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## ► To cite this version:

Sebastian Rezat. Extensions of number systems: continuities and discontinuities revisited. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02436283

**HAL Id: hal-02436283**

**<https://hal.science/hal-02436283>**

Submitted on 12 Jan 2020

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# Extensions of number systems: continuities and discontinuities revisited

Sebastian Rezat

Paderborn University, Institute for Mathematics, Paderborn, Germany; [srezat@math.upb.de](mailto:srezat@math.upb.de)

*The extension of number systems from natural to rational and real numbers and related arithmetic is a prominent theme in mathematics from primary to upper secondary education. In parallel to the development of the number concept and the extension of number systems, students need to proceed from arithmetic to algebra. Students' difficulties in mastering both, the extension from one number system to another and the progression from arithmetic to algebra are well documented. The paper focuses on the extension from natural numbers to integers with a particular interest in the relationship to the progression from arithmetic to algebra. Continuities and discontinuities in the alignment of these two parallel curricular developments are analyzed from three different perspectives, namely an epistemological, a psychological, and a pedagogical perspective. This analysis will include work from TWG02 "Arithmetic and Number Systems", which gives a flourishing account of the multifaceted issues related to the teaching and learning of different number systems since its foundation at CERME7 in 2011 and also draws on the work of TWG03 "Algebraic Thinking". Finally, conclusions will be drawn from the analysis of the relationship between the extension from natural numbers to integers and algebraic thinking in terms of the construction of a more coherent curriculum regarding these two developments.*

*Keywords: number systems, integers, negative numbers, algebraic thinking, coherence*

## Introduction

From the beginning of their lives and throughout schooling, students have to develop their number concept and related number sense. The extension of number systems from natural numbers to rational and real numbers<sup>1</sup> and related arithmetic is an endeavor that students are involved in from primary to upper secondary education in mathematics. In parallel to the development of the number concept and the extension of number systems, students need to proceed from arithmetic to algebra, i.e. from operating with known quantities to operating with unknowns, from the particular to the general, from numbers to symbols.

A large body of research shows that learners experience gaps and discontinuities related to both, the learning of number systems (Van Dooren, Lehtinen, & Verschaffel, 2015) and the learning of algebra (Hodgen, Oldenburg, & Strømskag, 2018; Kaput, 2008). In terms of the development of the number concept the transition from natural numbers to non-negative rational numbers (i.e. fractions and decimals) has received much attention. Research has unraveled the problems students encounter at

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<sup>1</sup> There are ambiguities in the use of the terms *natural number*, *rational number*, and *integers* in the literature. I use the term *natural number* to refer to set  $\mathbb{N} = \{1, 2, 3, \dots\}$ . The term *integer* is used to refer to the set  $\mathbb{Z}$  comprising positive natural numbers including zero and their additive inverses, while with the term *rational number*, I refer to the set  $\mathbb{Q}$  comprising all positive and negative fractions and decimals respectively (and therefore also  $\mathbb{N}$  and  $\mathbb{Z}$ ).

this transition. In particular, it shows that the extension from natural numbers to non-negative rational numbers requires changes in the basic understanding of what numbers can do and what effect operations have on numbers. The problems at this transition exemplify that natural number knowledge on the one hand is a prerequisite for the learning of other number systems, but on the other hand has manifold adverse effects (Vamvakoussi, 2015). This phenomenon is so pervasive that it has been termed the *whole* or *natural number bias* (Ni & Zhou, 2005). Consequently, students' prior knowledge that was developed with respect to the set of natural numbers has to be reorganized in such a substantial way that Vosniadou and Verschaffel (2004) speak of a "conceptual change" that is necessary at the transition from natural numbers to non-negative rational numbers.

In terms of the transition from arithmetic to algebra, the difficulties that students experience with operating on the unknown led Linchevski and Herscovics (1996) to speak of a "cognitive gap" between arithmetic and algebra. Kaput (2008) calls the separation of arithmetic and algebra in terms of "a computational approach to school arithmetic and an accompanying isolated and superficial approach to algebra" the "algebra problem". The answer to this problem has been to foster what has been termed *algebraic thinking*, *relational thinking*, *functional thinking*, or *early algebra* in the context of arithmetic, in order to provide students with ways of thinking that are crucial for school algebraic contexts.

Both of these long-term developments require a careful construction of the curriculum in order to facilitate students' understanding of these central mathematical ideas. However, researchers criticize the lack of a coherent vision for the teaching and learning of number systems and related transitions (Bruno & Martinon, 1999). This also seems to be the case for the link of the extension of number systems and the progression from arithmetic to algebra. Both, the discourse on the extension of number systems and the discourse on algebraic thinking only seem to be loosely related. This is also noticeable in CERME in that negative numbers are rarely considered in TWG03 on algebraic thinking.

Therefore, the goal of this paper is to unfold the relationship between the extension of number systems and the progression from arithmetic to algebra in order to analyze continuities and discontinuities in the alignment of these two curricular progressions. For reasons that will become obvious later, I will focus on the extension from natural numbers to integers in my analysis. In particular, I seek to answer the following questions:

- 1) What is relationship between the development of the negative number concept and algebra?
- 2) How could the teaching and learning of the extension from natural numbers to integers be aligned with the teaching and learning of algebraic thinking?

In order to answer these questions, I will analyze the transition from natural numbers to integers from different perspectives: a) epistemological; b) pedagogical; and c) psychological. The epistemological perspective will mainly serve to answer the first question. I will show that the development of the negative number concept and the development of algebra were mutually related. In fact, negative numbers play a crucial role in these two developments. The pedagogical and psychological perspective will yield answers to the second question.

Based on my analysis of the relation between the extension from natural numbers to integers and the progression from arithmetic to algebra from the three different perspectives, I will draw conclusions regarding the construction of curricula, in which both, the extension of number systems and the progression from arithmetic to algebra are more coherently aligned.

## **Epistemological perspective**

In this section, I will show that the same cognitive achievements which underlie the development of algebra were also crucial for the development and acceptance of the negative number concept. I start with a summary of the main cognitive achievements that have been pointed out as characteristic and crucial for the development of algebraic thinking. I will then show, how these cognitive achievements also have been crucial for the development of the negative number concept. However, I will not be able to give a comprehensive overview of this mutual related historical development of the negative number concept and algebra. An analysis of the obstacles in the historical development of the negative number concept was provided by Glaser (1981). Schubring (2005) gives an overview of the historical development of the negative number concept and its relation to the development of algebra.

Hodgen et al. (2018) regard algebraic thinking as the human activity from which algebra emerges. It focuses on generalization and the expression of generalization in increasingly systematic and conventional symbol systems as one core aspect of algebra rather than on syntactically guided actions on symbols (Kaput, 2008). The main cognitive achievements that have been pointed out as being characteristic and crucial for the development of algebraic thinking and algebra are:

1. Algebra deals with objects of *indeterminate* nature (unknowns, variables, parameters) (Radford, 2010)
2. Indeterminate objects are dealt with in *analytic* manner (Radford, 2010).
3. The development of algebraic thinking is characterized by a transition from an operational to a structural or relational perspective, i.e. by “reification“ (Sfard 1995) or “objectification“ (Radford, 2010) of processes into mathematical objects.
4. The new mathematical objects are detached from their original content meanings and achieve a formal character.

In the development of algebra and algebraic thinking, these four aspects are mutually related and difficult to consider in isolation. However, for the sake of clarity I elaborate on them separately.

### **Algebra deals with objects of indeterminate nature**

The epistemological development of algebra is closely related to dealing with objects of indeterminate nature. These indeterminate objects yielded the concepts of variable and parameter. It was also in the realm of dealing with indeterminate objects in the context of solving (systems of) equations that negative numbers became relevant (Gallardo, 2002; Glaser, 1981; Hefendehl-Hebeker, 1991). According to Damerow (2007, p. 49) the use of variables “opens a potential means of representation for a higher level of meta-cognitive insights such as the recognition that the natural numbers can be complemented with negative numbers to the system of whole numbers”. Diophantus was one of the first who solved equations based on transformation methods. Applying these methods also yielded

negative solutions. For example, in his *Arithmetica*, Diophantus referred to the equation  $4 = 4x + 20$  as absurd, since it would give the solution  $x = -4$ .

It was not until the second half of the 19<sup>th</sup> century that negative numbers were accepted as autonomous mathematical objects. However, in solving equations they were already accepted as auxiliary means, which had to be interpreted correctly after the equation was solved. A famous example is the problem discussed by D'Alembert in the article *Négatif* from Diderot's *Encyclopedia*:

suppose that we are looking for the value of a number  $x$  which when added to 100 yields 50. According to the rules of algebra; we have  $x + 100 = 50$ , so that  $x = -50$ . This shows that the magnitude  $x$  is 50 and that instead of being added to 100 it must be subtracted. This means that the problem should have been formulated as follows: find a magnitude  $x$  which when subtracted from 100 leaves the remainder 50; if the problem had been formulated in this manner, then we would have  $100 - x = 50$  and  $x = 50$ , and the negative form of  $x$  would cease to exist. Thus, in computations, negative magnitudes actually stand for positive magnitudes that were guessed to be in the wrong position. The sign “-“ before a magnitude is a reminder to eliminate and to correct an error made in the assumption, as the example just given demonstrates very clearly. (D'Alembert as cited in Hefendehl-Hebeker, 1991).

D'Alembert does not except the existence of a magnitude with value “-50” in his argumentation. To him, this solution only indicated an error made that had to be corrected.

Another way of making sense of negative solutions was to interpret them metaphorically by the opposite magnitude. This metaphorical interpretation of negative solutions by the opposite magnitude developed into the concept of opposite quantities, which cancel each other out:

“Quantities of the same kind which are considered under conditions that one diminishes the other shall be called opposite quantities. E.g., assets and debts, walking forward and walking backward. One of these quantities, as one likes, shall be called positive or affirmative, and its opposite negative or denying” (Kästner as cited in Schubring 2005, 134).

A similar understanding as “something being opposed to something familiar” was also crucial for the development of the concept of variable. There, it was the dualistic opposition to a known or constant quantity. This concept of variable was first overcome by Euler, who replaced this dichotomy by the universal concept of variable (Schubring, 2005). Similarly, the concept of negative quantity was for a long time conceptualized as an opposite quantity and had to be overcome by the formal concept of number.

### **Indeterminate quantities are dealt with in analytic manner**

As a second important characteristic of algebraic thinking, Radford (2010) points out that indeterminate quantities are dealt with in an analytic manner, i.e. it is calculated with these indeterminate quantities as if they were known by referring to their mutual relationships and the relationships to known quantities. (Hefendehl-Hebeker & Rezat, 2015; Radford, 2010).

As negative numbers occurred in the context of solving equations they result from calculations in which indeterminate quantities were treated as if they were known by referring to their mutual relationships and the relationships to known quantities. This aspect is especially apparent in the

formulation of the rules for calculating with negative numbers. For example, although not accepting negative solutions, Diophantus formulated the rules *minus times minus gives plus* and *plus times minus gives minus* (Tropfke, 1980). However, these rules were derived from calculating with complex expressions, such as  $(a - b)(c - d)$ , by analyzing the internal relationships. The explanation of the rule for subtracting negative numbers by Bernard Lamy (1640–1715) is a revealing example. He explains that when subtracting complex quantities like  $c + f$  and  $b - d$ . One did not want to subtract from  $c + f$  the entire  $b$ , thus  $c + f - b$ , but somewhat less. One thus had to change the algebraic sign of  $d$  from  $-$  into  $+$ , so as to perform the operation  $c + f - b + d$  (Schubring, 2005, p. 76). Consequently, the expression  $-(-b) = +b$  in solving the brackets is given sense in its relations to  $b$  and of  $(b - d)$  to  $(c + f)$ .

### **Reification and Objectification**

It has been pointed out that “reification” (Sfard, 1994, 1995) or “objectification” (Radford, 2010) of operations or—more generally—processes into mathematical objects had been crucial in the development of algebra. Just as the expression  $4x + 20$  can be seen either as a sequence of operations or as a mathematical object such as a representation of a number or as a function, the expression  $a - b$  can be seen as the operation of subtracting  $b$  from  $a$  or as the number resulting from this subtraction. From this perspective, the construction of negative numbers as autonomous mathematical objects required a transformation of mathematical processes into mathematical objects, which is visible in the step from carrying out the “hypothetical” subtraction  $a - b$  omitting the restriction  $b \leq a$  and accepting the negative number as an own entity. In fact, it may be regarded as one of the cognitive roots of negative numbers (Hefendehl-Hebeker & Rezat, 2015; Sfard, 1994, 1995) that carrying out fictive operations such  $50 - 100$  was actually considered as a possibility. Peacock refers to this generalization of operations as the move from arithmetical to symbolical algebra (Chiappini, 2011). While in arithmetical algebra the operations on symbols underlie the same restrictions as in arithmetic (of natural numbers), the operations with symbols in symbolic algebra are defined according to the properties of the operations.

According to Schubring (2005), it was Euler who first constructed the series of negative numbers by “perpetually subtracting unity”. Later, Hankel used the term  $a - b$  to construct negative numbers (Tropfke, 1980). It was also Hankel who noticed that it is sufficient to use  $a = 0$  and name the resulting number by  $-b$ .

### **The new mathematical objects are detached from content meanings and achieve a formal character.**

Descartes’ achievement to develop a symbolic language, which primarily relied on the relationships among the symbols and not on justifications through arithmetic or geometry is regarded to be a crucial step in the development of the symbolic language of algebra (Scholz, 1990). This can be understood as a detachment of the mathematical objects from content meanings.

Although at a very different time, the detachment of the number concept from content meanings was crucial for the acceptance of negative numbers. Schubring (2005) points out that the separation between numbers on the one hand and quantities or magnitudes on the other is decisive for understanding the historical development of negative numbers. He shows that in the history of

mathematics the epistemological limitation of the concepts of quantity and magnitude impeded the generalization of operations and thus the development of the negative number concept. As long as the concept of negative numbers was subordinated to that of magnitude, negative magnitudes were not accepted as autonomous mathematical objects. As already shown in the example by D'Alembert, the problems and their solutions were reinterpreted in the domain of positive numbers in order to avoid negative solutions. This view restricted the applicability of negative numbers to operations with "subtractive" or "opposite" quantities, which have a "natural element of opposition as giving and taking" (Schubring, 2005, p. 106). According to Schubring (2005), Euler was the first to consistently present algebra as a science of numbers and to conceptually separate numbers clearly from quantities and magnitudes. However, it was not until the 19th century that these obstacles imposed by the subordination of number to the concept of quantity and magnitude were overcome by a shift of view: The change consisted in the transition from the concrete to the formal viewpoint, which was advanced by Ohm, Peacock and Hankel. Subsequently, the concept of number could be introduced in a purely formal manner without consideration of the concept of magnitude. Hankel, who advanced this viewpoint argues:

Thus, the condition for the construction of a general arithmetic is that it be a purely intellectual mathematics detached from all intuition, a pure science of forms in which what are combined are not quanta or their number images but intellectual objects to which actual objects, or relations of actual objects, may, but need not, correspond (Hankel as cited in Hefendehl-Hebeker, 1991)

Accordingly, the underlying epistemology of justification changed from realism to that of internal consistency (Pierson Bishop et al., 2014). While operating with negative quantities and the related rules were known and used confidently already for a long time, the consistent system of rules for manipulating negative numbers is not deduced from reality, but from the basic rules of natural number arithmetic based on the permanence principle.

Hefendehl-Hebeker (1991) argues that the "separation of the construction of number systems from content considerations did not mean that the extended number systems were detached from content meanings" (p. 31) and provides some examples where negative numbers were successfully applied to real phenomena. The difference is that in these cases the concept of magnitude is subordinated to that of number and numbers are used as modelling tools for real-life situations.

In summary, I argued that the development of the negative number concept and the development of algebra are mutually related. The cognitive achievements, which have been emphasized as being crucial for the development of algebra, also underlie the development of the negative number concept. In particular, these are the possibility of carrying out fictive operations with indeterminate quantities in analytic ways, the "reification" (Sfard, 1994, 1995) and "objectification" (Radford, 2010) of mathematical processes, and the detachment of mathematical objects from content meanings and related formalization. According to Schubring (2005, p. 149), negative numbers "challenged the traditional first understanding of mathematics, its first 'paradigm' in Kuhn's terms, its understanding of being a science of quantities: of quantities that, while being abstracted to attain some autonomy from objects of the real world, continued at the same time to be epistemologically legitimized by the latter" until a formal algebraic introduction and justification of these numbers and their operations

was achieved. Therefore, from the epistemological perspective, negative numbers seem to play a crucial role in the development of number systems aligned with the development of algebraic thinking. This is the reason, why I focus on the case of negative numbers in my analysis of the relation between the extension of number systems and the transition from arithmetic to algebra.

## **Psychological perspective**

Although negative magnitudes are nowadays a natural part of our daily life as relative magnitudes on thermometers and other scales, research has shown that students struggle with the same obstacles that characterize the epistemology of negative numbers. Gallardo (2002) observes the same levels of acceptance of negative numbers that she found in the historical development in students understanding of the concept. However, she points out that the levels of acceptance do not follow a strict chronological order in the students and that the same student might show different levels of understanding dependent on the context of the task. In line with the historical development of negative numbers, Pierson Bishop et al. (2014) identify the magnitude-based perspective on numbers together with their understanding as cardinal numbers as an obstacle, because then negative numbers are perceived in a non-tangible way as less than nothing. This is also an obstacle for the understanding of operations with negative numbers.

Different studies show that students have difficulties with the order of integers. Students exhibit more difficulties in tasks that involve only negative numbers than in tasks with positive and negative numbers (Bofferding & Farmer, 2018). The main problem is that the size of a negative number is determined based on the absolute value or the opposite magnitude respectively. For example, students regard  $-10$  as 'bigger' than  $-5$ , because  $-10$  is colder than  $-5$ . This mirrors the understanding of negative numbers as opposite magnitudes, which was persistent throughout the historical development of the negative number concept. The extent to which these difficulties are shown also depends on the language used in comparison tasks (hottest/most hot/least cold vs. coldest/most cold/least hot) (Bofferding & Farmer, 2018). Schindler, Hußmann, Nilsson, and Bakker (2017) and Yilmaz and Isiksal-Bostan (2017) argue that it is important to consider students reasoning related to their answers. Their studies show that even correct answers might be based on faulty reasoning, which builds on prior experiences from the natural numbers.

Students also have difficulties with the different meanings of the minus sign. While in the set of natural numbers the minus sign only denotes subtraction, it additionally obtains a unary function as a structural signifier to denote a relative number and a symmetrical function as an operational signifier to denote the inverse in the set of integers. Vlassis (2004) shows that students do not assign any other meaning to the minus sign than that of subtraction in polynomial expressions. Their procedures of simplifying polynomials can be understood as strategies of making sense of the expressions and being able to carry out simplifications by adhering to this one meaning of the minus sign that originates from the natural numbers. When solving equations students have difficulties to find negative solutions in particular in situations with two successive signs, such as  $-6 \cdot x = 24$  (Vlassis, 2008).

In summary, as in the case of the transition from natural numbers to fractions, a natural number bias is also apparent at the transition from natural numbers to integers. The empirical findings of students' difficulties in understanding the negative number concept mirror the obstacles that characterize the



epistemological development of the negative number concept. However, the studies aiming at identifying students' obstacles vary in the degree, in which they consider students' prior experiences when learning negative numbers. The participants of many studies already learned integer arithmetic. Consequently, how they were introduced to negative numbers might have an effect on their understanding of them. In order to unveil continuities and discontinuities in the learning of the number concept, it would be important to understand how the obstacles that students experience in the learning of the negative number concept relate to the way they have been introduced to the concept and to their prior experiences in the set of natural numbers. Referring to the results from the epistemological analysis, the effects of early algebra on the learning of the negative number concept would be a matter of particular interest. However, studies analyzing students' understanding of negative numbers usually do not control for prior experiences.

### **Pedagogical perspective**

In the epistemological analysis I have shown that there are parallels in the development of algebraic thinking and the negative number concept. Therefore, it seems natural to align the teaching and learning of the extension from natural numbers to integers with ideas of algebraic thinking. From the psychological perspective, it is not yet clear, if the teaching and learning of the negative number concept aligned with ideas of algebraic thinking has positive effects on students' understanding of the concept.

Algebraic thinking is promoted in mathematics curricula for the elementary grades across the world and thus has become an important goal in the teaching and learning of natural number arithmetic (Cai, Ng, & Moyer, 2011; Venkat et al., 2018). The vast majority of tasks, learning trajectories and studies related to algebraic thinking is carried out in the domain of natural numbers. Only rarely are integers and other number systems considered related to algebraic thinking. As already mentioned in the introduction, the scientific discourses on the extension of number systems and on algebraic thinking only rarely seem to be related to each other.

Approaching the transition from natural numbers to integers from a pedagogical perspective, I seek to answer the question how the teaching of the transition from natural numbers to integers could be aligned with the teaching of algebraic thinking. In my presentation, I focus on two aspects: 1) didactic models, which align the extension of number systems and algebraic thinking; and 2) number sense as an important goal related to natural number arithmetic and algebraic thinking.

### **Didactic models for integers and algebraic thinking**

Within the scope of this article, I can only briefly sketch my understanding of didactic models. Space does not allow to elaborate on the rich theory behind it. I refer to an understanding of didactic models as representations of abstract mathematical concepts or structures. Thus, they reflect essential aspects of the mathematical concepts or structures. By allowing students to act upon the tangible or symbolically represented mathematical objects, they are used as tools in the meaning of cultural artifacts (Wartofsky, 1979; Wertsch, 1998) to foster students' conceptual development. I adopt the broad notion of models in Realistic Mathematics Education, in which models have different manifestations. From this perspective hands-on-materials, sketches, paradigmatic situations, schemes, or diagrams can serve as models (Van den Heuvel-Panhuizen, 2003). An important aspect

with regard to transitions is that didactic models need to be flexible in order to be applied to more advanced, sophisticated or abstract levels and thus support vertical mathematization (Van den Heuvel-Panhuizen, 2003).

Different categorizations of didactic models for integers have been suggested. Janvier (1983) distinguishes number-line-models and equilibrium models. While number-line models depict negative numbers and related operations on one continuous number line, equilibrium models introduce two separate (magnitude) representations for positive and negative numbers, e.g. black and red stones. These equilibrium models refer to the meaning of negative numbers as opposite quantities. Operations in equilibrium models are based on the principle of compensation between positive and negative numbers, i.e. the neutralization of equal amounts of opposites, e.g. of black and red stones. Another categorization of didactic models for integers has been suggested by Steinbring (1994). He distinguishes three categories: 1) real-life context as modelling structures; 2) models based on geometric or arithmetic permanence; 3) models providing autonomous representations of negative numbers. Temperatures, assets and depths, elevation, and the elevator model are typical examples of the first category. Freudenthal (1983) was a proponent of models of the second category, which aim to provide plausible reasons for the expansion of rules from the natural numbers to integers. The two categories distinguished by Janvier (1983) both belong to Steinbring's (1994) third category.

As Fischbein (2002) points out, there is no didactic model of negative numbers, which at the same time is intuitive and consistently represents all the algebraic properties of negative numbers. A model, which consistently represents the algebraic properties of negative numbers always needs to build on artificial conventions. Therefore, research has tried to identify the affordances and constraints of particular models in learning integer arithmetic (Hativa & Cohen, 1995; Linchevski & Williams, 1999; Stephan & Akyuz, 2012). Many researchers prefer the number line model for representing operations with integers (Altıparmak & Özdoğan, 2010; Bruno & Martinon, 1999; Hativa & Cohen, 1995). Other research highlights the power and importance of contextual knowledge from real-life situations for the understanding of negative numbers (e.g. Linchevski & Williams, 1999; Stephan & Akyuz, 2012). However, in real-life situations the understanding of negative numbers as opposite magnitudes is also identified as an obstacle (Schindler & Hußmann, 2013). Students need well developed mental networks that relate the opposite magnitudes and their order (Schindler & Hußmann, 2013; Yilmaz & Isiksal-Bostan, 2017).

Due to the artificial conventions that are necessary to develop didactic models for negative numbers (Fischbein, 2002), there is a tendency to develop games (e.g. Hattermann & vom Hofe, 2015; Linchevski & Williams, 1999) and artificial contexts (Altıparmak & Özdoğan, 2010; Streefland, 1996) as didactic models. As opposed to real-life contexts, artificial contexts and games facilitate the implementation of formal rules. On the contrary, most recent textbooks introduce negative numbers in real-life contexts such as temperature, assets and debts, or elevation in order to support the understanding of the negative number concept (Whitacre et al., 2015). Whitacre et al. (2015) show that students solve problems in the context of assets and debts without using negative numbers, but most of them were capable to relate negative numbers to the context if asked to do so.

Only rarely are didactical models for integers discussed in terms of their affordances to foster algebraic thinking. A few examples are Chiappini (2011); Gallardo (2002); Peled and Carraher (2008); Rezat (2014); and Schumacher and Rezat (in press). These authors exploit the potential of particular didactic models for the learning of integers aligned with algebraic thinking. Linchevski and Williams (1999) do not explicitly relate to algebraic thinking, but address reification as a main problem of learning integers, which was shown as being equally important for both, algebraic thinking and understanding the negative number concept. Therefore, their approach is also relevant in the present context. I will exemplify the different approaches by providing an example of each.

Gallardo (2002) builds on the historical-critical method as described by Filloy, Puig, and Rojano (2008). This approach is characterized by recurrent movements between the analysis of historical texts and empirical work in the classroom. Learning sequences are developed based on the historical analysis of the development of concepts. In her study, Gallardo (2002) uses word problems from historical sources, e.g. D'Alembert's problem.

Peled and Carraher (2008) criticize that most word problems, which involve negative numbers do not require the formal rules for manipulating negative numbers and consequently can be solved correctly while circumventing operations with negative numbers. Their main approach may be characterized by generalizing arithmetic problems in the realm of real-life contexts as modeling structures. They adjust problems using real-life contexts in order to foster algebraic thinking when learning integers and illustrate how these algebraic problems are more suited than arithmetic problems to promote meaningful learning of negative numbers.

An example from Peled and Carraher (2008, p. 309f) may illustrate their approach. The example juxtaposes to formulations of the same problem: an arithmetical formulation and an algebraic formulation.

*An arithmetical trip:* Anne drove 40 kilometers north from her home to an out of town meeting. She then drove back going 60 kilometers out to another meeting. After both meetings were over, she called home asking her husband, Ben to join her.

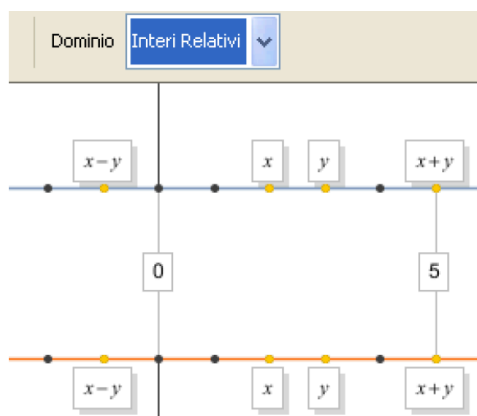
- a) How far will Ben have to go and in what direction?
- b) Write an expression for writing the length of Bens' trip.

*An algebraic trip:* Anne drove *a certain number of* kilometers north from her home to an out of town meeting. She then drove back going 60 kilometers out to another meeting. After both meetings were over, she called home asking her husband, Ben to join her.

- a) Write an expression for writing the length of Bens' trip.
- b) *Could Anne have driven less than 60 kilometers north on her first trip? If not, explain why. If she could have, give an example and explain its meaning.*

The example shows that the main idea is to formulate real-life problems in a generalized way in order to foster algebraic solutions which comprise negative numbers. Referring back to my epistemological analysis, their algebraic didactical model seems to be closer to the epistemological roots of negative numbers than the arithmetical counterpart. There, I pointed out that negative numbers became meaningful in the context involving subtractions with variables such as  $x - 60$ .

Chiappini (2011) exemplifies how the number line model can be used in a digital environment to foster algebraic thinking. He presents the algebraic line in the digital tool AlNuSet as a didactic model, which is supposed to mediate reification (Sfard, 1994) or objectification (Radford, 2010) of negative numbers in the context of the operation  $a - b$ . Different values of  $a$  and  $b$  can be chosen by dragging the corresponding points on the number line. The corresponding value of the expression  $a - b$  is shown by the system as a point on the number line (Fig. 2).



**Fig 2: Algebraic line from AlNuSet (Chiappini, 2011, p. 433)**

This didactical model focuses on the core operation, which led to the introduction of negative numbers in the history of mathematics and offers a representation for the result. However, it is not clear how this model is to be extended to other operations with integers.

Linchevski and Williams (1999) use the double abacus related to a real-life problem and a dice game. They conclude that by recording scores of a dice game on the double abacus and by operating on them „integers are encountered as objects in social activity, before they are symbolized mathematically, thus intuitively filling the gap formerly considered a major obstacle to reification“ (Linchevski & Williams, 1999, p. 144).

Rezat (2014) exploits the potential of a model building on the permanence principle. In an ongoing design research project, Rezat (2014) and Schumacher & Rezat (in press) developed a learning trajectory for the learning of integers and the operations with them. In the learning trajectory, they aim to implement a didactical model based on the permanence principle building on pattern generalization tasks. In this learning trajectory, negative numbers are introduced through the idea of counting backwards beyond zero. Based on this idea and the representation of negative numbers on the number line, the order of integers and the operations with integers are consecutively introduced in the following order:

1. Introduction of negative numbers
2. Order of negative numbers
3. Subtraction of positive numbers from negative numbers
4. Addition
5. Subtraction of negative numbers from negative numbers
6. Multiplication

Each section has an analogous structure. It begins with pattern generalization tasks followed by analyzing and exploring task relations within and among operations. Finally, the rules of calculating with integers are abstracted from these explorations. I will provide a more detailed account of this structure using the example of subtracting negative numbers.

Each chapter starts with pattern generalization tasks such as represented in Fig. 3.

<p>a)</p> $\begin{array}{rclcl} 3 & - & 2 & = & \underline{\hspace{1cm}} \\ 3 & - & 1 & = & \underline{\hspace{1cm}} \\ 3 & - & 0 & = & \underline{\hspace{1cm}} \\ 3 & - & \underline{\hspace{1cm}} & = & \underline{\hspace{1cm}} \\ 3 & - & \underline{\hspace{1cm}} & = & \underline{\hspace{1cm}} \\ 3 & - & \underline{\hspace{1cm}} & = & \underline{\hspace{1cm}} \end{array}$	<p>e)</p> $\begin{array}{rclcl} 3 & - & (-1) & = & \underline{\hspace{1cm}} \\ 2 & - & (-1) & = & \underline{\hspace{1cm}} \\ 1 & - & (-1) & = & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & - & (-1) & = & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & - & (-1) & = & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & - & (-1) & = & \underline{\hspace{1cm}} \end{array}$
...	

**Fig. 3: Pattern generalization tasks for the subtraction of integers**

Pattern generalization tasks are widely incorporated in German textbooks for primary level. The aim of these tasks is to foster students' recognition of patterns and their understanding of number relations. By generalizing the patterns, students can discover basic rules of arithmetic, e.g. that a difference remains constant when both, minuend and subtrahend are lowered or increased by the same number. Pattern generalization tasks are widely acknowledged to foster algebraic thinking (Radford, 2008; Rivera, 2013). Freudenthal (1983) also proposed to use them for learning the operations with integers. Although they are already used in this context, their full potential has only rarely been exploited. In their learning trajectory, Rezat (2014) and Schumacher & Rezat (in press) make intensive use of these tasks in order to support the learning of integer arithmetic aligned with algebraic thinking.

Students are supposed to analyze the relation of the tasks, complete the empty fields in the tasks, solve the tasks, and describe the structure of the pattern. The first three tasks in Fig. 3 a) should be easily solvable for students since they ask students to perform a simple natural number subtraction. Whereas in the fourth task, students encounter the subtraction of a negative number from another number for the first time. It is expected that students draw on their prior knowledge of this task type and derive the solution of this unfamiliar task from the pattern that the minuend remains constant, the subtrahend is lowered by one, and consequently, the result increases by one. In these patterns, students encounter the difficult situations, which incorporate two successive signs (Vlassis, 2008). Drawing on the structure of the pattern, they should be able to conjecture that  $3 - (-1)$  equals 4. In that, the prior knowledge of the task type provides a tool for solving tasks involving the subtraction of negative numbers. After more exploration and related reflection, a conjecture about the rule for the subtraction of negative numbers that is consistent with their prior knowledge should be possible. Deeper reflection reveals also new insights, such as that subtraction does not always lower the result. In other structures of task sequences students can encounter the different cases of the subtraction of negative numbers, such as the subtraction of a negative number from a negative number as shown in Fig. 3 e).

In our empirical investigation of students' behavior when working on these tasks we find both, students who solve the tasks correctly based on the pattern (Fig. 4, left) and students, who solve each

task separately and show a natural number bias, which relates to the lowering effect of subtraction (Fig. 4, middle and right):

Fig. 4. Three exemplary student solutions of the task in Fig. 3.

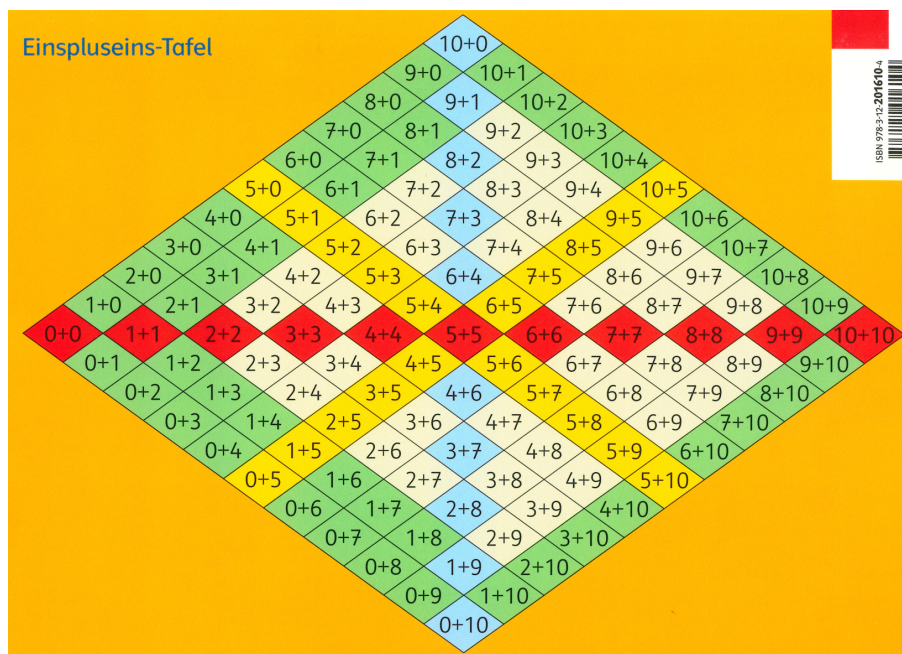
After completing these pattern generalization tasks students are supposed to transfer these tasks into a table such as depicted in Fig. 5.

-												
						5 - 0 = 5	5 - 1 = 4	5 - 2 = 3	5 - 3 = 2	5 - 4 = 1	5 - 5 = 0	...
	4 - (-5) =	4 - (-4) =	4 - (-3) =	4 - (-2) =	4 - (-1) =	4 - 0 = 4	4 - 1 = 3	4 - 2 = 2	4 - 3 = 1	4 - 4 = 0	4 - 5 = -1	...
						3 - 0 = 3	3 - 1 = 2	3 - 2 = 1	3 - 3 = 0	3 - 4 = -1	3 - 5 = -2	...
						2 - 0 = 2	2 - 1 = 1	2 - 2 = 0	2 - 3 = -1	2 - 4 = -2	2 - 5 = -3	...
						1 - 0 = 1	1 - 1 = 0	1 - 2 = -1	1 - 3 = -2	1 - 4 = -3	1 - 5 = -4	...
						0 - 0 = 0	0 - 1 = -1	0 - 2 = -2	0 - 3 = -3	0 - 4 = -4	0 - 5 = -5	...
						-1 - 0 = -1	-1 - 1 = -2	-1 - 2 = -3	-1 - 3 = -4	-1 - 4 = -5	-1 - 5 = -6	...
						-2 - 0 = -2	-2 - 1 = -3	-2 - 2 = -4	-2 - 3 = -5	-2 - 4 = -6	-2 - 5 = -7	...
						-3 - 0 = -3	-3 - 1 = -4	-3 - 2 = -5	-3 - 3 = -6	-3 - 4 = -7	-3 - 5 = -8	...
						-4 - 0 = -4	-4 - 1 = -5	-4 - 2 = -6	-4 - 3 = -7	-4 - 4 = -8	-4 - 5 = -9	...
						-5 - 0 = -5	-5 - 1 = -6	-5 - 2 = -7	-5 - 3 = -8	-5 - 4 = -9	-5 - 5 = -10	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Fig. 5: Subtraction table for negative numbers

The table shown in Fig. 5 is an extension of the rotated diagram shown in Fig. 6 to the negative case. Tables such as the one shown in Fig. 6 are recommended by German mathematics educators in order to display the relations between all the tasks with summands up to 10 and to support effective memorization of the basic tasks in the early primary grades (Schipper, Ebeling, & Dröge, 2015; Wittmann & Müller, 1997). The structure of the table is supposed to foster students' understanding

of task relations. Based on their understanding of this structure, students should be able to derive the result of a task from a task, which is memorized as a basic fact. For example, the solution of the task  $5 + 4$  might be derived from the (memorized) doubling task  $5 + 5$  by diminishing the result by one, since one of the summands is also diminished by one.



**Fig. 6: Basic addition-table (Wittmann & Müller, 2012)**

Similarly, students are supposed to complete the table in Fig. 5 by drawing on number relations and thus deriving the contents of the empty fields from adjacent fields. Again, the tasks relate to familiar tasks from the set of natural numbers and students can relate the “new” numbers to their prior knowledge by relying on number relations.

In order to explicate the rules for subtracting negative numbers, students explore number relations in excerpts from different tables as shown in Fig. 7.

$5 - (-5)$ $= 10$	$5 - (-4)$ $= 9$	$5 - (-3)$ $= 8$	$5 + 3$ $= 8$	$5 + 4$ $= 9$	$5 + 5$ $= 10$
$4 - (-5)$ $= 9$	$4 - (-4)$ $= 8$	$4 - (-3)$ $= 7$	$4 + 3$ $= 7$	$4 + 4$ $= 8$	$4 + 5$ $= 9$
$3 - (-5)$ $= 8$	$3 - (-4)$ $= 7$	$3 - (-3)$ $= 6$	$3 + 3$ $= 6$	$3 + 4$ $= 7$	$3 + 5$ $= 8$

**Fig. 7: Comparison of subtraction and addition table excerpts**

By comparing tasks and results, they can find, for example, that the two tasks  $5 - (-5)$  and  $5 + 5$  both equal ten. By exploring and analyzing adjacent tasks they can identify the same phenomenon. Therefore, they can conjecture that  $5 - (-5) = 5 + 5$ . After confirming this relation with more tasks,

they are asked to generalize their findings and formulate a rule for the subtraction of negative numbers.

The learning trajectory by Rezat (2014) and Schumacher and Rezat (in press) was evaluated in a comparative study with a learning trajectory based on real-life contexts as modelling structures. Results are going to be published elsewhere.

Comparing the presented didactical models that introduce negative numbers aligned with algebraic thinking reveals that all of them relate to important cognitive achievements in the epistemology of negative numbers. Chiappini (2011) directly relates to the generalization of the expression  $a - b$  and its representation on the number line, which played a crucial role in the epistemological development of negative numbers. Gallardo (2002) uses historical problems and Peled and Carraher (2008) argue for generalizations of word problems. As was shown in the epistemological analysis, problems of this type and their solution played an important role in the historical development of negative numbers. Rezat (2014) and Schumacher and Rezat (in press) relate to the permanence principle, which was the cognitive achievement that led to the formal understanding of negative numbers and their final acceptance as numbers.

While Chiappini (2011) and Peled and Carraher (2008) only present some isolated examples, which might be incorporated in a learning trajectory for negative numbers, Gallardo (2002) as well as Schumacher and Rezat (in press) present didactic models, which consistently make use of one core idea to foster the learning of negative numbers aligned with algebraic thinking.

The epistemological analysis has shown that it was crucial in the development of the negative number concept to overcome the understanding of negative numbers as magnitudes. The understanding of negative numbers as opposite magnitudes was persistent whenever negative numbers appeared in the solution of real-life-problems. It was not until the formalization of the number concept in the 19<sup>th</sup> century that negative numbers were accepted as autonomous quantities. Therefore, it is questionable if the teaching of the negative number concept solely based on real-life contexts as modelling structures is an appropriate approach to develop an algebraic understanding of negative numbers. According to genetic epistemology it might be appropriate to introduce negative numbers in such contexts, but it equally seems important to proceed towards an algebraic understanding of negative numbers and their operations based on the permanence principle as Rezat (2014) and Schumacher and Rezat (in press) suggest. However, learning trajectories that coherently align the development of the negative number concept and algebraic thinking are still missing in the research literature.

### **Number sense**

So far, I have analyzed the epistemological relation between the negative number concept and algebra as well as didactic models for negative numbers that explicitly relate to algebraic thinking. I will now turn to *number sense*, a construct that is of interest for the scope of this article for two reasons: 1. Like early algebra, number sense usually relates to children's abilities with natural numbers and is rarely used in the context of other number systems; 2. Number sense and algebraic thinking share some commonalities, which are rarely related. In order to unveil continuities and discontinuities in the learning of the number concept, I will firstly elaborate on the question, whether it makes sense to



consider number sense in other number domains. Secondly, I will briefly analyze the relation between number sense and algebraic thinking.

The development of number sense is a commonly shared goal for the learning of natural numbers. It is mentioned about 50 times in the publication of the 23<sup>rd</sup> ICMI Study on whole numbers in the primary grades (Bartolini Bussi & Hua Sun, 2018). Many papers in TWG02 “Arithmetic and number systems” at CERME stress that the presented research is devoted to the development of number sense.

The very *number sense* is used to denote different concepts (Rezat & Rye Ejersbo, 2018). Number sense in the meaning that is commonly shared in the psychological community refers to a persons’ foundational innate core systems to process quantities. Verschaffel elaborated on the facets of this psychological notion of number sense in his plenary talk at CERME 10 (Verschaffel, Torbeyns, & De Smedt, 2017). In mathematics education, number sense broadly refers to “the well-organized conceptual network that enables one to relate number and operation properties and to solve number problems in flexible and creative ways” (Sowder, 1992, p. 381).

There is a fundamental difference between the two perspectives. While the psychological perspective considers children’s innate abilities, which are not subject to learning, the perspective on number sense in mathematics education relates to abilities that children can develop through learning. Sayers and Andrews (2015) integrate three different perspectives on number sense and offer a model that comprises different conceptualizations of number sense at different stages in children’s learning history. In this paper, I refer to the didactical perspective on number sense.

An aspect that has been discussed repeatedly in TWG02 at CERME related to number sense is flexible and adaptive use of strategies in mental calculation (Carvalho & da Ponte, 2013; Morais & Serrazina, 2013; Rezat & Rye Ejersbo, 2018). Flexibility and adaptiveness in mental calculation require a deep understanding of number and operation relationships and knowledge of basic facts. These are core aspects of number sense (Threlfall, 2002; Rathgeb-Schnierer & Green, 2013). Therefore, number sense is regarded as both, a prerequisite and a goal for flexible and adaptive strategy use in mental calculation (Rezat & Rye Ejersbo, 2018).

Number sense and flexible and adaptive mental calculation usually relate to children’s abilities related to natural numbers. According to the definition by McIntosh, Reys, and Reys (1992) it seems desirable to develop number sense in other number domains. In their framework of number sense, McIntosh et al. (1992) include the understanding of the effect of operations with fractions and decimals. However, flexible mental calculation and number sense have been rarely investigated in other number domains. A slightly increasing interest in these issues related to fractions is noticeable (Markovits & Pang, 2007), which was also discussed in TWG02 at CERME (e.g. Carvalho & da Ponte, 2013). In terms of mental calculation related to number sense these studies differentiate between rule-based or instrumental / procedural strategies and number-sense or conceptual strategies, which are based on equivalence, numerical relationships and properties of operations (Lemonidis, Tsakiridou, & Meliopolou, 2018; Yang, Hsu, & Huang, 2004). Reys, Reys, Nohda, & Emori (1995) and Carvalho & da Ponte (2013) find that students tend to apply rule based strategies, where students perform the formal rule mentally. On the contrary, Yang et al. (2004) show in an intervention study how students are able to develop number-sense mental calculation strategies, which are based on

equivalence, numerical relationships and properties of operations. Rezat (2011) also explored students' strategies in mental calculation tasks with integers. He also finds that students transform the problem with integers into a problem with natural numbers and determine the sign of the result separately applying a procedural and rule-based strategy for calculation with integers. Consequently, all mental calculation tasks including integers were solved referring to mental calculation strategies from the set of natural numbers.

These findings give rise to the question of the relevance of mental calculation in other number sets than the natural numbers. If mental calculation in other number sets is reduced to the mental application of the rules for calculating in these domains and by transformation to problems with natural numbers the relevance for fostering number sense has to be questioned. However, Yang et al.'s findings indicate that it is possible to foster students' number-sense based mental calculation strategies related to fractions. Further investigation of mental calculation strategies, which are associated with number sense is needed. In general, the meaning and conceptualization of number sense related to fractions and integers requires further clarification and differentiation.

I will now turn to the relation of *algebraic thinking* and *number sense*. These two constructs are rarely related. While number sense is situated in the discourse of the development of the number concept, algebraic thinking is situated in the discourse of the development of algebra. However, in the latter context, constructs such as *structure sense* (Hoch & Dreyfus, 2004, 2006) and *symbol sense* (Arcavi, 1994, 2005), which seem to relate to number sense, have been suggested. Within the scope of this article, I am not able to analyze the relationship between number sense, structure sense, and symbol sense. I can only briefly outline some similarities between number sense and algebraic thinking.

McIntosh et al. (1992) have provided a framework of generally agreed components of basic number sense, which gives an account of the richness of the construct. This framework distinguishes between three major areas of number sense: 1. Knowledge of and facility with numbers; 2. Knowledge of and facility with operations; and 3. Applying knowledge of and facility with numbers and operations to computational settings. Each of these areas is divided into several categories.

Among the categories that characterize basic number sense we find several aspects that are repeatedly used to characterize algebraic thinking. In particular, these are related to knowledge of and facility with operations. A number of studies on early algebra focuses on students understanding of operations in terms of relational thinking (Bastable & Schifter, 2008; Carpenter, Colm, & Franke, 2003; Empson, Levi, & Carpenter, 2011; Russell, Schifter, & Bastable, 2011), i.e. "using fundamental properties of number and operations to transform mathematical expressions rather than simply calculating an answer following a prescribed sequence of procedures" (Carpenter, Levi, Franke, & Zeringue, 2005). An example, Carpenter et al. (2003, p. 4) provide to illustrate relational thinking is Robin's solution of the open number sentence  $18 + 27 = [ ] + 29$ : "29 is two more than twenty 27, so the number in the box has to be two less than 18 to make the two sides equal. So it's 16". This way of applying associativity in this context is exactly what would be expected from a child, who exhibits number sense.

Due to the similarities of number sense and algebraic thinking in terms of relational thinking, Pittalis, Pitta-Pantazi, and Christou (2016, 2018) argue that number sense has an innate algebraic dimension.

They have empirically validated a model of the structure and development of basic number sense by incorporating an algebraic dimension, which refers to algebraic arithmetic and quantitative relations. In their study, they validate their model, in which number sense is conceptualized as a second order theoretical construct made up of three first order latent factors, namely (a) elementary number sense, (b) conventional arithmetic, and (c) algebraic arithmetic.

This is a first and important step in understanding the relationship between number sense and algebraic thinking. Further research needs to deepen this understanding by also taking constructs like structure sense and symbol sense and their relation to number sense into consideration.

## **Conclusions**

I have focused on the relationship between the extension from natural numbers to integers and the transition from arithmetic to algebra. From an epistemological perspective on this transition, I have shown that the development of the negative number concept is closely linked to core algebraic ideas, such as indeterminate objects and their analytic treatment, reification and objectification, and a detachment from content meanings in order to proceed to a formalized view.

The psychological analysis showed that the same obstacles characterize students' learning of the negative number concept regardless of their prior knowledge and their prior experiences in the set of natural numbers. So far, the effects of early algebra on students' understanding of negative numbers has not been investigated.

The analysis from the pedagogical perspective has shown that two important goals related to the learning of natural numbers, namely the development of algebraic thinking and the development of number sense, are rarely considered in the domain of integers. I presented a few didactical models that have been suggested in the research literature in order to align the learning of integers and the development of algebraic thinking. However, the alignment seems to be quite loose so far and is rarely developed into coherent learning trajectories for the learning of integers. An analysis of didactic models of negative numbers and their potential to foster algebraic thinking in textbooks might complement the analysis and draw a more comprehensive picture.

In the introduction, I mentioned that the extension of number systems and the transition from arithmetic to algebra are two long-term developments that require a careful construction of the curriculum and related learning-trajectories. Many scholars stress the importance of curricular coherence in the construction and implementation of curricula in general (Confrey, Gianopulos, McGowan, Shah, & Belcher, 2017) and in particular related to goals of the number curriculum (Bruno & Martinon, 1999; Van den Heuvel-Panhuizen, 2008). Curricular coherence is defined differently according to the principles that are used in order to provide it (Confrey et al., 2017). While, for example, Bruner (1960) and Schmidt, Wang, and McKnight (2005) refer to the structure of the discipline as the means to provide curricular coherence, Confrey et al. (2017) argue for learner-centered curricular coherence, which they define as

an organizational means to promote a high likelihood that each learner traverses one of many possible paths to understanding target disciplinary ideas. The goal is that students achieve demonstrable and justifiable proficiency in the meanings, relationships, and utility of those target

ideas by building on and continuously broadening and modifying their ideas and experiences. (p. 719)

Looking at the curriculum in terms of the extension of number systems and the transition from arithmetic to algebra yields that there is a close relation of arithmetic and algebraic thinking in the set of natural numbers. However, this close alignment does not seem to be coherently continued in the extension of number systems. Further development of algebraic thinking in the domain of integers (and also in the domain of fractions) seems to be almost suspended. In terms of curricular coherence, it could be important for the learning of the negative number concept and for the learning of algebra to continuously foster algebraic thinking throughout the extension of number systems. Continuously unfolding number sense aligned with algebraic thinking throughout the extension of number systems might also be a means to provide continuity in a content domain where students' experience of discontinuity has been substantiated by a large body of research.

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