# Almost Everywhere Convergence of Prolate Spheroidal Series

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**Abstract**

In this paper, we show that the expansions of functions from $L^p$-Paley-Wiener type spaces in terms of the prolate spheroidal wave functions converge almost everywhere for $1 < p < \infty$, even in the cases when they might not converge in $L^p$-norm. We thereby consider the classical Paley-Wiener spaces $PW_c^p \subset L^p(\mathbb{R})$ of functions whose Fourier transform is supported in $[-c, c]$ and Paley-Wiener like spaces $B_{\alpha,c}^p \subset L^p(0, \infty)$ of functions whose Hankel transform $\mathcal{H}^\alpha$ is supported in $[0, c]$. As a side product, we show the continuity of the projection operator $P_c^\alpha f := \mathcal{H}^\alpha(\chi_{[0,c]} \cdot \mathcal{H}^\alpha f)$ from $L^p(0, \infty)$ to $L^q(0, \infty)$, $1 < p \leq q < \infty$.

*Keywords:* Prolate spheroidal wave functions, almost everywhere convergence, Paley-Wiener type spaces, Hankel transform, spherical Bessel functions
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## 1. Introduction

The prolate spheroidal wave functions (PSWF) form an orthonormal basis of $L^2(\mathbb{R})$ that is best concentrated in the time-frequency plane. They were first studied in the seminal work of Landau, Pollak, and Slepian [12, 13, 22, 21] at Bell Labs which inspired extensive research due to their efficiency

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as a tool for signal processing. They have since found further applications in various fields such as random matrix theory (*e.g.* [9, 14]), spectral estimation [23, 1] or numerical analysis (*e.g.* [26, 25]).

While the natural setting for prolate spheroidal wave functions is $L^2(\mathbb{R})$, the question of convergence of their series expansions arises in $L^p(\mathbb{R})$. This issue has been solved by Barceló and Cordoba [2] who showed that the expansion of an $L^p$-band limited function $f$ in the PSWF-basis converges to $f$ whenever $4/3 < p < 4$, and that this range of $p$'s is optimal. That result was recently extended in [3] to several natural variants of the prolates like the Hankel prolates.

The aim of this paper is to continue this work by investigating almost-everywhere convergence properties of PSWFs expansions of functions in $L^p(\mathbb{R})$. This question is very natural in view of Carleson's fundamental result about Fourier series [6] which was extended to several orthonormal bases of polynomials, *e.g.* by Pollard for Legendre series [17]. Their almost everywhere convergence results follow from very delicate analysis of the maximal operator associated to the expansions considered. On the other hand, the situation is much better for expansions in terms of spherical Bessel functions (*see* [7, 8]) where almost-everywhere convergence is surprisingly simple once mean convergence has been established. The key here is the fast decay of Bessel functions with respect to its parameter. Our main result is to show that the situation is similar for PSWF series since the PSWF-basis can be nicely expressed in terms of spherical Bessel functions.

Let us now be more precise and introduce some notation before giving the exact statements. For $f \in L^1(\mathbb{R})$, the Fourier transform is given by

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t}\,\mathrm{d}t,$$

and the definition extends to $L^2(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ (the space of tempered distributions) in the usual way. The Paley-Wiener spaces of band-limited functions are denoted by

$$PW_c^p = \left\{ f \in L^p(\mathbb{R}) : \ \mathrm{supp}\left(\widehat{f}\,\right) \subseteq [-c, c] \right\},$$

where the Fourier transform is to be understood in the distributional sense whenever $p > 2$. The projection $P_c f = \mathcal{F}^{-1}\big(\chi_{[-c,c]} \cdot \mathcal{F}(f)\big)$ is a continuous operator from $L^p(\mathbb{R})$ to $PW_c^p$, $1 < p < \infty$.

The Prolate Spheroidal Wave Functions (PSWFs) are eigenfunctions of an integral operator and, using the min-max theorem, can be defined by the following extremal problem

$$\psi_{n,c} := \operatorname{argmax}\left\{ \frac{\|f\|_{L^2(-1,1)}}{\|f\|_{L^2(\mathbb{R})}} \,:\, f \in PW_c^2,\ f \in \operatorname{span}\{\psi_{k,c} :\ k < n\}^{\perp} \right\}.$$

The family $(\psi_{n,c})_{n\geq 0}$ forms an orthonormal basis for $PW_c^2$ and satisfies also

$$\int_{-1}^{1} \psi_{n,c}(t)\overline{\psi_{m,c}(t)}dt = \lambda_n \delta_{n,m},$$

which is often referred to as double orthogonality.

The central object that we study in this paper is given by

$$\Psi_N f := \sum_{n=0}^{N} \langle f, \psi_{n,c}\rangle \psi_{n,c}.$$

It was shown in [2] that $\Psi_N f \to f$, $N \to \infty$, in $L^p(\mathbb{R})$-norm for every $f \in PW_c^p$ if and only if $\frac{4}{3} < p < 4$. Our main contribution in this paper is that the series $\Psi_N f$ converges also almost everywhere, and that the range of convergence extends to $1 \leq p < \infty$. Moreover, for general functions $f \in L^p(\mathbb{R})$, $\Psi_N f$ converges almost everywhere to $P_c f$.

**Theorem 1.1.** *If $1 < p < \infty$, and $f \in L^p(\mathbb{R})$, then $\Psi_N f \to P_c f$ almost everywhere. For $p = 1$, we have that $\Psi_N f \to f$ almost everywhere for every $f \in PW_c^1$.*

For $f \in L^1(0,\infty)$, the Hankel transform is given by

$$\mathcal{H}^{\alpha} f(x) := \int_0^{\infty} f(y)\sqrt{xy}J_{\alpha}(xy)\,\mathrm{d}y,$$

where $J_\alpha$ denotes the Bessel function of first kind and order $\alpha > -\frac{1}{2}$. Like the Fourier transform, the Hankel transform extends to a unitary operator on $L^2(0,\infty)$ and in a distributional sense to $L^p(0,\infty)$, $p > 2$. Similar to the Fourier transform, a function cannot be confined to a finite interval in both time and Hankel domain. See [4, 10, 18], for further uncertainty principles for the Hankel transform.

The Paley-Wiener type spaces $B^p_{\alpha,c}$ are defined as

$$B^p_{\alpha,c} := \left\{ f \in L^p(0,\infty) :\ \operatorname{supp}\left(\mathcal{H}^{\alpha}(f)\right) \subseteq [0,c] \right\},$$

and the band-limiting projection operator is given by $P_c^{\alpha} f = \mathcal{H}^{\alpha}(\chi_{[0,c]} \cdot \mathcal{H}^{\alpha} f)$. We will show in Theorem 4.1 that this operator is bounded on $L^p(0,\infty)$, for $1 < p < \infty$. Finally, the Circular (Hankel) Prolate Spheroidal Wave Functions (CPSWFs) were first introduced and studied in [19]. They are defined by

$$\varphi_{n,c}^{\alpha} := \operatorname{argmax}\left\{ \frac{\|f\|_{L^2(0,1)}}{\|f\|_{L^2(0,\infty)}} : f \in B_{\alpha,c}^2,\ f \in \operatorname{span}\{\varphi_{k,c}^{\alpha} :\ k < n\}^{\perp} \right\}.$$

The family $(\varphi_{n,c}^{\alpha})_{n\geq 0}$ forms an orthonormal basis of $B_{\alpha,c}^2$. Note also that when $\alpha = 1/2$, these are usual PSFWs, more precisely, $\varphi_{n,c}^{1/2} = \psi_{2n,c}$. Again we are interested in expansions of $f \in B_{\alpha,c}^p$ with respect to the CPSWFs, *i.e.* in the convergence of

$$\Phi_N f := \sum_{n=0}^{N} \langle f, \varphi_{n,c}^{\alpha}\rangle \varphi_{n,c}^{\alpha}.$$

It is shown in [3, Theorem 5.6] that $\Phi_N f \to f$ in $L^p(0,\infty)$-norm for every $f \in B_{\alpha,c}^p$ if and only if $\frac{3}{4} < p < 4$. Like for the classical Paley-Wiener space, the series converges also almost everywhere and the range of convergence extends to $1 \leq p < \infty$ for functions in $B_{\alpha,c}^p$.

**Theorem 1.2.** *Let $\alpha > -\frac{1}{2}$. If $1 < p < \infty$, then $\Phi_N f \to P_c^{\alpha} f$ almost everywhere for every $f \in L^p(0,\infty)$. Moreover, if $p = 1$, then $\Phi_N f \to f$ almost everywhere for every $f \in B_{\alpha,c}^p$.*

This paper is organized as follows. In Section 2, we collect some properties on spherical Bessel functions and the Muckenhoupt class that we will need for the proof of our main results. We then prove Theorem 1.1 in Section 3, and conclude with illustrating how this proof has to be adapted to show Theorem 1.2 in Section 4.

As usual, we will write $A \lesssim B$ if there is a constant $C$ such that $A \leq CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

## 2. Preliminaries

### *2.1. Properties of spherical Bessel functions*

In this section we define two families of spherical Bessel functions tailored to form orthonormal bases for the spaces $PW_c^2$ and $B_{\alpha,c}^2$ respectively.

For $\alpha > -1/2$, the Bessel functions of the first kind can be defined through the Poisson representation, *see e.g.* [15, 10.9.4],

$$J_{\alpha}(x) = \frac{x^{\alpha}}{\pi^{1/2} 2^{\alpha} \Gamma(\alpha + \frac{1}{2})} \int_0^1 (1-t^2)^{\alpha - \frac{1}{2}} \cos(xt)\, \mathrm{d}t.$$

Recall that $J_\alpha$ satisfies the pointwise bound

$$|J_\alpha(x)| \lesssim \frac{|x|^{\alpha+1/2}}{2^\alpha \Gamma(\alpha+\frac{1}{2})}. \tag{2.1}$$

It is shown in [2] and [3] that for $1 < p < \infty$

$$\left\| x^{-1/2} J_{n+1/2} \right\|_{L^p(\mathbb{R})} \sim \begin{cases} n^{-1+\frac{1}{p}} & \text{when } 1 < p < 4 \\ n^{-\frac{3}{4}} \log n & \text{when } p = 4 \\ n^{-\frac{5}{6}+\frac{1}{3p}} & \text{when } p > 4 \end{cases}. \tag{2.2}$$

Note that the nature of the right hand side shows that $\left\| x^{-1/2} J_{2n+\alpha+1} \right\|_{L^p(0,\infty)}$ satisfies the same bounds though with different constants.

The classical (dilated) spherical Bessel functions are defined by

$$j_{n,c}(x) := \sqrt{\frac{2n+1}{2}} \frac{J_{n+1/2}(cx)}{\sqrt{x}}. \tag{2.3}$$

They satisfy the orthogonality relations

$$\int_{\mathbb{R}} j_{n,c}(x) j_{m,c}(x)\, \mathrm{d}x = \delta_{n,m},$$

and their Fourier transforms are given by

$$\widehat{j_{n,c}}(\xi) = (-1)^n \sqrt{\frac{2n+1}{\pi c}} P_n\left(\frac{\xi}{c}\right) \cdot \chi_{[-c,c]}(\xi),$$

where $P_n$ denotes the Legendre polynomial of degree $n$. For $p > 1$, one has that $j_{n,c} \in L^p(\mathbb{R})$ and consequently $j_{n,c} \in PW_c^p$. Moreover, by (2.2) there exists $\gamma_p < \frac{1}{2}$ such that $\|j_{n,c}\|_{L^p(\mathbb{R})} \lesssim n^{\gamma_p}$, for every $1 < p < \infty$, and (2.1) implies that, for $x$ fixed,

$$|j_{n,c}(x)| \lesssim \frac{\sqrt{2n+1}}{n!} \left(\frac{c|x|}{2}\right)^n \lesssim n^{-2}. \tag{2.4}$$

The second family of spherical Bessel functions given by

$$k_{n,c}^\alpha(x) := \sqrt{2(2n+\alpha+1)} \frac{J_{2n+\alpha+1}(cx)}{\sqrt{x}}, \tag{2.5}$$

obeys the orthogonality relation

$$\int_0^\infty k_{n,c}^\alpha(x) k_{m,c}^\alpha(x)\, \mathrm{d}x = \delta_{n,m}.$$

Their Hankel transforms are given by

$$\mathcal{H}^\alpha(k^\alpha_{n,c})(x) = \sqrt{\frac{2(2n+\alpha+1)}{c}} \left(\frac{x}{c}\right)^{\alpha+\frac{1}{2}} P_n^{(\alpha,0)}\left(1-2\left(\frac{x}{c}\right)^2\right)\chi_{[0,c]}(x), \tag{2.6}$$

where $P_n^{(\alpha,0)}$ denotes the Jacobi polynomials of degree $n$ and parameter $\alpha$, normalized so that $P_n^{(\alpha,0)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$, see for example [20]. As before, for $1 < p < \infty$, the $L^p(0,\infty)$ norms of $k^\alpha_{n,c}$ are bounded like $\left\|k^\alpha_{n,c}\right\|_{L^p(0,\infty)} \lesssim n^{\gamma_p}$ for some $0 \le \gamma_p < 1$. Hence, $k^\alpha_{n,c} \in B^p_{\alpha,c}$. By (2.1) we also get that

$$|k^\alpha_{n,c}(x)| \lesssim \frac{\sqrt{2n+\alpha+1}}{\Gamma(2n+\alpha+1)}\left(\frac{c|x|}{2}\right)^{2n} \lesssim n^{-2}, \tag{2.7}$$

where the last estimate holds for $x$ fixed with a constant that depends on $x$ (which can be chosen uniformly if $x$ is in a compact set).

*2.2. The Hilbert Transform on Weighted $L^p$-spaces*

Let $J \subset \mathbb{R}$ be an interval. The class of Muckenhoupt weights $A^p(J)$, $1 < p < \infty$, consists of all functions $\omega : J \to \mathbb{R}_+$ such that

$$[\omega]_p := \sup_K \left(\frac{1}{|K|}\int_K \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|K|}\int_K \omega(x)^{-\frac{q}{p}}\,\mathrm{d}x\right)^{\frac{p}{q}} < \infty,$$

where the supremum is taken over all finite length intervals $K \subset J$.

The Hilbert transform is defined as

$$Hf(x) = \frac{1}{\pi}\int_J \frac{f(y)}{x-y}\,\mathrm{d}y,$$

where the integral has to be taken in the principal value sense.

Hunt, Muckenhaupt and Wheeden [11] showed that the Hilbert transform extends to a bounded linear operator on $L^p(J,\omega) \to L^p(J,\omega)$ if and only if $\omega$ is an $A^p(J)$ weight, and the sharp dependence of the operator norm on $[\omega]_p$ was established by Petermichl [16]. Let us also recall the well known fact that $x^\beta \in A^p(0,\infty)$ if and only if $-1 < \beta < p-1$.

## 3. Proof of Theorem 1.1

The prolate spheroidal wave functions $(\psi_{n,c})_{n\ge 0}$ can be expressed in terms of the spherical Bessel functions $(j_{n,c})_{n\ge 0}$ as

$$\psi_{n,c} = \sum_{k\ge 0} b^n_k\, j_{k,c}.$$

It follows from [2, Eqs. (8) & (9)] that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the numbers $b_k^n$ satisfy

(i) $|b_0^n| \lesssim n^{-2}$,

(ii) for $k \geq 1$, $|b_k^n| \lesssim n^{-|k-n|}$.

Note that indexes of the base change coefficients in [2] are shifted, and that these estimates can also be obtained from [3]. It was proven in [3, Lemma 2.6] that if the above conditions are satisfied, then $\|\psi_{n,c}\|_{L^p(\mathbb{R})} \lesssim n^{\gamma_p}$.

**Lemma 3.1.** *If $1 < p < \infty$, then $\Psi_N f$ converges absolutely in every point (and uniformly on compact sets) to some function $h$ for every $f \in L^p(\mathbb{R})$.*

Proof. Let us first rewrite $\psi_{n,c}$ as

$$\begin{aligned}\psi_{n,c} &= \sum_{k\geq 0} b_k^n j_{k,c} \\ &= b_0^n j_{0,c} + b_n^n j_{n,c} + b_{n-1}^n j_{n-1,c} + b_{n+1}^n j_{n+1,c} + \sum_{k=1}^{n-2} b_k^n j_{k,c} + \sum_{k\geq n+2} b_k^n j_{k,c}.\end{aligned}$$

Subsequently, we estimate each of the summands separately. First, by (i) we have $|b_0^n j_{0,c}(x)| \lesssim n^{-2}$, and $|b_n^n j_{n,c}(x)| \leq |j_{n,c}(x)| \lesssim n^{-2}$ by (2.4). If $n \geq 2$, the last two terms may be bounded using (ii) and (2.4) as

$$\sum_{k\geq n+2} |b_k^n j_{k,c}(x)| \lesssim \sum_{k\geq n+2} n^{-|k-n|} k^{-2} \leq n^{-2} \sum_{k\geq 0} n^{-(k+2)} \lesssim n^{-2},$$

and

$$\sum_{k=1}^{n-2} |b_k^n j_{k,c}(x)| \lesssim \sum_{k=1}^{n-2} n^{-|k-n|} k^{-2} \leq \sum_{k=2}^{n-1} n^{-k} = n^{-2} \sum_{k=0}^{n-3} n^{-k} \lesssim n^{-2}.$$

The third and fourth term may also easily be bounded using (ii):

$$|b_{n-1}^n j_{n-1,c}(x)| \lesssim n^{-1}(n-1)^{-2} \lesssim n^{-2},$$

and

$$|b_{n+1}^n j_{n+1,c}(x)| \lesssim n^{-1}(n+1)^{-2} \lesssim n^{-2}.$$

Therefore, $|\psi_{n,c}(x)| \lesssim n^{-2}$ which implies by Hölder's inequality that

$$\begin{aligned}\sum_{n=0}^{N} |\langle f, \psi_{n,c}\rangle \psi_{n,c}(x)| &\leq \sum_{n=0}^{N} \|f\|_{L^p(\mathbb{R})} \|\psi_{n,c}\|_{L^q(\mathbb{R})} |\psi_n(x)| \\ &\lesssim \|f\|_{L^p(\mathbb{R})} \sum_{n=0}^{N} n^{\gamma_q - 2} < \infty.\end{aligned}$$

$\square$

The following lemma is well-known to the majority of our readers. We will nevertheless include a proof here.

**Lemma 3.2.** *If $1 \leq p \leq q < \infty$, then $PW_c^p \hookrightarrow PW_c^q$ where the inclusion map is continuous and dense. Moreover, the projection $P_c$ maps $L^p(\mathbb{R})$ boundedly into $PW_c^p$ if $1 < p < \infty$.*

PROOF. Let $\vartheta_1, \vartheta_2$ be two functions from the Schwartz space such that $\operatorname{supp}(\widehat{\vartheta}_1) \subseteq [-c, c]$, $\widehat{\vartheta}_2(\xi) = 1$, for every $\xi \in [-c, c]$, and $\|\vartheta_1 - \vartheta_2\|_{L^1(\mathbb{R})} \leq \varepsilon$. If $f \in PW_c^p$, then $f * \vartheta_2 = f$ and, by Young's convolution inequality, one has

$$\|f\|_{L^q(\mathbb{R})} = \|f * \vartheta_2\|_{L^q(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|\vartheta_2\|_{L^{\frac{pq}{pq+p-q}}(\mathbb{R})}.$$

This shows that $PW_c^p$ is continuously embedded into $PW_c^q$.

Let us now show the density. For every $\varepsilon > 0$, and $f \in PW_c^q$ we may choose $f_\varepsilon \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, such that $\|f - f_\varepsilon\|_{L^q(\mathbb{R})} = \varepsilon$. Again by Young's convolution inequality we thus obtain

$$\begin{aligned} \|f_\varepsilon * \vartheta_1 - f\|_{L^q(\mathbb{R})} &\leq \|f_\varepsilon * \vartheta_1 - f * \vartheta_1\|_{L^q(\mathbb{R})} + \|f * \vartheta_1 - f * \vartheta_2\|_{L^q(\mathbb{R})} \\ &\leq \|f_\varepsilon - f\|_{L^q(\mathbb{R})} \|\vartheta_1\|_{L^1(\mathbb{R})} + \|f\|_{L^q(\mathbb{R})} \|\vartheta_1 - \vartheta_2\|_{L^1(\mathbb{R})} \lesssim \varepsilon. \end{aligned}$$

Finally, $f_\varepsilon * \vartheta_1 \in PW_c^p$ as $\operatorname{supp}\big(\mathcal{F}(f_\varepsilon * \vartheta_1)\big) \subseteq \operatorname{supp}(\widehat{\vartheta}_1)$, and $\|f_\varepsilon * \vartheta_1\|_{L^p(\mathbb{R})} \leq \|f_\varepsilon\|_{L^p(\mathbb{R})} \|\vartheta_1\|_{L^1(\mathbb{R})}$.

Concerning the continuity of $P_c$, we observe that for almost every point $x \in \mathbb{R}$, $P_c f(x)$ may be written as the sum of Hilbert transforms

$$\begin{aligned} P_c f(x) = f * 2c \operatorname{sinc}(2c\,\cdot)(x) &= \int_{\mathbb{R}} f(t) \frac{\sin(2\pi c(x-t))}{\pi(x-t)} dt \\ &= \sin(2\pi c x)\, \text{p. v.} \int_{\mathbb{R}} f(t) \frac{\cos(2\pi c t)}{\pi(x-t)} dt \\ &\quad - \cos(2\pi c x)\, \text{p. v.} \int_{\mathbb{R}} f(t) \frac{\sin(2\pi c t)}{\pi(x-t)} dt \\ &= \sin(2\pi c x) H\big(f \cdot \cos(2\pi c\,\cdot)\big)(x) - \cos(2\pi c x) H\big(f \cdot \sin(2\pi c\,\cdot)\big)(x). \end{aligned}$$

By the continuity of the Hilbert transform on $L^p(\mathbb{R})$ in the range $1 < p < \infty$, it thus follows that

$$\|P_c f\|_{L^p(\mathbb{R})} \leq \|H\big(f \cdot \cos(2\pi c\,\cdot)\big)\|_{L^p(\mathbb{R})} + \|H\big(f \cdot \sin(2\pi c\,\cdot)\big)\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

$\square$

**Proposition 3.3.** *If $\frac{4}{3} < p < 4$, then $\Psi_N f \to P_c f$ almost everywhere for every $f \in L^p(\mathbb{R})$.*

Proof. First, as $P_c$ is continuous for $\frac{4}{3} < p < 4$, it follows that $\langle f, j_{n,c}\rangle = \langle f, P_c j_{n,c}\rangle = \langle P_c f, j_{n,c}\rangle$ and therefore $\Psi_N f = \Psi_N P_c f$. Since $\Psi_N P_c f \to P_c f$ in $L^p(\mathbb{R})$ by [2, Theorem 1], there exists a subsequence $(\Psi_{N_k} P_c f)_{k\geq 0}$ that converges almost everywhere to $P_c f$. From Lemma 3.1 we know that $\Psi_N f$ converges pointwise for every $x \in \mathbb{R}$ to a function $h$. Consequently, $P_c f = h$ almost everywhere. $\square$

We have now gathered all ingredients to prove our main theorem.

Proof of Theorem 1.1. If $1 < p \leq 2$, then $P_c f \in PW_c^2$ for every $f \in L^p(\mathbb{R})$ by Lemma 3.2. In this case, the result follows from Proposition 3.3 and the identity $\Psi_N f = \Psi_N P_c f$. Similarly, the statement for $p = 1$ holds as $PW_c^1 \subset PW_c^2$. If $2 < p < \infty$, and $f \in L^p(\mathbb{R})$, then there exists a sequence $(f_k)_{k\geq 0} \subset PW_c^2$ by Lemma 3.2 such that $f_k \to P_c f \in PW_c^p$ and $\|f_k\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}$. Without loss of generality, we may assume that $f_k \to P_c f$ almost everywhere and define $X_f := \{x \in \mathbb{R} : \lim_{k\to\infty} f_k(x) = P_c f(x)\}$, as well as

$$X_k := \Big\{x \in \mathbb{R} : \lim_{N\to\infty} \Psi_N f_k(x) = f_k(x)\Big\}.$$

From Proposition 3.3 we know that each $\mathbb{R}\backslash X_k$ has Lebesgue measure zero and, consequently, so has $\mathbb{R}\backslash\big(\bigcap_{k\geq 0} X_k \cap X_f\big) = \bigcup_{k\geq 0}(\mathbb{R}\backslash X_k) \cup (\mathbb{R}\backslash X_f)$.

Using (2.4) and the estimate on the $L^p$-norms of $j_{n,c}$ we derive

$$\sum_{n\geq 0} |\langle f_k, \psi_n\rangle \psi_n(x)| \lesssim \|f\|_{L^p(\mathbb{R})} \sum_{n\geq 0} n^{\gamma_q - 2} < \infty.$$

Now, if $x \in \bigcap_{k\geq 0} X_k \cap X_f$, we may conclude by the dominated convergence theorem that

$$\begin{aligned} P_c f(x) &= \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \lim_{N\to\infty} \Psi_N f_k(x) = \lim_{N\to\infty} \lim_{k\to\infty} \Psi_N f_k(x) \\ &= \lim_{N\to\infty} \Psi_N P_c f(x) = \lim_{N\to\infty} \Psi_N f(x). \end{aligned}$$

$\square$

## 4. Adaptations for Circular PSWF

By a combination of several lemmas in [3] (in particular Lemmas 2.5, 5.3 & 5.4) one can deduce that the change of base coefficients $b_m^n$ in

$$\varphi_{n,c}^{\alpha} = \sum_{m\geq 0} b_m^n \, k_{m,c}^{\alpha}$$

satisfy the conditions (i) and (ii). Hence, $\|\varphi^{\alpha}_{n,c}\|_{L^p(0,\infty)} \lesssim n^{\gamma_p}$, for some $0 \le \gamma_p < 1$. The proof of Lemma 3.1 can therefore be transferred one-to-one to show that $\Phi_N f$ converges pointwise to a function $h$ for every $f \in L^p(0,\infty)$, $1 < p < \infty$.

The analogue of Lemma 3.2 needs some preparation. In particular, we need to show the continuity of the projection operator $P^{\alpha}_c$. In [24] the author shows that for the following convention of the Hankel transform

$$\mathcal{F}^{\alpha} f(x) := \frac{x^{-\alpha/2}}{2} \int_0^{\infty} f(t) J_{\alpha}\big(\sqrt{xt}\big) t^{\alpha/2} dt,$$

the projection $\mathcal{F}^{\alpha}(\chi_{[0,c]} \cdot \mathcal{F}^{\alpha})(x)$ acts as a continuous operator on the weighted Lebesgue spaces $L^p((0,\infty), x^{\alpha})$ whenever $4\frac{\alpha+1}{2\alpha+3} < p < 4\frac{\alpha+1}{2\alpha+1}$. Although different definitions of the Hankel transform can be related to each other and their $L^2$-theory coincides, this is no longer true for general $L^p$-spaces, $p \neq 2$.

For instance, when considering the $L^p\big((0,\infty), x^{\alpha}\big)$-convergence of spherical Bessel expansions, Varona [24] showed that these series converge in $L^p$ if and only if $\max\left\{\frac{4}{3}, 4\frac{\alpha+1}{2\alpha+3}\right\} < p < \min\left\{4, 4\frac{\alpha+1}{2\alpha+1}\right\}$. The restriction with respect to $\alpha$ however disappears in [3] due to a different convention for the Hankel transform and $L^p(0,\infty)$-convergence holds if and only if $\frac{4}{3} < p < 4$.

As a second example, we show in the following that for the convention chosen in this paper, the projection $P^{\alpha}_c$ is continuous for $1 < p < \infty$. We will follow the arguments of the proof of [24, Theorem 2].

**Theorem 4.1.** *If $\alpha > -\frac{1}{2}$, and $1 < p \le q < \infty$, then*

$$\|P^{\alpha}_c f\|_{L^q(0,\infty)} \lesssim \|f\|_{L^p(0,\infty)}, \qquad f \in L^p(0,\infty).$$

Proof. Let us first write down $P^{\alpha}_c$ explicitly

$$\begin{aligned} P^{\alpha}_c f(x) &= \mathcal{H}^{\alpha}\big(\chi_{[0,c]} \cdot \mathcal{H}^{\alpha} f\big)(x) = \int_0^c \mathcal{H}^{\alpha} f(t) \sqrt{tx} J_{\alpha}(tx)\, \mathrm{d}t \\ &= \int_0^c \int_0^{\infty} f(y) \sqrt{ty} J_{\alpha}(ty) dy \sqrt{tx} J_{\alpha}(tx)\, \mathrm{d}t \\ &= \int_0^{\infty} f(y) \sqrt{xy} \int_0^c J_{\alpha}(tx) J_{\alpha}(ty) t \, \mathrm{d}t \, \mathrm{d}y \\ &= \int_0^{\infty} f(y) \sqrt{xy} c \frac{y J_{\alpha-1}(cy) J_{\alpha}(cx) - x J_{\alpha-1}(cx) J_{\alpha}(cy)}{x^2 - y^2}\, \mathrm{d}y, \end{aligned}$$

where we have used the explicit expression for Lommel's integrals [5, p. 101] to derive the last equality. By a change of variables one then obtains

$$\begin{aligned}
&P_c^\alpha f(\sqrt{x})\\
&= \int_0^\infty f(\sqrt{y})(xy)^{1/4} c \frac{\sqrt{y} J_{\alpha-1}(c\sqrt{y}) J_\alpha(c\sqrt{x}) - \sqrt{x} J_{\alpha-1}(c\sqrt{x}) J_\alpha(c\sqrt{y})}{x-y} \frac{\mathrm{d}y}{2y^{1/2}}\\
&= W_1 f(x) - W_2 f(x),
\end{aligned}$$

where

$$W_1 f(x) := \frac{\pi c x^{1/4} J_\alpha(c\sqrt{x})}{2} H\Big(f(\sqrt{y}) y^{1/4} J_{\alpha-1}(c\sqrt{y})\Big)(x),$$

and

$$W_2 f(x) := \frac{\pi c x^{3/4} J_{\alpha-1}(c\sqrt{x})}{2} H\Big(f(\sqrt{y}) y^{-1/4} J_\alpha(c\sqrt{y})\Big)(x).$$

Note that both $x^{-1/2}$, and $x^{p/2-1/2}$ are Muckenhoupt $A^p(0,\infty)$ weights by the remark in Section 2.2. Hence, the Hilbert transform is continuous on the respective weighted Lebesgue spaces for every $1 < p < \infty$. As $x^{1/4} J_\alpha(c\sqrt{x}) \lesssim 1$, we thus have

$$\begin{aligned}
\int_0^\infty |W_1 f(x)|^p x^{-1/2}\,\mathrm{d}x &\lesssim \int_0^\infty \left| H\Big(f(\sqrt{y}) y^{1/4} J_{\alpha-1}(c\sqrt{y})\Big)(x)\right|^p x^{-1/2}\,\mathrm{d}x\\
&\lesssim \int_0^\infty \left| f(\sqrt{y}) y^{1/4} J_{\alpha-1}(c\sqrt{y})\right|^p y^{-1/2}\,\mathrm{d}y\\
&\lesssim \int_0^\infty |f(\sqrt{y})|^p\, y^{-1/2}\,\mathrm{d}y = 2\int_0^\infty |f(y)|^p\,\mathrm{d}y,
\end{aligned}$$

as well as

$$\begin{aligned}
\int_0^\infty |W_2 f(x)|^p x^{-1/2}\,\mathrm{d}x &\lesssim \int_0^\infty \left| H\Big(f(\sqrt{y}) y^{-1/4} J_\alpha(c\sqrt{y})\Big)(x)\right|^p x^{p/2-1/2}\,\mathrm{d}x\\
&\lesssim \int_0^\infty \left| f(\sqrt{y}) y^{-1/4} J_\alpha(c\sqrt{y})\right|^p y^{p/2-1/2}\,\mathrm{d}y\\
&\lesssim \int_0^\infty |f(\sqrt{y})|^p\, y^{-1/2}\,\mathrm{d}y = 2\int_0^\infty |f(y)|^p\,\mathrm{d}y.
\end{aligned}$$

This proves the case $p = q$ using $\|P_c^\alpha f\|_{L^p(0,\infty)} \leq \|W_1 f(x^2)\|_{L^p(0,\infty)} + \|W_2 f(x^2)\|_{L^p(0,\infty)}$ and a change of variables.

Now, notice that for every $1 \leq p \leq \infty$ one has

$$\|P_c^\alpha f\|_{L^\infty(0,\infty)} \lesssim \|\chi_{[0,c]} \mathcal{H}^\alpha f\|_{L^1(0,\infty)} \lesssim \|\mathcal{H}^\alpha f\|_{L^p(0,\infty)}.$$

Setting $p = 2$ gives $\|P_c^\alpha f\|_{L^\infty(0,\infty)} \lesssim \|f\|_{L^2(0,\infty)}$ by Plancherel's theorem, and $p = \infty$ yields $\|P_c^\alpha f\|_{L^\infty(0,\infty)} \lesssim \|f\|_{L^1(0,\infty)}$. It thus follows by the Riesz-Thorin theorem that $\|P_c^\alpha f\|_{L^\infty(0,\infty)} \lesssim \|f\|_{L^p(0,\infty)}$, for every $1 \le p \le 2$. Moreover, as $\|P_c^\alpha f\|_{L^p(0,\infty)} \lesssim \|f\|_{L^p(0,\infty)}$, $1 < p < \infty$, it follows again by the Riesz-Thorin theorem that

$$\|P_c^\alpha f\|_{L^q(0,\infty)} \lesssim \|f\|_{L^p(0,\infty)}, \quad 1 < p \le 2, \quad p \le q < \infty. \tag{4.8}$$

Since the adjoint of $P_c^\alpha : L^p(0,\infty) \to L^q(0,\infty)$ is given by $P_c^\alpha : L^{q'}(0,\infty) \to L^{p'}(0,\infty)$, we conclude that also

$$\|P_c^\alpha f\|_{L^{p'}(0,\infty)} \lesssim \|f\|_{L^{q'}(0,\infty)}, \quad 1 < q' \le p', \quad 2 \le p' < \infty. \tag{4.9}$$

Combining (4.8) and (4.9) thus yields the continuity of $P_c^\alpha : L^p(0,\infty) \to L^q(0,\infty)$ for the whole scale $1 < p \le q < \infty$. □

**Corollary 4.2.** *If $1 < p \le q < \infty$, then $B^p_{\alpha,c} \hookrightarrow B^q_{\alpha,c}$ with the inclusion map being continuous and dense. Moreover, if $1 \le p \le \infty$, then $B^1_{\alpha,c} \subset B^p_{\alpha,c}$.*

Proof. If $f \in B^p_{\alpha,c}$ one has $P_c^\alpha f = f$ and consequently $\|f\|_{L^q(0,\infty)} \lesssim \|f\|_{L^p(0,\infty)}$ by Theorem 4.1, which shows that $B^p_{\alpha,c}$ is continuously embedded into $B^q_{\alpha,c}$. Furthermore, if $f \in B^1_{\alpha,c}$, then $f \in B^p_{\alpha,c}$ as $f \in L^1(0,\infty) \cap L^\infty(0,\infty) \subset L^p(0,\infty)$.

It remains to show that the inclusion is dense. This however follows from the density properties of $L^p(0,\infty)$ and Theorem 4.1: Take $f \in B^q_{\alpha,c}$ and a sequence $(f_k)_{k\ge 0} \subset L^p(0,\infty) \cap L^q(0,\infty)$ such that $f_k \to f$ in $L^q(0,\infty)$. By the continuity of the projection operator, it follows that $(P_c^\alpha f_k)_{k \ge 0} \subset B^p_{\alpha,c} \cap B^q_{\alpha,c}$ and

$$\|f - P_c^\alpha f_k\|_{L^q(0,\infty)} = \|P_c^\alpha (f - f_k)\|_{L^q(0,\infty)} \lesssim \|f - f_k\|_{L^q(0,\infty)} \to 0.$$

□

Thus, as in Proposition 3.3, it follows from [3, Theorem 5.6] that $\Phi_N f$ converges almost everywhere to $P_c^\alpha f$ for every $f \in L^p(0,\infty)$ if $\frac{3}{4} < p < 4$.

This leaves us with all ingredients in place to prove Theorem 1.2 which is shown along the same line of argument as in the proof of Theorem 1.1.

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