Embodied instrumentation: combining different views on using digital technology in mathematics education

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The potential of digital technology for mathematics education has been widely investigated in recent decades. Still, much remains unknown about how to use tools to foster mathematics learning. To address this issue, I first consider the didactical functionalities of digital technology in mathematics education, and the overall modest effects of using these tools for learning. Next, to find possible explanations of these findings I address three relevant views: (1) a Realistic Mathematics Education (RME) view on tool use, (2) an instrumental approach to tool use, and (3) an embodied view on cognition. As a conclusion, I claim that all three lenses share a focus on mathematical meaning. Whereas the RME view provides important general guidelines, an integrative approach to tool use, which I label embodied instrumentation*, and which includes the careful alignment of embodied and instrumental experiences, seems promising to generate powerful learning activities.*

Keywords: Digital technology, Embodied instrumentation, Embodiment, Instrumental approach, Mathematics education, Realistic mathematics education.

An introduction to tool use

Since the origin of mankind, humans have been using tools to extend their scope and to carry out tasks more easily and more efficiently. A wide range of tools has been developed over time. The most basic ones, such as a stone axe for chopping wood, enabled their users to go beyond their physical limitations to achieve specific goals. The tools were not always designed as such. In some cases, people – or animals, as tool use is not limited to the human species – appropriated objects for a specific task, and in this way 'turned objects into tools'. For example, one could use a tree branch to hit somebody harder than one could do with bare hands, but the branch did not grow from the tree to facilitate beating.

Over time, tools have become more sophisticated and have been designed to address cognitive tasks. Think, for example, of clay tablets to capture calculations (Proust, 2012). Writing down calculations assumes ways to represent numbers and operations, which are quite abstract mathematical notions; these representations themselves can be considered tools already (Monaghan, Trouche, & Borwein, 2016). Since clay tablets, many other tools for mathematics have been designed and used over the centuries. Physical artefacts such as the abacus and compasses respectively facilitated calculations and geometrical constructions.

Gradually, new types of tools emerged, such as mechanical tools – think of Pascal's Pascaline (Maschietto & Soury-Lavergne, 2013) – and digital tools. Nowadays, digital technology such as calculators, tablets (but no longer made from clay), smartphones and smart watches, gives access to a wide range of mathematical features, including sophisticated computer algebra engines and statistical packages. In many cases, the mathematics embedded in the latter types of software has a non-transparent black-box character. In addition to this, the role of mathematics under the hood of advanced tools, such as search engines, navigation tools and credit cards, to mention just some examples, is becoming more and more invisible.

As education prepares for future private and professional life, the development and widespread availability of sophisticated mathematical tools affects mathematics education. These tools transform mathematical activity (Hoyles, 2018). However, much is still unknown about how to exploit the potential of these powerful technologies for mathematics learning. In spite of the available body of literature (for overviews see Ball et al., 2018; Hoyles & Lagrange, 2010; Trgalová, Clark-Wilson, & Weigand, 2018), the mathematics education community is still struggling with the integration of digital technology in teaching and learning. The question of how the use of digital technology may foster mathematics learning and which theoretical lenses may guide us, is waiting to be answered.

To address this question, I will first globally address the didactical functionalities of digital technology in mathematics education, and the overall modest effects of using these tools for learning. To consider possible explanations of these findings, I will then address three relevant theoretical views in more detail: (1) a Realistic Mathematics Education view on tool use, (2) an instrumental approach to tool use, and (3) an embodied view on cognition. Finally, I will claim that these three lenses share a focus on mathematical meaning. Whereas the RME view provides important general guidelines, an integrative approach to tool use, which I will label embodied instrumentation, and which includes the careful alignment of embodied and instrumental experiences, seems promising to generate powerful learning activities.

Digital tools in mathematics education

A taxonomy of digital tools

In the last decades, a myriad of digital tools for mathematics education has been developed. These tools show a wide variety with respect to mathematical focus, didactical functionality, user-friendliness, and other features. All, however, come with affordances and limitations, with opportunities and constraints. Let me try to sketch an overview of the fragmented landscape of digital technology in mathematics education. A first dimension, of course, is the tool's mathematical functionality. A categorization of the mathematical functionality of a tool can be close to a categorization of the field of mathematics itself. Digital tools can carry out algebraic work, graphing tasks, statistical analyses, calculus procedures, and geometric jobs. The traditional domains of school mathematics (e.g., number, ratio, algebra, geometry, calculus, statistics) may do for globally classifying the mathematical functionality of digital tools for mathematics education. It goes without saying that a specific digital tool may cover a range of these domains and as such serve more than one mathematical functionality, but this mathematical categorization still seems to work.

Slightly more complicated is a taxonomy of the didactical functionality of a digital tool, all the more as this is not just a matter of the tool itself, but also highly depends on the type of tasks and on the way the use is embedded and orchestrated in the teaching and learning processes. In spite of this evident limitation, I do feel that the very global model presented in Figure 1 (Drijvers, Boon, & Van Reeuwijk, 2011; Drijvers, 2018b) may help teachers and educators to prepare their teaching with technology, and to be explicit about their main goals and corresponding choices with respect to the tool to use. The first didactical functionality in Figure 1 is to “do mathematics”. This functionality does not target the heart of the mathematical activity itself, but concerns outsourcing part of the work to relieve the student’s mind. In this way, energy can be saved for the core matter; a division of labour between student and machine, so to say. Next, Figure 1 shows two types of didactical functionality that focus on learning. With respect to learning through practicing mathematical skills, digital tools may offer variation and randomization of tasks, and automated and intelligent feedback. As such, the digital tools form a personal environment in which one can safely make mistakes and learn from them. Finally, tool use for concept development involves using a digital tool to explore phenomena that invite conceptual development. This is probably the most challenging and subtle didactical functionality to exploit, as concept development can be considered a higher-order learning goal.

Of course, the categories in this didactical functionality taxonomy are not mutually exclusive; in many cases, the “developing concepts” didactical functionality rests on the outsourcing function for doing mathematics. Also, the didactical function of a digital tool is just a tool feature to a lesser extent than the mathematical functionality is; it also depends on the type of tasks and student activity, and the educational setting. This being said, the model may help to identify some main roles of digital technology in the learning of mathematics.

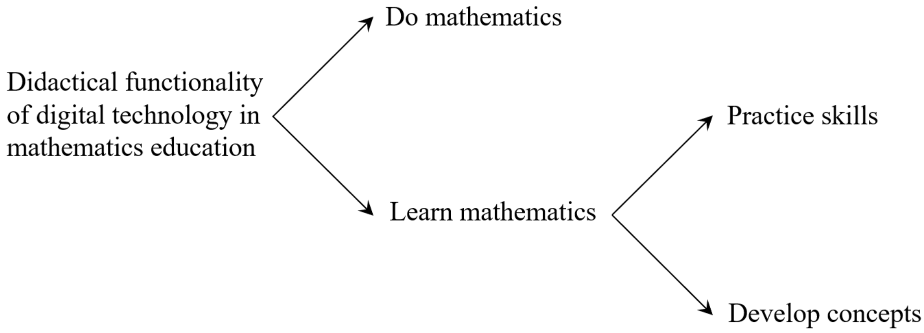


Figure 1: Didactical functionality of digital technology in mathematics education (Drijvers, Boon, & Van Reeuwijk, 2011; Drijvers, 2018b)

The benefits of tool use

After this global sketch of the mathematical and didactical landscape, one might wonder about the benefits of using digital technology in mathematics education. How much evidence is there for the learning gains? Recently, OECD was not very optimistic about this evidence:

> Despite considerable investments in computers, internet connections and software for educational use, there is little solid evidence that greater computer use among students leads to better scores in mathematics and reading. (OECD, 2015, p. 145)

To further investigate this, I revisited some review studies in this domain (Drijvers, 2018a). While doing so, a main source was a second-order meta-analysis carried out by Young (2017), who, interestingly enough, took the didactical functionality typology shown in Figure 1 as a starting point. Including 19 meta studies, Young finds a significant positive effect of the use of technology in mathematics education with a small to moderate average effect size of 0.38 (Cohen, 1988). In his calculation, Cohen’s *d* and Hedges *g* are considered comparable. This average varies slightly over the three different didactical functions: 0.47 for the “do mathematics” role, 0.42 for the “practice skills” role, and 0.36 for the “develop concepts” role. This not surprising to me, as the latter functionality usually requires more student reflection than the other two do. For studies in which the different didactical functionalities are combined, however, the average effect size is lower, namely 0.21.

An interesting finding by Young (ibid.) is that the reported average effect size seems to decrease with the increasing quality of the meta-analyses included. Quality here refers to both the meta-analysis itself and to the quality of the studies included in it. For example, the three review studies mentioned in Table 1 are the only ones rated high quality and they show relatively low effect sizes. A more detailed look at these studies also reveals that the effect sizes reported in the different research reports do not significantly increase over time, whereas one might hope that technological tools are improving, along with teachers’ ability to exploit them in teaching. A possible explanation might be that a possible positive development over time is compensated by other factors, such as more rigorous study designs and methods, and bigger sample sizes.

Study	Number of effect sizes	Average effect size	Global conclusion according to the authors
Li & Ma, 2010	85	$d = 0.28$ (weighted)	Moderate significant positive effects
Cheung & Slavin, 2013	74	$d = 0.16$	A positive, though modest effect
Steenbergen-Hu & Cooper, 2013	61	g range 0.01 – 0.09	No negative and perhaps a small positive effect

Table 1: Effect sizes reported in in three high-quality meta review studies (based on Drijvers, 2018a)

Of course, this zooming out approach suffers from important limitations. The review studies are based on older research, so the picture might have changed since then. Also, the review studies only include experimental, quantitative studies and neglect qualitative or design-based research. And, finally, overviews such as these do not distinguish educational levels, types of technology used, and other educational factors that may be decisive.

Still, it would be too easy to ignore the above findings because of these limitations. The effect sizes are not overwhelming and the OECD quote at the start of this section seems appropriate. Why is the integration of digital technology in mathematics education not the success that one might have hoped for? In my opinion, the question “does ICT work in mathematics education?” is too broad. If we would replace “ICT” by “textbook”, for example, no one would be surprised to get an answer like “it just depends on the quality of the textbook”. In a similar way, exploiting the full potential of digital tools in mathematics education is a complex issue that requires more detailed insights into the learning processes that play a role, into the targeted mathematical content, and in the ways in which the mathematical activity is affected by the use of the tool. Therefore, I will now zoom in on three theoretical and more nuanced views on the use of digital tools in mathematics education, that may offer principles and frameworks to better tackle the subtlety of the topic.

A Realistic Mathematics Education view

Even though the theory of Realistic Mathematics Education (RME) applies to mathematics education in general, it might also shed some light on the possible benefits of using digital technology in mathematics education. Let me first explain some general RME features. RME is an instruction theory for the teaching and learning of mathematics that was developed in the Netherlands. A starting point was Freudenthal's (1973) view on mathematics as a human activity, i.e., mathematics should be experienced as meaningful, authentic, sensemaking and real by the students. The following quote stresses that the word "realistic" should not be understood as "real world":

> Although 'realistic' situations in the meaning of 'real-world' situations are important in RME, 'realistic' has a broader connotation here. It means students are offered problem situations which they can imagine. This interpretation of 'realistic' traces back to the Dutch expression 'zich REALISEren', meaning 'to imagine'. It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to students can come from the real world, but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the student's mind. (Van den Heuvel-Panhuizen & Drijvers, 2014, p. 521).

This starting point is elaborated in some key concepts, including the activity principle, mathematization, and didactical phenomenology. Let me briefly elaborate on each of these three.

- The *activity principle* links to the view of mathematics as a human activity and highlights that students should have the opportunity to explore and to re-invent mathematics, and in this way build up their mathematical knowledge.
- In line with this, *mathematization* refers to the activity of doing mathematics. Treffers (1987) distinguishes horizontal and vertical mathematization. Horizontal mathematization concerns mathematizing reality and the process of formulating a mathematical description, involving the transfer between different domains. Vertical mathematization concerns mathematizing mathematics and the process of reorganization within the mathematical system, involving the genesis of mathematical objects and relations between them.
- A *didactical phenomenology* is an analysis of "how mathematical thought objects can help organizing and structure phenomena" (Van den Heuvel-Panhuizen, 2014, p. 175). It identifies phenomena that beg to be organized with the specific mathematical means that are the topic of the learning, and as such may "show the teacher the places where the learner might step into the learning process of mankind" (Freudenthal, 1983, p. ix). It invites the development of the mathematics at stake and gives meaning to it. As said before, these phenomena can come from different "worlds" as long as they are experientially real to the students (Gravemeijer & Doorman, 1999). Such an analysis on the one hand asks for a thorough analysis of the mathematical topic, and on the other hand for a clear view on the targeted audience of the teaching.

How do these RME principles inform the use of digital technology in mathematics education? First, the interpretation of the word "realistic" in the sense of experientially real suggests that students should experience the activity with the digital technology as meaningful. In line with Ainley, Pratt and Hansen (2006), students may perceive an activity as meaningful if they are aware of its purpose and its utility, where purpose refers to the activity leading to a “meaningful outcome for the pupil, in terms of an actual or virtual product, or the solution of an engaging problem” (p. 29), and utility to “the ways in which those mathematical ideas are useful” (p. 30). I expect that a certain level of transparency of the tool would foster meaning in terms of experienced purpose and utility.

Second, the activity principle and the human activity view suggest that the digital tool should offer the students opportunities to explore, and to be an actor rather than a passive user. I expect that a degree of ownership and the feeling of being in control may invite this. From a mathematization perspective, being in control also includes the opportunity to easily express yourself mathematically with the amount of freedom that one also has while doing paper-and-pen mathematics. This requires a sound mathematical basis for the tool in use.

Third and final, taking a didactical phenomenology perspective leads me to expect that the phenomena may change in a technology-rich classroom: the digital environment itself may be a meaningful phenomenon to study. For example, if students regularly use digital tools like graphing calculators or software for dynamic geometry, these environments really become part of the classroom environment and as such may elicit inspiring phenomena that invite further investigation. Also, as many students nowadays are familiar with games and tools, digital environments may be quite natural and authentic to them, which offers opportunities to better realize this RME principle.

Let me illustrate these principles through the example of an online lesson series on arrow chains and functions for grade 8 (14-year-old students), implemented in the Freudenthal Institute’s Digital Mathematics Environment[1] (Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012; Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). In this lesson series, students first explore chains of operations in meaningful contexts. As an example, they figure out how the breaking distances of different types of vehicles, the distance needed to stop in case of emergency, depend on their velocity. Next, they act out the chaining by standing next to each other, creating an input-output-chain in which each student is responsible for performing one of the operations, to prepare for the work in the digital environment.

Figure 2 shows some snapshots of the work in the digital environment that follows. The first row shows how students can chain operations to calculate the breaking distance in meters of a scooter with an initial velocity of 40 km/hour. Of course, after their previous experience with series of numerical calculations, the construction of these chains should be experienced by the student as a meaningful way to organize these calculations. In the second row, the breaking distance is investigated as function of the initial velocity, and a graph is added. In the third row, these breaking distance functions are compared for scooters, cars and lorries. As the window got too full, the user has collapsed the function chains into single boxes, allowing for a good comparison of the three graphs.

Even if these activities are described only briefly, some RME principles can be recognized. As for the reality principle, the students have been introduced to arrow chains to organize calculations in

[1] See https://app.dwo.nl/en/student/ -> Old -> Secondary education -> Algebra -> Lesson function and arrow chains.

whole-class and paper-and-pen activities, so these have become meaningful ways to capture calculations. The open character of the environment, with a big empty window as exploration room for students, is meant to provide the students with means to freely build, change and organize arrow chains and to be in control of what is happening. Horizontal mathematization is addressed through the task of modelling the arrow chains for the case of the breaking distances, and vertical mathematization comes into play as soon as the three quadratic functions and their relationships are compared, independently from the initial breaking distance problem situation. The option to collapse function chains into boxes is intended to support an object view on function. A didactical phenomenology lens led the designers to consider the breaking distance context as a phenomenon that can very well be organized through mathematical functions and arrow chains.

To summarize, the example illustrates how the general RME principles can be applied to the specific situation of using digital tools in a meaningful way, and as such provide guidelines and criteria for sensible tool use. In this way, the RME lens may offer a nuanced view on using digital technology in mathematics education, even if it is not dedicated to this particular case.

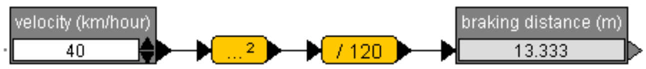
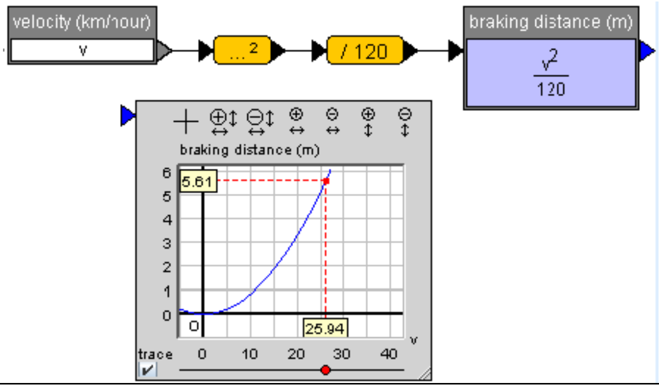
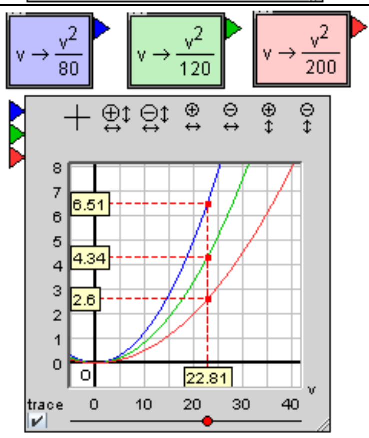
	A function as an input-output assignment: braking distance as a function of velocity
	Investigation of the co-variation of velocity and braking distance through tracing the graph
	Investigation of a family of functions, representing braking distances for three different vehicles

Figure 2: Snapshots from the Function and Arrow Chains material (from Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013). See https://youtu.be/OMDjC5yVlr0 for an animation.

An instrumental approach

In addition to the general guidelines offered by RME, a more detailed view on the interplay between mathematics and tool use is needed. A first, somewhat naive view on digital tools and their use in mathematics education might be that tools are just "objective" mathematical assistants that help us to carry out tasks, to "do the jobs": using them reduces solving mathematical tasks to pressing buttons, and as such simplifies our lives. However, things turn out to be not that straightforward. Tools are not as neutral as one might hope, but come with affordances and constraints, with opportunities and obstacles, and as such guide the user's mathematical practices:

> Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures. (Hoyles & Noss, 2003, p. 341)

For example, drawing a circle with physical compasses is quite different from drawing it in a dynamic geometry environment such as GeoGebra. In the first case, one really experiences a circular movement, after deciding on the centre and the radius. In the latter case, the focus is on setting the centre and the radius, but while enlarging the radius, the circle is growing, and the circular movement is no longer needed. Different tools lead to different techniques, and as such to different views on the same underlying mathematical concept.

Indeed, in line with Vygotsky (1978), tools mediate between human activity and the environment. As a consequence, using digital tools for learning and doing mathematics is not just a matter of directly transforming mathematical thinking into tool commands. On the one hand, the user shapes the techniques for using the tool, but on the other hand the tool shapes and transforms the user's mathematical practice. These considerations gave rise to the development of a new theoretical view, called the instrumental approach to tool use. Key in this approach are the notions of artefact, instrument, instrumental genesis, scheme, and technique. Let me briefly explain these notions.

A starting point in instrumental approaches is the distinction between artefact and instrument (Rabardel, 2002; Vérillon & Rabardel, 1995). The artefact is the object that is used as a tool. In our case, graphing calculators or dynamics geometry software are artefacts, even if we also might want to look in more detail, and consider the graphing window in GeoGebra an artefact, or the Solver option in a graphing calculator. An instrument consists of an artefact and "one or more associated utilization schemes" (Verillon & Rabardel, 1995, p. 87). So, besides the artefact, the instrument also involves the schemes that the user develops and applies while using the artefact for a specific class of instrumented activity situations, in our case often involving a type of mathematical tasks. To summarize this in a somewhat simplified 'formula': Instrument = Artefact + Scheme. The process of an artefact becoming part of an instrument is called instrumental genesis (Artigue, 2002; Trouche & Drijvers, 2010).

What are these schemes, key in instrumental genesis? Based on the work by Piaget (1985) and others, a scheme is considered a more or less stable way to deal with specific situations or tasks. Vergnaud claims that "the sequential organization of activity for a certain situation is the primitive and prototypical reference for the concept of scheme" (Vergnaud, 2009, p.84). Referring to the scheme of counting in particular, Vergnaud (1987, p.47) speaks of "a functional and organized sequence of rule-governed actions, a dynamic totality whose efficiency requires both sensorimotor skills and cognitive competencies.” Later, Vergnaud (2009) prefers to speak about percepto-gestual schemes rather than of sensorimotor schemes, as to go beyond the purely biological level and to highlight the close relationship between perception and gesture on the one hand, and conceptualization on the other. In agreement with these ideas, the term sensorimotor scheme in this rest of this text should be interpreted in this wider sense.

Scheme development involves the intertwined development of sensorimotor skills and cognition. As we see a scheme here as part of an instrument, we speak of an instrumentation scheme. Artigue (2002) highlights the pragmatic and epistemic value of schemes: the pragmatic value in the sense of their productive potential to “get things done”, and the epistemic value in the sense of contributing to the meaning and understanding of the mathematics involved.

The observable parts of an instrumentation scheme, the concrete interactions between user and artefact, are called instrumented techniques. Instrumented techniques are more or less stable sequences of technical interactions between the user and the artefact with a particular goal. As such, an instrumentation scheme consists of one or more observable instrumented techniques, that are guided by the opportunities and constraints the artefact offers, and by the students’ knowledge. In the meantime, the techniques may also contribute to the development of this knowledge. As such, techniques can be seen as actions that reflect students’ knowledge. And, even more important, techniques and knowledge may co-emerge. It is this co-emergence that forms the heart of instrumental genesis and that reflects the main educational potential of using the artefact in a given situation.

In the instrumental approach, a scheme depends on the subject, the artefact and the task. Three comments should be made here. First, this implies that carrying out a similar task with different artefacts is likely to lead to different schemes. The compasses case described above shows that different instrumental geneses will take place. It is interesting to use different artefacts for similar tasks and to confront and compare the different schemes that emerge (Maschietto & Soury-Lavergne, 2013). As a consequence, mathematical practices transform through the use of digital artefacts (Hoyles, 2018). Second, instrumental genesis is not just an individual process, but is part of social learning processes and institutionalization within the specific educational context. Through teachers’ instrumental orchestration (Trouche, 2004), a collective instrumental genesis is taken care of, to assure the convergence towards shared instruments and shared mathematical knowledge. In fact, teachers are involved in a double instrumental genesis, including their personal development of schemes on the one, and schemes for use in teaching their students on the other. Third, some artefacts are more suitable for specific types of instrumental genesis than others. Haspekian (2014) introduced the notion of instrumental distance to stress the change in mathematical practice that may emerge as a result of some type of tool use. If the distance between regular or targeted mathematical practices on the one hand, and techniques invited by the artefact on the other is too big, instrumental genesis might be not productive for the learning process.

The importance of this instrumental view for the use of mathematical tools in mathematics education lies not only in the acknowledgement of the subtlety and complexity of the issue, but also in the concrete guidelines it offers for a fruitful use: as teachers and educational designers, we should set up activities for students and choose appropriate artefacts that together lead to instrumental genesis processes in which the targeted mathematical knowledge is developed in a meaningful and natural way. A mismatch between the task, the affordances of the artefact, and the mathematical knowledge at stake will not be effective. Outlining these three elements is the game to play; it includes being explicit about the instrumental genesis and scheme development that is aimed for.

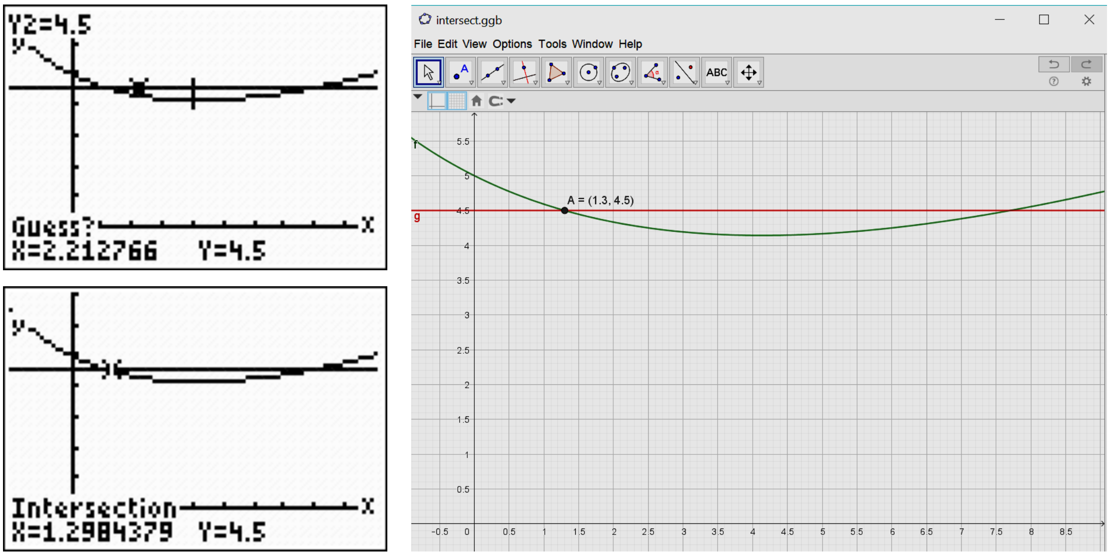


Figure 3: Intersecting graphs to solving equations with two different tools

As an example of such scheme development, Figure 3 shows two ways in which an equation can be solved graphically through an Intersect procedure, one using a graphing calculator (screens on the left) and one using GeoGebra (screen on the right). This equation, $\frac{x^2+100}{2x+20} = 4.5$, appeared in a realistic problem situation in a national examination task in the Netherlands, in which students used a graphing calculator. Technically speaking, the procedure comes down to entering the left-hand side and the right-hand side of the equation separately as functions in the graphing tool. Next, a viewing window should be set so that the two graphs show an intersection point. Then the intersect procedure is called, through selecting the two graphs. This will lead to the coordinates of such an intersection point. Finally, its first coordinate is a solution of the equation. Phrased this way, the technique sounds very straightforward and procedural.

However, several conceptual elements are involved, First, the student needs to be aware of the relationship between solutions of an equation and graphical intersection points. Second, while setting the viewing window dimensions, the student needs to have an idea of where intersection points can be found, which requires some reasoning (or some trial-and-error behaviour). Third, the result consists of not one but two numerical values, and a solution is a numerical, approximated value. Fourth and final, the procedure leads to one, single solution; the procedure needs to be repeated for equations with multiple solutions, that can be visible in the current viewing window, but may also exist outside its boundaries. Again, some reasoning is needed to consider the option of other intersection points outside the current view. Eventually, this technique can be complemented by zooming in at intersection points, by zooming out to get an overview, or by generating tables of function values.

It is this intertwinement of technical and conceptual elements that makes me speak of an Intersect instrumentation scheme. The development of this scheme impacts on students' view on equation solving in a subtle and somewhat implicit way. Solving an equation is no longer a matter of exact algebraic manipulation while maintaining equivalence, but is replaced by a functional, graphical view that leads to approximated values. As such, the tool use affects the mathematical content. The Intersect scheme, therefore, integrates techniques and mathematical ideas, and this is exactly what instrumental genesis is about.

The instrumentation schemes that students develop depend on the digital tool in use. In Figure 3, the left part shows two screens of a graphing calculator, here a Texas Instruments model (Drijvers & Barzel, 2012). The right-hand side shows a similar screen in GeoGebra, which offers a larger screen and higher resolution. The techniques are also slightly different: GeoGebra does not ask for a starting value, but immediately comes up with a point. This makes the procedure more efficient, but it also makes it harder to find the coordinates of the second intersection point. Also, whether the coordinates are displayed depends on the settings in GeoGebra. The two tools – again, in the default setting – provide the results with different accuracy.

To summarize, this example illustrates the interplay between techniques for using a digital tool, and the related mathematical knowledge involved; an interplay that fundamentally affects the mathematics, but in the meantime is subtle to study. The example shows, to rephrase the quotation by Hoyles and Noss (2003) earlier in this section, that tools and techniques are not neutral, but may highlight or even require specific mathematical views on the task at stake. As a consequence, tool use is less simple than it might seem. Instrumental views are helpful to become aware of this complexity and to identify the interplay between artefacts and tasks, between techniques and schemes. Recognizing instrumental genesis as a path to learning mathematics, and probably also different mathematics, is an important step forward to fostering learning while using digital tools.

An embodied view on cognition

The instrumental approach to tool use has proved valuable in understanding the interplay between the technical and the conceptual when learners use artefacts. So far, however, it focused mostly on higher-level mathematics such as pre-university streams, and on sophisticated digital tools such as computer algebra and dynamic geometry systems. It is maybe due to these foci that mathematics is approached as a cerebral activity, and that the bodily foundations of cognition tend to be neglected. To do justice to the latter aspect, I now consider an embodied view on cognition as a third lens to look at the use of digital technology in mathematics education.

A general starting point here is that body and mind cannot be separated and that a dualistic view on them is inappropriate. Cognition is not considered an exclusively mental affair, but based on bodily experiences, that take place in interaction with the physical and social world (e.g., see Radford, 2009; Lakoff & Núñez, 2000; de Freitas & Sinclair, 2014; Ferrara & Sinclair, 2016). Embodiment "is the surprisingly radical hypothesis that the brain is not the sole cognitive resource we have available to us to solve problems" (Wilson & Golonka, 2013, p.1). Phrased differently, Alibali and Nathan (2012) claim:

> According to this perspective, cognitive and linguistic structures and processes – including basic ways of thinking, representations of knowledge, and methods of organizing and expressing information – are influenced and constrained by the particularities of human perceptual systems and human bodies. Put simply, cognition is shaped by the possibilities and limitations of the human body. (p. 250)

Also for the case of mathematics, often considered a highly abstract and mental subject, cognition is more and more acknowledged to be rooted in sensorimotor activities, and mathematical objects to be grounded in sensorimotor schemes. Two special issues–57(3) and 70(2)–of *Educational Studies in Mathematics,* dedicated to embodiment in mathematics education, testify to the growing interest of mathematics education research in this perspective. Many embodied approaches take "the four E's" of embodied, extended, embedded and enactive cognition[2] as a starting point. In these terms, the initial views on tool use in this paper highlight the role of tools to extend the body; now the focus shifts towards the other E's, and to digital technology providing opportunities to create embodied experiences in particular.

At first glance, a tension might seem to exist between this embodied view on cognition and the use of digital technology in mathematics education. Digital tools such as spreadsheets, computer algebra software and dynamic geometry systems embed an impressive amount of mathematical knowledge. As these tools are not transparent, they seem to "hide this knowledge under the hood", which may create a distance between user and mathematics used. And, even more importantly, ways to interact with these tools have not so far been 'body-based', as the interaction mainly took place through keyboard strokes in the more remote past, and through mouse movements in the more recent past. Recent technological developments, however, open up new horizons to do justice to the multimodality of mathematical knowledge. Maybe partly due to the need for embodied experiences, improved user interfaces – think of multitouch screen technology, handwriting recognition, motion sensors, and virtual and augmented reality – have been developed, offering new opportunities to investigate an embodied approach to tool use. For example, some researchers studied students who "walk graphs" using a motion sensor, and in this way create embodied experiences of distance, speed and acceleration changing over time (Duijzer, Van den Heuvel-Panhuizen, Veldhuis, Doorman, & Leseman, 2019; Robutti, 2006).

In line with the didactical engineering tradition within my institute (Margolinas & Drijvers, 2015), the work done in this field by my colleagues an myself within my institute follows an embodied design approach (Abrahamson, 2009; Abrahamson, Shayan, Bakker, & Van der Schaaf, 2016). We use "design genres" (Abrahamson, 2014; Bakker, Shvarts, & Abrahamson, 2019) in which activities in digital environments. In these activities, students can engage in bodily experience and develop mathematical cognition. Let me illustrate this embodied design approach with two examples.

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[2] https://4ecognitiongroup.wordpress.com/

As a first example, Figure 4 shows a task designed by Shvarts (2018) in the Digital Mathematics Environment. The left screen shows a fixed base line and a small, fixed black point. The larger grey dot is projected onto the base line. The three points together define a triangle, which in the left screen is red. The student can move the big grey point with her finger, and the triangle changes accordingly. Suddenly, the triangle becomes green (right screen). The triangle turns green when it is isosceles, otherwise it is red. As a consequence, the triangle is green if the grey, moveable point lies on the parabola which has the base line as directrix and the fixed point as focus. This dashed parabola, however, is not shown to the student, but it is included here for the reader.

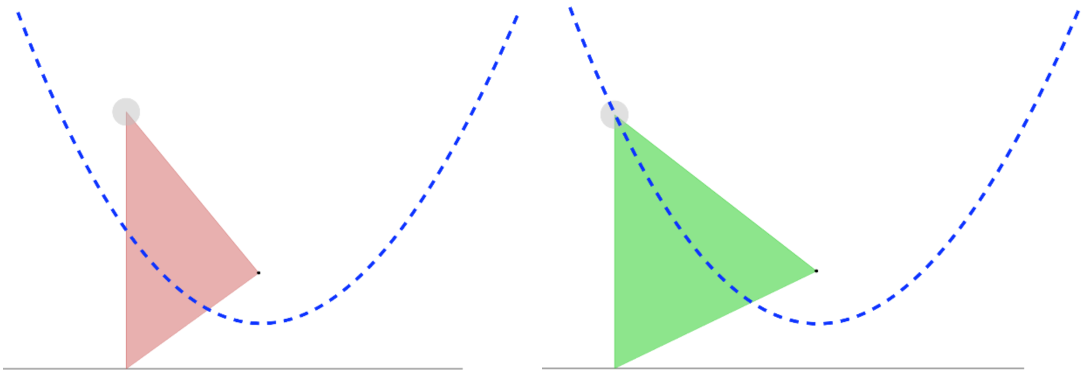

Figure 4: The parabola task (Shvarts, 2018; Shvarts & Abrahamson, 2019). See https://youtu.be/JHiHlFdUGtw for an animation.

The task, now, is to constantly move the grey point so that the triangle remains green. When the students become fluent in their movements, they are asked about the rule that determines the colour of the triangle. The task is challenging from different perspectives. From an embodiment perspective, the sensorimotor coordination is quite complex, as the direction to move should be constantly checked with the orientation of the triangle's base. Eye-tracking data indeed show many eye movements jumping between the movable grey point and the midpoint of the base, along the triangle's median (Shvarts & Abrahamson, 2019). These iterative jumps reveal the most important "attentional anchors" (Duijzer, Shayan, Bakker, van der Schaaf, & Abrahamson, 2017). From a mathematical perspective, the property that makes the triangle green is it being isosceles. As one vertex of the triangle is fixed to a point and another runs through a line, this task provides students with sensorimotor experiences that in the future learning might feed the notion of parabola. At a higher mathematical level, and this goes beyond how the example is presented here, one might elaborate this activity towards reflection on the locus of the grey point for the case that the triangle remains green (Shvarts, 2018). As the triangle is isosceles, and one of the sides is perpendicular to the base line (directrix), we can recognize the property of a conic: the distances to the focus and the directrix remain equal, so the locus is a parabola! This example shows how sensorimotor experiences may draw students' attention to the notion of a triangle being isosceles as preparation to the notion of parabola. Of course, many design decisions need to be taken, such as on how to phrase the task, which support to offer (do we display coordinates, or a grid), how to sequence different variations on this task, how to foster the notion of parabola, et cetera. The learning effect of such tasks may to an important extent depend on these subtle design decisions, but goes beyond the scope of the present example. Digital design is a relatively new phenomenon, which puts high demands on the designers (Leung & Baccaglini-Frank, 2017).

The second example concerns handwriting recognition. Writing mathematics by hand, whether it is on an old, dusty chalk board or on a tablet, involves hand movements and gestures that may provide a sensorimotor experience to students. Therefore, the Digital Mathematics Environment now has a handwriting recognition module, that allows for the integration of the human experience of hand movements while writing, and the software's intelligence to interpret the handwriting and to evaluate mathematical correctness for the sake of feedback.

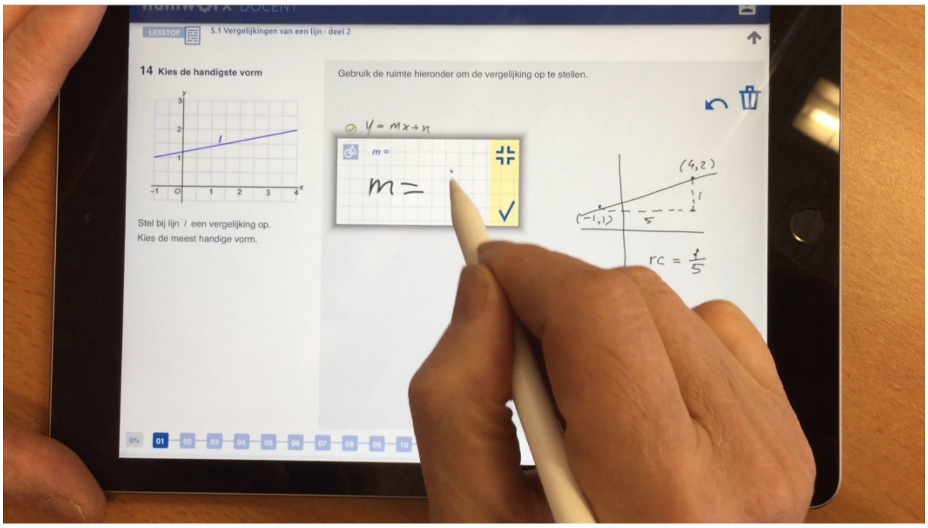

Figure 5: Handwriting recognition in the Digital Mathematics Environment. See https://youtu.be/YKtrr1IxWaA for an animation.

To summarize, the two examples illustrate how an embodied design approach may lead to tasks in which sensorimotor experiences form the basis of mathematical cognition, a view that is not explicitly present in the two views presented earlier.

Embodied instrumentation

Instrumental and embodied views on the use of digital tools in mathematics education may seem quite different. On the one hand, instrumental approaches in many cases focus on the development of individual, mental schemes – even if collective instrumental genesis is acknowledged –, on high-level conventional mathematics, and on sophisticated digital tools. Embodied views, on the other hand, focus on sensorimotor schemes, on bodily experiences, on basic mathematical ideas, and make use of dedicated software tools; the convergence to conventional mathematical cognition and techniques sometimes receives less attention. From a networking theory perspective (Bikner-Ahsbahs & Prediger, 2014), however, it seems interesting to compare, contrast, combine and coordinate these different views.

The claim I want to make here is that, in spite of these apparent differences, embodied and instrumental approaches both highlight the complexity of user-tool interaction, share some similar theoretical bases, and can be coordinated and aligned in a meaningful way. In line with researchers who in the past have been exploring the interface between embodied and instrumental approaches (e.g., Artigue, Cazes, Haspekian, Khanfour-Armale, & Lagrange, 2013; Arzarello, Paola, Robutti, & Sabena, 2009; Maschietto & Bartolini-Bussi, 2009), I argue for an embodied instrumentation approach, to reconcile the embodied nature of instrumentation schemes and the instrumental nature of sensorimotor schemes. As such, an embodied instrumentation approach explores the co-emergence of sensorimotor schemes, tool techniques and mathematical cognition, and offers a design heuristic for ICT activities which align the bodily foundations of cognition and the need for instrumental genesis.

As for the shared theoretical basis, embodied and instrumental approaches share a theoretical foundation in ideas from Vygotsky (1978) on tool use and from Piaget (1985) on schemes, and both approaches acknowledge the subtlety of tool mediation in meaningful mathematical activity. In the meantime, the two approaches can be complementary, in the sense that embodied approaches so far have not overstressed the convergence of tool techniques and conventional mathematical notions, whereas instrumental approaches have tended to neglect the sensorimotor view on schemes and the embodied nature of cognition.

Concerning the coordination and alignment of the two approaches, I can image productive learning trajectories on fundamental mathematical concepts, in which the development of sensorimotor schemes may gradually go hand in hand with, or even be part of instrumental genesis. Such a trajectory might lead to schemes in which embodied experiences still form the basis, and through a process of reflective abstraction (Abrahamson, Shayan, Bakker, & van der Schaaf, 2016) lead to instrumental genesis. In this way, embodied and instrumental approaches might be aligned: the process of instrumental genesis is fostered by embodied activities. As students advance in a learning trajectory, the tool techniques and mathematical knowledge emerge from an instrumental genesis process, and the embodied basis may move more to the background. Ensuring coordination between the development of sensorimotor schemes and instrumental genesis might be a strong design heuristic for technology-rich tasks in mathematics teaching.

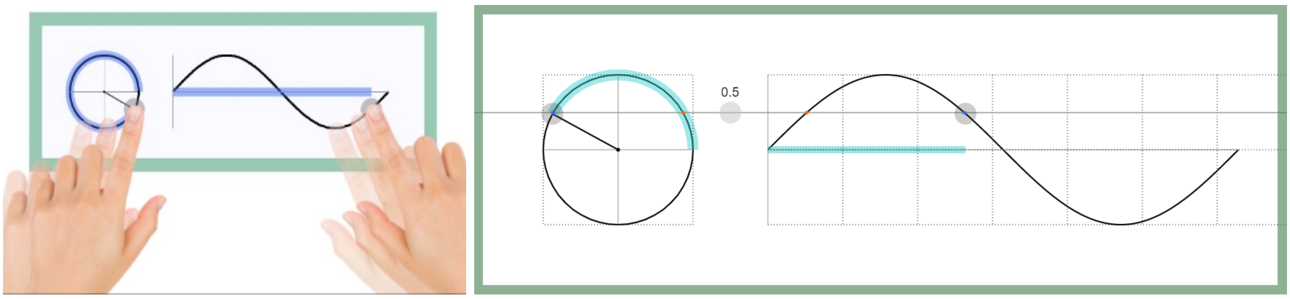

Figure 6: MIT-T tasks (Alberto, Bakker, Walker-van Aalst, Boon, & Drijvers, 2019). See https://youtu.be/1eOU4XyyHmg for an animation.

Let me illustrate this embodied instrumentation approach in a final example. Figure 6's left screen shows the so-called MIT-T app, where the abbreviation stands for Mathematics Imagery Trainer – Trigonometry (Alberto, Bakker, Walker-van Aalst, Boon, & Drijvers, 2019). It shows a unit circle and a sine graph, with a movable point on each of them. As a first task, students use the multitouch screen to simultaneously move the two points and, similar to the case in the parabola example presented above, to explore when the frame around the graphs becomes green. This is the case if a correct match is made between the sine of an angle in the unit circle and the function value of the sine in a point on the horizontal axis, so if the two points are at equal height. The activity of "keeping the frame green" clearly invites appropriate sensorimotor coordination of keeping two points at the same height and is expected to induce a "feeling" for the coordination of the two movements. As was the case for the parabola task (Fig. 4), this may lead to many follow-up activities, each requiring design decisions and subtle arrangements of tasks and tools. As a possible end point of such a sequence, Figure 6's right-hand screen shows the additional tool of a horizontal line, which can be moved up and down through the central big grey point with label its height, 0.5, thus implementing the coordination that just was enacted. Now both the unit circle and the sine graph can be used to solve the equation sin $a = 0.5$: in the unit circle, one can move the point to meet the intersection of circle and line, and similarly in the graph. Of course, these two techniques need to be coordinated as well: if the two intersection points do not match, the feedback frame will not become green. In a later phase, one might want to drop the unit circle and focus on the sine graph. At that stage, the task in fact comes down to graphically solving equations of the form $f(x) = c$, which is exactly the example shown at the end of the instrumental view section (see Figure 3).

To summarize, this example illustrates how embodied and instrumental approaches may be coordinated and aligned in a learning trajectory. In this design, the embodied experiences mediated by digital tools prepare for instrumental genesis. Of course, more research is needed to decide whether such alignments would lead to higher effect sizes than the ones reported earlier. Speaking in general, both the common theoretical bases of embodied and instrumental views, and their complementarity make exploring their potential alignment, as expressed in the notion of embodied instrumentation, a highly interesting enterprise.

Conclusion

In this paper, I first outlined a taxonomy for the didactical functionality of digital technology in mathematics education. This taxonomy guided a second-order meta-analysis, the results of which suggest that the effect sizes of technology-rich interventions are significantly positive, but small to moderate. This led to the idea of looking in more detail at three views on tool use in mathematics education. As a first lens, Realistic Mathematics Education theory highlights that students should experience mathematics as meaningful. Applied to tool use, this implies that tools should be transparent and should provide the students with authentic ways to express themselves mathematically. More specifically focusing on tool use, the second lens of instrumentation stresses the intertwinement of techniques for using the tool and the mathematical knowledge involved. Techniques and mathematical meaning co-emerge in processes of instrumental genesis. The third lens of embodied cognition claims that sensorimotor activities form the basis of cognition, and, more than the other two approaches, highlights the need to root mathematical knowledge in bodily experiences.

A key guiding principle shared by all three approaches is mathematical meaning, even if each approach stresses different aspects of it: RME highlights the idea of mathematics being "experientially real", instrumentation theory points to the mathematical meaning embedded in techniques for using tools, and embodied views see sensorimotor schemes as the foundation of mathematical meaning. This stress on meaning makes sense: if we do not manage to incorporate digital technology in students' mathematical practices in a way that they experience as meaningful, it is a useless enterprise.

As a conclusion, in my opinion the three views have much to offer for technology-rich mathematics education. The RME view provides some important general guidelines, which may inform instrumental and embodied approaches. As for the interplay between embodied and instrumental views, I strongly believe that the two can be coordinated and aligned in a so-called embodied instrumentation approach. Embodied and enacted experiences, so present in the embodied cognition approach, can form the basis for learning. As such, these experiences are the foundations on which instrumental genesis can build; a bodily-based instrumental genesis during which tool techniques and mathematical cognition co-emerge.

Of course, this argument for an embodied instrumentation approach needs further elaboration on several aspects. First, the RME lens may reveal possible tensions between its reality principle and the instrumental approach's focus on tool techniques. Similarly, the alignment of sensorimotor schemes and tool techniques is a subtle one, even if the examples in Figures 4, 5 and 6 provide some possible approaches. Also, a body of empirical evidence for positive effects of such an embodied instrumentation approach is lacking so far. In spite of these limitations, I do believe that the three lenses do justice to three main elements in the "landscape" of mathematics education: the world around us, our bodily interaction with it, and the tools we use to facilitate this interaction. As such, I suggest that embodied instrumentation, seen as an integrated embodied and instrumental approach in which sensorimotor schemes, tool techniques and mathematical cognition co-emerge, deserves priority in the research agenda of those interested in the use of digital technology in mathematics education.

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