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# Teaching mathematics in an international class: Designing a path towards productive disposition

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*How can we design a teaching environment for an international class of refugees seizing its natural variety to encourage a path towards productive disposition? The article discusses and evaluates a potential answer by sketching a multiply implemented teaching approach to algebraic expressions and their manipulation via figurate numbers. The intended goal of that approach is twofold: On the one hand challenging pupils' beliefs of mathematics and its teaching, and on the other hand providing a suitable substantial learning environment for the introduction to elementary algebra. As multicultural classrooms, where the language of education differs from the learners' mother tongue, ask for special, language-sensitive teaching approaches, this approach considers the linguistic accessibility and tends to serve as a catalyst for further linguistic and mathematical development.*

*Keywords: Elementary algebra, refugees, genetic principle, figurate numbers, heterogeneity.*

## Introduction

In Germany, IVKs (German abbreviation for International Preparation Class) have been established since 2015 to structure the education of foreign pupils who migrated to Germany. Most pupils are refugees from conflict and war zones all over the world, with medium to non-existent German language skills. In general, pupils can attend these classes for up to two years (depending on the state), and the goals of these classes can be very diverse: different school degrees, transition into the regular educational system or qualifying for the start of an apprenticeship. As in every class there is a natural diversity with respect to the pupils' mathematical giftedness and abilities. However, this natural heterogeneity is intensified by a larger age spectrum (of at least 8 years) and previous experiences in the pupils' various homelands and the conditions of their flight to Germany.

## Modelling strands of heterogeneity

In the tradition of Wittmann's view of mathematics education as a 'design science' (Wittmann, 1995), I identified, in cooperation with various experienced teachers, different strands of heterogeneity in a design research project with multiple implementation cycles (Sauerwein, 2020). A few of these (provisional) strands serve as a basic framework for the reasoning in the article to emphasise the crucial differences between teaching the IVK and regular classes:

- *Communicative strand:* There is no common daily language - neither for informal communication nor for discussions in the classroom. Learning German as a foreign language thus is not only restricted to the classrooms: It encompasses the complete daily routine. The pupils' knowledge of the German language is in-between the levels A1 and B2 (with respect to the CEFRL); some do not even cover A1.
- *Emotional strand:* The linguistic problems are not separated from daily life. This can contribute to an overwhelming frustration of feeling inadequate in nearly every respect of

everyday life. Even prior to our project, we observed many pupils approaching mathematics with more courage than other subjects, where they remained passive or even missed lessons.

- *Mathematical strand:* The prior knowledge of mathematics is very individual and reaches from a rudimentary number sense in the number range 1–10 up to twelve years of mathematics education. Moreover, the primary focus in the pupils' previous mathematical education seems to lie on the symbolic languages of arithmetic and algebra.
- *Cultural strand:* Regular school attendance is not part of every culture. In some countries, the pupils are expected to contribute to their families' income. Nevertheless, all pupils had a positive attitude towards school and were grateful for every help. However, for some pupils, a structured week with a time schedule and the learning commitments were new and resulted in an overextension. Rapid changes between gratitude and frustration were observable.
- *Institutional strand:* The school system in Germany probably differs from the pupils' experiences in their respective homelands. According to what some pupils tell me, teaching here is less authoritarian and contains more independent working phases; learning by heart is less dominating. This implies that lessons for these pupils are more demanding as they ask for creativity, self-initiative or mathematical discourse.

### **Problematic consequences**

Although these strands are not explicitly addressed in class, they influence teaching in many subtle ways. For example, different beliefs of mathematics and its teaching seemed to appear in the IVK. These expressed themselves in the following observed misconceptions in practice:

- In mathematics, there is only one correct solution and it's either the teacher's or the calculator's job to decide.
- Mathematics is a finished product and consists mainly of mimicking the teacher's solutions in a formal way with no deeper insights or discussions (in class).

This experience cannot certainly be generalised towards a statement about non-European school systems. These attitudes towards school in general and views on mathematics can prevent the pupils from building sustainable mathematical concepts. On top of that, they can cause challenges with respect to the integration into regular classes.

### **Research question**

The description of different strands of heterogeneity does not imply that these could be addressed separately, as posed in Gorgorió and Planas (2001, p. 28): "It is very difficult to disentangle the social and cultural conflicts of a multiethnic mathematics classroom from the language issues."

As argued before, the above-mentioned intertwined strands influence the teaching. For simplicity we assume that they characterise the heterogeneity in the IVK. The context of an IVK is fundamentally different to bilingual classrooms (e.g., in Malta, Farrugia, 2017; in Catalonia-Spain, Planas & Setati, 2009; in Germany Schüler-Meyer, 2017). In general, one cannot expect to have a common language. The broad research question from the beginning can be rephrased to: Can figurate numbers help to address the special situation within an IVK in a *holistic way* to stimulate a meaningful learning of elementary algebra? In the next section, I address the topic of figurate numbers by building partly on data and findings of my doctoral dissertation (Sauerwein, 2020).

## Figurate numbers

“If human actors invite the object to be a part of the discourse, objects can challenge the students to find a new language [...]. This aspect is of special interest from a didactical point of view. Objects cannot only relieve students on the language level, but also challenge them. They can be ‘supporters’ in the development of mathematical thinking and of a precise mathematical language.” (Fetzer & Tiedemann, 2015, p. 1392)

Inspired by typical generalising approaches to algebra (e.g. Mason, 1996) we define a *figurate number* as a sequence of dot patterns including the corresponding integer sequence:



Figure 1: "3, 6, 9, ..."

As the definition is based on sequences and thus incorporating the (hidden) rule of construction, Figurate numbers demand naturally for a pursuit to find this rule. In keeping with Bruner's (1966) three modes of representation of knowledge, enactive – iconic – symbolic, figurate numbers possess iconic and symbolic components. The iconic nature allows the pupils of the IVK to find linguistically more accessible common ground for mathematical discussions and development.

Such workshop languages [or intermediate languages] which are characterised by a high vividness, situatedness and grammatical looseness, which are often supported by gestures and which may hardly be comprehensible to non-participants, will emerge inevitably from themselves and their communicative importance must be highly appreciated. (Winter, 1978, p. 13, my translation)

It should be noted that, although the language is more accessible, it does not result in a cutback with respect to the richness of its mathematical content. Similarly, Moschkovich (1999) implicitly criticises some teaching practices which limit the pupils' opportunities by the early introduction of compulsory mathematical vocabulary:

“[...] [L]anguage learners can and do participate in discussions where they grapple with important mathematical content. [...] But it is only possible to uncover the mathematical content in what students say if students have the opportunity to participate in a discussion and if this discussion is a mathematical one.” (Moschkovich, 1999, pp. 18-19)

## Practical perspective

Each of the four implementations of the learning environment *Figurate Numbers* started with the dot pattern in Figure 1 which was easily continued by all pupils in an iconic way. In almost all cases – with one interesting exception mentioned below – the resulting dot pattern was a 3 · 4-rectangle. Since not all pupils knew the word for rectangle, they used gestures to distinguish between columns and rows while saying a number, meaning the number of dots in that particular part of the dot pattern. In this way, they were eventually able to describe the pattern and its underlying rule of construction:

- The next dot pattern consists of three horizontal rows with four dots each.

- The next dot pattern consists of four of these columns.
- One always adds the first dot pattern.
- The right dot pattern consists of the preceding two.

The first three descriptions imply that the next dot pattern consists of 12 dots. From there on, many pupils continued the number sequences in a symbolic mode, thus identifying it with the multiples of 3. Although this change of mode of representation seems to be economic, one loses some structure: The expressions  $3 \cdot 4$ ,  $3 + 3 + 3 + 3$  and  $9 + 3$  all yield 12, but each of them also encodes a certain perspective onto the iconic representation, namely how it is structured and consequently counted.

What is the meaning of the fourth description? The pupil's idea incorporates the triple Fibonacci-Numbers. Cutting the dot pattern horizontally it is easily be seen that the pupil's construction is the law of construction of the Fibonacci-Numbers 1, 1, 2, 3, 5, .... Pupils were confronted with the fact, that different views on the given data permitted different and equally legitimate solutions; while the multiples of three later offered an obvious description as a closed term the pupils discovered the essential truth that mathematical observation, if taken seriously, might lead to open ends which cannot be immediately tied up on the basis of available knowledge. Therefore, it is fair to say that this insight was achievable due to the different perspective on the innocent looking figurate number.

Consider now the *filling glass*:

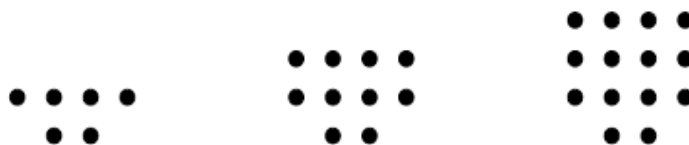


Figure 2: "6, 10, 14, ..."

The filling glass with its richer iconic structure allows more different approaches to counting. Pupils introduced the notion of "negative points" (the missing points in the lowest row) to arrive at "recipes" like  $4 \cdot 4 - 2$ , they observed symmetry (arriving at  $2 \cdot (2 \cdot 3 + 1)$ ). To communicate these ideas the pupils used gestures and inscriptions into the dot patterns, thus enriching them – as diagrams - with a basis for socially accepted reasoning (in the sense of Dörfler, 2006). All these views are potential starting points for generalisations. When changing the iconic pattern is allowed too (for instance changing the position of the two points to a position below the first two columns), more terms can be found, and the observation of equivalence can be generalised to consider terms as objects for further studies and in this context understanding equality among terms on a level beyond syntactical limits.

When in a later state, pupils manipulate terms according to algebraic rules they can, if necessary, find in adapted iconic representations and the study of their geometric transformability into each other a reliable basis for their mathematical discourse.

### Meta perspective

In the beginning of each implementation of our project we were met with the pupils' objection that these dot patterns were nothing but a puzzle for children and no mathematics at all. This pupils' prejudice, which can well be understood from the mathematical and institutional strands explained above, in the course of our project changed into a qualified appreciation of the subject.

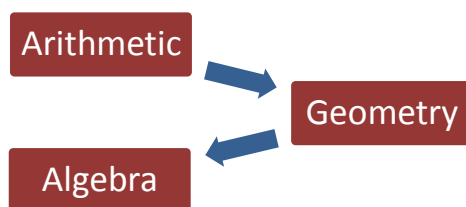
Presenting patterns of small dots (i.e. the iconic representation of the first few elements of a figurate number) proved to offer several possibilities and – referring to our intentions – advantages, which are closely related to their “simplicity”:

- Each pupil can count the dots; hence the figurate numbers create a productive entry for different levels. Since in our project every pupil had a certain fluency in counting activities and basic arithmetic (even if limited to a certain range of numbers) the low-threshold accessibility of our project could be taken for granted.
- No tools such as calculators or other digital tools are needed. Only a piece of paper, a pencil and a rubber are needed to gain access to the dot patterns. Taking up on Mason’s (1996) distinction of *looking at* vs. *looking through*, the plainness combined with a thorough investigation offers the chance to look through the particular pattern and grasp the general rule. Thus, the picture becomes literally a “diagram” (Dörfler, 2006) and a tool for new insights.
- All but one pattern which we used in our design project corresponded to linear counting expressions. From a mathematical viewpoint these expressions are rather elementary. As this form of representation is new to the pupils, they have to accept and discover the iconic representations as worthwhile objects of further studies and later the possibility to switch between different representations, for example between the dot pattern and the number (maybe written as sum or product) as a tool to prove the validity of an algebraic argument.

This outlined simplicity yields a situation where pupils can express their own ideas even though their skills of the German language would not allow them to do so. The stimulated discussions between the pupils were making use of a certain *intermediate language* (Winter, 1978): The iconic representation becomes the basis for discussion enriched with gestures and rudimental symbolic notations. Winter emphasises the necessity of this intermediate language as a foundation for the symbolic language. Thus, he argues, one should not rush through this stage.

### Epistemological perspective

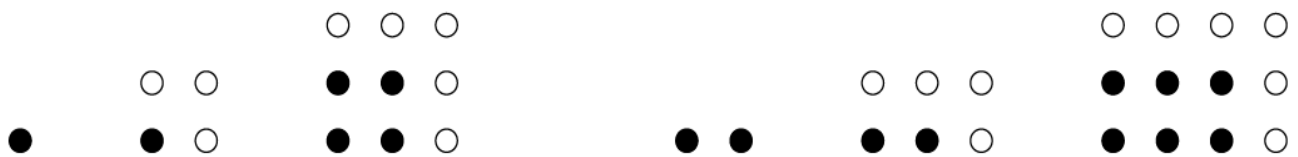
An historical and epistemological analysis of Kvasz (2008) suggests that figurate numbers were a crucial step for the development of the symbolic language of algebra. Kvasz points out that there was no direct development from arithmetic to algebra as Frege proposed (1891). Instead, there were two instances of a certain *pattern of change* which he calls *re-coding*:



**Figure 3: The *Re-coding* Arithmetic – Geometry – Algebra**

Considering arithmetic and algebra as symbolic languages and geometry as an iconic language, *re-coding* can simplistically be seen as change of the mode of representation. The first *re-coding* is strongly related to the Pythagoreans who invented figurate numbers by using  $\psi\eta\varphi\omicron\iota$  (calculi). A

first detailed account on figurate numbers can be found in Nikomachos' *Introduction to Arithmetic* (D'Ooge, 1926) where he introduces polygonal and polyhedral numbers apart from the two fundamental sequences of quadratic and oblong numbers (Figure 4). By the inscribed gnomons it is easily possible to continue each sequence. The second *re-coding* is the symbolisation of the notion of a variable which was only implicitly used as a line segment of indefinite length, thus freeing the variable from three-dimensional space. The analysis of Kvasz and his view on the language of mathematics with its different stratifications where each layer has its own *potentials* (or *cognitive aspects*), yield the necessity to widen the definition of algebra as calculating with letters – it is more than just arithmetic with letters. The introduction of the symbol notation is only the first step to a new symbolic language (Kvasz, 2012). It also involves new ways of operating with the newly introduced symbols. Therefore, one can conclude that typical mathematical activities such as generalising, conjecturing, proving, etc. are part of the use of the language of algebra.



**Figure 4: The quadratic and the oblong numbers**

### **Communicative and linguistic connotations of figurate numbers**

The notion of language occurs in different contexts within this report. These contexts are strongly related to different aspects of figurate numbers. On the one hand, there is language as a medium of communication. According to the communicative strand this function influences all teaching in the IVK. In this sense of language, figurate numbers serve as a tool for discussion of pupils without a common language. On the other hand, they act as a vehicle in a process of conceptualisation, that is, the conceptualisation of the iconic manipulations which lead to language of elementary integer algebra in the sense of Kvasz (2008).

In contrast to the perspective of Kvasz which describes the development of mathematics, the individual process of learning algebra is not linear and straightforward. From a didactical point of view, the pattern of re-coding should be used in both ways. Consequently, figurate numbers become a mediating tool between the languages. In particular, the interplay between the iconic and symbolic components of a figurate number turned out to be fruitful. Figurate numbers opened a context where pupils could conjecture, generalise, prove, discuss, etc.

### **Résumé**

The aforementioned examples from practice showed that figurate numbers can address the special situation in an IVK by seizing the mathematical and communicative strands: The appearance of different counting strategies led to different counting expressions for the quantity of dots and ultimately to the quest to prove equivalence of different solutions on a legitimate and more concise basis than merely counting dots in examples. The iconic form of the figurate numbers served as a sustainable and workable representation compensating the lack of a common language in the class.

Furthermore, the other strands of heterogeneity were confronted: The pupils in the IVK were not familiar with the possibility that there could exist various correct solutions and that mathematics could consist of more than numbers and symbols. Thus, the opportunity arose to tackle the

mentioned misconceptions. Moreover, the correctness of a solution was not judged by the teacher; it followed from a consistent argument based on different perspectives of the structure of figurate numbers. It could be observed that the sketched approach to figurate numbers was not only addressing the strands of heterogeneity and some of their consequences but also contributing to the ambitious goal of the pupils' empowerment.

The strands of heterogeneity cannot be treated separately as they are strongly intertwined. Nevertheless, the initially observed positive attitude within the IVK towards mathematics seemed to be an appropriate starting point: This inclination of the pupils encouraged me to develop it into a "productive disposition" (NRC, 2001, p. 116).

The figurate numbers here act as a generic example for the context of elementary algebra. It could be worthwhile looking for other subject matters with similar potential with respect to mathematical and linguistic development in this social and cultural context of an international class for refugees.

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