname{Re} \int_{\mathbf{E}} \lambda_{1,J}(\mathbf{I} - \mathbf{B})\bar{w} w &= \frac{1}{2} \int_{\mathbf{E}} [\lambda_{1,J}(\bar{w} - \mathbf{B}\bar{w})w + \lambda_{1,J}(w - \mathbf{B}w)\bar{w}] \\ &= \frac{1}{2} \int_{\mathbf{E}} [\lambda_1|w - \mathbf{B}w|^2 + \lambda_{1,J}(\bar{w} - \mathbf{B}\bar{w})\mathbf{B}f + \lambda_{1,J}(., -.) (\mathbf{B}w - w)\mathbf{B}\bar{w}] \\ &= \frac{1}{2} \int_{\mathbf{E}} \lambda_{1,J}|w - \mathbf{B}w|^2 + \frac{1}{2} \int_{\mathbf{E}} (\lambda_{1,J}(., -.) - \lambda_1)|\mathbf{B}w|^2 \\ &\quad + \int_{\mathbf{E}} (\lambda_1\bar{w}\mathbf{B}f - \lambda_1(., -.)w\mathbf{B}\bar{w}). \end{aligned}$$

Since  $\int_{\mathbf{E}} (\lambda_1\bar{w}\mathbf{B}f - \lambda_1(., -.)w\mathbf{B}\bar{w}) = 0$ , one has:

$$\operatorname{Re} \int_{\mathbf{E}} \lambda_{1,J}(\mathbf{I} - \mathbf{B})\bar{w} w = \frac{1}{2} \int_{\mathbf{E}} \lambda_{1,J}|w - \mathbf{B}w|^2 + \frac{1}{2} \int_{\mathbf{E}} (\lambda_{1,J} - \lambda_{1,J}(., -.))|w|^2.$$

Using (3) and the definition of  $\lambda_1$ , it holds:

$$\frac{1}{2} \int_{\mathbf{E}} (\lambda_{1,J} - \lambda_{1,J}(., -.))|w|^2 = \int_{\mathbf{E}} v \cdot \partial_x U |w|^2 = -\operatorname{Re} \int_{\mathbf{E}} (-v \cdot \mathbf{d}_{U,h}\bar{w})w.$$

This concludes the proof of Lemma 3. ■

In all this work, we denote by  $\mathbf{P}_h$  (resp.  $\mathbf{L}_h$ ) for either  $\mathbf{P}_h^{\text{BPS}}$  and  $\mathbf{P}_h^{\text{ZZ}}$ ,

$$\mathbf{P}_h \in \{\mathbf{P}_h^{\text{BPS}}, \mathbf{P}_h^{\text{ZZ}}\} \quad (\text{resp. } \mathbf{L}_h \in \{\mathbf{L}_h^{\text{BPS}}, \mathbf{L}_h^{\text{ZZ}}\}).$$

Let us define

$$D(\mathbf{P}_h) = \{f \in L^2(\mathbf{E}), v \cdot \partial_x f \in L^2(\mathbf{E})\}.$$

The choice of the domain  $D(\mathbf{P}_h)$  follows from the fact that  $\mathbf{P}_h$  is the sum of the unbounded operator  $-v \cdot \partial_x f$  and a bounded operator on  $L^2(\mathbf{E})$  (because the jump rates are bounded on  $\mathbf{E}$  as well as  $\lambda_r$  on  $\mathbf{M}$ ). The following lemma is the starting point of our analysis.

**Proposition 4.** *Let  $h > 0$  be fixed. The space  $C_x(\mathbf{E})$  is dense in  $D(\mathbf{P}_h)$ . Moreover, the operator  $(\mathbf{P}_h, D(\mathbf{P}_h))$  is  $m$ -accretive and its adjoint is the operator  $(\mathbf{P}_h^*, D(\mathbf{P}_h^*))$  with  $D(\mathbf{P}_h^*) = D(\mathbf{P}_h)$ . In addition,  $C^\infty(\mathbf{E})$  is a core for both  $\mathbf{P}_h$  and  $\mathbf{P}_h^*$ .*

The proof of this result is postponed to Section 2.

Proposition 4 is required for two reasons. The first one is that we need some regularity on test functions to perform computations for  $\mathbf{P}_h$  (see Lemma 3 for instance and the proof of Proposition 10) and then pass to the limit, to extend these estimates on  $D(\mathbf{P}_h)$  in order to obtain resolvent estimates. The second one is that we need to be able to identify the adjoint of

$(P_h, D(P_h))$  and to do the same computations for the adjoint as those we did for  $P_h$ . This is indeed needed to justify that the resolvent of the operator defined by (29) below exists<sup>1</sup> in Step 1 in the proof of Proposition 11.

Let us now state the main assumption of this work:

**Assumption (Morse).** *The function  $U : M \rightarrow \mathbb{R}$  is a  $C^\infty$  Morse function with  $m_0$  local minima in  $M$ .*

The first main results of this work is the following.

**Theorem 1.** *Let us assume that (Morse) holds. Then, there exists  $\alpha_0 > 0$  such that for all  $\alpha \in (0, \alpha_0)$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,  $P_h$  has exactly  $m_0$  eigenvalues  $\{\lambda_1, \dots, \lambda_{m_0}\}$  (counted with algebraic multiplicity), with  $\lambda_1 = 0$ , in the set  $\{\operatorname{Re} z \leq \alpha h^2\}$ . Moreover, for all  $h$  small enough and for all  $i \in \{1, \dots, m_0\}$ ,  $\lambda_i \in \mathbb{R}$ , the algebraic multiplicity of  $\lambda_i$  equals its geometric multiplicity, and there exists  $c > 0$  such that for  $h$  small enough,  $\lambda_i \leq e^{-\frac{c}{h}}$ . Finally, the eigenvalue 0 has algebraic multiplicity 1 for  $P_h$  and  $\operatorname{Ker} P_h = \operatorname{Span}(e^{-\frac{1}{h}U} \mathbf{1}_V)$ . The same holds for  $P_h^*$ .*

The fact that the eigenvalue 0 is simple and isolated for both  $P_h$  and  $P_h^*$  actually holds for all  $h > 0$ , see indeed Proposition 14 in the appendix. The proof mixes different techniques of semiclassical analysis from [24, 28, 35]. Our analysis is also inspired from non semiclassical hypoocoercive techniques from the original papers [23, 25] which were later generalized in [13] (see also [1]).

From Proposition 4, the Hille-Yosida Theorem implies that  $-P_h$  generates a strongly continuous contraction semigroup  $(e^{-tP_h})_{t \geq 0}$  on  $L^2(\mathbf{E})$ . The second main results of this work is the following which characterizes the convergence of the semigroup  $(e^{-tP_h})_{t \geq 0}$  in the small temperature regime.

**Theorem 2.** *Assume that (Morse) hold. Denote by  $\lambda_1, \dots, \lambda_{m_0}$  the  $m_0$  smallest eigenvalues of  $P_h$ , which are real and exponentially small when  $h \rightarrow 0$  (see Theorem 1). Let  $\Pi_{\lambda_j}(P_h)$ ,  $j = 1, \dots, m_0$ , be the spectral projection associated with  $\lambda_j$  for  $P_h$ . Then there exist  $\gamma > 0$ ,  $C > 0$ , and  $h_0 > 0$  such that for all  $h \in (0, h_0)$ , it holds for all  $t \geq 0$ :*

$$\left\| e^{-tP_h} - \sum_{j=1}^{m_0} e^{-t\lambda_j} \Pi_{\lambda_j}(P_h) \right\| \leq C e^{-\gamma t h^2},$$

and for all  $j = 1, \dots, m_0$ ,  $\|\Pi_{\lambda_j}(P_h)\| \leq C$ .

Here and in the following  $\|K\|$  denotes the norm of  $K \in \mathcal{L}(L^2(\mathbf{E}))$  when  $L^2(\mathbf{E})$  is endowed with its natural Hermitian inner product.

## 2 Proof of Proposition 4

Let us first prove the following result.

**Lemma 5.** *Let  $f \in L^2(\mathbb{R}_x^d \times \mathbf{V})$ . such that  $v \cdot \partial_x f \in L^2(\mathbb{R}_x^d \times \mathbf{V})$ . Then, there exists a sequence  $(f_n)_{n \geq 0} \in C_x(\mathbb{R}_x^d \times \mathbf{V})^{\mathbb{N}}$  such that  $f_n \rightarrow f$  and  $v \cdot \partial_x f_n \rightarrow v \cdot \partial_x f$  as  $n \rightarrow +\infty$  in  $L^2(\mathbb{R}_x^d \times \mathbf{V})$ . Finally, if  $f$  is moreover compactly supported in  $\mathbf{B}(0, r) \times \mathbf{V}$  where  $\mathbf{B}(0, r)$  is the open ball  $\mathbf{B}(0, r)$  of radius  $r > 0$  in  $\mathbb{R}_x^d$  centred in 0, the function  $f_n$ , for all  $n \geq 0$ , can be chosen compactly supported in  $\mathbf{B}(0, r) \times \mathbf{V}$ .*

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<sup>1</sup>Using the identity  $\operatorname{Ran}(K - z) = \operatorname{Ker}(K^* - \bar{z})^\perp$  valid for an accretive, closed, and densely defined operator  $K$  on a Hilbert space.

*Proof.* Let  $f \in L^2(\mathbb{R}_x^d \times \mathbf{V})$ . Let us consider a sequence of mollifier  $(\rho_n)_{n \in \mathbb{N}^*} \in C^\infty(\mathbb{R}_x^d)^{\mathbb{N}^*}$ , i.e. for all  $n \geq 1$ ,  $\rho_n$  is non negative,  $\int_{\mathbb{R}_x^d} \rho_n = 1$ , and  $\rho_n$  is supported in  $\mathbf{B}(0, 2n^{-1})$ . Define, for all  $v \in \mathbf{V}$ , the function

$$x \in \mathbb{R}_x^d \mapsto f_n(x, v) = \int_{\mathbb{R}_x^d} \rho_n(y) f(x - y, v) dy.$$

For almost every  $v \in \mathbf{V}$ ,  $\int_{\mathbb{R}_x^d} |f_n(x, v)|^2 dx < +\infty$  because  $f \in L^2(\mathbb{R}_x^d \times \mathbf{V})$ . Thus, from [10, Proposition 4.20], for almost every  $v \in \mathbf{V}$ ,  $x \in \mathbb{R}_x^d \mapsto f_n(x, v)$  is  $C^\infty$ . Using the Young inequality, one has for all  $n \geq 1$  and for a.e  $v \in \mathbf{V}$ :

$$\|f_n(\cdot, v)\|_{L^2(\mathbb{R}_x^d)} \leq \|\rho_n\|_{L^1(\mathbb{R}_x^d)} \|f(\cdot, v)\|_{L^2(\mathbb{R}_x^d)} = \|f(\cdot, v)\|_{L^2(\mathbb{R}_x^d)} \in L^2(\mathbf{V}).$$

and thus,  $f_n \in L^2(\mathbb{R}_x^d \times \mathbf{V})$ . Since for almost every  $v \in \mathbf{V}$ ,  $f(\cdot, v) \in L^2(\mathbb{R}_x^d)$ , it holds from [10, Theorem 4.22],

$$\text{for almost every } v \in \mathbf{V}, F_n(v) := \|f_n(\cdot, v) - f(\cdot, v)\|_{L^2(\mathbb{R}_x^d)}^2 \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Since  $F_n(v) \leq 4\|f(\cdot, v)\|_{L^2(\mathbb{R}_x^d)}^2 \in L^1(\mathbf{V})$ , the dominated convergence theorem implies that  $F_n \rightarrow 0$  in  $L^1(\mathbf{V})$ , i.e.  $f_n \rightarrow f$  in  $L^2(\mathbb{R}_x^d \times \mathbf{V})$ . In the sense of distribution in  $\mathbb{R}_x^d$ , it holds, for all  $n \in \mathbb{N}$  and  $i \in \{1, \dots, d\}$ ,

$$\partial_{x_i} f_n = \int_{\mathbb{R}_x^d} \rho_n(y) \partial_{x_i} f(x - y, v) dy,$$

and therefore for almost every  $v \in \mathbf{V}$ ,

$$v \cdot \partial_x f_n = \int_{\mathbb{R}_x^d} \rho_n(y) [v \cdot \partial_x f(x - y, v)] dy,$$

which actually belongs in  $L^2(\mathbb{R}_x^d)$  for almost every  $v \in \mathbf{V}$  (because  $v \cdot \partial_x f \in L^2(\mathbb{R}_x^d \times \mathbf{V})$  implies that for almost every  $v \in \mathbf{V}$ ,  $v \cdot \partial_x f \in L^2(\mathbb{R}_x^d)$ ). Repeating the previous argument for  $v \cdot \partial_x f_n$  instead of  $f_n$  and  $v \cdot \partial_x f$  instead of  $f$ , we obtain that  $v \cdot f_n \in L^2(\mathbb{R}_x^d \times \mathbf{V})$  and  $v \cdot f_n \rightarrow v \cdot f$  in  $L^2(\mathbb{R}_x^d \times \mathbf{V})$  when  $n \rightarrow +\infty$ . This proves the first claim in Lemma 5. The second one follows from the fact that, for almost every  $v \in \mathbf{V}$ , the support of  $f_n(\cdot, v)$  is equal to  $\text{supp } f(\cdot, v) + \text{supp } \rho_n = \text{supp } f(\cdot, v) + 2n^{-1}$  which is included in  $\mathbf{B}(0, r)$  for all  $n$  small enough. This concludes the proof of the lemma.  $\blacksquare$

**Corollary 6.** *The set  $C_x(\mathbf{E})$  is dense in  $D(\mathbf{P}_h)$ .*

*Proof.* By considering a finite number of open charts covering the compact manifold  $\mathbf{M}$  and a partition of unity on  $\mathbf{M}$  subordinate to this open cover (with compact supports and indexed by the open cover, which is possible because  $\mathbf{M}$  is compact), Lemma 5 implies that  $C_x(\mathbf{E})$  is dense in  $D(\mathbf{P}_h)$ .  $\blacksquare$

Let us now end the proof of Proposition 4

*Proof of Proposition 4.* The operator  $(\mathbf{P}_h, D(\mathbf{P}_h))$  is accretive on  $C_x(\mathbf{E})$  (see Lemma 3) and from Corollary 6, it is accretive on  $D(\mathbf{P}_h)$ . It is clear that the operator  $(\mathbf{P}_h, D(\mathbf{P}_h))$  is closed. The same clearly holds for  $(\mathbf{P}_h^*, D(\mathbf{P}_h))$ : it is accretive and closed.

Let us denote by  $(\mathbf{P}_h^\dagger, D(\mathbf{P}_h^\dagger))$  the adjoint of  $(\mathbf{P}_h, D(\mathbf{P}_h))$  which is defined by:

$$D(\mathbf{P}_h^\dagger) = \{f \in L^2(\mathbf{E}), \exists g \in L^2(\mathbf{E}) \text{ s.t } \forall \phi \in D(\mathbf{P}_h), \langle \mathbf{P}_h \phi, f \rangle_{L^2(\mathbf{E})} = \langle \phi, g \rangle_{L^2(\mathbf{E})}\},$$



and  $P_h^\dagger f := g$ . Let  $f \in D(P_h^\dagger)$ . Let  $\phi \in C^\infty(\mathbb{E})$ . Then, it holds  $\langle P_h \phi, f \rangle_{L^2(\mathbb{E})} = \langle \phi, g \rangle_{L^2(\mathbb{E})}$ , which implies that the distribution  $v \cdot \partial_x f$  belongs to  $L^2(\mathbb{E})$  and thus:

$$\langle P_h \phi, f \rangle_{L^2(\mathbb{E})} = \langle \phi, P_h^* f \rangle_{L^2(\mathbb{E})}.$$

In particular, for all  $\phi \in C^\infty(\mathbb{E})$ ,  $\langle \phi, g \rangle_{L^2(\mathbb{E})} = \langle \phi, P_h^* f \rangle_{L^2(\mathbb{E})}$ , which leads to  $g = P_h^* f$ . We thus have

$$(P_h^\dagger, D(P_h^\dagger)) \subset (P_h^*, D(P_h)). \quad (7)$$

Therefore, since  $(P_h^*, D(P_h))$  is accretive, so is  $(P_h^\dagger, D(P_h^\dagger))$ . Because a closed accretive operator with accretive adjoint is  $m$ -accretive, one deduces that  $P_h$  with domain  $D(P_h)$  is  $m$ -accretive. For the same reasons,  $(P_h^\dagger, D(P_h^\dagger))$  is also  $m$ -accretive. The same arguments shows that  $P_h^*$  with domain  $D(P_h)$  is also  $m$ -accretive. Then, Equation (7) implies that,

$$(P_h^\dagger, D(P_h^\dagger)) = (P_h^*, D(P_h)).$$

It remains to show that  $C^\infty(\mathbb{E})$  is a core for both  $P_h$  and  $P_h^*$ . Let us denote by  $(P_h, C^\infty(\mathbb{E}))^\dagger$  the adjoint of the closable operator  $(P_h, C^\infty(\mathbb{E}))$ . On the one hand, by definition of  $(P_h, C^\infty(\mathbb{E}))^\dagger$  and reasoning as we proved Equation (7), one has:  $(P_h, C^\infty(\mathbb{E}))^\dagger \subset (P_h^*, D(P_h^*)) = (P_h^\dagger, D(P_h^\dagger))$ . Taking the adjoint, leads to

$$(P_h, D(P_h)) \subset \overline{(P_h, C^\infty(\mathbb{E}))}.$$

Since the reverse inclusion clearly holds, one gets that  $(P_h, D(P_h)) = \overline{(P_h, C^\infty(\mathbb{E}))}$ , i.e.  $C^\infty(\mathbb{E})$  is a core for  $(P_h, D(P_h))$ . The same holds of course for  $(P_h^*, D(P_h))$ . This concludes the proof of Proposition 4.  $\blacksquare$

**Remark 7.** In [14, 26], it is shown that when considering the semigroup on  $L^\infty(\mathbb{R}^d \times \mathbb{V})$ , the space of smooth compactly supported functions on  $\mathbb{R}^d \times \mathbb{V}$  is a core for  $L_h$ . Proposition 4 is concerned with the  $L^2(\mathbb{E})$ -setting and also provides a full characterisation of the adjoint of  $L_h$ , which is required in our analysis.

### 3 Witten Laplacian associated with $U$ on $M$

Our analysis is based on the spectral properties of the Witten Laplacian associated with  $U$  on  $M$  in the limit  $h \rightarrow 0$ . Let us recall that the Witten Laplacian associated with  $U$  on  $M$  is the operator

$$\Delta_{U,h} = -e^{\frac{1}{h}U} h \operatorname{div}_x e^{-\frac{1}{h}U} \circ d_{U,h} = -h^2 \Delta_x + |\partial_x U|^2 - h \Delta_x V,$$

where  $-e^{\frac{1}{h}U} h \operatorname{div}_x e^{-\frac{1}{h}U}$  is the adjoint of  $d_{U,h}$  in  $L^2(M)$ . The operator  $(\Delta_{U,h}, H^2(M))$  is self-adjoint with compact resolvent on  $L^2(M)$ . Moreover, it is the closure of the Friedrichs extension of the quadratic form

$$Q_{U,h} : w \in H^1(M) \mapsto \|d_{U,h} w\|_{L^2(M)}^2.$$

When **(Morse)** holds, from [19, 21], there exist  $h_0 > 0$  and  $\gamma_0 > 0$  such that for all  $h \in (0, h_0)$ ,

$$\Delta_{U,h} \text{ has exactly } m_0 \text{ eigenvalues smaller than } \gamma_0 h. \quad (8)$$

Moreover, these  $m_0$  smallest eigenvalues are exponentially small for  $h$  small enough, i.e. if  $\Phi_{j,h}$  is a  $L^2(M)$ -normalized eigenfunction associated with the  $j$ -th eigenvalue of  $\Delta_{U,h}$  ( $j \in \{1, \dots, m_0\}$ ), it holds:

$$\|d_{U,h} \Phi_{j,h}\|_{L^2(M)} \leq e^{-\frac{\epsilon}{h}}, \quad (9)$$

where  $c > 0$  is a constant independent of  $h$ . The  $\Phi_{j,h}$ 's are two by two orthogonal and from the standard elliptic regularity, for all  $j \in \{1, \dots, m_0\}$ ,  $\Phi_{j,h} \in \mathcal{C}^\infty(\mathbf{M})$ . We have the following lemma which is a direct consequence of the spectral theorem and (8).

**Lemma 8.** *Assume that (Morse) holds. Then, there exists  $\delta_0 > 0$  such that for  $h$  small enough and for all  $w \in H^2(\mathbf{M})$  such that  $w \in \{\Phi_{1,h}, \dots, \Phi_{m_0,h}\}^\perp_{L^2(\mathbf{M})}$ ,*

$$\langle w, [1 + \Delta_{U,h}]^{-1} \Delta_{U,h} w \rangle_{L^2(\mathbf{M})} \geq \delta_0 h \|w\|_{L^2(\mathbf{M})}^2.$$

Here and in the following, the space  $\{\Phi_{1,h}, \dots, \Phi_{m_0,h}\}^\perp_{L^2(\mathbf{M})}$  denotes the orthogonal of the span of  $\{\Phi_{1,h}, \dots, \Phi_{m_0,h}\}$  in  $L^2(\mathbf{M})$ . Let us define the vector space

$$\mathbf{G} := \text{Span}(\Phi_{1,h} \mathbf{1}_V, \dots, \Phi_{m_0,h} \mathbf{1}_V),$$

and the orthogonal projection  $\pi_{\mathbf{G}}$  on  $\mathbf{G}$  in  $L^2(\mathbf{E})$ , i.e. for all  $u \in L^2(\mathbf{E})$ ,

$$\pi_{\mathbf{G}} u = \sum_{j=1}^{m_0} \langle \Phi_{j,h} \mathbf{1}_V, u \rangle_{L^2(\mathbf{E})} \Phi_{j,h} \mathbf{1}_V. \quad (10)$$

Let us now give estimates of the terms  $P_h(\Phi_{j,h} \mathbf{1}_V)$  and  $P_h^*(\Phi_{j,h} \mathbf{1}_V)$  when  $h \rightarrow 0$ . Since  $P_h(\Phi_{j,h} \mathbf{1}_V) = -v \cdot \mathbf{d}_{U,h}(\Phi_{j,h} \mathbf{1}_V) = -v \cdot (\mathbf{d}_{U,h} \Phi_{j,h}) \mathbf{1}_V$  and  $P_h^*(\Phi_{j,h} \mathbf{1}_V) = v \cdot \mathbf{d}_{U,h}(\Phi_{j,h}) \mathbf{1}_V$ , from (9), for all  $j \in \{1, \dots, m_0\}$  and  $h$  small enough, it holds:

$$\|P_h(\Phi_{j,h} \mathbf{1}_V)\|_{L^2(\mathbf{E})} \leq e^{-\frac{c}{h}} \text{ and } \|P_h^*(\Phi_{j,h} \mathbf{1}_V)\|_{L^2(\mathbf{E})} \leq e^{-\frac{c}{h}}, \quad (11)$$

for some  $c > 0$  independent of  $h$ . Equation (11) implies that for all  $u \in L^2(\mathbf{E})$ , and  $h$  small enough,

$$\|P_h \pi_{\mathbf{G}} u\|_{L^2(\mathbf{E})} \leq e^{-\frac{c}{h}} \|\pi_{\mathbf{G}} u\|_{L^2(\mathbf{E})} \text{ and } \|P_h^* \pi_{\mathbf{G}} u\|_{L^2(\mathbf{E})} \leq e^{-\frac{c}{h}} \|\pi_{\mathbf{G}} u\|_{L^2(\mathbf{E})}. \quad (12)$$

We end this section with an important identity which will be used in the next section. For all  $u \in \mathcal{C}^\infty(\mathbf{E})$ , it holds:

$$\pi_v \Delta_{U,h} \pi_v u = \mathbf{m}_2^{-1} \pi_v (v \cdot \mathbf{d}_{U,h})^* \circ (v \cdot \mathbf{d}_{U,h}) \pi_v u, \quad (13)$$

where

$$\mathbf{m}_2 := \int_V v_1^2 d\nu,$$

and where we have used that  $\int_V v_i v_j d\nu = 0$  for all  $i, j \in \{1, \dots, d\}$ ,  $i \neq j$  (when  $V = \mathbb{S}^{d-1}$ , this can be proved using the standard polar decomposition, see for instance the proof of [1, Lemma 36]).

#### 4 Resolvent estimates on $P_h$

Since the computations are exactly the same for  $P_h^{\text{BPS}}$  and  $P_h^{\text{ZZ}}$ , we set, for ease of notation,

$$P_h = P_h^{\text{BPS}}.$$

It will be clear from our analysis that the results stated in the remainder of this work also holds for  $P_h^{\text{ZZ}}$ .

#### 4.1 The operator $A_h$

In this section we introduce an operator  $A_h$  which play a crucial role in our analysis. Let

$$\mathbb{T}_h = \frac{1}{2}(\mathbb{P}_h - \mathbb{P}_h^*) = -v \cdot \mathbf{d}_{U,h} + \frac{1}{2}(v \cdot \partial_x V)(I - \mathbb{B})$$

be the antisymmetric part of  $\mathbb{P}_h$ , and

$$\mathbb{S} = \frac{1}{2}(\mathbb{P}_h + \mathbb{P}_h^*) = \frac{1}{2}(\lambda_{1,J} + \lambda_{1,J}(\cdot, -)) (I - \mathbb{B}) + \lambda_r(I - \pi_v),$$

be the symmetric part of  $\mathbb{P}_h$ , both with domain  $\mathcal{C}^\infty(\mathbf{E})$  on  $L^2(\mathbf{E})$ . Notice that  $\mathbb{S}$  is independent  $h$ . We have the following direct properties:

- The operator  $\mathbb{S}$  with domain  $\mathcal{C}^\infty(\mathbf{E})$  is closable on  $L^2(\mathbf{E})$  and its closure is a self-adjoint bounded operator on  $L^2(\mathbf{E})$ . Moreover,

$$\mathbb{S}\pi_v = 0. \quad (14)$$

Furthermore, since  $\lambda_{1,J} \in L^\infty(\mathbf{E})$  and  $\lambda_r \in L^\infty(\mathbf{M})$ , there exists  $C > 0$  such that for all  $h$ ,

$$\|\mathbb{S}\| \leq C. \quad (15)$$

- For all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,  $\mathbb{T}_h\pi_v u = -v \cdot \mathbf{d}_{U,h}\pi_v u$  and consequently,  $(\mathbb{T}_h\pi_v)^*u = -\pi_v(v \cdot \mathbf{d}_{U,h})^*u$ , where we recall that  $(v \cdot \mathbf{d}_{U,h})^* = v \cdot (-h\partial_x + \partial_x U)$ . Therefore, from (13), it holds on  $\mathcal{C}^\infty(\mathbf{E})$ :

$$(\mathbb{T}_h\pi_v)^*(\mathbb{T}_h\pi_v) = \mathbf{m}_2\pi_v\Delta_{U,h}\pi_v = \mathbf{m}_2\Delta_{U,h}\pi_v. \quad (16)$$

- For all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,  $\pi_v\mathbb{T}_h\pi_v u = 0$ , which follows from  $\int_{\mathbf{V}} v_i d\nu = 0$  for all  $i \in \{1, \dots, d\}$ .

Moreover, we have the following result.

**Proposition 9.** *Let  $h > 0$ . It holds as an equality between bounded operators on  $L^2(\mathbf{E})$ ,*

$$[1 + (\mathbb{T}_h\pi_v)^*(\mathbb{T}_h\pi_v)]^{-1}\pi_v = [1 + \mathbf{m}_2\Delta_{U,h}]^{-1}\pi_v. \quad (17)$$

Let us introduce the operator

$$A_h := [1 + (\mathbb{T}_h\pi_v)^*(\mathbb{T}_h\pi_v)]^{-1}(\mathbb{T}_h\pi_v)^* = -[1 + \mathbf{m}_2\Delta_{U,h}]^{-1}\pi_v(v \cdot \mathbf{d}_{U,h})^*$$

with domain  $\mathcal{C}^\infty(\mathbf{E})$  on  $L^2(\mathbf{E})$ . The operator  $A_h$  with domain  $\mathcal{C}^\infty(\mathbf{E})$  is closable on  $L^2(\mathbf{E})$  and its closure is a bounded operator on  $L^2(\mathbf{E})$  with norm smaller than 1. Moreover, for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,  $\pi_v A_h u = A_h u$  and

$$\|\mathbb{T}_h\pi_v A_h u\|_{L^2(\mathbf{E})} \leq \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}. \quad (18)$$

Finally, one has for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,  $A_h^*u = -v \cdot \mathbf{d}_{U,h}[1 + \mathbf{m}_2\Delta_{U,h}]^{-1}\pi_v u$ , and

$$\|A_h^*u\|_{L^2(\mathbf{E})} \leq \mathbf{m}_2^{-1/2}\|\pi_v u\|_{L^2(\mathbf{E})}, \quad (19)$$

which extends to all  $u \in L^2(\mathbf{E})$ .

*Proof.* The proof of Proposition 9 is very similar to those made in [1, 13]. We recall it in our setting for the sake of completeness.

On the one hand, the operator  $\mathbb{T}_h \pi_v = -v \cdot \mathbf{d}_{U,h} \pi_v$  with domain  $\mathcal{C}^\infty(\mathbf{E})$  is closable and densely defined on  $L^2(\mathbf{E})$ . Thus, from [33, Theorem 5.1.9] (see also [1, Proposition 26]),  $1 + (\mathbb{T}_h \pi_v)^* (\mathbb{T}_h \pi_v)$  is a positive self-adjoint operator from  $D((\mathbb{T}_h \pi_v)^* (\mathbb{T}_h \pi_v))$  to  $L^2(\mathbf{E})$  and  $[1 + (\mathbb{T}_h \pi_v)^* (\mathbb{T}_h \pi_v)]^{-1}$  is a bounded operator on  $L^2(\mathbf{E})$ . On the other hand, it is standard that  $[1 + \mathbf{m}_2 \Delta_{U,h}]^{-1}$  is a bounded operator on  $L^2(\mathbf{M})$ . Consider  $u \in L^2(\mathbf{E})$ . Let  $f_1 \in L^2(\mathbf{E})$  be such that  $f_1 = [1 + (\mathbb{T}_h \pi_v)^* (\mathbb{T}_h \pi_v)]^{-1} \pi_v u$ , i.e.  $[1 + \mathbf{m}_2 \Delta_{U,h} \pi_v] f_1 = \pi_v u$  (see (16)). Then, it holds,

$$\pi_v f_1 = \pi_v^2 u - \mathbf{m}_2 \pi_v \Delta_{U,h} \pi_v f_1 = \pi_v u - \mathbf{m}_2 \Delta_{U,h} \pi_v f_1 = f_1.$$

Therefore  $f_1$  is independent of  $v \in \mathbf{V}$  and thus,  $[1 + \mathbf{m}_2 \Delta_{U,h}] f_1 = \pi_v u$ . This implies that  $f_1 = [1 + \mathbf{m}_2 \pi_v \Delta_{U,h}]^{-1} \pi_v u$ . This proves (17).

Let us now prove the statements concerning  $\mathbf{A}_h$ . To this end, let  $u \in \mathcal{C}^\infty(\mathbf{E})$ . Set

$$f := \mathbf{A}_h u = [1 + (\mathbb{T}_h \pi_v)^* (\mathbb{T}_h \pi_v)]^{-1} (\mathbb{T}_h \pi_v)^* u = -[1 + \mathbf{m}_2 \Delta_{U,h}]^{-1} \pi_v (v \cdot \mathbf{d}_{U,h})^* u.$$

Then, it holds:  $f = \pi_v f \in \mathcal{C}^\infty(\mathbf{M})$  (by elliptic regularity). The fact that  $[1 + (\mathbb{T}_h \pi_v)^* (\mathbb{T}_h \pi_v)] f = (\mathbb{T}_h \pi_v)^* u$  leads to,

$$\|f\|_{L^2(\mathbf{E})}^2 + \|v \cdot \mathbf{d}_{U,h} f\|_{L^2(\mathbf{E})}^2 = -\langle \pi_v (v \cdot \mathbf{d}_{U,h})^* u, f \rangle_{L^2(\mathbf{E})} = -\langle u, v \cdot \mathbf{d}_{U,h} f \rangle_{L^2(\mathbf{E})}.$$

Recall that  $\pi_v (v \cdot \mathbf{d}_{U,h} f) = \pi_v \mathbb{T}_h \pi_v f = 0$ . Consequently  $\langle u, v \cdot \mathbf{d}_{U,h} f \rangle_{L^2(\mathbf{E})} = \langle (1 - \pi_v) u, v \cdot \mathbf{d}_{U,h} f \rangle_{L^2(\mathbf{E})}$ . Using in addition the Cauchy-Schwarz inequality, one deduces that

$$\max(\|f\|_{L^2(\mathbf{E})}, \|v \cdot \mathbf{d}_{U,h} f\|_{L^2(\mathbf{E})}) \leq \|(1 - \pi_v) u\|_{L^2(\mathbf{E})}.$$

Thus, the closure of  $(\mathbf{A}_h, \mathcal{C}^\infty(\mathbf{E}))$  exists and is a bounded operator on  $L^2(\mathbf{E})$  with norm smaller than 1. Moreover,  $\|v \cdot \mathbf{d}_{U,h} f\|_{L^2(\mathbf{E})} = \|(\mathbb{T}_h \pi_v) f\|_{L^2(\mathbf{E})} \leq \|(1 - \pi_v) u\|_{L^2(\mathbf{E})}$ , which proves (18).

Let us now prove the statements concerning  $\mathbf{A}_h^*$ . It is clear, using an integration by parts, that for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,  $\mathbf{A}_h^* u = -v \cdot \mathbf{d}_{U,h} [1 + \mathbf{m}_2 \Delta_{U,h}]^{-1} \pi_v u$ . Set  $f := [1 + \mathbf{m}_2 \Delta_{U,h}]^{-1} \pi_v u$  which is a function independent of  $v \in \mathbf{V}$  and belongs to  $\mathcal{C}^\infty(\mathbf{E})$ . It thus holds  $\mathbf{A}_h^* u = -v \cdot \mathbf{d}_{U,h} f$  and  $\|\mathbf{A}_h^* u\|_{L^2(\mathbf{E})} = \|v \cdot \mathbf{d}_{U,h} f\|_{L^2(\mathbf{E})} \leq \|\mathbf{d}_{U,h} f\|_{L^2(\mathbf{M})}$ . The function  $f$  is solution to

$$f + \mathbf{m}_2 \Delta_{U,h} f = \pi_v u \text{ on } \mathbf{M}. \quad (20)$$

This implies that

$$\|f\|_{L^2(\mathbf{M})}^2 + \mathbf{m}_2 \|\mathbf{d}_{U,h} f\|_{L^2(\mathbf{M})}^2 \leq |\langle \pi_v u, f \rangle_{L^2(\mathbf{M})}|,$$

and therefore,  $\|f\|_{L^2(\mathbf{M})} \leq \|\pi_v u\|_{L^2(\mathbf{M})}$  and  $\|\mathbf{d}_{U,h} f\|_{L^2(\mathbf{M})} \leq \mathbf{m}_2^{-1/2} \|\pi_v u\|_{L^2(\mathbf{M})}$ . This proves (19) and concludes the proof of Proposition 9.  $\blacksquare$

The operator  $\mathbf{A}_h$  is the corner stone of our analysis. This operator was used in [1, 13] to prove the convergence to the equilibrium measure when  $h > 0$  is fixed. A similar operator was introduced in [20, 23, 35] to study the Boltzmann equation and the Kramers-Fokker-Planck equation. We also refer to [25] in this connexion.

## 4.2 Semi-classical hypo coercive estimate for $P_h$

The aim of this section is to prove Proposition 10 which gives a semiclassical hypo coercive estimate on  $P_h$ . Such estimates were first derived in [23] (see also [35]) for the Boltzmann operator.

**Proposition 10.** *Let us assume that (Morse) holds. There exist  $c_0 > 0$  and  $h_0 > 0$  such that for all  $h \in (0, h_0)$  and all for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ , it holds:*

$$\operatorname{Re} \langle P_h u, [1 + \varepsilon_0 h(\mathbf{A}_h + \mathbf{A}_h^*)] u \rangle_{L^2(\mathbf{E})} \geq c_0 h^2 \|u\|_{L^2(\mathbf{E})}^2. \quad (21)$$

*This implies that for all  $c_1 \in (0, c_0)$ , if  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  and for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ :*

$$\|(P_h - z)u\|_{L^2(\mathbf{E})} \geq \frac{c_1}{2} h^2 \|u\|_{L^2(\mathbf{E})}. \quad (22)$$

*This equality extends to all  $u \in D(P_h) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ . Finally, Equation (22) holds for  $P_h^*$  (with different constants).*

Here and in the following, the space  $\mathbf{G}^{\perp L^2(\mathbf{E})}$  denotes the orthogonal of  $\mathbf{G}$  in  $L^2(\mathbf{E})$ .

Let us prove Proposition 10.

*Proof of Proposition 10.* Let  $\varepsilon > 0$ . For  $u \in \mathcal{C}^\infty(\mathbf{E})$ , one has from Lemma 3 and (2):

$$\begin{aligned} \operatorname{Re} \langle P_h u, [1 + \varepsilon(\mathbf{A}_h + \mathbf{A}_h^*)] u \rangle_{L^2(\mathbf{E})} &= \operatorname{Re} \langle P_h u, u \rangle_{L^2(\mathbf{E})} + \varepsilon \operatorname{Re} \langle P_h u, (\mathbf{A}_h + \mathbf{A}_h^*) u \rangle_{L^2(\mathbf{E})} \\ &\geq r_0 \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2 + \varepsilon \operatorname{Re} \langle P_h u, (\mathbf{A}_h + \mathbf{A}_h^*) u \rangle_{L^2(\mathbf{E})}. \end{aligned} \quad (23)$$

In addition,

$$\begin{aligned} \operatorname{Re} \langle P_h u, (\mathbf{A}_h + \mathbf{A}_h^*) u \rangle_{L^2(\mathbf{E})} &= \operatorname{Re} \langle S u, (\mathbf{A}_h + \mathbf{A}_h^*) u \rangle_{L^2(\mathbf{E})} + \operatorname{Re} \langle T_h u, (\mathbf{A}_h + \mathbf{A}_h^*) u \rangle_{L^2(\mathbf{E})} \\ &= \operatorname{Re} \langle \mathbf{A}_h S u, u \rangle_{L^2(\mathbf{E})} + \operatorname{Re} \langle \mathbf{A}_h T_h (1 - \pi_v) u, u \rangle_{L^2(\mathbf{E})} \\ &\quad + \operatorname{Re} \langle S u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} + \operatorname{Re} \langle T_h u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} \\ &\quad + \operatorname{Re} \langle \mathbf{A}_h T_h \pi_v u, u \rangle_{L^2(\mathbf{E})}. \end{aligned}$$

We will now estimate each of the above terms. In the following  $C > 0$  is a constant independent of  $h$  and  $u$  which can change from one occurrence to another.

**Step 1.** Lower bound on  $\varepsilon \operatorname{Re} \langle \mathbf{A}_h T_h \pi_v u, u \rangle_{L^2(\mathbf{E})}$ . Because  $\pi_v \mathbf{A}_h = \mathbf{A}_h$  (see Proposition 9) it holds

$$\operatorname{Re} \langle \mathbf{A}_h T_h \pi_v u, u \rangle_{L^2(\mathbf{E})} = \operatorname{Re} \langle \mathbf{A}_h T_h \pi_v u, \pi_v u \rangle_{L^2(\mathbf{E})}.$$

From (16) and (17), we recall that it holds:

$$\mathbf{A}_h T_h \pi_v = [1 + \mathbf{m}_2 \Delta_{U,h}]^{-1} \mathbf{m}_2 \Delta_{U,h} \pi_v \text{ on } \mathcal{C}^\infty(\mathbf{E}),$$

Assume that (Morse) holds. From Lemma 8, there exists  $\delta_0 > 0$  such that for  $h$  small enough and for all  $w \in \mathcal{C}^\infty(\mathbf{M}) \cap \{\Phi_{1,h}, \dots, \Phi_{m_0,h}\}^{\perp L^2(\mathbf{M})}$ ,

$$\langle \mathbf{m}_2 [1 + \mathbf{m}_2 \Delta_{U,h}]^{-1} \Delta_{U,h} w, w \rangle_{L^2(\mathbf{M})} \geq \delta_0 h \|w\|_{L^2(\mathbf{M})}^2.$$

Let  $u \in \mathcal{C}^\infty(\mathbf{E})$ . Assume that  $u \in \mathbf{G}^{\perp L^2(\mathbf{E})}$ , i.e. that  $\pi_v u \in \{\Phi_{1,h}, \dots, \Phi_{m_0,h}\}^{\perp L^2(\mathbf{M})}$ . Using the previous inequality, one deduces that for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ :

$$\varepsilon \operatorname{Re} \langle \mathbf{A}_h T_h \pi_v u, u \rangle_{L^2(\mathbf{E})} = \varepsilon \operatorname{Re} \langle \mathbf{A}_h T_h \pi_v u, \pi_v u \rangle_{L^2(\mathbf{E})} \geq \varepsilon \delta_0 h \|\pi_v u\|_{L^2(\mathbf{E})}^2.$$

**Step 2.** Upper bound on the term  $\operatorname{Re} \langle \mathbf{A}_h \mathbf{S}u, u \rangle_{L^2(\mathbf{E})} + \operatorname{Re} \langle \mathbf{A}_h \mathbf{T}_h(1 - \pi_v)u, u \rangle_{L^2(\mathbf{E})}$ .

Let  $u \in \mathcal{C}^\infty(\mathbf{E})$ . Using (14), it holds  $\langle \mathbf{A}_h \mathbf{S}u, u \rangle_{L^2(\mathbf{E})} = \langle (1 - \pi_v)u, \mathbf{S}\mathbf{A}_h^*u \rangle_{L^2(\mathbf{E})}$ . Thus from (15), one has for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\langle \mathbf{A}_h \mathbf{S}u, u \rangle_{L^2(\mathbf{E})} \leq C \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\mathbf{S}\| \|\mathbf{A}_h^*u\|_{L^2(\mathbf{E})} \leq C \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\mathbf{A}_h^*u\|_{L^2(\mathbf{E})}. \quad (24)$$

Using (19), it then holds for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\langle \mathbf{A}_h \mathbf{S}u, u \rangle_{L^2(\mathbf{E})} \leq C \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\pi_v u\|_{L^2(\mathbf{E})}.$$

Let us now deal with the term  $\langle \mathbf{A}_h \mathbf{T}_h(1 - \pi_v)u, u \rangle_{L^2(\mathbf{E})}$ . One has for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\langle \mathbf{A}_h \mathbf{T}_h(1 - \pi_v)u, u \rangle_{L^2(\mathbf{E})} = -\langle (1 - \pi_v)u, \mathbf{T}_h \mathbf{A}_h^*u \rangle_{L^2(\mathbf{E})} \leq \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\mathbf{T}_h \mathbf{A}_h^*u\|_{L^2(\mathbf{E})},$$

where we used that  $\mathbf{T}_h^* = -\mathbf{T}_h$  on  $\mathcal{C}^\infty(\mathbf{E})$  and  $\mathbf{A}_h^*u \in \mathcal{C}^\infty(\mathbf{E})$  by elliptic regularity. Let  $f := [1 + \mathbf{m}_2 \Delta_{U,h}]^{-1} \pi_v u$ . Then, it holds:

$$\begin{aligned} \mathbf{T}_h \mathbf{A}_h^*u &= \left[ v \cdot \mathbf{d}_{U,h} - \frac{1}{2}(v \cdot \partial_x V)(1 - \mathbf{B}) \right] v \cdot \mathbf{d}_{U,h} f \\ &= \left[ v \cdot h \partial_x + v \cdot \partial_x U - \frac{1}{2}(v \cdot \partial_x V)(1 - \mathbf{B}) \right] v \cdot \mathbf{d}_{U,h} f \\ &= v \cdot h \partial_x (v \cdot h \partial_x f + f v \cdot \partial_x U) + \left[ v \cdot \partial_x U - \frac{1}{2}(v \cdot \partial_x V)(1 - \mathbf{B}) \right] v \cdot \mathbf{d}_{U,h} f. \end{aligned}$$

Therefore, for  $h$  small enough, it holds:

$$C \|\mathbf{T}_h \mathbf{A}_h^*u\|_{L^2(\mathbf{E})} \leq h^2 \|f\|_{H^2(\mathbf{M})} + h \|\partial_x f\|_{L^2(\mathbf{M})} + \|f\|_{L^2(\mathbf{M})}.$$

From (20) and the lines after, one has,

$$\|f\|_{L^2(\mathbf{M})} + \|\mathbf{d}_{U,h} f\|_{L^2(\mathbf{M})} \leq C \|\pi_v u\|_{L^2(\mathbf{M})}.$$

Moreover, Equation (20) writes

$$-\mathbf{m}_2 h^2 \Delta_x f = \mathbf{m}_2 (h \Delta_x V - |\partial_x V|^2) f - f + \pi_v u.$$

Thus, multiplying by  $f$  and using an integration by parts, for all  $h$  such that  $|h| \leq 1$ , it holds:

$$h^2 \|\partial_x f\|_{L^2(\mathbf{M})}^2 \leq C \|\pi_v u\|_{L^2(\mathbf{M})}^2.$$

Let us recall that from the standard elliptic regularity on  $\mathbf{M}$ , one has,

$$\|f\|_{H^2(\mathbf{M})} \leq C (\|\Delta_x f\|_{L^2(\mathbf{M})} + \|f\|_{L^2(\mathbf{M})}).$$

Thus for  $|h| \leq 1$  small enough, it holds

$$\|f\|_{H^2(\mathbf{M})} \leq C [h^{-2} (\|f\|_{L^2(\mathbf{M})} + \|\pi_v u\|_{L^2(\mathbf{M})}) + \|f\|_{L^2(\mathbf{M})}] \leq Ch^{-2} \|\pi_v u\|_{L^2(\mathbf{M})}.$$

In conclusion, for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\langle \mathbf{A}_h \mathbf{T}_h u, u \rangle_{L^2(\mathbf{E})} \leq C \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\pi_v u\|_{L^2(\mathbf{E})}.$$

**Step 3.** Upper bound on the term  $\operatorname{Re} \langle \mathbf{S}u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} + \operatorname{Re} \langle \mathbf{T}_h u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})}$ .

On the one hand, from Proposition 9, it holds  $\pi_v \mathbf{A}_h = \mathbf{A}_h$ , and since from (14),  $\pi_v \mathbf{S} = 0$ , one has, for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\langle \mathbf{S}u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} = \langle \pi_v \mathbf{S}u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} = 0.$$

On the other hand, since  $\pi_v \mathbf{T}_h \pi_v = 0$ , one has, for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\begin{aligned} \langle \mathbf{T}_h u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} &= \langle \mathbf{T}_h u, \pi_v \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} \\ &= \langle \pi_v \mathbf{T}_h (1 - \pi_v)u, \pi_v \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} \\ &= -\langle (1 - \pi_v)u, \mathbf{T}_h \pi_v \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} \leq \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\mathbf{T}_h \pi_v \mathbf{A}_h\|_{L^2(\mathbf{E})}. \end{aligned}$$

Since  $\|\mathbf{T}_h \pi_v \mathbf{A}_h\|_{L^2(\mathbf{E})} \leq \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}$  (see (18)), one deduces that for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ ,

$$\langle \mathbf{T}_h u, \mathbf{A}_h u \rangle_{L^2(\mathbf{E})} \leq C \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2.$$

**Step 4.** We have proved that there exist  $C > 0$  and  $\delta_0 > 0$  such that for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$  and  $h$  small enough, it holds:

$$\begin{aligned} \operatorname{Re} \langle \mathbf{P}_h u, [1 + \varepsilon (\mathbf{A}_h + \mathbf{A}_h^*)]u \rangle_{L^2(\mathbf{E})} &\geq r_0 \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2 + \varepsilon \delta_0 h \|\pi_v u\|_{L^2(\mathbf{E})}^2 \\ &\quad - C\varepsilon \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\pi_v u\|_{L^2(\mathbf{E})} \\ &\quad - C\varepsilon \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2. \end{aligned}$$

Let  $\lambda_0 := \delta_0/C$ . Define  $\lambda_v := r_0/C$  and  $\lambda_x(h) := \lambda_0 h$ . Then for  $h$  small enough, one has

$$\begin{aligned} C^{-1} \operatorname{Re} \langle \mathbf{P}_h u, [1 + \varepsilon (\mathbf{A}_h + \mathbf{A}_h^*)]u \rangle_{L^2(\mathbf{E})} &\geq (\lambda_v - \varepsilon) \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2 + \varepsilon \lambda_x(h) \|\pi_v u\|_{L^2(\mathbf{E})}^2 \\ &\quad - \varepsilon \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\pi_v u\|_{L^2(\mathbf{E})} \\ &= X_h(u)^t M_h X_h(u), \end{aligned}$$

where the vector field  $X_h(u)$  equals to

$$X_h(u) := (\|\pi_v u\|_{L^2(\mathbf{E})}, \|(1 - \pi_v)u\|_{L^2(\mathbf{E})})^t,$$

and the symmetric  $2 \times 2$  real matrix  $M_h$  equals to:

$$M_h := \begin{pmatrix} \varepsilon \lambda_x(h) & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \lambda_v - \varepsilon \end{pmatrix}.$$

According to [1, Section 3.1 and Lemma 23] (here  $\lambda_x(h) < 1$  for  $h$  small enough), for all  $h$  small enough, the smallest eigenvalue  $\Lambda_0(\varepsilon)$  of  $M_h$  is non negative providing that:

$$\varepsilon \leq \frac{4\lambda_v \lambda_x(h)}{4\lambda_x(h) + 1}$$

and equals

$$2\Lambda_0(\varepsilon) = \lambda_v - \varepsilon(1 - \lambda_x(h)) - \sqrt{[\lambda_v - \varepsilon(1 - \lambda_x(h))]^2 - 4\varepsilon \lambda_x(h)(\lambda_v - \varepsilon) + \varepsilon^2}.$$

When  $h > 0$  fixed, from [1, item (b) Lemma 24],  $\varepsilon \in [0, 4\lambda_x(h)\lambda_v/(4\lambda_x(h) + 1)] \mapsto \Lambda_0(\varepsilon)$  (see [1, item (b) Lemma 24]) attains its maximum at a unique point  $\varepsilon_{max}$  whose expression is given by [1, Lemma 23]<sup>2</sup>:

$$\varepsilon_{max} = \lambda_v \times \frac{1 + \lambda_x(h) - (1 - \lambda_x(h)) \left[ 1/(1 + 4\lambda_x(h)) \right]^{\frac{1}{2}}}{(1 + \lambda_x(h))^2 + 1}. \quad (25)$$

<sup>2</sup>This follows from the fact that, with the notation of [1],  $\Lambda_0(\varepsilon) = \lambda_v \Lambda(\varepsilon/\lambda_v)$ .

An asymptotic expansion when  $h \rightarrow 0$  of  $\varepsilon_{\max}$  then leads to  $\varepsilon_{\max} = \varepsilon_0 h + O(h^2)$  where  $\varepsilon_0 = \lambda_v \lambda_0$ . Then, one has in the limit  $h \rightarrow 0$ :

$$2\Lambda_0(\varepsilon_0 h) = 2\varepsilon_0 \lambda_0 \left(1 - \frac{\varepsilon_0}{4\lambda_v \lambda_0}\right) h^2 + O(h^3) = \frac{3}{2} \lambda_v \lambda_0^2 h^2 + O(h^3). \quad (26)$$

Indeed, one has:

$$2\Lambda_0(\varepsilon_0 h) = \lambda_v - \varepsilon_0 h + \varepsilon_0 \lambda_0 h^2 - \sqrt{\lambda_v^2 - 2\varepsilon_0 \lambda_v h + \mathbf{p} h^2} + O(h^3),$$

with  $\mathbf{p} = 2\varepsilon_0 \lambda_0 \lambda_v + \varepsilon_0^2 - 4\varepsilon_0 \lambda_0 \lambda_v + \varepsilon_0^2 = -2\varepsilon_0 \lambda_0 \lambda_v + 2\varepsilon_0^2$ . Then, when  $h \rightarrow 0$ ,

$$\begin{aligned} -\lambda_v \sqrt{1 - \frac{2\varepsilon_0}{\lambda_v} h + \frac{\mathbf{p}}{\lambda_v^2} h^2 + O(h^3)} &= -\lambda_v \left[1 + \frac{1}{2} \left(-\frac{2\varepsilon_0}{\lambda_v} h + \frac{\mathbf{p}}{\lambda_v^2} h^2\right) - \frac{1}{8} \frac{4\varepsilon_0^2}{\lambda_v^2} h^2 + O(h^3)\right] \\ &= -\lambda_v + \varepsilon_0 h + \left(-\frac{\mathbf{p}}{2\lambda_v} + \frac{\varepsilon_0^2}{2\lambda_v^2}\right) h^2 + O(h^3). \end{aligned}$$

This implies that

$$2\Lambda_0(\varepsilon_0 h) = \left[\varepsilon_0 \lambda_0 - \frac{\mathbf{p}}{2\lambda_v} + \frac{\varepsilon_0^2}{2\lambda_v^2}\right] h^2 + O(h^3) = \left(2\varepsilon_0 \lambda_0 - \frac{\varepsilon_0^2}{2\lambda_v}\right) h^2 + O(h^3),$$

which is (26). Thus, there exists  $c_0 > 0$  such that for  $h$  small enough and for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ , one has:

$$\operatorname{Re} \langle \mathbf{P}_h u, [1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] u \rangle_{L^2(\mathbf{E})} \geq c_0 h^2 \|u\|_{L^2(\mathbf{E})}^2.$$

This concludes the proof of the first statement in Proposition 10, namely (21). Let us now prove (22). Let  $z \in \mathbb{C}$  and  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ . Then, it holds:

$$\|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} \|(1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*))u\|_{L^2(\mathbf{E})} \geq c_0 h^2 \|u\|_{L^2(\mathbf{E})}^2 - \operatorname{Re} (z \langle u, [1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] u \rangle_{L^2(\mathbf{E})}).$$

Since  $1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)$  is a (bounded) self-adjoint operator, it holds  $\langle u, [1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] u \rangle_{L^2(\mathbf{E})} \in \mathbb{R}$ . Moreover, for all  $w \in \mathcal{C}^\infty(\mathbf{E})$ :

$$\begin{aligned} \langle w, [1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] w \rangle_{L^2(\mathbf{E})} &= \|w\|_{L^2(\mathbf{E})}^2 + \varepsilon_0 h \langle w, (\mathbf{A}_h + \mathbf{A}_h^*) w \rangle_{L^2(\mathbf{E})} \\ &\geq \|w\|_{L^2(\mathbf{E})}^2 - \varepsilon_0 h \|w\|_{L^2(\mathbf{E})}^2 (\|\mathbf{A}_h\| + \|\mathbf{A}_h^*\|) \\ &\geq (1 - 2\varepsilon_0 h) \|w\|_{L^2(\mathbf{E})}^2 \geq 0, \end{aligned}$$

for  $h > 0$  such that  $2\varepsilon_0 h < 1$  and where we have used that  $\|\mathbf{A}_h^*\| = \|\mathbf{A}_h\| \leq 1$ , see Proposition 9. Thus, it holds:

$$0 \leq \langle w, [1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] w \rangle_{L^2(\mathbf{E})} \leq \|w\|_{L^2(\mathbf{E})} \|[1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] w\|_{L^2(\mathbf{E})}.$$

Thus, it holds, for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ ,

$$\begin{aligned} \|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} \|(1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*))u\|_{L^2(\mathbf{E})} &\geq c_0 h^2 \|u\|_{L^2(\mathbf{E})}^2 \\ &\quad - (\operatorname{Re} z)_+ \|u\|_{L^2(\mathbf{E})} \|[1 + \varepsilon_0 h (\mathbf{A}_h + \mathbf{A}_h^*)] u\|_{L^2(\mathbf{E})}, \end{aligned}$$

where we recall that  $(\operatorname{Re} z)_+ = \max(\operatorname{Re} z, 0)$ . Consequently,

$$\|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} [1 + \varepsilon_0 h (\|\mathbf{A}_h\| + \|\mathbf{A}_h^*\|)] \|u\|_{L^2(\mathbf{E})} \geq \left[ c_0 h^2 - (\operatorname{Re} z)_+ [1 + \varepsilon_0 h (\|\mathbf{A}_h\| + \|\mathbf{A}_h^*\|)] \right] \|u\|_{L^2(\mathbf{E})}^2.$$



and therefore,

$$\|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} \geq \left[ \frac{c_0 h^2}{1 + 2\varepsilon_0 h} - (\operatorname{Re} z)_+ \right] \|u\|_{L^2(\mathbf{E})}.$$

Let  $c_1 \in (0, c_0)$ . Then, for  $h$  small enough and  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ ,  $\|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} \geq [c_1 h^2 - (\operatorname{Re} z)_+] \|u\|_{L^2(\mathbf{E})}^2$ . Let us assume that  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ . Then,  $\|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} \geq \frac{c_1}{2} h^2 \|u\|_{L^2(\mathbf{E})}^2$ . This proves (22).

Let us show that this equality extends to all  $u \in D(\mathbf{P}_h) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ . Let  $u \in D(\mathbf{P}_h) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ . Let  $(u_n)_{n \in \mathbb{N}} \in \mathcal{C}^\infty(\mathbf{E})^{\mathbb{N}}$  such that  $u_n \rightarrow u$  and  $\mathbf{P}_h u_n \rightarrow \mathbf{P}_h u$  in  $L^2(\mathbf{E})$  when  $n \rightarrow +\infty$ . Notice that  $\pi_{\mathbf{G}} w \in \mathcal{C}^\infty(\mathbf{E})$  for all  $w \in L^2(\mathbf{E})$ , because  $\pi_{\mathbf{G}}$  is the orthogonal projection on a finite number of smooth functions, namely  $(\Phi_{j,h} \mathbf{1}_V)_{j=1, \dots, m_0}$ . Therefore,  $(1 - \pi_{\mathbf{G}})u_n \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$  and thus, for all  $n$ , it holds:

$$\|(\mathbf{P}_h - z)(1 - \pi_{\mathbf{G}})u_n\|_{L^2(\mathbf{E})} \geq \frac{c_1}{2} h^2 \|(1 - \pi_{\mathbf{G}})u_n\|_{L^2(\mathbf{E})}^2. \quad (27)$$

Moreover,  $\pi_{\mathbf{G}} u_n \rightarrow \pi_{\mathbf{G}} u$  in  $L^2(\mathbf{E})$  when  $n \rightarrow +\infty$  and  $\mathbf{P}_h \pi_{\mathbf{G}} u_n \rightarrow \mathbf{P}_h \pi_{\mathbf{G}} u$  in  $L^2(\mathbf{E})$  when  $n \rightarrow +\infty$ , because  $\mathbf{P}_h \pi_{\mathbf{G}}$  is a finite rank operator. Thus passing to the limit in (27), and since  $(1 - \pi_{\mathbf{G}})u = u$ , it holds:

$$\|(\mathbf{P}_h - z)u\|_{L^2(\mathbf{E})} \geq \frac{c_1}{2} h^2 \|(1 - \pi_{\mathbf{G}})u\|_{L^2(\mathbf{E})}^2.$$

The same analysis leads to the same estimate for  $\mathbf{P}_h^*$ . This concludes the proof of Proposition 10.

■

We now proceed to the proof of Theorems 1 and 2.

## 5 Proofs of Theorems 1 and 2

Let us recall that

$$\mathbf{P}_h \in \{\mathbf{P}_h^{\text{BPS}}, \mathbf{P}_h^{\text{ZZ}}\}.$$

### 5.1 Proof of Theorem 1

In this section we prove Theorem 1. To this end, we prove the following result.

**Proposition 11.** *Assume that (Morse) holds. Let  $c_1 > 0$  be as in (22) and  $c_2 < c_1/2$ . Then,*

1. *There exists  $K > 0$  and  $h_0 > 0$  such that for all  $z \in \mathbb{C}$  such that  $|z| \geq c_2 h^2$  and  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ , and for all  $h \in (0, h_0)$ , it holds,*

$$\|(\mathbf{P}_h - z)^{-1}\| \leq \frac{K}{h^2}.$$

*Moreover, there exists  $\delta > 0$ , such that for  $h$  small enough  $\sigma(\mathbf{P}_h) \cap \{\operatorname{Re} z \leq \frac{c_1}{2} h^2\} \subset \{|z| \leq e^{-\frac{\delta}{h}}\}$ . The same holds for  $\mathbf{P}_h^*$  (with different constants). For  $\eta \in (c_2, c_1/2)$ , the spectral Riesz projection*

$$\pi_{\eta h^2}(\mathbf{P}_h) = -\frac{1}{2i\pi} \int_{|z|=\eta h^2} (\mathbf{P}_h - z)^{-1} dz, \quad (28)$$

*is thus well defined for  $h$  small enough.*

2. *There exists  $\eta_1 > 0$  such that for all  $\eta \in (0, \eta_1)$  it holds for all  $h$  small enough,*

$$\dim \operatorname{Ran} \pi_{\eta h^2}(\mathbf{P}_h) = m_0.$$

*The same holds for  $\pi_{\eta h^2}(\mathbf{P}_h^*) = \pi_{\eta h^2}(\mathbf{P}_h)^*$ .*

*Proof.* Assume that **(Morse)** holds. The proof is inspired by [28, Section 2.2]. The main idea is to build a Grushin problem. We refer to [36] for a review on this topic. Let us define the operator

$$\hat{P}_h = (1 - \pi_G)P_h(1 - \pi_G) : D(P_h) \cap \mathbf{G}^{\perp L^2(\mathbb{E})} \subset \mathbf{G}^{\perp L^2(\mathbb{E})} \rightarrow \mathbf{G}^{\perp L^2(\mathbb{E})}, \quad (29)$$

which is well defined because  $\pi_G L^2(\mathbb{E}) \subset \mathcal{C}^\infty(\mathbb{E}) \subset D(P_h)$ . We equip  $\mathbf{G}^{\perp L^2(\mathbb{E})}$  with the Hermitian inner product of  $L^2(\mathbb{E})$ . This is a Hilbert space. Notice that  $\hat{P}_h$  is a closed (it is the sum of  $P_h$  and the finite rank operator  $-\pi_G P_h - P_h \pi_G + \pi_G P_h \pi_G$ ) and densely defined on  $\mathbf{G}^{\perp L^2(\mathbb{E})}$  (indeed, for  $u \in \mathbf{G}^{\perp L^2(\mathbb{E})}$ , there exists  $(u_n)_{n \geq 0} \in \mathcal{C}^\infty(\mathbb{E})$  such that  $u_n \rightarrow u$  in  $L^2(\mathbb{E})$  and thus  $\mathbf{G}^{\perp L^2(\mathbb{E})} \cap \mathcal{C}^\infty(\mathbb{E}) \ni (1 - \pi_G)u_n \rightarrow (1 - \pi_G)u = u$  in  $L^2(\mathbb{E})$ ).

**Step 1.** Let us prove that the operator  $\hat{P}_h - z$  is invertible for  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and  $h$  small enough (up to choosing  $c_1$  smaller, where  $c_1$  is as in (22)).

From (22) and the Pythagorean Theorem, if  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  and for all  $u \in D(P_h) \cap \mathbf{G}^{\perp L^2(\mathbb{E})}$ :

$$\|(\hat{P}_h - z)u\|_{L^2(\mathbb{E})} \geq \frac{c_1}{2}h^2 \|u\|_{L^2(\mathbb{E})}.$$

This proves that  $(\hat{P}_h - z)$  is injective with closed range. Let us now prove that its range is dense. To this end, we need to identify the adjoint  $\hat{P}_h^\dagger$  of  $\hat{P}_h$  to make use of the relation

$$\operatorname{Ran}(\hat{P}_h - z) = \operatorname{Ker}(\hat{P}_h^\dagger - z)^\perp,$$

which holds because  $\hat{P}_h$  is a closed accretive operator. Recall that  $\hat{P}_h$  is the sum of  $P_h$  and the bounded operator  $-\pi_G P_h - P_h \pi_G + \pi_G P_h \pi_G$ . Therefore, the adjoint of  $(\hat{P}_h, D(P_h^*) \cap \mathbf{G}^{\perp L^2(\mathbb{E})})$  on  $\mathbf{G}^{\perp L^2(\mathbb{E})}$  is the operator

$$\hat{P}_h^* = (1 - \pi_G)P_h^*(1 - \pi_G)$$

with domain  $D(P_h) \cap \mathbf{G}^{\perp L^2(\mathbb{E})}$  (recall that  $D(P_h^*) = D(P_h)$ , see Proposition 4).

To prove that the range of  $\hat{P}_h - z$  is dense, it is thus sufficient to prove that  $\hat{P}_h^*$  is injective. But this follows from the fact that since  $P_h^*(1 - \pi_G) = \hat{P}_h^*$  also satisfies a resolvent estimate (22) on  $D(P_h^*) \cap \mathbf{G}^{\perp L^2(\mathbb{E})}$  for all  $z$  such that  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  up to choosing  $c_1 > 0$  smaller (see Proposition 10),  $(1 - \pi_G)P_h^*(1 - \pi_G) - z$  is injective. In conclusion,  $\hat{P}_h - z$  is invertible for all  $z \in \mathbb{C}$  such that  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and it holds:

$$\|(\hat{P}_h - z)^{-1}\| \leq \frac{2}{c_1 h^2} \quad (30)$$

Here  $\|(\hat{P}_h - z)^{-1}\|$  denotes the norm of  $(\hat{P}_h - z)^{-1} \in \mathcal{L}(\mathbf{G}^{\perp L^2(\mathbb{E})})$ .

**Step 2.** Grushin problem.

We define the operators (see (10)):

$$\mathbf{R}_- : \mathbb{C}^{m_0} \rightarrow L^2(\mathbb{E}), \quad (\mu_k)_{j=1, \dots, m_0} \mapsto \sum_{j=1}^{m_0} \mu_j \Phi_{j,h} \mathbf{1}_V,$$

and

$$\mathbf{R}_+ : L^2(\mathbb{E}) \rightarrow \mathbb{C}^{m_0}, \quad u \mapsto (\langle \Phi_{j,h} \mathbf{1}_V, u \rangle_{L^2(\mathbb{E})})_{j=1, \dots, m_0}.$$

We equip  $\mathbb{C}^{m_0}$  with the  $\ell_2$  norm. Notice that

$$\mathbf{R}_+ \mathbf{R}_- = \mathbf{I}_{\mathbb{C}^{m_0}} \text{ and } \mathbf{R}_- \mathbf{R}_+ = \pi_G. \quad (31)$$

In addition, it holds:

$$\|R_+\| \leq \sqrt{m_0}, \quad \|R_-\| \leq \sqrt{m_0}, \quad \|R_+P_h\| \leq e^{-\frac{c}{h}} \quad \text{and} \quad \|P_hR_-\| \leq e^{-\frac{c}{h}}, \quad (32)$$

where the two previous inequalities follows from (11) and hold for  $h$  small enough ( $c > 0$  is independent of  $h$ ). For  $z \in \mathbb{C}$ , let us denote by  $\mathcal{P}_h(z)$  the linear operator defined by

$$\mathcal{P}_h(z) : (u, u_-) \in D(P_h) \times \mathbb{C}^{m_0} \mapsto \begin{pmatrix} (P_h - z)u + R_-u_- \\ R_+u \end{pmatrix}.$$

**Step 2a.** The Grushin problem is well posed.

Let us prove that the previous Grushin problem is well posed, i.e. let us prove that  $\mathcal{P}_h(z) : D(P_h) \times \mathbb{C}^{m_0} \rightarrow L^2(\mathbf{E}) \times \mathbb{C}^{m_0}$  is invertible for  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and  $h$  small enough. Let  $z \in \mathbb{C}$  such that  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and  $(f, y) \in L^2(\mathbf{E}, \mu) \times \mathbb{C}^{m_0}$ . Assume that  $(u, u_-) \in D(P_h) \times \mathbb{C}^{m_0}$  satisfies

$$\mathcal{P}_h(z)(u, u_-) = (f, y).$$

By definition, the previous equality means that

$$(P_h - z)u + R_-u_- = f \quad \text{and} \quad R_+u = y.$$

Applying  $R_+$  to the first equation and  $R_-$  to the second leads to  $R_+(P_h - z)u + u_- = R_+f$  and  $\pi_G u = R_-y$  (see (31)). Write  $u = \pi_G u + (1 - \pi_G)u = R_-y + (1 - \pi_G)u$ . It thus remains to find  $(1 - \pi_G)u$ . One has  $(P_h - z)(1 - \pi_G)u + (P_h - z)R_-y + R_-u_- = f$ , and applying  $(1 - \pi_G)$  leads to

$$\hat{P}_h(1 - \pi_G)u = (1 - \pi_G)f - (1 - \pi_G)P_hR_-y,$$

where we have used that  $(1 - \pi_G)R_- = 0$ . Thus, it holds:

$$u = R_-y + v \quad \text{and} \quad u_- = R_+f - R_+(P_h - z)[R_-y + v],$$

where

$$v = (\hat{P}_h - z)^{-1}(1 - \pi_G)f - (\hat{P}_h - z)^{-1}(1 - \pi_G)P_hR_-y.$$

Let us now assume choose  $z$  such that  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and  $h$  small enough. Take  $(f, y) \in L^2(\mathbf{E}, \mu) \times \mathbb{C}^{m_0}$  and choose  $(u, u_-)$  as above. Then  $v \in D(P_h) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$  (according to the previous step,  $(\hat{P}_h - z)^{-1}$  is well defined on  $\mathbf{G}^{\perp L^2(\mathbf{E})}$  and its range is  $D(P_h) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$ ). Therefore  $u \in D(P_h)$ . Since  $R_+v = 0$ , it holds  $R_+u = y$ . Moreover, using (31), it is straightforward to check that  $(P_h - z)u + R_-u_- = f$ . This proves that the previous Grushin problem is well posed. Write the inverse of  $\mathcal{P}_h(z)$  as, for  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and  $h$  small enough,

$$(f, y) \in L^2(\mathbf{E}) \times \mathbb{C}^{m_0} \mapsto \begin{pmatrix} \mathcal{E}(z) & \mathcal{E}_+(z) \\ \mathcal{E}_-(z) & \mathcal{E}_{-+}(z) \end{pmatrix} \begin{pmatrix} f \\ y \end{pmatrix},$$

where the operators  $\mathcal{E}(z)$ ,  $\mathcal{E}_+(z)$ ,  $\mathcal{E}_-(z)$ , and  $\mathcal{E}_{-+}(z)$  equal:

1.  $\mathcal{E}(z) = (\hat{P}_h - z)^{-1}(1 - \pi_G)$  is holomorphic for  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ . Using (30), it holds for  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ ,

$$\|\mathcal{E}(z)\| \leq \frac{2}{c_1 h^2}. \quad (33)$$

2.  $\mathcal{E}_{-+}(z) = -R_+(P_h - z)R_- + R_+(P_h - z)(\hat{P}_h - z)^{-1}(1 - \pi_G)P_hR_-$ .

3.  $\mathcal{E}_+(z) = R_- - (\hat{P}_h - z)^{-1}(1 - \pi_G)P_h R_-$ .
4.  $\mathcal{E}_-(z) = R_+ - R_+ P_h (\hat{P}_h - z)^{-1}(1 - \pi_G)$  (notice that we have used here that  $u_- = R_+ f - R_+(P_h - z)R_- y - R_+ P_h v$  because  $R_+ v = R_+(1 - \pi_G)v = 0$ ).

**Step 2b.** End of the proof of Proposition 11.

In the following  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$  and  $h$  is small enough. Let us recall that  $P_h - z$  is invertible if and only if  $\mathcal{E}_{-+}(z)$  is invertible (see [36]) and in this case,

$$(P_h - z)^{-1} = \mathcal{E}(z) - \mathcal{E}_+(z)\mathcal{E}_{-+}(z)^{-1}\mathcal{E}_-(z). \quad (34)$$

From (30) and (32), there exists  $c > 0$  such that for  $h$  small enough:

$$\|\mathcal{E}_-(z) - R_+\| \leq e^{-\frac{c}{h}} \text{ and } \|\mathcal{E}_+(z) - R_-\| \leq e^{-\frac{c}{h}}.$$

Let us now estimate  $\mathcal{E}_{-+}(z)$ . First, one has using (32),

$$-R_+(P_h - z)R_- = z|_{\mathbb{C}^{m_0}} + O(e^{-\frac{c}{h}}).$$

Secondly,  $R_+(P_h - z)(\hat{P}_h - z)^{-1}(1 - \pi_G)P_h R_- = R_+ P_h (\hat{P}_h - z)^{-1}(1 - \pi_G)P_h R_-$  because  $R_+ = 0$  on  $G^\perp L^2(\mathbb{E})$ . Thus, using (30) and (32),

$$\|R_+(P_h - z)(\hat{P}_h - z)^{-1}(1 - \pi_G)P_h R_-\| \leq e^{-\frac{c}{h}}.$$

Thus,  $\mathcal{E}_{-+}(z) = z|_{\mathbb{C}^{m_0}} + O(e^{-\frac{c}{h}})$ . Let  $c_2 \in (0, c_1/2)$ . Then, the operator  $\mathcal{E}_{-+}(z)$  is invertible for  $z \in \mathbb{C}$  such that  $|z| \geq c_2 h^2$  and  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ , and for  $h$  small enough. In this case, it holds  $\mathcal{E}_{-+}(z)^{-1} = z^{-1}(1 + O(e^{-\frac{c}{h}}))$ . Hence, from (34), since  $R_+ R_- = \pi_G$ ,  $\|\pi_G\| = 1$ ,  $\|R_+\| \leq \sqrt{m_0}$ , and  $\|R_-\| \leq \sqrt{m_0}$ , the previous estimates on  $\mathcal{E}_+(z)$ ,  $\mathcal{E}_-(z)$ , and  $\mathcal{E}_{-+}(z)$  imply that there exists  $c > 0$  such that:

$$(P_h - z)^{-1} = \mathcal{E}(z) - z^{-1}(\pi_G + O(e^{-\frac{c}{h}})), \quad (35)$$

for all  $h$  small enough. From (33), if  $|z| \geq c_2 h^2$  and  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ , one has for  $h$  small enough:

$$\|(P_h - z)^{-1}\| \leq \frac{2}{c_1 h^2} + \frac{(1 + O(e^{-\frac{c}{h}}))}{|z|} \leq \frac{K}{h^2},$$

where  $K > 0$  is a constant independent of  $z$  and  $h$ . The same estimates also holds for  $P_h^*$  with different constants  $c_1$  and  $c_2$ . Notice also that there exists  $\delta > 0$  (not too large) such that if  $|z| \geq e^{-\frac{\delta}{h}}$  and  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ , for  $h$  small enough,  $\mathcal{E}_{-+}(z)$  is invertible and thus so is  $P_h - z$  (see (34)). Therefore, for  $h$  small enough:

$$\sigma(P_h) \cap \left\{ \operatorname{Re} z \leq \frac{c_1}{2}h^2 \right\} \subset \left\{ |z| \leq e^{-\frac{\delta}{h}} \right\}.$$

This concludes the proof of item 1 in Proposition 11. Moreover, from (28) and (35), one has

$$\pi_{\eta h^2}(P_h) = \pi_G + O(e^{-\frac{c}{h}}).$$

This concludes the proof of Proposition 11 because  $\pi_G$  is a  $m_0$  dimensional projection.

Let us finally give an estimate which will be used in the proof of Theorem 2 below. Let  $z$  be such that  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ . For all  $(u, u_-) \in D(P_h) \times \mathbb{C}^{m_0}$ , consider  $(f, y) \in L^2(\mathbb{E}) \times \mathbb{C}^{m_0}$  such that  $P_h(z)(u, u_-) = (f, y)$ . Then,  $u = \mathcal{E}(z)f + \mathcal{E}_+(z)y$ . The previous estimates (see (28)) imply that for all  $z$  such that  $\operatorname{Re} z \leq \frac{c_1}{2}h^2$ , for all  $h$  small enough and all  $u \in D(P_h)$ ,

$$\|u\|_{L^2(\mathbb{E})} \leq \frac{2}{c_1 h^2} \|f\|_{L^2(\mathbb{E})} + (\sqrt{m_0} + e^{-\frac{c}{h}}) \|y\|_{\ell_2}, \quad (36)$$

for some  $c > 0$  independent of  $z$ ,  $h$  and  $u$ . ■

To conclude the proof of Theorem 1, in view of Proposition 11, it remains to show that in a disk of radius smaller than  $\eta h^2$ , the spectrum of  $P_h$  is made of real eigenvalues. To this end we will use the fact that the operator  $P_h$  admits a PT-symmetry property. We refer to [24, 35] for more details and references on this topic.

Let us define the operator,

$$X : u \in L^2(\mathbf{E}) \mapsto Xu(x, v) = u(x, -v) \in L^2(\mathbf{E}).$$

The operator  $X$  is unitary, self-adjoint, and  $X^{-1} = X$  in  $L^2(\mathbf{E})$ . Moreover, it holds (see Proposition 4),

$$P_h^* = X^{-1}P_hX. \quad (37)$$

Let us finally define the bilinear form,

$$u, v \in L^2(\mathbf{E}) \mapsto \langle u, v \rangle_X = \langle Xu, v \rangle_{L^2(\mathbf{E})}.$$

We have the following result.

**Lemma 12.** *Assume that (Morse) hold. Then  $u, v \in L^2(\mathbf{E}) \mapsto \langle u, v \rangle_X$  restricted to the Range of  $\pi_{\eta h^2}(P_h)$  (see (28)) is an Hermitian inner product uniformly in  $h$  small enough.*

*Proof.* We just have to check that  $u, v \in \text{Ran}(\pi_{\eta h^2}(P_h)) \mapsto \langle u, v \rangle_X$  is positive-definite uniformly in  $h$  small enough. From (11), there exists  $c > 0$  such that for  $h$  small enough, it holds for all  $j \in \{1, \dots, m_0\}$ :

$$(P_h - z)\Phi_{j,h}\mathbf{1}_V = -z\Phi_{j,h}\mathbf{1}_V + O(e^{-\frac{c}{h}}).$$

Let  $z \in \mathbb{C}$  such that  $|z| \geq c_2 h^2$  and  $\text{Re } z \leq \frac{c_1}{2} h^2$ . Using item 1 in Proposition 11, one deduces that for  $h$  small enough:

$$(P_h - z)^{-1}\Phi_{j,h}\mathbf{1}_V = -z^{-1}[\Phi_{j,h}\mathbf{1}_V + O(e^{-\frac{c}{h}})].$$

Thus (28) implies that

$$\pi_{\eta h^2}(P_h)(\Phi_{j,h}\mathbf{1}_V) = \Phi_{j,h}\mathbf{1}_V + O(e^{-\frac{c}{h}}) \text{ (and the same holds for } \pi_{\eta h^2}(P_h^*) = \pi_{\eta h^2}(P_h)^*). \quad (38)$$

Thus, for  $h$  small enough, one has:

$$\langle \pi_{\eta h^2}(P_h)\Phi_{j,h}\mathbf{1}_V, \pi_{\eta h^2}(P_h)\Phi_{i,h}\mathbf{1}_V \rangle_{L^2(\mathbf{E})} = \delta_{i,j} + O(e^{-\frac{c}{h}}). \quad (39)$$

Since  $\pi_{\eta h^2}(P_h)$  is of rank  $m_0$  for  $h$  small enough, one deduces that for  $h$  small enough,

$$\{\pi_{\eta h^2}(P_h)\Phi_{1,h}, \dots, \pi_{\eta h^2}(P_h)\Phi_{m_0,h}\} \text{ is a basis of } \text{Ran } \pi_{\eta h^2}(P_h). \quad (40)$$

Let  $j \in \{1, \dots, m_0\}$ . Since  $X(\Phi_{j,h}\mathbf{1}_V) = \Phi_{j,h}\mathbf{1}_V$ , from (39), one has:

$$\langle \pi_{\eta h^2}(P_h)\Phi_{j,h}\mathbf{1}_V, \pi_{\eta h^2}(P_h)\Phi_{i,h}\mathbf{1}_V \rangle_X = \delta_{i,j} + O(e^{-\frac{c}{h}}).$$

Therefore, for  $h$  small enough and for all  $w \in \text{Ran}(\pi_{\eta h^2}(P_h))$ , writing

$$w = \sum_{j=1}^{m_0} w_{j,h} \pi_{\eta h^2}(P_h)\Phi_{j,h}\mathbf{1}_V,$$

where  $w_{j,h} \in \mathbb{R}$ , one deduces that for  $h$  small enough

$$\langle w, w \rangle_{\mathcal{X}} = (1 + O(e^{-\frac{c}{h}})) \sum_{j=1}^{m_0} |w_{j,h}|^2,$$

where  $c > 0$  is independent of  $h$  and  $w$ . Therefore, uniformly in  $h$  small enough,  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$  is an Hermitian inner product when restricted to  $\text{Ran}(\pi_{\eta h^2}(\mathbf{P}_h))$ . This concludes the proof of the lemma. Notice that the same holds true for  $\langle w, w \rangle_{L^2(\mathbb{E})}$  and thus for  $h$  small enough, the Hermitian inner products

$$w \mapsto \langle w, w \rangle_{\mathcal{X}} \text{ and } w \mapsto \langle w, w \rangle_{L^2(\mathbb{E})} \text{ are equivalent on } \text{Ran}(\pi_{\eta h^2}(\mathbf{P}_h)), \quad (41)$$

where the equivalent constants are of order  $1 + O(e^{-\frac{c}{h}})$ .  $\blacksquare$

Let us now end the proof of Theorem 1. According to item 2 in Proposition 11 and from the Riesz decomposition theorem of the spectrum (see for instance [37, Theorem 3.14.10] or [27, Theorem 6.17]), the spectrum of  $\mathbf{P}_h$  lying inside the disk of radius  $\eta h^2$  is the spectrum of the square matrix  $\mathbf{M}_h$  of size  $m_0$  of

$$\mathbf{P}_h : \text{Ran}(\pi_{\eta h^2}(\mathbf{P}_h)) \rightarrow \text{Ran}(\pi_{\eta h^2}(\mathbf{P}_h)),$$

computed for instance in an orthonormal basis of  $\text{Ran}(\pi_{\eta h^2}(\mathbf{P}_h))$  for the the Hermitian inner product  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$  (see Lemma 12). This implies that the spectrum of  $\mathbf{P}_h$  lying inside the disk of radius  $\eta h^2$  is composed of a finite number of eigenvalues of finite algebraic multiplicities. Furthermore, from (37),  $\mathbf{M}_h$  is symmetric. Thus, these eigenvalues are real and their algebraic multiplicity equals their geometric multiplicity. From item 1 in Proposition 11, these  $m_0$  eigenvalues are actually exponentially small in the limit  $h \rightarrow 0$ . The same arguments also apply to  $\mathbf{P}_h^*$ . This concludes the proof of Theorem 1 (as already mentioned, the statement concerning the eigenvalue 0 in Theorem 1 is true for all  $h > 0$ , see Proposition 14 below).

## 5.2 Proof of Theorem 2

In this section we prove Theorem 2. Let us assume that **(Morse)** holds. Let us denote by  $\lambda_1, \dots, \lambda_{m_0}$  the  $m_0$  smallest eigenvalues of  $\mathbf{P}_h$ , which are real and exponentially small when  $h \rightarrow 0$  according to Theorem 1. Let  $\Pi_{\lambda_j}(\mathbf{P}_h)$ ,  $j = 1, \dots, m_0$ , be the spectral projection associated with  $\lambda_j$  for  $\mathbf{P}_h$ .

**Remark 13.** In [31], in a one-dimensional case, it is shown that the spectrum of  $\mathbf{P}_h$  contains a sequence  $(\lambda_n)_{n \geq 0}$  such that  $\text{Re } \lambda_n$  is bounded and  $|\text{Im } \lambda_n| \rightarrow \infty$  as  $n \rightarrow +\infty$ . This suggests that the contour deformation procedure made in [20, 25] on the semigroup  $(e^{-t\mathbf{P}_h})_{t \geq 0}$  might not be successful for  $\mathbf{P}_h$ . Thus, we rather use a resolvent estimate on  $\mathbf{P}_h(1 - \pi_{\eta h^2}(\mathbf{P}_h))$  and a quantitative version of the Gearhart-Prüss Theorem, see [18, Section 13] and [22].

Let us recall that from Proposition 10, for all  $c_1 \in (0, c_0)$ , if  $\text{Re } z \leq \frac{c_1}{2} h^2$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  and for all  $u \in D(\mathbf{P}_h) \cap \mathbf{G}^{\perp L^2(\mathbb{E})}$ :

$$\|(\mathbf{P}_h - z)u\|_{L^2(\mathbb{E})} \geq \frac{c_1}{2} h^2 \|u\|_{L^2(\mathbb{E})}.$$

Let us show that this implies a similar resolvent estimate for  $\mathbf{P}_h(1 - \pi_{\eta h^2}(\mathbf{P}_h))$ . Let  $w \in (1 - \pi_{\eta h^2}(\mathbf{P}_h))D(\mathbf{P}_h)$ . Then, one has, using (38):

$$\langle w, \Phi_{j,h} \mathbf{1}_{\mathbf{V}} \rangle_{L^2(\mathbb{E})} = \langle (1 - \pi_{\eta h^2}(\mathbf{P}_h))w, \Phi_{j,h} \mathbf{1}_{\mathbf{V}} \rangle_{L^2(\mathbb{E})} = \langle w, (1 - \pi_{\eta h^2}(\mathbf{P}_h))^* \Phi_{j,h} \mathbf{1}_{\mathbf{V}} \rangle_{L^2(\mathbb{E})} \leq e^{-\frac{c}{h}} \|w\|_{L^2(\mathbb{E})}.$$

This implies that  $\|R_+ w\|_{\ell_2} \leq e^{-\frac{c}{h}} \|w\|_{L^2(\mathbb{E})}$ . Let  $z$  be such that  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ . Then, setting  $u_- = 0$  and  $u = w$  in (36), it holds  $y = R_+ w$  and  $f = (P_h - z)w$ , and then:

$$\|w\|_{L^2(\mathbb{E})} \leq \frac{2}{c_1 h^2} \|f\|_{L^2(\mathbb{E})} + (\sqrt{m_0} + e^{-\frac{c}{h}}) e^{-\frac{c}{h}} \|w\|_{L^2(\mathbb{E})},$$

for some  $c > 0$  independent of  $z$ ,  $h$  and  $w$ . This implies that for all  $u \in D(P_h)$ , for  $h$  small enough,

$$\frac{1}{2} \|(1 - \pi_{\eta h^2}(P_h))u\|_{L^2(\mathbb{E})} \leq \frac{2}{c_1 h^2} \|(P_h - z)(1 - \pi_{\eta h^2}(P_h))u\|_{L^2(\mathbb{E})}. \quad (42)$$

Let us denote by  $\tilde{P}_h$  the operator  $P_h$  with domain  $(1 - \pi_{\eta h^2}(P_h))D(P_h)$  on the closed subspace  $F := (1 - \pi_{\eta h^2}(P_h))L^2(\mathbb{E})$  of  $L^2(\mathbb{E})$ . Let us recall that, for  $h > 0$ ,  $\tilde{P}_h$  is the generator of the strongly continuous contraction semigroup  $(e^{-t\tilde{P}_h}|_F)_{t \geq 0}$  (see for instance [37, Theorem 3.14.10]) and is thus  $m$ -accretive from the Hille-Yosida Theorem. Let us show that  $\tilde{P}_h - z$  is invertible for all  $z$  such that  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$  and for all  $h$  small enough. Equation (42) implies that  $\tilde{P}_h - z$  is injective with closed range for all  $h$  small enough and for all  $z$  such that  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ . In particular, it is a semi-Fredholm operator. Since  $\tilde{P}_h$  is  $m$ -accretive for  $h > 0$ ,  $\tilde{P}_h - z$  is invertible when  $z \in \mathbb{R}_-^*$  and thus its index is 0. This implies that, since the index is constant on the connected set  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$  (see [27, Theorem 5.17]), that the index of  $\tilde{P}_h - z$  is 0 when  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ . Thus,  $\operatorname{Ran}(\tilde{P}_h - z) = F$ . Consequently, for all  $h$  small enough and for all  $z$  such that  $\operatorname{Re} z \leq \frac{c_1}{2} h^2$ ,  $\tilde{P}_h - z$  is invertible and one has the resolvent estimate

$$\|(\tilde{P}_h - z)^{-1}\| \leq \frac{4}{c_1 h^2}.$$

The previous resolvent estimate for  $\tilde{P}_h$  implies, applying [22, Proposition 2.1] to  $e^{-t\tilde{P}_h}|_F$  (see also [18, Proposition 13.31]), that for all  $h > 0$  small enough, one has for all  $u \in L^2(\mathbb{E})$  and all  $t \geq 0$ :

$$\|e^{-t\tilde{P}_h}(1 - \pi_{\eta h^2}(P_h))u\| \leq \left[1 + 2\frac{c_1 h^2/2}{c_1 h^2/4}\right] e^{-\frac{c_1}{2} h^2 t} \left(1 + \|\pi_{\eta h^2}(P_h)\|\right) \|u\|_{L^2(\mathbb{E})}.$$

Moreover, we have:

$$\pi_{\eta h^2}(P_h) = \sum_{j=1}^{m_0} \Pi_{\lambda_j}(P_h) \text{ and for all } j \text{ and for all } t \geq 0, e^{-t\tilde{P}_h} \Pi_{\lambda_j}(P_h) = e^{-t\lambda_j} \Pi_{\lambda_j}(P_h).$$

In addition, from (41), for all  $j = 1, \dots, m_0$  and  $h$  small enough,  $\|\Pi_j\| \leq C \|\Pi_{\lambda_j}(P_h)\|_{\mathbb{X}}$  where  $\|\Pi_{\lambda_j}(P_h)\|_{\mathbb{X}}$  denotes the norm of  $\Pi_{\lambda_j}(P_h)$  when  $\operatorname{Ran}(\pi_{\eta h^2}(P_h))$  is equipped with  $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ . Because  $P_h$  is self-adjoint on  $\operatorname{Ran}(\pi_{\eta h^2}(P_h))$  for the Hermitian inner product  $\langle \cdot, \cdot \rangle_{\mathbb{X}}$  (see Lemma 12),  $\|\Pi_{\lambda_j}(P_h)\|_{\mathbb{X}} = 1$ . This concludes the proof of Theorem 2.

### 5.3 Other scalings for the refreshment operator

In this section, we investigate the effect of a different scaling for the refreshment operator. More precisely, let  $\beta \in \mathbb{R}$  be fixed and consider the the operators

$$P_h^{\text{BPS}}(\beta) = -v \cdot d_{U,h} + \lambda_{1,J}(I - B) + h^\beta \lambda_r(I - \pi_v),$$

and

$$P_h^{\text{ZZ}}(\beta) = -v \cdot d_{U,h} + \sum_{k=1}^d \lambda_{1,J}^{(k)}(I - B^{(k)}) + h^\beta \lambda_r(I - \pi_v).$$

In the following, let

$$\mathbf{P}_h(\beta) \in \{\mathbf{P}_h^{\text{BPS}}(\beta), \mathbf{P}_h^{\text{ZZ}}(\beta)\}.$$

With this notation, the operator  $\mathbf{P}_h^{\text{BPS}}(0)$  is the operator  $\mathbf{P}_h$  which has been studied in the previous sections. The antisymmetric part of  $\mathbf{P}_h(\beta)$  is still equal to  $\mathbf{T}_h$  (i.e. antisymmetric part of  $\mathbf{P}_h$ , see the beginning of Section 4.1) whereas its symmetric part now depends on  $h$  and equals

$$\mathbf{S}_h(\beta) := \frac{1}{2}(\mathbf{P}_h(\beta) + \mathbf{P}_h^*(\beta)) = \frac{1}{2}(\lambda_{1,\mathbf{J}} + \lambda_{1,\mathbf{J}}(\cdot, \cdot))(1 - \mathbf{B}) + h^\beta \lambda_r(1 - \pi_v).$$

Let us recall that Theorems 1 and 2 deal with the case  $\beta = 0$ . When  $\beta \neq 0$ , the statements of these theorems are changed as follows.

**The case when  $\beta \geq 0$ .**

Roughly speaking, when  $\beta > 0$  (resp.  $\beta = 0$ ), the refreshment operator  $h^\beta \lambda_r(1 - \pi_v)$  is smaller (resp. of the same order) than the operators  $\lambda_{1,\mathbf{J}}(1 - \mathbf{B})$  or  $\sum_{k=1}^d \lambda_{1,\mathbf{J}}^{(k)}(1 - \mathbf{B}^{(k)})$ . Let  $\beta \geq 0$ . When  $\beta \geq 0$ , for  $h \in (0, 1]$ , the symmetric part  $\mathbf{S}_h(\beta)$  of  $\mathbf{P}_h(\beta)$  still satisfies  $\|\mathbf{S}_h\| \leq c$  for some  $c > 0$  independent of  $h$ . Hence, in view of the three first steps of the proof of Proposition 10 (where only  $r_0$  in (23) is changed into  $r_0 h^\beta$ ), there exists  $\lambda_v > 0$ ,  $C > 0$ , and  $\lambda_0 > 0$ , such that for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$  and  $h \in (0, 1]$ ,

$$\begin{aligned} C^{-1} \operatorname{Re} \langle \mathbf{P}_h(\beta)u, [1 + \varepsilon(\mathbf{A}_h + \mathbf{A}_h^*)]u \rangle_{L^2(\mathbf{E})} &\geq (\lambda_v h^\beta - \varepsilon) \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2 + \varepsilon \lambda_x(h) \|\pi_v u\|_{L^2(\mathbf{E})}^2 \\ &\quad - \varepsilon \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\pi_v u\|_{L^2(\mathbf{E})} \\ &= X_h(u)^t M_h(\beta) X_h(u), \end{aligned}$$

where

$$X_h(u) := (\|\pi_v u\|_{L^2(\mathbf{E})}, \|(1 - \pi_v)u\|_{L^2(\mathbf{E})})^t, \quad M_h(\beta) := \begin{pmatrix} \varepsilon \lambda_x(h) & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \lambda_v h^\beta - \varepsilon \end{pmatrix}, \quad \text{with } \lambda_x(h) = \lambda_0 h.$$

Notice that  $M_h(0)$  is equal to the matrix  $M_h$  defined in the fourth step of the proof of Proposition 10. Let us recall that according to [1, Section 3.1 and Lemma 23], the smallest eigenvalue  $\Lambda_0^{(\beta)}(\varepsilon)$  of  $M_h(\beta)$  is non negative if  $\varepsilon \leq 4\lambda_v h^\beta \lambda_x(h) / (4\lambda_x(h) + 1)$ . From (25) and the lines just before, where one just has to change  $\lambda_v$  there by  $\lambda_v h^\beta$ , it holds in the limit  $h \rightarrow 0$ :

$$\varepsilon_{\max} = \lambda_v h^\beta \times [\lambda_0 h + O(h^2)] = \varepsilon_0 h^{\beta+1} + O(h^{\beta+2}), \quad \text{where we recall } \varepsilon_0 = \lambda_v \lambda_0.$$

An asymptotic expansion when  $h \rightarrow 0$  then leads to

$$2\Lambda_0^{(\beta)}(\varepsilon_0 h^{\beta+1}) = 2\varepsilon_0 \lambda_0 \left(1 - \frac{\varepsilon_0}{4\lambda_v \lambda_0}\right) h^{\beta+2} + o(h^{\beta+2}) = \frac{3}{2} \lambda_v \lambda_0^2 h^{\beta+2} + o(h^{\beta+2}).$$

Therefore, when  $\beta > 0$ , Proposition 10 remains valid for  $\mathbf{P}_h(\beta)$  by changing in its statement all the  $h^2$  by  $h^{\beta+2}$ . The same holds for Proposition 11. In conclusion, when  $\beta > 0$ , Theorem 1 remains valid for  $\mathbf{P}_h(\beta)$  by changing in its statement  $\alpha h^2$  by  $\alpha h^{\beta+2}$ . Finally, Theorem 2 then holds true for  $\mathbf{P}_h(\beta)$  if one changes there  $e^{-\gamma t h^2}$  by  $e^{-\gamma t h^{\beta+2}}$ .

**The case when  $\beta < 0$ .**

Roughly speaking, in this case, the refreshment operator  $h^\beta \lambda_r(1 - \pi_v)$  is larger than the operators  $\lambda_{1,\mathbf{J}}(1 - \mathbf{B})$  or  $\sum_{k=1}^d \lambda_{1,\mathbf{J}}^{(k)}(1 - \mathbf{B}^{(k)})$ . When  $\beta < 0$ , the symmetric part  $\mathbf{S}_h(\beta)$  of  $\mathbf{P}_h(\beta)$  is no more uniformly bounded when in  $h \rightarrow 0$  but satisfies, for  $h$  small enough,  $\|\mathbf{S}_h\| \leq c h^\beta$  for some  $c > 0$  independent of  $h$ . Therefore, in view of the three first steps of the proof of Proposition 10 (where



only  $r_0$  in (23) is changed into  $r_0 h^\beta$  and  $\|\mathbf{S}_h\|$  is changed into  $Ch^\beta$  in (24)), there exists  $\lambda_v > 0$ ,  $C > 0$ , and  $\lambda_0 > 0$ , such that for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \mathbf{G}^{\perp L^2(\mathbf{E})}$  and  $h$  small enough, it holds:

$$\begin{aligned} C^{-1} \operatorname{Re} \langle \mathbf{P}_h(\beta)u, [1 + \varepsilon(\mathbf{A}_h + \mathbf{A}_h^*)]u \rangle_{L^2(\mathbf{E})} &\geq (\lambda_v h^\beta - \varepsilon) \|(1 - \pi_v)u\|_{L^2(\mathbf{E})}^2 + \varepsilon \lambda_x(h) \|\pi_v u\|_{L^2(\mathbf{E})}^2 \\ &\quad - \varepsilon h^\beta \|(1 - \pi_v)u\|_{L^2(\mathbf{E})} \|\pi_v u\|_{L^2(\mathbf{E})} \\ &= X_h(u)^t M_h(\beta) X_h(u), \end{aligned}$$

where

$$X_h(u) := (\|\pi_v u\|_{L^2(\mathbf{E})}, \|(1 - \pi_v)u\|_{L^2(\mathbf{E})})^t, \quad M_h(\beta) := \begin{pmatrix} \varepsilon \lambda_x(h) & -\frac{\varepsilon}{2} h^\beta \\ -\frac{\varepsilon}{2} h^\beta & \lambda_v h^\beta - \varepsilon \end{pmatrix}, \quad \text{and } \lambda_x(h) = \lambda_0 h.$$

According to [1, Section 3.1 and Lemma 23] (with  $R_0 = h^\beta$  there), for all  $h$  small enough, the smallest eigenvalue  $\Lambda_0(\varepsilon)$  of  $M_h(\beta)$  is non negative providing that  $\varepsilon \leq 4\lambda_v h^\beta \lambda_x(h) / (4\lambda_x(h) + R_0^2)$  and equals

$$2\Lambda_0^{(\beta)}(\varepsilon) = \lambda_v h^\beta - \varepsilon(1 - \lambda_x(h)) - \sqrt{[\lambda_v h^\beta - \varepsilon(1 - \lambda_x(h))]^2 - 4\varepsilon \lambda_x(h)(\lambda_v h^\beta - \varepsilon) + \varepsilon^2 R_0^2}.$$

When  $h > 0$  fixed, from [1, item (b) Lemma 24], the function  $\varepsilon \in [0, 4\lambda_x(h)\lambda_v h^\beta / (4\lambda_x(h) + R_0^2)] \mapsto \Lambda_0^{(\beta)}(\varepsilon)$  (see [1, item (b) Lemma 24]) attains its maximum at a unique point  $\varepsilon_{max}$  where

$$\varepsilon_{max} = \lambda_v h^\beta \times \frac{1 + \lambda_x(h) - (1 - \lambda_x(h)) \left[ R_0^2 / (R_0^2 + 4\lambda_x(h)) \right]^{\frac{1}{2}}}{(1 + \lambda_x(h))^2 + R_0^2} = 2\lambda_0 \lambda_v h^{-3\beta+1} + o(h^{-3\beta+1}),$$

when  $h \rightarrow 0$ . Set  $\varepsilon_1 = 2\lambda_0 \lambda_v$ . Then, in the limit  $h \rightarrow 0$ , it holds:

$$2\Lambda_0^{(\beta)}(\varepsilon_1 h^{-3\beta+1}) = 2\lambda_0 \varepsilon_1 h^{-3\beta+2} + o(h^{-3\beta+2}) = 4\lambda_0^2 \lambda_v h^{-3\beta+2} + o(h^{-3\beta+2}).$$

Therefore, when  $\beta < 0$ , Theorem 1 (resp. Theorem 2) remains valid for  $\mathbf{P}_h(\beta)$  by changing in its statement  $\alpha h^2$  by  $\alpha h^{-3\beta+2}$  (resp. by changing  $e^{-\gamma th^2}$  by  $e^{-\gamma th^{-3\beta+2}}$ ).

In conclusion, when adding too much or not enough refreshment (i.e  $|\beta| > 0$ ), our analysis provides a separation when  $h \rightarrow 0$  of the spectrum between the  $m_0$  smallest eigenvalues of  $\mathbf{P}_h(\beta)$  and the rest of its spectrum of order  $h^{\beta+2}$  if  $\beta \geq 0$  or  $h^{-3\beta+2}$  if  $\beta < 0$ . The better separation is thus obtained when  $\beta = 0$ .

## Appendix

In this appendix, we prove the following non semiclassical result (i.e. when  $h > 0$  is fixed).

**Proposition 14.** *Let  $h > 0$  be fixed. The eigenvalue 0 is isolated and has algebraic multiplicity 1 for both  $\mathbf{P}_h$  and  $\mathbf{P}_h^*$ . Moreover, the spectral projection  $\pi_0(\mathbf{P}_h)$  associated with  $\mathbf{P}_h$  and 0 equals*

$$u \in L^2(\mathbf{E}) \mapsto \pi_0(\mathbf{P}_h)u = \frac{\langle e^{-\frac{1}{h}U} \mathbf{1}_V, u \rangle_{L^2(\mathbf{E})}}{\|e^{-\frac{1}{h}U} \mathbf{1}_V\|_{L^2(\mathbf{E})}^2} e^{-\frac{1}{h}U} \mathbf{1}_V,$$

and is thus orthogonal in  $L^2(\mathbf{E})$ .

*Proof.* Since  $h > 0$  is fixed in what follows, we set  $h = 1$ . Then the computations to prove (21) (which are basically those of [1, 13]) imply that for all  $\varepsilon_0 > 0$  small enough, there exists  $c(\varepsilon_0) > 0$  such that for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \{e^{-U} \mathbf{1}_V\}^{\perp L^2(\mathbf{E})}$ , it holds:

$$\operatorname{Re} \langle \mathbf{P}_1 u, [1 + \varepsilon_0(\mathbf{A}_1 + \mathbf{A}_1^*)] u \rangle_{L^2(\mathbf{E})} \geq c(\varepsilon_0) \|u\|_{L^2(\mathbf{E})}^2. \quad (43)$$

Choosing  $\varepsilon_0 > 0$  such that  $1 + \varepsilon_0(\|\mathbf{A}_1\| + \|\mathbf{A}_1^*\|) < 2$ , Equation (43) implies that for all  $u \in \mathcal{C}^\infty(\mathbf{E}) \cap \{e^{-U} \mathbf{1}_V\}^{\perp L^2(\mathbf{E})}$  and  $z \in \mathbb{C}$ , it holds:

$$\|(\mathbf{P}_1 - z)u\|_{L^2(\mathbf{E})} \geq \left[ \frac{1}{2}c(\varepsilon_0) - (\operatorname{Re} z)_+ \right] \|u\|_{L^2(\mathbf{E})}. \quad (44)$$

Set  $Z := \|e^{-U} \mathbf{1}_V\|_{L^2(\mathbf{E})}^2$ . Then, since  $\mathbf{P}_1(e^{-U} \mathbf{1}_V) = \mathbf{P}_1^*(e^{-U} \mathbf{1}_V) = 0$ , for all  $u \in \mathcal{C}^\infty(\mathbf{E})$ , one has:

$$\left\| (\mathbf{P}_1 - z) \left( u - \frac{1}{Z} \langle e^{-U} \mathbf{1}_V, u \rangle_{L^2(\mathbf{E})} e^{-U} \mathbf{1}_V \right) \right\|_{L^2(\mathbf{E})}^2 = \|(\mathbf{P}_1 - z)u\|_{L^2(\mathbf{E})}^2 - \frac{|z|^2}{Z} \left| \langle e^{-U} \mathbf{1}_V, u \rangle_{L^2(\mathbf{E})} \right|^2.$$

Using (44), it then holds for all  $u \in \mathcal{C}^\infty(\mathbf{E})$  and  $z \in \mathbb{C}$ :

$$\begin{aligned} \|(\mathbf{P}_1 - z)u\|_{L^2(\mathbf{E})} &\geq \left[ \frac{1}{2}c(\varepsilon_0) - (\operatorname{Re} z)_+ \right] \left\| u - \frac{1}{Z} \langle e^{-U} \mathbf{1}_V, u \rangle_{L^2(\mathbf{E})} e^{-U} \mathbf{1}_V \right\|_{L^2(\mathbf{E})} \\ &\quad + \frac{|z|^2}{Z} \left| \langle e^{-U} \mathbf{1}_V, u \rangle_{L^2(\mathbf{E})} \right|^2, \end{aligned} \quad (45)$$

which extends to all  $u \in D(\mathbf{P}_1)$ . The same estimate holds for  $\mathbf{P}_1^*$  choosing if necessary  $\varepsilon_0 > 0$  smaller. Let  $z \in \mathbb{C}$  such that  $\operatorname{Re}(z) < c(\varepsilon_0)/2$  and  $z \neq 0$ . Then,  $\mathbf{P}_1 - z$  is injective with closed and dense range, i.e.  $\mathbf{P}_1 - z$  is invertible. It is in particular a Fredholm operator (with index 0). We claim that  $\mathbf{P}_1$  is also a Fredholm operator. Let us prove it. We have

$$\operatorname{Ker} \mathbf{P}_1 = \operatorname{Ker} \mathbf{P}_1^* = \operatorname{Span}(e^{-U} \mathbf{1}_V). \quad (46)$$

It is clear that  $\operatorname{Span}(e^{-U} \mathbf{1}_V) \subset \operatorname{Ker} \mathbf{P}_1 \cap \operatorname{Ker} \mathbf{P}_1^*$ . If now  $\mathbf{P}_1 w = 0$ , then from Lemma 3 and (2),  $\pi_v w = w$  is thus independent of  $v \in V$ . Then,  $\mathbf{P}_1 w = 0$  writes  $v \cdot \mathbf{d}_{U,h} w = 0$  for all  $(x, v) \in \mathbf{M} \times V$ . If  $\mathbf{d}_{U,h} w(x) \neq 0$  then choose  $v = \mathbf{d}_{U,h} w(x) / |\mathbf{d}_{U,h} w(x)|$  which leads to  $\mathbf{d}_{U,h} w = 0$  on  $\mathbf{M}$ . The same holds for  $\mathbf{P}_1^*$ . This proves (46). Furthermore, Equation (45) with  $z = 0$  implies that the range of  $\mathbf{P}_1$  is closed. Let us recall that  $(\operatorname{Ran} \mathbf{P}_1)^{\perp L^2(\mathbf{E})} = \operatorname{Ker} \mathbf{P}_1^* = \operatorname{Span}(e^{-U} \mathbf{1}_V)$ . Thus,  $\operatorname{Ran} \mathbf{P}_1 = \operatorname{Span}(e^{-U} \mathbf{1}_V)^{\perp L^2(\mathbf{E})}$ , which leads to

$$\dim \operatorname{coKer} \mathbf{P}_1 = 1.$$

Hence,  $\mathbf{P}_1$  is a Fredholm operator. In conclusion, for all  $z \in \mathbb{C}$  such that  $\operatorname{Re}(z) < c(\varepsilon_0)/2$ ,  $\mathbf{P}_1 - z$  is a Fredholm operator. Since it is invertible for  $z \in \mathbb{R}_-^*$  ( $\mathbf{P}_1$  is  $m$ -accretive), by the analytic Fredholm Theorem, the function

$$z \in \{\tilde{z} \in \mathbb{C}, \operatorname{Re}(\tilde{z}) < c(\varepsilon_0)/2\} \mapsto (\mathbf{P}_1 - z)^{-1}$$

is meromorphic with poles of finite rank. The only pole of this function in this region is  $z = 0$  and has therefore finite algebraic multiplicity. Let us assume that for  $\mathbf{P}_1$  this algebraic multiplicity is strictly larger than 1. Since  $\operatorname{Ker}(\mathbf{P}_1) = \operatorname{Span}(e^{-U} \mathbf{1}_V)$ , this implies that there exists a generalized eigenfunction  $f \in D(\mathbf{P}_1)$  such that  $\mathbf{P}_1 f = e^{-U} \mathbf{1}_V$ . Consequently  $0 < \|e^{-U} \mathbf{1}_V\|_{L^2(\mathbf{E})}^2 = \langle \mathbf{P}_1 f, e^{-U} \mathbf{1}_V \rangle = \langle f, \mathbf{P}_1^*(e^{-U} \mathbf{1}_V) \rangle = 0$  which leads to a contradiction. The same reasoning also applies to  $\mathbf{P}_1^*$ . This concludes the proof of Proposition 14.  $\blacksquare$

Let us end this work with a short remark on how one can easily deduce the convergence of  $(e^{-tP_1})_{t \geq 0}$  to  $\pi_0(P_1)$  as  $t \rightarrow \infty$  from (44) and with the Gearhart-Prüss Theorem. Using (44), for all  $u \in C^\infty(\mathbf{E}) \cap \{e^{-U} \mathbf{1}_V\}^{\perp L^2(\mathbf{E})}$  and  $z \in \mathbb{C}$  such that  $\operatorname{Re} z \leq c(\varepsilon_0)/4$ , it holds:

$$\|(P_1 - z)u\|_{L^2(\mathbf{E})} \geq \frac{1}{4}c(\varepsilon_0)\|u\|_{L^2(\mathbf{E})}.$$

This implies that the resolvent of  $P_1|_{(1-\pi_0(P_1))L^2(\mathbf{E})}$  is uniformly bounded on the set  $\{z, \operatorname{Re} z < c(\varepsilon_0)/4\}$ . Hence, applying the Gearhart-Prüss Theorem to  $e^{-tP_1}(1-\pi_0(P_1))$  on  $\operatorname{Ran}(1-\pi_0(P_1))$ , it holds for  $t \geq 0$ ,

$$\|e^{-tP_1}(1-\pi_0(P_1))\| \leq C e^{-\frac{c(\varepsilon_0)}{4}t},$$

for some  $C > 0$  independent of  $t \geq 0$ . In conclusion, this shows that starting from (43), we recover the (non semiclassical) results of [1, 13] with the Gearhart-Prüss Theorem together with uniform resolvent estimates.

**Acknowledgement.** The second author is grateful to Dorian Le Peutrec for his suggestion in the proof of Proposition 4. This work is supported by the ANR-17-CE40-0030 - EFI - Entropy, flows, inequalities.

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