An explicit formula for the free exponential modality of linear logic

Paul-André Melliès, Nicolas Tabareau, Christine Tasson

Under consideration for publication in Math. Struct. in Comp. Science

An explicit formula for the free exponential modality of linear logic

Paul-André Melliès Nicolas Tabareau Christine Tasson

Received March 2010

The exponential modality of linear logic associates a commutative comonoid !A to every formula A, this enabling to duplicate the formula in the course of reasoning. Here, we explain how to compute the free commutative comonoid !A as a sequential limit of equalizers in any symmetric monoidal category where this sequential limit exists and commutes with the tensor product. We apply this general recipe to a series of models of linear logic, typically based on coherence spaces, Conway games and finiteness spaces. This algebraic description unifies for the first time the various constructions of the exponential modality in spaces and games. It also sheds light on the duplication policy of linear logic, and its interaction with classical duality and double negation completion.

Contents

Introduction

Linear logic is based on the principle that every hypothesis A_i should appear exactly once in a proof of the sequent

$$A_1, \ldots, A_n \quad \vdash \quad B. \tag{1}$$

This logical restriction enables to represent the logic in monoidal categories, along the idea that every formula denotes an object of the category, and every proof of the sequent (1) denotes a morphism

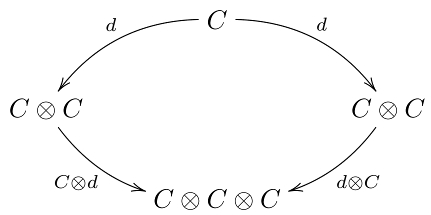
$$A_1 \otimes \cdots \otimes A_n \quad \longrightarrow \quad B$$

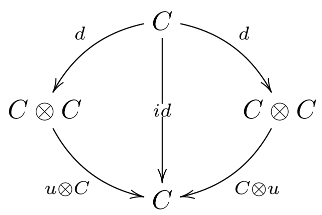
where the tensor product is thus seen as a linear kind of conjunction. Here, for clarity's sake, we use the same notation for a formula A and for its interpretation (or denotation) in the monoidal category.

However, this linearity policy on proofs is far too restrictive in order to reflect traditional forms of reasoning, where it is customary to repeat or to discard an hypothesis in the course of a logical argument. This point is nicely resolved by providing linear logic with an exponential modality, whose task is to strengthen every formula A into a formula $!A$ which may be repeated or discarded. From a semantic point of view, the formula $!A$ is most naturally interpreted as a *comonoid* of the monoidal category. Recall that a comonoid (C, d, u) in a monoidal category $\mathscr{C}$ is defined as an object C equipped with two morphisms

$$d \quad : \quad C \quad \longrightarrow \quad C \otimes C \qquad\qquad u \quad : \quad C \quad \longrightarrow \quad \mathbf{1}$$

where $\mathbf{1}$ denotes the monoidal unit of the category. The morphism d and u are respectively called the *multiplication* and the *unit* of the comonoid. The two morphisms d and u are supposed to satisfy *associativity* and *unitality* properties, neatly formulated by requiring that the two diagrams





commute. Note that we draw our diagrams as if the category were *strictly* monoidal, although the usual models of linear logic are only *weakly* monoidal.

The comonoidal structure of the formula $!A$ enables to interpret the *contraction rule* and the *weakening rule* of linear logic

$$\dfrac{\dfrac{\pi}{\Gamma, !A, !A, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\ \text{Contraction} \qquad\qquad \dfrac{\dfrac{\pi}{\Gamma, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\ \text{Weakening}$$

by pre-composing the interpretation of the proof π with the multiplication d in the case of contraction

$$\Gamma \otimes !A \otimes \Delta \xrightarrow{d} \Gamma \otimes !A \otimes !A \otimes \Delta \xrightarrow{\pi} B$$

and with the unit u in the case of weakening

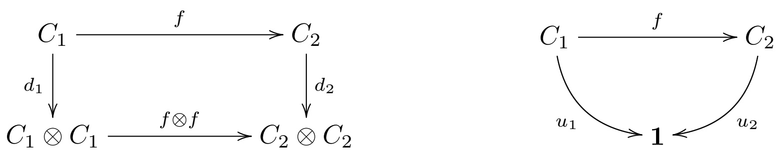
$$\Gamma \otimes !A \otimes \Delta \xrightarrow{u} \Gamma \otimes \Delta \xrightarrow{\pi} B.$$

Linear logic also enables to permute the order of the hypothesis $A_1, \ldots, A_n$ in a context, and is thus interpreted in a *symmetric* monoidal category. Accordingly, one requires that the comonoid $!A$ is commutative, this meaning that the following equality holds:

$$A \xrightarrow{d} A \otimes A \xrightarrow{symmetry} A \otimes A \quad = \quad A \xrightarrow{d} A \otimes A\,.$$

When linear logic was introduced by Jean-Yves Girard, twenty years ago, it was soon realized by Robert Seely (among a few others) that the multiplicative fragment of the logic should be interpreted in a $*$-autonomous category, or at least, a symmetric monoidal closed category $\mathscr{C}$; and that the category should have finite products in order to interpret the additive fragment of the logic, see (Seely 1989). A more difficult question was to understand what categorical properties of the exponential modality " ! " were exactly required, in order to define a model of propositional linear logic... that is, a model including the multiplicative, the additive and the exponential components of the logic.

However, Yves Lafont formulated in his PhD thesis (Lafont 1988) a simple and quite general way to define a model of linear logic. Recall that a comonoid morphism between two comonoids (C_1, d_1, u_1) and (C_2, d_2, u_2) is defined as a morphism $f : C_1 \to C_2$ such that the two diagrams



commute. One says that the commutative comonoid $!A$ is freely generated by an object A when there exists a morphism

$$\varepsilon \quad : \quad !A \quad \to \quad A$$

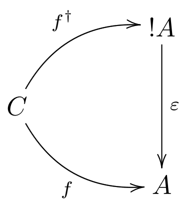
such that for every morphism

$$f \quad : \quad C \quad \to \quad A$$

from a commutative comonoid C to the object A, there exists a unique comonoid morphism

$$f^\dagger \quad : \quad C \quad \to \quad !A$$

such that the diagram



(2)

commutes. So, from the point of view of provability, $!A$ is the largest comonoid below the object A. Lafont noticed that a model of propositional linear logic follows automatically from the existence of a free commutative comonoid $!A$ for every object A of a symmetric monoidal closed category $\mathscr{C}$. Remember that this is not the only way to construct a model of linear logic. A folklore example is the coherence space model, which admits two alternative interpretations of the exponential modality: the original one, formulated by Girard (Girard 1987) where the coherence space $!A$ is defined as a space of *cliques*, and the free construction, where $!A$ is defined as a space of *multicliques* (cliques with multiplicity) of the original coherence space A.

In this paper, we explain how to construct the free commutative comonoid in the symmetric monoidal categories $\mathscr{C}$ typically encountered in the semantics of linear logic. To that purpose, we start from the well-known formula defining the *symmetric algebra*

$$SA \quad = \quad \bigoplus_{n\in\mathbb{N}} \quad A^{\otimes n} \ / \ \sim_n \tag{3}$$

generated by a vector space A. Recall that the formula (3) computes the free commutative monoid associated to the object A in the category of vector spaces over a given field $\Bbbk$. The group Σ_n of permutations on $\{1,\ldots,n\}$ acts on the vector space $A^{\otimes n}$, and the vector space $A^{\otimes n}/\sim_n$ of equivalence classes (or orbits) modulo the group action is defined as the coequalizer of the $n!$ symmetries

$$A^{\otimes n} \underset{symmetry}{\overset{symmetry}{\underset{\cdots}{\rightrightarrows}}} A^{\otimes n} \xrightarrow{\ coequalizer\ } A^{\otimes n}/\sim_n$$

in the category of vector spaces. Since a comonoid in the category $\mathscr{C}$ is the same thing as a monoid in the opposite category $\mathscr{C}^{op}$, it is tempting to apply the *dual* formula to (3) in order to define the free commutative comonoid $!A$ generated by an object A in the monoidal category $\mathscr{C}$. Although the idea is extremely naive, the resulting formula is surprisingly close to the solution we are aiming at. Indeed, one key observation of the article is that the equalizer A^n of the $n!$ symmetries

$$A^n \xrightarrow{\ equalizer\ } A^{\otimes n} \underset{symmetry}{\overset{symmetry}{\underset{\cdots}{\rightrightarrows}}} A^{\otimes n} \tag{4}$$

exists in many traditional models of linear logic, and that it provides there the n-th layer of the free commutative comonoid $!A$ generated by the object A. This general principle will be nicely illustrated in Section 3 by the equalizer A^n in the category of coherence spaces, which contains the multicliques of cardinality n in the coherence space A ; and in Section 4 by the equalizer A^n in the category of Conway games, which defines the game where Opponent may open up to n copies of the game A, one after the other, in a sequential order.

Of course, the construction of the free exponential modality does not stop here: one still needs to combine the layers A^n together in order to define $!A$ properly. As we already mentioned, one obvious solution is to apply the dual of formula (3) and to define $!A$ as the infinite cartesian product

$$!A \quad = \quad \mathop{\&}_{n\in\mathbb{N}} \quad A^n \ / \ \sim_n \ . \tag{5}$$

This formula works perfectly well in any symmetric monoidal category $\mathscr{C}$ where the infinite product commutes with the tensor product, in the sense that the canonical morphism

$$X \quad \otimes \quad \Big(\ \mathop{\&}_{n\in\mathbb{N}} \quad A^n \ \Big) \quad \to \quad \mathop{\&}_{n\in\mathbb{N}} \ (\ X \ \otimes \ A^n \ ) \tag{6}$$

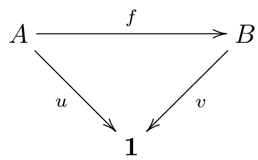
is an isomorphism. This logical degeneracy occurs in particular in the relational model of linear logic, where the free exponential $!A$ is thus defined according to formula (5) as the set of finite multisets of A, each equalizer A^n describing the set of multisets of cardinality n.

On the other hand, the formula (5) is far too optimistic in general, because the canonical morphism (6) is *not* reversible in the typical models of linear logic, based on coherence spaces, or on sequential games. In order to understand better what this means, it is quite instructive to apply the formula (5) to the category of Conway games: it defines a game $!A$ where the first move by Opponent selects a component A^n, and thus decides the number n of copies of the game A played subsequently. This departs from the free commutative comonoid $!A$ which we shall define in Section 4, where Opponent is allowed to open a new copy of the game A at any point of the interaction.

So, there remains to understand how the various layers A^n should be combined together inside $!A$ in order to perform this particular copy policy. This puzzle has a very simple solution: one should "glue" every layer A^n inside the next layer A^{n+1} so as to enable the computation to transit from one layer to the next in the course of interaction. As we will see, one simple way to perform this "glueing" operation is to introduce the notion of pointed (or affine) object. A pointed object in a monoidal category $\mathscr{C}$ is defined as a pair (A, u) consisting of an object A and of a morphism $u : A \to \mathbf{1}$ to the monoidal unit. So, a pointed object is something like a comonoid without a multiplication. The category $\mathscr{C}_\bullet$ has the pointed objects as objects, and as morphisms

$$(A, u) \quad \to \quad (B, v)$$

the morphisms $f : A \to B$ of the category $\mathscr{C}$ making the diagram below commute:



So, the category $\mathscr{C}_\bullet$ of pointed objects coincides in fact with the slice category $\mathscr{C} \downarrow \mathbf{1}$. It is folklore that the category $\mathscr{C}_\bullet$ is symmetric monoidal, with monoidal structure inherited from the category $\mathscr{C}$, and more specifically, the fact that the monoidal unit $\mathbf{1}$ is a commutative monoid. The category $\mathscr{C}_\bullet$ is moreover *affine* in the sense that its monoidal unit $\mathbf{1}$ is terminal in the category.

Plan of the paper. The main purpose of this paper is to compute the free commutative comonoid $!A$ as a sequential limit of equalizers in various models of linear logic. The construction is excessively simple and works every time the sequential limit exists, and commutes with the tensor product in the underlying category $\mathscr{C}$. We start by describing the limit construction (in Section 1) and by explaining how this formula computes the free commutative comonoid $!A$ as a Kan extension along a change of symmetric monoidal theory (in Section 2). Then, we establish that the construction works in the expected way in the category of coherence spaces (in Section 3) and in the category of Conway games (in Section 4). This establishes that the two modalities are defined in *exactly* the same way in the two models of linear logic. So, the two modalities have the same genotype, although their phenotypes differ. The discovery is unexpected, but immediately counterbalanced by the observation that the limit construction does not work in every model of linear logic. As a matter of fact, we exhibit the instructive counterexample of the finiteness space model introduced by Thomas Ehrhard (Ehrhard 2005). We explain (in Section 5) why the limit construction fails in this particular model of linear logic. This counterexample leads us to study more closely the interaction (and possible interference) between the free construction of the exponential modality on the one hand, and the double negation completion to classical linear logic on the other hand. In particular, we introduce (in Section 6) a symmetric monoidal category of configuration spaces whose full subcategory of objects coincides with the category of coherence spaces. We explain (in Section 7) how the free exponential modality of the coherence space model is inherited from the free exponential modality of configuration space model by a series of monoidal adjunctions. We then repeat the procedure (in Section 8) for finiteness spaces, this enabling us to recover the free exponential modality of the finiteness space model as the double negation completion of the free exponential modality of a configuration space model. We conclude and indicate future research directions (in Section 9).

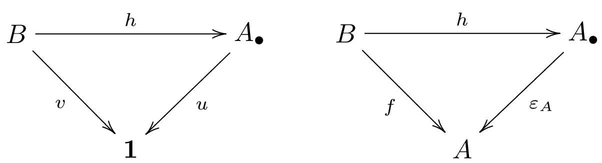
1. The limit construction in three easy steps

The construction of the free exponential modality proceeds along a simple recipe in three steps, which we choose to describe here in the most direct and pedestrian way. We then turn in the next Section to the conceptual reasons which justify the construction.

First step. The first step of the construction requires to make the mild hypothesis that the object A of the category $\mathscr{C}$ generates a *free pointed object* $(A_\bullet, u)$ in the category $\mathscr{C}_\bullet$. This means that there exists a morphism

$$\varepsilon_A \quad : \quad A_\bullet \quad \to \quad A$$

such that for every pointed object (B, v) and every morphism $f : B \to A$, there exists a unique morphism $h : B \to A_\bullet$ making the two diagrams below commute:



This typically happens when the forgetful functor $\mathscr{C}_\bullet \to \mathscr{C}$ has a right adjoint. Informally speaking, the purpose of the pointed object $A_\bullet$ is to contain one copy of the object A, or no copy at all. This free pointed object exists in most models of linear logic, in particular when the underlying category has finite products: in the case of coherence spaces, it is the space $A_\bullet = A \,\&\, \mathbf{1}$ obtained by adding a point to the web of A with $u : A \,\&\, \mathbf{1} \to \mathbf{1}$ defined as the second projection ; in the case of Conway games, it is the game $A_\bullet = A$ itself, at least when the category of Conway games is restricted to Opponent-starting games.

Second step. The object $A^{\leq n}$ is then defined (when it exists) as the equalizer $(A_\bullet)^n$ of the diagram

$$A^{\leq n} \xrightarrow{\text{equalizer}} A_\bullet^{\otimes n} \underset{symmetry}{\overset{symmetry}{\rightrightarrows}} A_\bullet^{\otimes n} \tag{7}$$

in the category $\mathscr{C}$. Note that $A^{\leq n}$ is computed as the equalizer on the n-th tensorial power of the *pointed* object $A_\bullet$. The purpose of $A^{\leq n}$ is thus to describe *at the same time* all the layers A^k of k copies of A, for $k \leq n$. Typically, the equalizer $A^{\leq n}$ computed in the category of coherence spaces coincides with the space of all multicliques in A of cardinality less than or equal to n; the equalizer $A^{\leq n}$ computed in the category of Opponent-starting Conway games coincides with the game where n copies of the game A are played in parallel, and where Opponent is not allowed to play an opening move in the copy A_{i+1} until all the previous copies $A_1, \ldots, A_i$ have been opened.

Third step. The universal property of equalizers in the category $\mathscr{C}$ ensures that there exists a canonical morphism

$$A^{\leq n} \longleftarrow A^{\leq n+1}$$

induced by the unit $u : A_\bullet \to \mathbf{1}$ of the pointed object $A_\bullet$, for every natural number n. This enables to define the object A^∞ (when it exists) as the sequential limit of the sequence

$$\mathbf{1} \longleftarrow A^{\leq 1} \longleftarrow A^{\leq 2} \longleftarrow \cdots \longleftarrow A^{\leq n} \longleftarrow A^{\leq n+1} \longleftarrow \cdots \tag{8}$$

with limiting cone defined by projection maps

$$A^\infty \xrightarrow{\text{projection}} A^{\leq n}.$$

This recipe in three steps defines the free commutative comonoid $!A$ as the sequential limit A^∞, at least when the object A satisfies the limit properties described in the next statement.

Proposition 1. Consider an object A in a symmetric monoidal category $\mathscr{C}$. Suppose that the object A generates a free pointed object $(A_\bullet, u)$ in the sense explained above. Suppose moreover that the equalizer (7) and the sequential limit (8) exist in the category $\mathscr{C}$, and that they commute with the tensor product, in the sense that for every object X of the category $\mathscr{C}$, (a) the morphism

$$X \otimes A^{\leq n} \xrightarrow{X\otimes\ \text{equalizer}} X \otimes A_\bullet^{\otimes n}$$

defines an equalizer of the diagram

$$X \otimes A_\bullet^{\otimes n} \underset{X\otimes\ symmetry}{\overset{X\otimes\ symmetry}{\rightrightarrows}} X \otimes A_\bullet^{\otimes n}$$

for every natural number n, and (b) the family of morphisms

$$X \otimes A^\infty \xrightarrow{X\otimes\ \text{projection}} X \otimes A^{\leq n}$$

defines the limiting cone of the diagram

$$X \otimes \mathbf{1} \longleftarrow X \otimes A^{\leq 1} \longleftarrow X \otimes A^{\leq 2} \longleftarrow \cdots \longleftarrow X \otimes A^{\leq n} \longleftarrow X \otimes A^{\leq n+1} \longleftarrow \cdots .$$

In that case, the object A generates a free commutative comonoid $!A$ in the symmetric monoidal category $\mathscr{C}$, defined as the sequential limit A^{∞}.

2. A model-theoretic account of the construction

Although Proposition 1 is extremely simple to state, one needs to climb a few steps in the conceptual ladder in order to establish the property, using the methods of functorial model theory initiated by Lawvere (Lawvere 1963). To that purpose, one starts by identifying the limit construction $A \mapsto A^{\infty}$ as the computation of a right Kan extension of the object A (seen as a functor) along a change of symmetric monoidal theory going from the trivial theory to the theory of commutative comonoids. This formulation enables then to deduce Proposition 1 from a general result on free constructions of algebraic theories recently established by Melliès and Tabareau in a suitable 2-categorical framework, see (Melliès & Tabareau 2008). Let us briefly explain the argument establishing the proposition.

Symmetric monoidal theories. A symmetric monoidal theory $\mathbb{T}$ (also called PROP in the literature) is defined as a symmetric monoidal category whose objects are the natural numbers, and whose tensor product on objects coincides with the sum on natural numbers: $m \otimes n = m + n$. A model M of the theory $\mathbb{T}$ in a symmetric monoidal category $\mathscr{C}$ is defined as a monoidal functor $M : \mathbb{T} \to \mathscr{C}$, understood in the strong sense. Finally, the category $Mod(\mathbb{T}, \mathscr{C})$ has such models as objects, and monoidal natural transformations as morphisms between them.

The simplest possible PROP is provided by the category $\mathbb{B}$ with finite ordinals $[n] = \{0, \ldots, n-1\}$ as objects, and with bijections as morphisms. Note that $\mathbb{B}$ is a the free symmetric monoidal category generated by the category with one object. Consequently, every object A of a symmetric monoidal category $\mathscr{C}$ induces a model (also noted A) of the theory $\mathbb{B}$ in the category $\mathscr{C}$, defined as

$$A \quad : \quad [n] \quad \mapsto \quad A^{\otimes n}.$$

In particular, one has $A([0]) = \mathbf{1}$, $A([1]) = A$, $A([2]) = A^{\otimes 2}$ and so on. The resulting functor from the category $\mathscr{C}$ to the category $Mod(\mathbb{B}, \mathscr{C})$ defines an equivalence of categories, this meaning that a model of the theory $\mathbb{B}$ is essentially the same thing as an object of the underlying category $\mathscr{C}$. This is the reason why the category $\mathbb{B}$ is often called the *trivial* symmetric monoidal theory.

Now, consider the category $\mathbb{F}$ with finite ordinals $[n] = \{0, \ldots, n-1\}$ as objects and with functions $[p] \to [q]$ as morphisms. It appears that its opposite category $\mathbb{F}^{op}$ defines the symmetric monoidal theory of commutative comonoids, in the sense that the category of commutative comonoids and comonoid morphisms in any symmetric monoidal category $\mathscr{C}$ is equivalent to the category $Mod(\mathbb{F}^{op}, \mathscr{C})$. In particular, every model M of the theory $\mathbb{F}^{op}$ induces a commutative comonoid $C = M([1])$ in the category $\mathscr{C}$, whose comultiplication

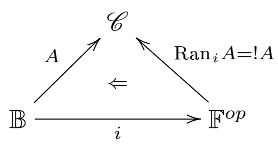
$d : C \to C^{\otimes 2}$ and counit $u : \mathbf{1} \to C$ are provided by the image of the (unique) morphisms $[1] \to [2]$ and $[1] \to [0]$ in the category $\mathbb{F}^{op}$.

Free models computed as Kan extensions. One substantial benefit of this functorial approach to model theory is that the forgetful functor U transporting a commutative comonoid (C, d, u) to its underlying object C in the category $\mathscr{C}$ is reformulated as the functor

$$U \quad : \quad Mod(\mathbb{F}^{op}, \mathscr{C}) \quad \longrightarrow \quad Mod(\mathbb{B}, \mathscr{C})$$

which transports (a) every model $M : \mathbb{F}^{op} \to \mathscr{C}$ to the model $M \circ i : \mathbb{B} \to \mathscr{C}$ obtained by precomposing M with the symmetric monoidal functor $i : \mathbb{B} \to \mathbb{F}^{op}$ defined as the identity on objects, and (b) every morphism $\theta : M \to N$ of models of $\mathbb{F}^{op}$ to the morphism $\theta \circ i : M \circ i \to N \circ i$ of models of $\mathbb{B}$.

Computing the free commutative comonoid $!A$ generated by an object A in the category $\mathscr{C}$ amounts then to computing the right Kan extension of the model $A : \mathbb{B} \to \mathscr{C}$ along the inclusion functor $i : \mathbb{B} \to \mathbb{F}^{op}$ in the 2-category **SMCat** of symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations.



The whole point of the construction is that, by definition of Kan extensions, the resulting commutative comonoid $!A$ satisfies the unique lifting property expressed in Diagram 2. Now, it is folklore that the right Kan extension of A along j may be computed in the 2-category **Cat** of categories and functors as the end formula

$$!A \quad = \quad \int_{[n] \in \mathbb{B}} \mathbb{F}^{op}([1], [n]) \circ A^{\otimes n} \quad = \quad \int_{[n] \in \mathbb{B}} A^{\otimes n} \tag{9}$$

where $E \circ C$ denotes the cotensor product of an object C of the category $\mathscr{C}$ by a set E, see (Kelly 1982) for details. This is precisely the reason why we considered this end formula in the introduction, formulated there as the infinite product of equalizers (5). The theorem established in (Melliès & Tabareau 2008) ensures that the right Kan extension $\mathrm{Ran}_i A$ computed in the 2-category **Cat** defines a right Kan extension in the 2-category **SMCat**, as long as one additional condition is satisfied: the tensor product of the category $\mathscr{C}$ should commute with the end formula. This means that the end formula (9) defines the free commutative comonoid generated by the object A when the canonical morphism

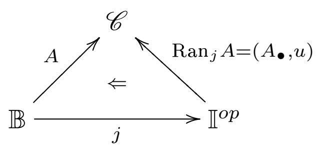
$$X \otimes \int_{[n] \in \mathbb{B}} A^{\otimes n} \quad \longrightarrow \quad \int_{[n] \in \mathbb{B}} X \otimes A^{\otimes n} \tag{10}$$

is an isomorphism for every object X of the category $\mathscr{C}$. This justifies to apply the limit formula (9) whenever this commutation property holds, typically in the category of sets and relations, or in the category of modules over a commutative ring.

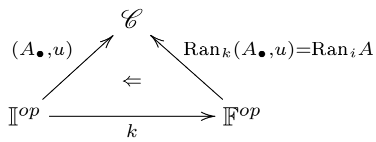
Performing the Kan extension in two steps. However, we observed in the introduction that the canonical morphism (10) is not an isomorphism in most models of linear logic. This difficulty is resolved by decomposing the computation of the right Kan extension of A in two independent steps, taking advantage of the fact that the functor $i : \mathbb{B} \to \mathbb{F}^{op}$ factors as

$$\mathbb{B} \quad \xrightarrow{j} \quad \mathbb{I}^{op} \quad \xrightarrow{k} \quad \mathbb{F}^{op}$$

where $\mathbb{I}$ denotes the category with finite ordinals $[n] = \{0, \ldots, n-1\}$ as objects, and injections $[p] \to [q]$ as morphisms, and j and k denote the obvious identity-on-object functors. Note that the opposite category $\mathbb{I}^{op}$ defines the PROP for pointed objects. In particular, its category of models $Mod(\mathbb{I}^{op}, \mathscr{C})$ is equivalent to the category $\mathscr{C}_\bullet$ of pointed object defined earlier. Hence, the first assumption of Proposition 1 means that the pointed object $(A_\bullet, u)$ defines the right Kan extension of the object A along the functor $j : \mathbb{B} \to \mathbb{I}^{op}$ in the 2-category **SMCat**, as depicted below:



Recall that Kan extensions may be composed in any 2-category, and in particular in the 2-category **SMCat**. From this follows that the right Kan extension $\mathrm{Ran}_k(A_\bullet, u)$ of the pointed object $(A_\bullet, u)$ along the functor k (when it exists in the 2-category **SMCat**) defines the right Kan extension of the object A along the functor $i = k \circ j$ in the same 2-category **SMCat**, as depicted below:



Since the right Kan extension of A along i in the 2-category **SMCat** defines the free commutative comonoid $!A$ generated by the object A, there remains to compute the Kan extension $\mathrm{Ran}_k(A_\bullet, u)$. In the same way as previously, it is well-known that the right Kan extension is computed in the 2-category **Cat** as the end formula:

$$A^\infty \quad = \quad \int_{n \in \mathbb{I}^{op}} \mathbb{F}^{op}([1],[n]) \circ (A_\bullet)^{\otimes n} \quad = \quad \int_{n \in \mathbb{I}^{op}} (A_\bullet)^{\otimes n}. \tag{11}$$

It is not difficult to see that this end exists and coincides with the sequential limit (8) when the equalizers (7) and the sequential limit (8) exist in the category $\mathscr{C}$. This is precisely the second assumption of Proposition 1. This enables to apply the general theorem of (Melliès & Tabareau 2008) which ensures that the Kan extension in the 2-category **Cat** coincides with the Kan extension in the 2-category **SMCat** as soon as the canonical morphism

$$X \otimes \int_{[n] \in \mathbb{I}^{op}} A^{\otimes n} \quad \to \quad \int_{[n] \in \mathbb{I}^{op}} X \otimes A^{\otimes n}$$

is an isomorphism for every object X of the category $\mathscr{C}$. This property follows from the last assumption of Proposition 1 which states that the tensor product commutes with the equalizers (7) and the sequential limit (8) in the category $\mathscr{C}$. This establishes that the end formula (11) and thus the sequential limit (8) computes the free commutative comonoid generated by the object A, whenever the assumptions of Proposition 1 are satisfied.

3. Coherence spaces

In this section, we apply the recipe described in Section 1 to the category of coherence spaces introduced by Jean-Yves Girard (Girard 1987) and establish that it does indeed compute the free exponential modality in this category. Recall that a *coherence space* $E = (|E|, \frown)$ consists of a set $|E|$ called its *web* equipped with a binary reflexive and symmetric relation $\frown$ called its *coherence relation*. A *clique* u of E is defined as a set of pairwise coherent elements of the web:

$$\forall e_1, e_2 \in u, \qquad e_1 \frown e_2.$$

The coherence relation induces an *incoherence relation* $\asymp$ defined as

$$e_1 \asymp e_2 \quad \iff \quad \neg(e_1 \frown e_2) \quad \text{or} \quad e_1 = e_2.$$

We recall below the definition of the category **Coh** of coherence spaces, together with the interpretation of the logical connectives which appear in the construction of the free exponential modality.

Product. The product $E_1 \,\&\, E_2$ of two coherence spaces E_1 and E_2 is defined as the coherence space whose web $|E_1 \,\&\, E_2| = |E_1| \sqcup |E_2|$ is the disjoint union of the two webs $|E_1|$ and $|E_2|$, and where two elements (e, i) and (e', j) for $i \in \{1, 2\}$ are coherent when $i \neq j$ or when $i = j$ and $e \frown_{E_i} e'$.

Tensor product. The tensor product $E_1 \otimes E_2$ of two coherence spaces E_1 and E_2 is defined as the coherence space whose web $|E_1 \otimes E_2| = |E_1| \times |E_2|$ is equal to the cartesian product of the two webs $|E_1|$ and $|E_2|$ with the following coherence relation:

$$e_1 \otimes e_2 \frown_{E_1 \otimes E_2} e_1' \otimes e_2' \quad \iff \quad e_1 \frown_{E_1} e_1' \quad \text{and} \quad e_2 \frown_{E_2} e_2'$$

where $e_1 \otimes e_2$ is a convenient notation for the pair (e_1, e_2) in the web $|E_1 \otimes E_2|$.

Linear implication. The linear implication $E_1 \multimap E_2$ of two coherence spaces E_1 and E_2 is defined as the coherence space whose web $|E_1 \multimap E_2| = |E_1| \times |E_2|$ is equal to the cartesian product of the two webs $|E_1|$ and $|E_2|$ with the following coherence relation:

$$e_1 \multimap e_2 \frown_{E_1 \multimap E_2} e_1' \multimap e_2' \quad \iff \quad \begin{cases} e_1 \frown_{E_1} e_1' & \Rightarrow \quad e_2 \frown_{E_2} e_2' \\ & \text{and} \\ e_2 \asymp_{E_2} e_2' & \Rightarrow \quad e_1 \asymp_{E_1} e_1' \end{cases}$$

where $e_1 \multimap e_2$ is a convenient notation for the pair (e_1, e_2) in the web $|E_1 \multimap E_2|$.

The category of coherence spaces. The category **Coh** of coherence spaces has coherence spaces as objects and cliques of $E_1 \multimap E_2$ as morphisms from E_1 to E_2. Note that the web of $E_1 \multimap E_2$ is equal to the cartesian product $|E_1| \times |E_2|$. This enables to see a morphism as a relation between the sets $|E_1|$ and $|E_2|$ satisfying additional consistency constraints. In particular, identity and composition are defined in the same way as in the category of sets and relations. The category **Coh** is $*$-autonomous and defines a model of the multiplicative and additive fragment of linear logic.

On the other hand, it is easy to see that the tensor product does not commute with finite products. Typically, the canonical morphism

$$E \otimes (\mathbf{1} \,\&\, \mathbf{1}) \quad \rightarrow \quad (E \otimes \mathbf{1}) \,\&\, (E \otimes \mathbf{1})$$

is not an isomorphism. This explains why formula (5) does not work in the category of coherence spaces. Hence, the construction of the free exponential modality proceeds along the recipe in three steps described in Section 1.

First step: compute the free affine object. Computing the free pointed object $(E_\bullet, u)$ generated by a coherence space E is straightforward because the category **Coh** has finite products: the pointed object is thus defined as

$$E_\bullet \quad = \quad E \,\&\, \mathbf{1}$$

equipped with the second projection $u : E \,\&\, \mathbf{1} \rightarrow \mathbf{1}$. Recall that a multiclique of the coherence space E is defined as a multiset on $|E|$ whose underlying set is a clique of E. It is useful to think of $E \,\&\, \mathbf{1}$ as the space of multicliques of E with *at most* one element, this defining (as we will see) the very first layer of the construction of the free exponential modality. In that respect, the unique element $*$ of the web of $\mathbf{1}$ denotes the empty clique, while every element e of E denotes the singleton clique $[e]$.

Second step: compute the symmetric tensor power. We would like to compute the equalizer of the $n!$ symmetries

$$(E \,\&\, \mathbf{1})^{\otimes n} \underset{symmetry}{\overset{symmetry}{\underset{\cdots}{\rightrightarrows}}} (E \,\&\, \mathbf{1})^{\otimes n} \tag{12}$$

on the coherence space $(E \,\&\, \mathbf{1})^{\otimes n}$ in the category of coherence spaces. We claim that this equalizer is provided by the coherence space $E^{\leq n}$ defined as follows:

— its web $|E^{\leq n}| = \mathcal{M}^{\leq n}_{\text{fin}}(|E|)$ contains the multicliques of E with at most n elements, seen equivalently as the multicliques of n elements in $E \,\&\, \mathbf{1}$,

— two elements u and v are coherent in $E^{\leq n}$ precisely when their union $u \uplus v$ is a multiclique

together with the morphism

$$E^{\leq n} \xrightarrow{\quad equalizer \quad} (E \,\&\, \mathbf{1})^{\otimes n}$$

defined as the clique containing all the elements of the form

$$[e_1, \cdots, e_n] \;\multimap\; e_1 \otimes \cdots \otimes e_n$$

where $[e_1, \cdots, e_n]$ is a multiclique of n elements in $E \,\&\, \mathbf{1}$. Here, we take advantage of the fact that every multiclique $[e_1, \cdots, e_n]$ may be seen alternatively as a clique of $p \leq n$ elements in E, completed by $n-p$ occurrences of the element $*$. It is not difficult to establish that $E^{\leq n}$ defines the expected equalizer. Simply observe that a clique

$$X \xrightarrow{R} (E \,\&\, \mathbf{1})^{\otimes n}$$

equalizes the $n!$ symmetries precisely when R factors as

$$X \xrightarrow{S} E^{\leq n} \xrightarrow{equalizer} (E \,\&\, \mathbf{1})^{\otimes n}$$

where S is defined as the clique

$$S \quad = \quad \{\, x \multimap [e_1, \ldots, e_n] \quad | \quad x \multimap e_1 \otimes \cdots \otimes e_n \;\in\; R \,\}.$$

This factorization is moreover unique, this establishing that $E^{\leq n}$ is the equalizer of the diagram (12). The recipe described in Section 1 requires also to check that this equalizer commutes with the tensor product, in the sense that the morphism

$$X \otimes E^{\leq n} \xrightarrow{X \otimes \; equalizer} X \otimes (E \,\&\, \mathbf{1})^{\otimes n} \tag{13}$$

defines the equalizer of the $n!$ symmetries on the coherence space $X \otimes (E \,\&\, \mathbf{1})^{\otimes n}$ for every coherence space X. So, suppose that a morphism

$$Y \xrightarrow{R} X \otimes (E \,\&\, \mathbf{1})^{\otimes n}$$

equalizes the $n!$ symmetries, in the sense that the $n!$ composite morphisms are equal in the diagram below:

$$Y \xrightarrow{R} X \otimes (E \,\&\, \mathbf{1})^{\otimes n} \underset{X \otimes symmetry}{\overset{X \otimes symmetry}{\rightrightarrows}} X \otimes (E \,\&\, \mathbf{1})^{\otimes n}$$

In that case, it is easy to see that the morphism R factors as

$$Y \xrightarrow{S} X \otimes E^{\leq n} \xrightarrow{X \otimes \; equalizer} X \otimes (E \,\&\, \mathbf{1})^{\otimes n}$$

where the morphism S is defined as the clique

$$S \quad = \quad \{ \quad y \multimap x \otimes [e_1, \cdots, e_n] \quad | \quad y \multimap x \otimes e_1 \otimes \cdots \otimes e_n \;\in\; R \,\}$$

where $e_1, \ldots, e_n$ are elements of the web $|E \,\&\, \mathbf{1}| = |E| \sqcup \{*\}$. This is moreover the unique way to factor R through the morphism (13). This concludes the proof that $E^{\leq n}$ defines the equalizer of $n!$ symmetries in (12).

Third step: compute the sequential limit. We compute the limit of the sequential diagram

$$E^{\leq 0} = \mathbf{1} \longleftarrow E^{\leq 1} = (E \,\&\, \mathbf{1}) \longleftarrow E^{\leq 2} \longleftarrow E^{\leq 3} \cdots$$

where each morphism $E^{\leq n+1} \to E^{\leq n}$ is defined as the clique

$$\{ \quad [e_1, \cdots, e_n, *] \quad \multimap \quad [e_1, \cdots, e_n] \quad | \quad [e_1, \cdots, e_n] \in |E^{\leq n}| \quad \}.$$

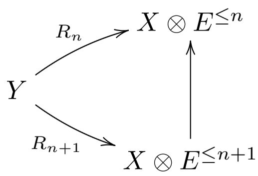
Note that the morphism enables to see the coherence space $E^{\leq n}$ as the coherence space $E^{\leq n+1}$ restricted to its multisets $[e_1, \cdots, e_n, *]$ containing $k \leq n$ elements of the original coherence space E. It is nearly immediate that this limit is provided by the coherence space $!E$ defined as follows:

— its web $|!E| = \mathcal{M}_{\text{fin}}(|E|)$ contains the finite multicliques of E,
— two elements u and v are coherent in $!E$ precisely when their union $u \uplus v$ is a multiclique

together with the family of projections $\pi_n : !E \to E^{\leq n}$ defined by restricting the coherence space $!E$ to its multisets containing $k \leq n$ elements of the original coherence space E. At this point, there simply remains to check that the sequential limit commutes with the tensor product. Consider a family of morphisms

$$Y \xrightarrow{R_n} X \otimes E^{\leq n}$$

which makes the diagram



commute for every natural number n. In that case, every morphism R_n factors as

$$Y \xrightarrow{S} X \otimes !E \xrightarrow{X \otimes \pi_n} X \otimes E^{\leq n}$$

where the morphism S is defined as the clique

$$S \quad = \quad \{ \quad y \multimap x \otimes [e_1, \cdots, e_n] \quad | \quad y \multimap x \otimes [e_1, \cdots, e_n] \quad \in \quad R_n \quad \}$$

where $e_1, \cdots, e_n$ are elements of the web of the original coherence space E. Moreover, there exists a unique such morphism S satisfying the equality $R_n = (X \otimes \pi_n) \circ S$ for every natural number n. This establishes that the assumptions of Proposition 1 are satisfied, and consequently, that the sequential limit $!E$ defines the free commutative comonoid generated by the coherence space E in the category **Coh**.

4. Conway games

In this section, we apply the recipe described in Section 1 to the category of Conway games introduced by André Joyal in (Joyal 1977) and establish that, just as in the case of coherence spaces, it computes the free exponential modality in this category.

Conway games. A *Conway game* A is an oriented rooted graph (V_A, E_A, λ_A) consisting of (a) a set V_A of vertices called the *positions* of the game; (b) a set $E_A \subset V_A \times V_A$ of edges called the *moves* of the game; (c) a function $\lambda_A : E_A \to \{-1, +1\}$ indicating whether a move is played by Opponent (-1) or by Proponent $(+1)$. We write $\star_A$ for the root of the underlying graph. A Conway game is called *negative* when all the moves starting from its root are played by Opponent.

A *play* $s = m_1 \cdot m_2 \cdot \ldots \cdot m_{k-1} \cdot m_k$ of a Conway game A is a path $s : \star_A \twoheadrightarrow x_k$ starting from the root $\star_A$

$$s \quad : \quad \star_A \xrightarrow{m_1} x_1 \xrightarrow{m_2} \ldots \xrightarrow{m_{k-1}} x_{k-1} \xrightarrow{m_k} x_k \quad .$$

Two paths are parallel when they have the same initial and final positions. A play is *alternating* when

$$\forall i \in \{1, \ldots, k-1\}, \qquad \lambda_A(m_{i+1}) = -\lambda_A(m_i).$$

We note Play_A the set of plays of a game A.

Dual. Every Conway game A induces a *dual* game A^* obtained simply by reversing the polarity of moves.

Tensor product. The tensor product $A \otimes B$ of two Conway games A and B is essentially the asynchronous product of the two underlying graphs. More formally, it is defined as:

— $V_{A \otimes B} = V_A \times V_B$,
— its moves are of two kinds :

$$x \otimes y \to \begin{cases} z \otimes y \text{ if } x \to z \text{ in the game } A \\ x \otimes z \text{ if } y \to z \text{ in the game } B, \end{cases}$$

— the polarity of a move in $A \otimes B$ is the same as the polarity of the underlying move in the component A or the component B.

The unique Conway game $\mathbf{1}$ with a unique position $\star$ and no move is the neutral element of the tensor product. As usual in game semantics, every play s of the game $A \otimes B$ can be seen as the interleaving of a play $s_{|A}$ of the game A and a play $s_{|B}$ of the game B.

Strategies. Remark that the definition of a Conway game does not imply that all the plays are alternating. The notion of alternation between Opponent and Proponent only appears at the level of strategies (i.e. programs) and not at the level of games (i.e. types). A *strategy* σ of a Conway game A is defined as a non empty set of *alternating plays* of even length such that (a) every non empty play starts with an Opponent move; (b) σ is closed by even length prefix; (c) σ is *deterministic*, i.e. for all plays s, and for all moves m, n, n',

$$s \cdot m \cdot n \in \sigma \ \wedge \ s \cdot m \cdot n' \in \sigma \ \Rightarrow \ n = n'.$$

The category of Conway games. The category **Conway** has Conway games as objects, and strategies σ of $A^* \otimes B$ as morphisms $\sigma : A \to B$. The composition is based on the usual "parallel composition plus hiding" technique and the identity is defined by a copycat strategy. The resulting category **Conway** is compact-closed in the sense of (Kelly & Laplaza 1980).

It appears that the category **Conway** does not have finite nor infinite products (Melliès 2005). For that reason, we compute the free exponential modality in the full subcategory $\mathbf{Conway}^{-}$ of negative Conway games, which is symmetric monoidal closed and has products. The linear implication $A \multimap B$ is obtained by restricting the plays of $A^* \otimes B$

to opponent starting plays. We explain in a later stage how the free construction on the subcategory **Conway⁻** induces the free construction on the whole category.

First step: compute the free pointed object. The monoidal unit **1** is terminal in the category **Conway⁻**. In other words, every negative Conway game may be seen as an affine object in a unique way, by equipping it with the empty strategy $t_A : A \to \mathbf{1}$. In particular, the free affine object $A_\bullet$ is simply A itself.

Second step: compute the symmetric tensor power. We would like to compute the equalizer of the $n!$ symmetries

$$A^{\otimes n} \underset{symmetry}{\overset{symmetry}{\rightrightarrows}} A^{\otimes n} \tag{14}$$

on the coherence space $A^{\otimes n}$ in the category of negative Conway games. We claim that this equalizer is provided by the Conway game $A^{\leq n}$ defined as follows

— the positions of the game $A^{\leq n}$ are the finite words $w = x_1 \cdots x_n$ of length n, whose letters are positions x_i of the game A, and such that for every $1 \leq i < n$, the position x_{i+1} is the root $\star_A$ of the game A whenever the position x_i is the root $\star_A$ of the game A,

— its root is the word $\star_{A^{\leq n}} = \star_A \cdots \star_A$ where the n the positions are at the root $\star_A$ of the game A,

— a move $w \to w'$ is a move played in one copy:

$$w_1 \; x \; w_2 \;\to\; w_1 \; y \; w_2$$

where $x \to y$ is a move of the game A. Note that the condition on the positions implies that when a new copy of A is opened (that is, when $x = \star_A$) no position in w_1 is at the root, and all the positions in w_2 are at the root,

— the polarities of moves are inherited from the game A in the obvious way.

It is equipped with the morphism

$$A^{\leq n} \xrightarrow{\;equalizer\;} A^{\otimes n}$$

defined as the strategy containing every even play s of $A^{\leq n} \multimap A^{\otimes n}$ such that

$$\forall t \prec^{even} s \;,\; t_{|A^{\leq n}} = \langle t_{|A^{\otimes n}} \rangle$$

where $\prec^{even}$ is the prefix order restricted to even plays and $\langle t_{|A^{\otimes n}} \rangle$ is the play of the game $A^{\leq n}$ obtained by reordering the copies of A so that every occurrence of $\star_A$ appears at the end of each position of the play.

It is not difficult to establish that $A^{\leq n}$ defines the expected equalizer. Simply observe that a strategy

$$X \xrightarrow{\;\sigma\;} A^{\otimes n}$$

equalizes the $n!$ symmetries precisely when σ factors as

$$X \xrightarrow{\;\tau\;} A^{\leq n} \xrightarrow{\;equalizer\;} A^{\otimes n}$$

where τ is defined as the strategy containing every even play s of $X \multimap A^{\leq n}$ such that

$$\forall t \prec^{even} s \ , \exists t' \in \sigma, \quad t_{|A^{\leq n}} = \langle t'_{|A^{\otimes n}} \rangle \qquad \text{and} \qquad t_{|X} = t'_{|X}.$$

This factorization is moreover unique, this establishing that $A^{\leq n}$ is the equalizer of the diagram (14). The recipe described in Section 1 requires also to check that this equalizer commutes with the tensor product. So, suppose that a morphism

$$Y \xrightarrow{\quad\sigma\quad} X \otimes A^{\otimes n}$$

equalizes the $n!$ symmetries. In that case, it is easy to see that the morphism σ factors as

$$Y \xrightarrow{\quad\tau\quad} X \otimes A^{\leq n} \xrightarrow{\quad X\otimes \ equalizer\quad} X \otimes A^{\otimes n}$$

where the morphism τ is defined as the strategy containing every even play s of the game $Y \multimap X \otimes A^{\leq n}$ such that

$$\forall t \prec^{even} s \ , \exists t' \in \sigma, \quad t_{|A^{\leq n}} = \langle t'_{|A^{\otimes n}} \rangle \qquad \text{and} \qquad t_{|X} = t'_{|X} \qquad \text{and} \qquad t_{|Y} = t'_{|Y}.$$

This is moreover the unique way to factor σ through the morphism $X \otimes$ *equalizer*. This concludes the proof that $A^{\leq n}$ defines the equalizer of $n!$ symmetries in (14).

Third step: compute the sequential limit. We compute the limit of the sequential diagram

$$A^{\leq 0} = \mathbf{1} \longleftarrow A^{\leq 1} = A \longleftarrow A^{\leq 2} \longleftarrow A^{\leq 3} \longleftarrow \cdots$$

where each morphism $A^{\leq n+1} \to A^{\leq n}$ is defined as the partial copycat strategies $A^{\leq n} \leftarrow A^{\leq n+1}$ identifying $A^{\leq n}$ as the subgame of $A^{\leq n+1}$ where only the first n copies of A can be played. It is easy to check that the limit of this diagram in the category **Conway**$^{-}$ is the Conway game A^{∞} defined as follows:

— the positions of the game A^{∞} are the infinite words $w = x_1 \cdot x_2 \cdots$ for which there exists an n such that

$$x_1 \cdots x_n \in V_{A^{\leq n}}$$

and x_i is the root $\star_A$ for all $i \geq n$,

— its root is the infinite word $\star_{A^{\infty}} = \star_A^{\omega}$ where every position is at the root of A,

— a move $w \to w'$ is a move played in one copy:

$$w_1 \ x \ w_2 \ \to \ \ w_1 \ y \ w_2$$

where $x \to y$ is a move of the game A,

— the polarities of moves are inherited from the game A in the obvious way

together with the family of projections $\pi_n : A^{\infty} \to A^{\leq n}$ defined as the strategy containing every even play s of $A^{\infty} \multimap A^{\leq n}$ such that

$$\forall t \prec^{even} s \ , \quad t_{|A^{\infty}} = \langle t_{|A^{\leq n}} \rangle_{\infty}.$$

In the equation above, we use the fact that the game $A^{\leq n}$ can be seen as a sub game of A^{∞} (obtained by expanding every finite word with $\star_A^{\omega}$) and we note $\langle - \rangle_{\infty}$ this embedding.

At this point, there simply remains to check that the sequential limit commutes with the tensor product. Consider a family of morphisms

$$Y \xrightarrow{\sigma_n} X \otimes A^{\leq n}$$

which makes the diagram

$$\begin{array}{ccc} & \overset{\sigma_n}{\nearrow} & X \otimes A^{\leq n} \\ Y & & \uparrow \\ & \underset{\sigma_{n+1}}{\searrow} & X \otimes A^{\leq n+1} \end{array}$$

commute for every natural number n. In that case, every morphism σ_n factors as

$$Y \xrightarrow{\tau} X \otimes A^{\infty} \xrightarrow{X \otimes \pi_n} X \otimes A^{\leq n}$$

where the morphism τ is defined as the strategy containing every even play s of the game $Y \multimap X \otimes A^{\infty}$ such that

$$\forall t \prec^{even} s \ , \exists t' \in \sigma, \quad t_{|A^{\infty}} = \langle t'_{|A^{\leq n}} \rangle_{\infty} \ \text{ and } \ t_{|X} = t'_{|X} \ \text{ and } \ t_{|Y} = t'_{|Y}.$$

Moreover, τ is the unique morphism satisfying the equality $\sigma_n = (X \otimes \pi_n) \circ \tau$ for every natural number n. This establishes that the assumptions of Proposition 1 are satisfied, and consequently, that the sequential limit A^{∞} defines the free commutative comonoid generated by the negative Conway game A in the category $\mathbf{Conway}^{-}$.

The free exponential in the whole category Conway. It is worth observing that the free construction in the category $\mathbf{Conway}^{-}$ extends in fact to the whole category of Conway games. The reason why is that every commutative comonoid K in the category of Conway games is in fact an Opponent-starting game. The proof is extremely simple: suppose that there exists an initial Player move m in a commutative comonoid (K, d, u). In that case, the equality

$$K \xrightarrow{d} K \otimes K \xrightarrow{K \otimes u} K \qquad = \qquad K \xrightarrow{id} K$$

ensures that the strategy $d : K_1 \to K_2 \otimes K_3$ reacts to the move m played in the component K_1 by playing a move n in the component K_2 or K_3. Then, the equality

$$K \xrightarrow{d} K \otimes K \xrightarrow{\sigma} K \otimes K \qquad = \qquad K \xrightarrow{d} K \otimes K$$

implies that the strategy d reacts by playing the same move n in the other component K_3 or K_2. This contradicts the fact that the strategy d is deterministic, and establishes that every commutative comonoid is negative. Moreover, the inclusion functor from $\mathbf{Conway}^{-}$ to **Conway** has a right adjoint, which associates to every Conway game A the negative Conway game A^{-} obtained by removing all the Proponent moves from the root $\star_A$. By combining these two observations, one obtains that the game $(A^{-})^{\infty}$ is the free commutative comonoid generated by a Conway game A in the category **Conway**.

5. An instructive counterexample: finiteness spaces

We have just established that the very same limit formula enables to compute the free exponential modality in the coherence space model as well as in the Conway game model. Interestingly, this does not mean that the formula works in every model of linear logic. This is precisely the purpose of this section: we explain why the formula does not work in the finiteness space model of linear logic, an important relational model introduced by Thomas Ehrhard (Ehrhard 2005). Our purpose is not only to analyze the reasons for the defect, but also to pave the way for the solution based on configuration spaces developed in the subsequent Sections 6 and 8.

5.1. *The category of finiteness spaces*

The definition of a finiteness space is based on the notion of an orthogonality relation, defined as follows. Let $\mathbb{E}$ be a countable set. Two subsets u, $u' \subseteq \mathbb{E}$ are called *orthogonal* precisely when their intersection $u \cap u'$ is finite:

$$u \perp_{\text{fin}} u' \iff u \cap u' \text{ finite.} \tag{15}$$

The orthogonal of a set of subsets $\mathcal{F} \subseteq \mathcal{P}(\mathbb{E})$ is then defined as:

$$\mathcal{F}^{\perp} \quad = \quad \{ \quad u' \subseteq \mathbb{E} \quad | \quad \forall u \in \mathcal{F},\ u \perp_{\text{fin}} u' \quad \}.$$

A *finiteness space* $E = (|E|, \mathcal{F}(E))$ consists of a countable $|E|$ called its *web* and of a set $\mathcal{F}(E) \subseteq \mathcal{P}(|E|)$ called its *finiteness structure*. One requires moreover that the finiteness structure is equal to its biorthogonal:

$$\mathcal{F}(E)^{\perp\perp} = \mathcal{F}(E).$$

The elements of $\mathcal{F}(E)$ (resp. $\mathcal{F}(E)^{\perp}$) are called *finitary* (resp. *antifinitary*).

Finite product. The product $E_1 \& E_2$ of two finiteness spaces E_1 and E_2 is defined by its web $|E_1 \& E_2| = |E_1| \sqcup |E_2|$ and by its finiteness structure $\mathcal{F}(E_1 \& E_2) = \mathcal{F}(E_1) \sqcup \mathcal{F}(E_2)$.

Tensor product. The tensor product $E_1 \otimes E_2$ of two finiteness spaces E_1 and E_2 is defined by $|E_1 \otimes E_2| = |E|_1 \times |E_2|$ and by

$$\mathcal{F}(E_1 \otimes E_2) = \left\{ w \subseteq |E|_1 \times |E_2| \;\middle|\; \begin{array}{l} \Pi_{E_1}(w) \in \mathcal{F}(E_1), \\ \Pi_{E_2}(w) \in \mathcal{F}(E_2) \end{array} \right\}$$

where $\Pi_{E_1}(w) = \{e_1 \in |E_1| \,|\, \exists e_2 \in |E_2|,\, (e_1, e_2) \in w\}$. The unit of the tensor is defined by $|1| = \{*\}$, and $\mathcal{F}(1) = \{\emptyset, \{*\}\}$. Note that the definition of the tensor product is valid because the set $\mathcal{F}(E_1 \otimes E_2)$ is equal to its biorthogonal, see (Ehrhard 2005) for details.

Linear implication. A finitary relation R between two finiteness spaces E_1 and E_2 is defined as a subset of $|E_1| \times |E_2|$ such that

$$\begin{array}{rcl} \forall u \in \mathcal{F}(E_1), \quad R(u) & = & \left\{ e_2 \in |E_2| \;\middle|\; \exists e_1 \in u,\ e_1 \, R \, e_2 \right\} \in \mathcal{F}(E_2), \\ \forall v' \in \mathcal{F}(E_2)^{\perp}, \quad {}^{t}R(v') & = & \left\{ e_1 \in |E_1| \;\middle|\; \exists e_2 \in v',\ e_1 \, R \, e_2 \right\} \in \mathcal{F}(E_1)^{\perp}. \end{array}$$

The linear implication $E_1 \multimap E_2$ is defined as the finiteness space with web $|E_1 \multimap E_2| = |E|_1 \times |E_2|$ and with finiteness structure $\mathcal{F}(E_1 \multimap E_2)$ the set of finitary relations.

The exponential modality. The exponential modality ! is defined as follows: given a finiteness space E, the finiteness space $!E$ has its web $|!E| = \mathcal{M}_{\text{fin}}(|E|)$ defined as the set of finite multisets $\mu : |E| \to \mathbb{N}$ and its finiteness structure defined as

$$\mathcal{F}(!E) \quad = \quad \{ \quad M \subseteq \mathcal{M}_{\text{fin}}(|E|) \quad | \quad \Pi_E(M) \in \mathcal{F}(E) \quad \},$$

where the support $\Pi_E(M)$ of a set of finite multisets $M \in \mathcal{M}_{\text{fin}}(|E|)$ is defined as

$$\Pi_E(M) \quad = \quad \{ \quad e \in |E| \quad | \quad \exists \mu \in M,\, \mu(e) \neq 0 \quad \}.$$

The category of finiteness spaces. The category **Fin** of finiteness spaces has finiteness spaces as objects and finitary relations as morphisms, composed in a relational way. Observe in particular that the identity relation on the web $|E|$ of a finiteness space E defines a finitary relation between E and itself, and that relational composition of two finitary relations defines a finitary relation. The category **Fin** of finiteness spaces is $*$-autonomous and provides a model of propositional linear logic.

5.2. *The counter-example*

Christine Tasson observes in her PhD thesis that the exponential modality ! defined by Ehrhard associates to every finiteness space E its free commutative comonoid $!E$ in the category **Fin**, see (Tasson 2009) for details. On the other hand, it appears that the finiteness space E^{∞} computed by the limit formula (9) does *not* coincide with the finiteness space $!E$, and in fact, does not define (in any obvious way) a commutative comonoid in the category **Fin**. So, let us proceed along the recipe explained in Section 1, and see where the construction goes wrong. The first step of the construction is to compute the free pointed object $E_{\bullet}$ generated by a finiteness space E. Since the category **Fin** has cartesian products, the object $E_{\bullet}$ is simply defined as

$$E_{\bullet} \quad = \quad E \,\&\, \mathbf{1}.$$

The second step of the construction is to compute the symmetric tensor power $E^{\leq n}$ of the finiteness space E, defined as the equalizer of the $n!$ symmetries over the finiteness space $(E_{\bullet})^{\otimes n}$. A simple computation shows that the web of $E^{\leq n}$ is equal to the set of multisets of elements of $|E|$ of cardinality less than n:

$$|E^{\leq n}| \quad = \quad \mathcal{M}^{\leq n}_{\text{fin}}(|E|)$$

and that its finiteness structure is equal to

$$\mathcal{F}(E^{\leq n}) \quad = \quad \{ \quad M_n \subseteq \mathcal{M}^{\leq n}_{\text{fin}}(|E|) \quad | \quad \Pi_E(M_n) \in \mathcal{F}(E) \quad \}.$$

Moreover, this equalizer commutes with the tensor product in the expected sense. This completes the second step of the construction.

The third and last step in order to compute the limit formula (9) is to take the sequential limit E^∞ of the finiteness spaces $E^{\leq n}$. The web of this sequential limit is equal to the set $|E^\infty| = \mathcal{M}_{\text{fin}}(|E|)$ of finite multisets of elements of $|E|$, and its finiteness structure to:

$$\mathcal{F}(E^\infty) \quad = \quad \left\{ \quad M \in \mathcal{M}_{\text{fin}}(|E|) \quad \middle| \quad \forall n \in \mathbb{N}, \quad \begin{array}{l} M_n = M \cap \mathcal{M}_{\text{fin}}^{\leq n}(|E|), \\ \Pi_E(M_n) \in \mathcal{F}(E). \end{array} \quad \right\}.$$

Note that the webs of $!E$ and of E^∞ are equal, and coincide in fact with the free exponential in the relational model. However, the finiteness structures of $!E$ and E^∞ do not coincide in general:

$$\mathcal{F}(!E) \quad \subsetneq \quad \mathcal{F}(E^\infty).$$

We illustrate that point on the finiteness space **Nat** whose web $|\mathbf{Nat}| = \mathbb{N}$ is the set of natural numbers and whose finiteness structure $\mathcal{F}(\mathbf{Nat})$ is the collection $\mathcal{P}_{\text{fin}}(\mathbb{N})$ of finite subsets. Note that **Nat** is the interpretation of the formula $(!\mathbf{1})^\perp$ in the category **Fin**. It is easy to see that the finiteness spaces **!Nat** and $\mathbf{Nat}^\infty$ have the same web $\mathcal{M}_{\text{fin}}(\mathbb{N})$, but different finiteness structures:

$$\begin{array}{rcl} \mathcal{F}(!\mathbf{Nat}) & = & \{M \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}) \mid \Pi_{\mathbf{Nat}}(M) \text{ finite}\} \\ & = & \{M \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}) \mid \exists N \in \mathbb{N}, M \subseteq \mathcal{M}_{\text{fin}}(\{0, \ldots, N\})\}, \\ \mathcal{F}(\mathbf{Nat}^\infty) & = & \{M \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}) \mid \forall n \in \mathbb{N}, \Pi_{\mathbf{Nat}}(M \cap \mathcal{M}_{\text{fin}}^n(\mathbb{N})) \text{ finite}\}. \end{array}$$

For instance, let $\mu_n = [0, \cdots, n]$ be the set of all natural numbers $k \leq n$ seen as a multiset. Then, the set of all these multisets

$$M \quad = \quad \{ \quad \mu_n \quad | \quad n \in \mathbb{N} \quad \}$$

is an element of $\mathcal{F}(\mathbf{Nat}^\infty)$ but not an element of $\mathcal{F}(!\mathbf{Nat})$. Considering the content of Proposition 1, the reason for the failure of the construction is that the sequential limit (8) does not commute with the tensor product. Let us illustrate that interesting point by comparing $\mathbf{Nat} \otimes \mathbf{Nat}^\infty$ with the sequential limit $\boldsymbol{lim}(\mathbf{Nat} \otimes \mathbf{Nat}^{\leq n})$ of the diagram of finiteness spaces:

$$\mathbf{Nat} \otimes \mathbf{1} \longleftarrow \mathbf{Nat} \otimes \mathbf{Nat}^{\leq 1} \longleftarrow \cdots \longleftarrow \mathbf{Nat} \otimes \mathbf{Nat}^{\leq n} \longleftarrow \mathbf{Nat} \otimes \mathbf{Nat}^{\leq n+1} \longleftarrow \cdots$$

The two finiteness spaces $\mathbf{Nat} \otimes \mathbf{Nat}^\infty$ and $\boldsymbol{lim}(\mathbf{Nat} \otimes \mathbf{Nat}^{\leq n})$ have the same web $\mathbb{N} \times \mathcal{M}_{\text{fin}}(\mathbb{N})$ but different finiteness structures:

$$\mathcal{F}(\mathbf{Nat} \otimes \mathbf{Nat}^\infty) = \{M \subseteq \mathbb{N} \times \mathcal{M}_{\text{fin}}(\mathbb{N}) \mid \exists N, M \subseteq \{0, \ldots, N\} \times \mathcal{M}_{\text{fin}}(\{0, \ldots, N\})\},$$

$$\mathcal{F}\left(\boldsymbol{lim}(\mathbf{Nat} \otimes \mathbf{Nat}^{\leq n})\right) = \{M \mid \forall n \in \mathbb{N}, \exists N_n, M_n \subseteq \{0, \ldots, N_n\} \times \mathcal{M}_{\text{fin}}(\{0, \ldots, N_n\})\},$$

where $M_n = M \cap (\mathbb{N} \times \mathcal{M}_{\text{fin}}^n(\mathbb{N}))$ denotes the subset of M made of pairs whose second component is a multiset containing exactly n elements. Typically, the set of pairs

$$M' \quad = \quad \{ \quad (n, \mu_n) \quad | \quad n \in \mathbb{N} \quad \}$$

is an element of $\mathcal{F}\left(\boldsymbol{lim}(\mathbf{Nat} \otimes \mathbf{Nat}^{\leq n})\right)$ but not of $\mathcal{F}(\mathbf{Nat} \otimes \mathbf{Nat}^\infty)$ because its projection on the first component **Nat** has the infinite support $\mathbb{N}$. This subtle phenomenon comes from the fact that an infinite directed union of finitary sets in a finiteness space E is not necessarily finitary in that space. This departs from the coherence space model where an infinite directed union of cliques of a space E is a clique of that space, this explaining the success of the recipe in the coherence model.

Remark. The interested reader will check that Formula (5) computes the same finiteness space E^∞ as Formula (9) because the finiteness space $E^{\leq n}$ coincides in fact with the cartesian product of E^k for $k \leq n$.

The next two sections are devoted to a resolution of that question, achieved by embedding the category of finiteness spaces in a larger category of configuration spaces, and performing the free commutative comonoid construction in that larger universe.

6. Configuration spaces

As a preliminary training exercise before attacking (in Section 8) the question of finiteness spaces, we come back to the coherence space model, and explain how the free exponential modality may be constructed in a larger universe of *configuration spaces* where negation is not involutive anymore. More specifically, we show that the limit formula described in Section 1 computes the free commutative comonoid in the category of configuration spaces. Then, we explain (in Section 7) how to recover the category of coherence spaces, its tensor product, its cartesian product, and its exponential modality, by restricting the category of configuration spaces to its self-dual objects. This provides an alternative construction of the exponential modality in the category of coherence spaces, as well as a precious guide towards the construction of the exponential modality in finiteness spaces, which will be performed along the same lines in Section 8.

The category of configuration spaces. A *configuration space* is defined as a pair

$$E \quad = \quad (|E|, \mathrm{Config}(E))$$

consisting of a countable set $|E|$ called the web of E and of a set $\mathrm{Config}(E) \subseteq \mathcal{P}(|E|)$ whose elements are called the configurations of E. Every configuration space is required moreover to satisfy the following covering condition:

$$\forall x \in |E|, \quad \exists u \in \mathrm{Config}(E) \quad \text{such that} \quad x \in u.$$

The category **Config** has configuration spaces as objects, and its morphisms

$$R \quad : \quad E_1 \to E_2$$

are the binary relations $R \subseteq |E_1| \times |E_2|$ satisfying the two properties:

— R *transports configurations forward*:

$$\forall u \in \mathrm{Config}(E_1), \quad R(u) \in \mathrm{Config}(E_2),$$

— R is *locally injective*:

$$\forall u \in \mathrm{Config}(E_1),\ \forall e_1, e_1' \in u,\ \forall e_2 \in |E_2|, \quad e_1\, R\, e_2 \text{ and } e_1'\, R\, e_2 \ \Rightarrow\ e_1 = e_1'.$$

Here, $R(u)$ is defined as

$$R(u) \quad = \quad \{ \quad e_2 \in |E_2| \quad | \quad \exists e_1 \in u, \quad e_1 \; R \; e_2 \quad \}.$$

The identity and composition laws are defined as in the category of sets and relations. Note in particular that the identity relation satisfies the two properties, just stated, about morphisms between configuration spaces, and that relational composition preserves them.

Finite product. The product $E_1 \,\&\, E_2$ of two configuration spaces E_1 and E_2 is defined by its web $|E_1 \,\&\, E_2| = |E_1| \sqcup |E_2|$ and by its configurations:

$$\mathrm{Config}(E_1 \,\&\, E_2) \quad = \quad \{ \quad u_1 \sqcup u_2 \,|\, u_1 \in \mathrm{Config}(E_1), \; u_2 \in \mathrm{Config}(E_2) \quad \}.$$

Its unit is the terminal object of the category **Config**, the configuration space $\top$ with an empty web, and the empty set as its unique configuration: $\mathrm{Config}(\top) = \{\emptyset\}$.

Finite coproduct. The coproduct $E_1 \oplus E_2$ of two configuration spaces E_1 and E_2 is defined by its web $|E_1 \oplus E_2| = |E_1| \sqcup |E_2|$ and by its configurations:

$$\mathrm{Config}(E_1 \oplus E_2) \quad = \quad \{ \quad u_1 \in \mathrm{Config}(E_1) \quad \} \quad \cup \quad \{ \quad u_2 \in \mathrm{Config}(E_2) \quad \}.$$

Its unit is the initial object of the category **Config**, the configuration space $\mathbf{0}$ with an empty web, and with no configuration: $\mathrm{Config}(\mathbf{0}) = \emptyset$.

Tensor product. The tensor product $E_1 \otimes E_2$ of two configuration spaces is defined by its web $|E_1 \otimes E_2| = |E_1| \times |E_2|$ and by its configurations:

$$\mathrm{Config}(E_1 \otimes E_2) \quad = \quad \{ \quad (u_1, u_2) \quad | \quad u_1 \in \mathrm{Config}(E_1), \; u_2 \in \mathrm{Config}(E_2) \quad \}.$$

The monoidal unit $\mathbf{1}$ is the configuration space with a singleton web $|\mathbf{1}| = \{*\}$ and two configurations: $\mathrm{Config}(\mathbf{1}) = \{\emptyset, \{*\}\}$. This equips the category **Config** with the structure of a symmetric monoidal category.

First step: compute the free affine object. Every configuration space E generates the free pointed object defined as

$$E_\bullet = E \,\&\, \mathbf{1}.$$

Its web contains all the elements of the web of E together with an additional point denoted $*$. Its configurations are the same as the configurations of E, except that every configuration $u \in \mathrm{Config}(E)$ is augmented with the point $*$:

$$\mathrm{Config}(E \,\&\, \mathbf{1}) \quad = \quad \{ \quad u \sqcup \{*\} \,|\, u \in \mathrm{Config}(E) \quad \}.$$

Second step: compute the symmetric tensor power. The equalizer $E^{\leq n}$ of the $n!$ symmetries on the configuration space $(E \,\&\, \mathbf{1})^{\otimes n}$ is the configuration space

— whose support is the set of multisets of cardinality at most n:

$$|E^{\leq n}| \quad = \quad \{ \quad [e_1, \ldots, e_n] \quad | \quad \exists u \in \mathrm{Config}(E) \text{ s.t. } \forall i \leq n, \; e_i \in u \sqcup \{*\} \quad \},$$

— whose configurations $u^{\leq n}$ are deduced from the configurations u of E,

$$\mathrm{Config}(E^{\leq n}) \quad = \quad \{ \quad u^{\leq n} \subseteq |E^{\leq n}| \,|\, u \in \mathrm{Config}(E) \quad \},$$

where the configuration $u^{\leq n}$ is defined as

$$\begin{array}{rcl} u^{\leq n} & = & \{ \quad [e_1, \ldots, e_n] \quad | \quad \forall i \leq n,\, e_i \in u \sqcup \{*\} \quad \} \\ & = & \{ \quad [e_1, \ldots, e_p, \underbrace{*, \ldots, *}_{n-p}] \quad | \quad \text{for some } p \in \mathbb{N},\, \forall i \leq p,\, e_i \in u \quad \}. \end{array}$$

Note in particular that the configurations of $E^{\leq n}$ are in a one-to-one relationship with the configurations of E.

Third step: compute the sequential limit. For every configuration space E, let $!E$ denote the configuration space whose web is the set of multisets whose support are included in the configurations:

$$|!E| \quad = \quad \{ \quad \mu \in \mathcal{M}_{\mathrm{fin}}(|E|) \quad | \quad \exists v \in \mathrm{Config}(E) \text{ such that } \Pi_E(\mu) \subseteq v \quad \}.$$

and whose configurations $u^\dagger$ are generated by the configurations of E:

$$\mathrm{Config}(!E) \quad = \quad \{ \quad u^\dagger \quad | \quad u \in \mathrm{Config}(E) \quad \},$$

where

$$u^\dagger \quad = \quad \{ \quad [e_1, \ldots, e_p] \quad | \quad \forall i \leq p,\, e_i \in u \quad \}.$$

We claim that the configuration space $!E$ is the limit of the sequential diagram in the category **Config**:

$$\mathbf{1} \xleftarrow{\iota_0} E_\bullet \xleftarrow{\iota_1} E^{\leq 2} \xleftarrow{\iota_2} E^{\leq 3} \xleftarrow{\iota_3} \cdots \tag{16}$$

where

$$[e_1, \ldots, e_p, \underbrace{*, \ldots, *}_{n+1-p}] \;\iota_n\; [f_1, \ldots, f_q, \underbrace{*, \ldots, *}_{n-q}] \iff p = q \quad \text{and} \quad [e_1, \ldots, e_p] = [f_1, \ldots, f_q].$$

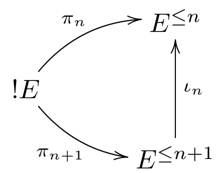
For every $n \in \mathbb{N}$, let $\pi_n : !E \to E^{\leq n}$ denote the following binary relation:

$$[e_1, \ldots, e_p] \;\pi_n\; [f_1, \ldots, f_q, \underbrace{*, \ldots, *}_{n-q}] \iff \left\{ \begin{array}{l} p = q \\ [e_1, \ldots, e_p] = [f_1, \ldots, f_q] \end{array} \right.$$

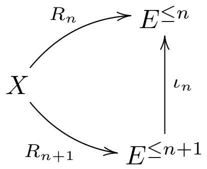
The relation π_n is locally injective and satisfies

$$\forall u \in \mathrm{Config}(E), \quad \pi_n(u^\dagger) \quad = \quad u^{\leq n}.$$

This shows that π_n is a morphism of configuration spaces. Moreover, the diagram



commutes in the category **Config**, for every natural number n. Now, consider another family of morphisms $R_n : X \to E^{\leq n}$ making the diagram



(17)

commute for every natural number n. The binary relation $S : |X| \to |!E|$ is defined as

$$x\, S\, [e_1, \cdots, e_n] \quad \iff \quad x\, R_n\, [e_1, \cdots, e_n]$$

where $e_1, \cdots, e_n$ are elements of the web of E. We establish that the binary relation S defines in fact a morphism $S : X \to !E$ of configuration spaces. We start by the easiest and less interesting part, and show that S is locally injective. Suppose that two elements $x_1, x_2 \in |X|$ of a configuration $w \in \mathrm{Config}(X)$ are related by the relation S to the same element $[e_1, \cdots, e_n] \in |!E|$. We know that, by definition of S, the two elements x_1 and x_2 are related to $[e_1, \cdots, e_n]$ by the relation R_n. The equality $x_1 = x_2$ follows from the local injectivity of R_n. This proves that the relation S is locally injective. The next step is the most interesting part of the proof: it consists in establishing that the relation S transports every configuration w of X to a configuration $R(w)$ of $!E$. So, let w be such a configuration of X. The relation $R_1 : X \to E$ is a morphism of configuration space, and thus transports the configuration w to a configuration $R_1(w)$ of E. By definition of $E \,\&\, \mathbf{1}$, the configuration $R_1(w)$ is of the form $u \sqcup \{*\}$ for a configuration u of space E. Now, we establish by induction on n that $R_n(w) = u^{\leq n}$ for all n. This is true for $n = 0$ because the singleton configuration is the unique configuration of the unit $\mathbf{1}$. This is also true for $n = 1$. Now, suppose that $R_n(w) = u^{\leq n}$ for a given natural number n. We establish that $R_{n+1}(w) = u^{\leq n+1}$ by observing that the relation $R_{n+1} : X \to !E$ is a morphism of configuration spaces, and this transports the configuration w to a configuration $R_{n+1}(w)$ of the space $E^{\leq n+1}$. By definition of $E^{\leq n+1}$, the configuration $R_{n+1}(w)$ is necessarily of the form $v^{\leq n+1}$ for a configuration v of E. Now, we apply our induction hypothesis together with the fact that the diagram (17) commutes, and deduce that $\iota_n(v^{\leq n+1}) = u^{\leq n}$. From this follows immediately that $u^{\leq n} = v^{\leq n}$, since $v^{\leq n} = \iota_n(v^{\leq n+1})$. Hence, $u = v$ since u and v may be recovered as the set of singleton multisets in $u^{\leq n}$ and $v^{\leq n}$. This concludes our proof by induction that $R_n(w) = u^{\leq n}$. From this follows that

$$S(w) \quad = \quad \bigcup_{n \in \mathbb{N}} R_n(w) \quad = \quad \bigcup_{n \in \mathbb{N}} u^{\leq n} \quad = \quad u^{\dagger}.$$

Then, we observe that every morphism R_n factors as $\pi_n \circ S$, and that every other relation T such that $R_n = \pi_n \circ T$ is equal to S. This concludes the proof that $!E$ is the limit of the sequential diagram (16). In order to complete the third step of the recipe, we also need to show that the sequential limit commutes with the tensor product. Consider a family of morphisms $R_n : Y \to X \otimes E^{\leq n}$ making the diagram

$$\begin{array}{ccc} & \overset{R_n}{\nearrow} & X \otimes E^{\leq n} \\ Y & & \uparrow{\scriptstyle X \otimes \iota_n} \\ & \underset{R_{n+1}}{\searrow} & X \otimes E^{\leq n+1} \end{array}$$

commute for every natural number n, and define the relation $S : |Y| \to |X \otimes !E|$ as follows:

$$y\ S\ (x \otimes [e_1, \ldots, e_n]) \iff y\ R_n\ (x \otimes [e_1, \ldots, e_n])$$

where $e_1, \cdots, e_n$ are elements of the web of E. It is easy to check that S defines the unique morphism $S : Y \to X \otimes !E$ of configuration spaces such that every morphism R_n factors as

$$R_n \quad : \quad Y \xrightarrow{\ S\ } X \otimes !E \xrightarrow{\ X \otimes \pi_n\ } E^{\leq n} \ .$$

This elementary argument concludes the proof that all the assumptions of Proposition 1 are satisfied, and thus, that the configuration space $!E$ defines the free commutative comonoid generated by E in the category **Config** of configuration spaces.

Remark. The careful reader will notice that the morphisms ι_n play a fundamental role in the construction of the space $!E$. They ensure in particular that a configuration of $!E$ is entirely determined by its projection on each level $E^{\leq n}$, and thus, that the configurations of the form $u^{\dagger}$ are the only configurations of the sequential limit $!E$. In particular, the computation of $!E$ would not work with the more primitive definition (5) of the exponential modality as an infinite product of symmetric powers. This unexpected discovery (together with its later application to finiteness spaces) is the main additional contribution of the article with respect to the extended abstract published in the ICALP conference (Melliès, Tabareau & Tasson 2009).

7. Configuration spaces and coherence spaces

Now that the symmetric monoidal category **Config** is equipped with a free exponential modality, we would like to provide it with a suitable notion of negation. The simplest way to achieve this is to deduce negation from a relevant choice of "false object" provided by a carefully selected configuration space $\perp$. As we will see, the resulting notion of negation ($A \mapsto \neg A$) enables to identify the category **Coh** as the full subcategory of self-dual objects in the category **Config**. The two categories are related by an adjunction

$$\mathbf{Config} \ \underset{\mathscr{R}}{\overset{\mathscr{L}}{\rightleftarrows}} \ \mathbf{Coh} \qquad (\perp) \tag{18}$$

where the embedding functor $\mathscr{R}$ is fully faithful and injective on objects, and the left adjoint functor $\mathscr{L}$ transports every configuration space E to its double negation $\neg\neg E$. This clarifies the categorical content of an old observation by Jean-Yves Girard, see for instance (Girard 2006), which has become folklore in the linear logic circles: it says that a coherence space can be alternatively described (a) as a web $|E|$ equipped with a coherence relation $\frown_E$ as in Section 3, or (b) as a web $|E|$ equipped with a set C of configurations $u \subseteq |E|$ closed under biorthogonality

$$\mathrm{C} \quad = \quad \mathrm{C}^{\perp\perp}$$

for the following notion of orthogonality:

$$\forall u, v \subseteq |E|, \qquad u \perp v \quad \Longleftrightarrow \quad \#(u \cap v) \leq 1 \tag{19}$$

where $\#(u \cap v)$ denotes the cardinality of the set $u \cap v$. Here, the orthogonal $X^\perp$ of a set X of subsets of $|E|$ is defined as

$$X^\perp \quad = \quad \{ \quad u \quad | \quad \forall v \in X, \quad u \perp v \quad \}.$$

The categorical construction may be also seen as a particular instance of the glueing construction described by Hyland and Schalk in (Hyland & Schalk 2002).

The dialogue category of configuration spaces. The configuration space $\perp$ is defined as the space with a singleton web $|\perp| = \{*\}$ and the two sets $\emptyset$ and $\{*\}$ as configurations, that is, $\mathrm{Config}(\perp) = \{\emptyset, \{*\}\}$. The configuration space $\perp$ is exponentiable in the category **Config**, in the sense that for every configuration space E the presheaf

$$F \quad \mapsto \quad \mathbf{Config}(E \otimes F, \perp)$$

on the category **Config** is representable by an object noted $\neg E$ together with a family of bijections

$$\varphi_{E,F} \quad : \quad \mathbf{Config}(E \otimes F, \perp) \quad \cong \quad \mathbf{Config}(F, \neg E)$$

natural in F. Here, $\neg E$ is defined as the configuration space with the same web as the configuration space E, and with set of configurations defined as:

$$\mathrm{Config}(\neg E) \quad = \quad \{ \quad u \quad | \quad \forall v \in \mathrm{Config}(E), \quad u \perp v \quad \} \quad = \quad \mathrm{Config}(E)^\perp$$

where orthogonality is defined as in (19). This induces a dialogue category where negation defines a functor

$$\neg \quad : \quad \mathbf{Config} \quad \to \quad \mathbf{Config}^{op} \tag{20}$$

in a canonical way, thanks to the Yoneda lemma. See (Melliès 2008) for a discussion on the alternative definitions of a dialogue category.

The functor $\mathscr{R}$ from coherence spaces to configuration spaces. It is easy to see that every coherence space $E = (|E|, \frown_E)$ induces a configuration space $\mathscr{R}(E)$ with the same web $|\mathscr{R}(E)| = |E|$ and with $\mathrm{Config}(\mathscr{R}(E))$ defined as the set of cliques of E. In particular, the configuration space $\mathscr{R}(E)$ satisfies the covering condition because every element of its web $|\mathscr{R}(E)|$ is an element of the configuration $\{e\} \in \mathrm{Config}(\mathscr{R}(E))$. This defines a functor

$$\mathscr{R} \quad : \quad \mathbf{Coh} \quad \to \quad \mathbf{Config}$$

which transports every morphism $S : E \to F$ of coherence spaces to the morphism $\mathscr{R}(S)$:

$\mathscr{R}(E) \to \mathscr{R}(F)$ of configuration space with the same underlying relation $S : |E| \to |F|$. Note in particular that the relation $S : |E| \to |F|$ defines a morphism $\mathscr{R}(E) \to \mathscr{R}(F)$ of configuration spaces because the relation S transports every clique u of E into a clique $S(u)$ of F, and because S satisfies the following local injectivity property:

$$\forall e_1, e_2 \in |E|, \forall f \in |F|, \quad e_1 \frown_E e_2 \text{ and } e_1 \, S \, f \text{ and } e_2 \, S \, f \;\Rightarrow\; e_1 = e_2.$$

The functor $\mathscr{R}$ is obviously faithful. Although this is less obvious, the functor $\mathscr{R}$ is also full, because every morphism $S : \mathscr{R}(E) \to \mathscr{R}(F)$ of configuration spaces is defined as a binary relation $R : |E| \to |F|$ which defines at the same time a morphism $R : E \to F$ of coherence spaces. Hence, $\mathscr{R}(R) = S$. Let us explain why. Suppose that $e_1 \multimap f_1$ and $e_2 \multimap f_2$ are two elements of a binary relation $R : |E| \to |F|$ underlying a morphism $S : \mathscr{R}(E) \to \mathscr{R}(F)$ of configuration spaces. We establish that $e_1 \multimap f_1$ and $e_2 \multimap f_2$ are coherent in the coherence space $E \multimap F$. First of all, the statement

$$e_1 \frown\!\!\!\!\smile_E e_2 \quad \Rightarrow \quad f_1 \frown\!\!\!\!\smile_F f_2$$

comes from the fact that the morphism S transports the configuration $u = \{e_1, e_2\}$ of $\mathscr{R}(E)$ into a configuration $S(u)$ of $\mathscr{R}(F)$ which contains the elements f_1 and f_2, which are thus coherent in F, since $S(u)$ is a clique of the coherence space F. The second statement

$$f_1 \asymp_E f_2 \quad \Rightarrow \quad e_1 \asymp_E e_2$$

follows from the fact that the morphism $S : \mathscr{R}(E) \to \mathscr{R}(F)$ of configuration spaces is locally injective. This establishes that the elements $e_1 \multimap f_1$ and $e_2 \multimap f_2$ of the relation R are pairwise coherent, and thus, that R is a clique in the coherence space $E \multimap F$. This concludes the proof that the functor $\mathscr{R}$ is fully faithful. The functor $\mathscr{R}$ being also injective on objects, this enables to see the category **Coh** as a full subcategory of the category **Config** of configuration spaces.

The functor $\mathscr{L}$ from configuration spaces to coherence spaces. There remains to characterize the coherence spaces among the configuration spaces. Interestingly, the solution will come from the notion of linear negation on configuration spaces defined above. Observe indeed that every configuration space $\neg E$ is of the form $\mathscr{R}(F)$ for the coherence space F with same web as E, and with coherence relation defined as:

$$f_1 \frown_F f_2 \quad \Longleftrightarrow \quad \forall v \in \text{Config}(E), \quad \{f_1, f_2\} \perp v. \tag{21}$$

This ensures in particular that the configurations of $\neg E$ which are defined as the sets orthogonal to Config(E), are the same as the cliques of F. This establishes that the image of the negation functor (20) lies in the full subcategory **Coh**. Applying the negation functor twice, one thus gets a functor

$$\mathscr{L} \quad : \quad \textbf{Config} \quad \to \quad \textbf{Coh}$$

which transports every configuration space E to the coherence space $\neg\neg E$. It appears moreover that this functor $\mathscr{L}$ is left adjoint to the functor $\mathscr{R}$. The reason is that double negation defines a monad in every dialogue category, with unit

$$\eta_E \quad : \quad E \quad \to \quad \neg\neg E.$$

In the particular case of the dialogue category **Config**, this double negation monad T is idempotent, this meaning that the multiplication $\mu : T \circ T \to T$ is an isomorphism. In the case of a dialogue category, this reduces to the fact that the morphism

$$\neg E \xrightarrow{\ \eta_{\neg E}\ } \neg\neg\neg E \xrightarrow{\ \neg\eta_E\ } \neg E$$

is equal to the identity for every object E. The category **Coh** coincides with the category of algebras of the double negation monad, which is equivalent to the kleisli category because the monad is idempotent.

A reconstruction of the coherence space model. It is possible to transfer along the right adjoint functor $\mathscr{R}$ the symmetric monoidal structure as well as the cocartesian structure of the category **Config** in order to recover the structure of symmetric monoidal category with finite sums described earlier (in Section 3) on the category **Coh** of coherence spaces. In this reconstruction, the tensor product of two coherence spaces E and F is recovered as:

$$E \otimes_{\mathbf{Coh}} F \quad = \quad \mathscr{L}(\mathscr{R}(E) \otimes \mathscr{R}(F)) \tag{22}$$

with associated unit $\mathbf{1}_{\mathbf{Coh}} = \mathscr{L}(\mathbf{1})$, while the coproduct of two coherence spaces E and F is recovered as:

$$E \oplus_{\mathbf{Coh}} F \quad = \quad \mathscr{L}(\mathscr{R}(E) \oplus \mathscr{R}(F))$$

with associated unit $\mathbf{0}_{\mathbf{Coh}} = \mathscr{L}(\mathbf{0})$. This enables to see the structure of $*$-autonomous category with finite coproducts of the category **Coh** as inherited from the category **Config**. In particular, in that reconstruction, the fact that the category **Coh** has finite products follows from the existence of finite coproducts, and self-duality. One distinctive point of the reconstruction is that the exponential modality of coherence spaces may be recovered in the same way as:

$$!_{\mathbf{Coh}}\, E \quad = \quad \mathscr{L}\ !\ \mathscr{R}(E). \tag{23}$$

There is a nice conceptual explanation behind that formula, which is that the free exponential modality ! on the category **Config** factors as $! = \mathscr{U} \circ \mathscr{F}$ where $\mathscr{F}$ is right adjoint to the forgetful functor $\mathscr{U}$ from the category **Comon** of commutative comonoids to the category of configuration spaces:

$$\mathbf{Comon} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathbf{Config} \qquad (\perp) \tag{24}$$

This adjunction may be composed with the adjunction (18) in the following way

$$\mathbf{Comon} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathbf{Config} \underset{\mathscr{R}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{Coh} \qquad (\perp,\ \perp) \tag{25}$$

this defining a third adjunction, establishing that the functor $\mathscr{L} \circ \mathscr{U}$ is left adjoint to the functor $\mathscr{F} \circ \mathscr{R}$. The exponential modality $!_{\mathbf{Coh}}$ coincides then with the comonad $\mathscr{L} \circ \mathscr{U} \circ \mathscr{F} \circ \mathscr{R}$ on the category of coherence spaces induced by the adjunction. Moreover, the category **Comon** is cartesian, with structure provided by the tensor product on configuration spaces, this ensuring that the adjunction $\mathscr{U} \dashv \mathscr{F}$ is symmetric monoidal. Hence, in order to establish that the adjunction (25) defines an exponential modality on the category of coherence spaces, it is sufficient to check that the adjunction (18) is symmetric monoidal. This follows from the definition (or the reconstruction) of the tensor product on coherence spaces as performed in (22). Alternatively, the reader may also observe that the adjunction (18) is symmetric monoidal because the double negation monad is idempotent in the category of configuration spaces.

This reconstruction of the coherence space model should be understood as a categorical counterpart of the double negation translation underlying the phase space model of linear logic (Girard 1987, Girard 2006) or the double orthogonal construction (Hyland & Schalk 2002). In particular, the transfer of structure may be reformulated in purely logical terms, as follows:

$$\begin{array}{rclcrcl} E \otimes_{\mathbf{Coh}} F & = & \neg\neg(E \otimes F) & \qquad & \mathbf{1}_{\mathbf{Coh}} & = & \neg\neg\mathbf{1} \\ E \oplus_{\mathbf{Coh}} F & = & \neg\neg(E \oplus F) & & \mathbf{0}_{\mathbf{Coh}} & = & \neg\neg\mathbf{0} \end{array}$$

together with:

$$!_{\mathbf{Coh}}\, E \quad = \quad \neg\neg\, !\, E.$$

This last equality reformulates equation (23) which we find useful to clarify before the end of the section. The equation relies on the definition of $\mathscr{L}$ as double negation, together with the explicit description of the coherence space $\neg E$ associated to a configuration space E provided in equation (21). This leads to the following description of the coherence space $\mathscr{L}(E)$ associated to a configuration space E:

$$e_1 \frown_{\mathscr{L}(E)} e_2 \quad \iff \quad \exists v \in \mathrm{Config}(E),\ \{e_1, e_2\} \subseteq v. \tag{26}$$

Let E be a coherence space. By definition, the configurations of $\mathscr{R}(E)$ are the cliques of E. The web of the configuration space $!\,\mathscr{R}(E)$ is thus defined as the set of multisets μ whose support is included in a clique of E. This establishes already that the two coherence spaces $!_{\mathbf{Coh}}E$ and $\mathscr{L}\ !\ \mathscr{R}(E)$ have the same web. There remains to check that their coherence relations coincide. By equation (26), two elements of $\mathscr{L}\ !\ \mathscr{R}(E)$ are coherent precisely when there exists a configuration $u^{\dagger}$ of $!\,\mathscr{R}(E)$ which contains them both, where u is a clique of E. This happens precisely when their support is contained in u, and thus precisely when they are coherent in the sense of $!_{\mathbf{Coh}}E$. This establishes equation (23).

8. Configuration spaces and finiteness spaces

In the two previous sections, we have shown how to reconstruct the coherence space model of linear logic, from the configuration space model of tensorial logic where negation is not involutive. This reconstruction includes in particular the exponential modality. In this section, we explain how to apply the same recipe to the finiteness space model. This enables us to recover the exponential modality of finiteness spaces defined by Ehrhard (Ehrhard 2005) from the free exponential modality computed in a relevant category of configuration spaces. The whole point of the approach is that it enables us to compute the free exponential modality on configuration spaces as the sequential limit described in Section 1, in contrast to what happens in the original category of finiteness spaces, as we observed in Section 5. We start the section by defining the category **Confin** of configuration spaces adapted to finiteness spaces.

The category of configuration spaces and locally finite relations. The category **Confin** has the configuration spaces as objects, and its morphisms $R : E_1 \to E_2$ are the relations $R \subseteq |E_1| \times |E_2|$ such that

— R *transports configurations forward*:

$$\forall u \in \mathrm{Config}(E_1), \quad R(u) \in \mathrm{Config}(E_2),$$

— R is *locally finite*:

$$\forall u \in \mathrm{Config}(E_1), \forall e_2 \in |E_2|, \qquad \{\ e_1 \in u \mid \ e_1\, R\, e_2\ \} \quad \text{is finite.}$$

The definition of the **tensor product**, of the **cartesian product**, of the **cartesian sum** and of the **exponential modality** are the same in the category **Confin** as in the category **Config** described in Section 6. This provides the category **Confin** with the structure of a symmetric monoidal category with finite products and coproducts. The proof of that last point works as in the case of the category **Config**, except that locally injective relations are replaced by locally finite relations.

Now, we would like to relate the categories **Fin** and **Confin** in the same way as we related the categories **Coh** and **Config** in the previous Section 7. To that purpose, we define the functor

$$\mathscr{R} \quad : \quad \mathbf{Fin} \quad \to \quad \mathbf{Confin}$$

which transports every finitary space $E = (|E|, \mathcal{F}(E))$ to the configuration space $\mathscr{R}(E)$ with the same web, and whose configurations are the finitary subsets of E. It appears that the functor $\mathscr{R}$ is full and faithful, just as it is the case with coherence spaces. This enables us to see the category **Fin** as a full subcategory of **Config**. Moreover, the functor $\mathscr{R}$ has a left adjoint, obtained by applying twice the negation functor defined below.

Negation. Negation is defined in the same way as in the category **Config**, but this time with respect to the orthogonality relation $\perp_{\mathrm{fin}}$ described in equation (15). Namely, the negation $\neg E$ of a configuration space E is the configuration space with the same web as E, and whose configurations are orthogonal to the configurations of E:

$$\mathrm{Config}(\neg E) \quad = \quad \{ \quad u \subseteq E \quad \mid \quad \forall v \in \mathrm{Config}(E), \quad u \perp_{\mathrm{fin}} v \quad \}.$$

This defines a dialogue category, with a negation functor from **Config** to its opposite category. By definition of finiteness spaces as double negated objects, every configuration space $\neg E$ is of the form $\mathscr{R}F$ for the finiteness space $F = \neg E$. The left adjoint functor

$$\mathscr{L} \quad : \quad \mathbf{Confin} \quad \to \quad \mathbf{Fin}$$

is then defined as the double negation functor $\mathscr{L} : E \mapsto \neg\neg E$. One obtains a symmetric monoidal adjunction

$$\mathbf{Confin} \underset{\mathscr{R}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{Fin} \qquad (\mathscr{L} \dashv \mathscr{R})$$

between the categories **Fin** and **Confin**, just as in the case of coherence spaces. In particular, one has the natural isomorphisms

$$\mathscr{L}(E) \otimes_{\mathbf{Fin}} \mathscr{L}(F) \;\cong\; \mathscr{L}(E \otimes_{\mathbf{Confin}} F) \qquad\qquad \mathbf{1}_{\mathbf{Fin}} \;\cong\; \mathscr{L}(\mathbf{1}_{\mathbf{Confin}})$$

This implies in particular that the tensor product $\otimes_{\mathbf{Fin}}$ on finiteness spaces E and F may be recovered from the tensor product $\otimes_{\mathbf{Confin}}$ in the same way as in the case of coherence spaces:

$$E \otimes_{\mathbf{Fin}} F \quad = \quad \mathscr{L}(\ \mathscr{R}(E) \otimes_{\mathbf{Confin}} \mathscr{R}(F)\ )$$

thanks to the equation $E = \mathscr{L} \circ \mathscr{R}\ (E)$. The finite coproducts on finiteness spaces E and F may be similarly recovered with the equations:

$$E \oplus_{\mathbf{Fin}} F \quad = \quad \mathscr{L}(\ \mathscr{R}(E) \oplus_{\mathbf{Confin}} \mathscr{R}(F)\ ) \qquad\qquad 0_{\mathbf{Fin}} \quad = \quad \mathscr{L}\ (\ 0_{\mathbf{Confin}}\ )$$

Exponential modality. The free commutative comonoid $!_{\mathbf{Confin}}\ E$ generated by a configuration space E is computed in the category **Confin** by applying the general recipe of Section 1. In particular, the configuration space $!_{\mathbf{Confin}}\ E$ is the limit of the same sequential diagram

$$\mathbf{1} \xleftarrow{\ \iota_0\ } E_{\bullet} \xleftarrow{\ \iota_1\ } E^{\leq 2} \xleftarrow{\ \iota_2\ } E^{\leq 3} \xleftarrow{\ \iota_3\ } \cdots$$

as in the category **Config**, see Section 6 for details. The key point is that this limit commutes with the tensor product in **Confin** for the same reasons that it commutes with the tensor product in **Config**. This induces a pair of symmetric monoidal adjunctions

$$\mathbf{Comon} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathbf{Confin} \underset{\mathscr{R}}{\overset{\mathscr{L}}{\rightleftarrows}} \mathbf{Fin}$$

where **Comon** denotes the category of commutative comonoids and homomorphisms associated to the category **Confin**. The symmetric monoidal adjunction

$$\mathbf{Comon} \underset{\mathscr{F} \circ \mathscr{R}}{\overset{\mathscr{L} \circ \mathscr{U}}{\rightleftarrows}} \mathbf{Fin}$$

obtained by composing the two adjunctions defines an exponential modality on the category of finiteness spaces. Moreover, the induced comonad

$$!_{\mathbf{Fin}}\ E \quad = \quad \mathscr{L}\ !_{\mathbf{Confin}}\ \mathscr{R}\,(E)$$

coincides with the exponential modality $!_{\mathbf{Fin}}$ described in (Ehrhard 2005).

9. Conclusion and future works

This investigation on the algebraic nature of the exponential modality leads to interesting remarks of a purely logical nature. First of all, the computation of the free commutative comonoid $!A$ as a sequential limit

$$\mathbf{1} \longleftarrow A^{\leq 1} \longleftarrow A^{\leq 2} \longleftarrow \cdots \longleftarrow A^{\leq n} \longleftarrow A^{\leq n+1} \longleftarrow \cdots$$

where the space $A^{\leq n}$ of $k \leq n$ copies of the space A is "glued" inside the space $A^{\leq n+1}$ of $k \leq n+1$ copies reflects the fact that an intuitionistic proof (or a recursive program) opens new copies of its argument A on the fly, in the course of interaction. In particular, the number of copies of A is chosen dynamically, and not statically at the beginning of the interaction, as it would be the case with the definition of $!A$ as the infinite cartesian product:

$$!A \quad = \quad \mathop{\&}_{n\in\mathbb{N}} \quad A^n \ / \ \sim_n .$$

It is also quite puzzling that the sequential limit is not expressible (at least apparently) as the construction of a recursive type. However, there should be a way to extend type theory in order to incorporate such constructions, possibly starting from the model-theoretic approach described in Section 2. This is an interesting topic for future work.

Another fascinating issue enlightened in this work is the status of duality in logic. The following slogan appears at the end of the survey (Melliès 2008):

$$\textbf{logic} \quad = \quad \textbf{data structure} \quad + \quad \textbf{duality.}$$

Here, the exponential modality is obviously on the side of data structure. As such, its construction has no reason to interfere with negation and duality. This drastic philosophy of logic provided a surprisingly fruitful guideline in this work. On the one hand, it offered a conceptual explanation for the failure of the sequential limit construction in the self-dual category of finiteness spaces. On the other hand, it led to the resolution of this issue in the larger category of configuration spaces, where the exponential construction and the negation are carefully separated. Much remains to be clarified on negation and duality at this point, and it is certainly a bit too soon to judge. For instance, it is nearly immediate to adapt the construction of Section 8 to the vectorial version of finiteness spaces defined by Ehrhard (Ehrhard 2002). Adapting the approach to Köthe spaces requires much more care, and is left for future work. More generally, one would like to understand the status of negation on topological vector spaces, and more specifically the relationship to topological completion, without necessarily starting from a basis on the vector space, as in the current presentation of finiteness spaces and Köthe spaces.

Acknowledgements. The authors would like to thank Thomas Ehrhard and Martin Hyland for the lively discussions and inspiring comments.

References

Ehrhard, T. (2002), 'On Köthe sequence spaces and linear logic', *Mathematical Structures in Computer Science* **12**(05), 579–623.

Ehrhard, T. (2005), 'Finiteness spaces', *Mathematical. Structures in Comp. Sci.* **15**(4).

Girard, J.-Y. (1987), 'Linear logic', *Theoretical Computer Science* **50**, 1–102.

Girard, J.-Y. (2006), 'Le point aveugle: Cours de logique: Tome 1, vers la perfection', *Vision des Sciences.* .

Hyland, M. & Schalk, A. (2002), Games on graphs and sequentially realizable functionals, *in* 'Logic in Computer Science 02', IEEE Computer Society Press, Kopenhavn, pp. 257–264.

Joyal, A. (1977), 'Remarques sur la théorie des jeux à deux personnes', *Gazette des Sciences Mathématiques du Québec* **1**(4), 46–52. English version by Robin Houston.

Kelly, M. (1982), *Basic Concepts of Enriched Category Theory*, Vol. 64 of *Lecture Notes in Mathematics*, Cambridge University Press.

Kelly, M. & Laplaza, M. (1980), 'Coherence for compact closed categories', *Journal of Pure and Applied Algebra* **19**, 193–213.

Lafont, Y. (1988), Logique, catégories et machines, Thèse de doctorat, Université de Paris 7, Denis Diderot.

Lawvere, W. (1963), Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories, PhD thesis, Columbia University.

Melliès, P.-A., Tabareau, N. & Tasson, C. (2009), An explicit formula for the free exponential modality of linear logic, *in* 'ICALP '09: Proceedings of the 36th Internatilonal Collogquium on Automata, Languages and Programming', Springer-Verlag, pp. 247–260.

Melliès, P.-A. (2005), 'Asynchronous Games 3: An Innocent Model of Linear Logic', *Electronic Notes in Theoretical Computer Science* **122**, 171–192.

Melliès, P.-A. (2008), Categorical Semantics of Linear Logic. À paraître dans *Panoramas et Synthèses*, Société Mathématique de France.

Melliès, P. & Tabareau, N. (2008), Free models of T-algebraic theories computed as Kan extensions. unpublished, http://hal.archives-ouvertes.fr/hal-00339331/fr/.

Seely, R. (1989), Linear logic, ∗-autonomous categories and cofree coalgebras, *in* 'Applications of categories in logic and computer science', number 92, Contemporary Mathematics.

Tasson, C. (2009), Sémantiques et syntaxes vectorielles de la logique linéaire, PhD thesis, Université Paris Diderot.