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Thierry Leviandier

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Cliff edge stability in a 4-dimension map

Thierry Leviandier

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33 Avenue de la République 94260 Fresnes, France. thierry@leviandier.fr

Abstract

A previous paper delivered a discrete-time dynamic system of dimension 4 and discussed the stability of its 4-cycles. The present paper is focused on a subset of 2-cycles, indefinite on a whole variety of the space unless a small perturbation is added, nevertheless attractive and generally stable in a sense adapted to its specificity. 3 out of 4 eigenvalues of the Jacobian matrix are found equal to ± 1 and eigenvectors are used to interpret simulations of a large number of iterations.

résumé

Un précédent papier a introduit un système dynamique de dimension 4 à temps discret et discuté de la stabilité de ses 4-cycles. On se concentre ici sur un 2-cycle qui présente la particularité d'être indéfini sur toute une variété de l'espace d'origine, si on ne lui injecte pas une perturbation, tout en étant attractif et généralement stable, sous condition d'adaptation de la définition de ces propriétés. On calcule les valeurs propres du gradient, dont 3 sont égales à ± 1 et on utilise les vecteurs propres pour interpréter les simulations sur un très grand nombre d'itérations.

keywords: nonlinear system; stability; discrete-time; eigenvectors

1 Introduction

An entry to discrete-time nonlinear systems is to look at their fixed points and cycles, then to study the stability of the cycles, Pac (2016). A system of dimension 4 rich in 4-cycles derived from 2-D geometry was presented in a previous work, Leviandier (2019), from which we extract its equations giving $(\epsilon^+, \eta^+, \theta^+, \phi^+)$ in function of $(\epsilon, \eta, \theta, \phi)$, with auxiliary variables X to V .

$$4X = (1 + \epsilon - \theta)(-\epsilon - \theta - 1) + u^2(1 - \eta + \phi)(\phi + \eta - 1) \quad (1)$$

$$4Y = -(1 + \epsilon - \theta)(1 - \eta - \phi) + (1 - \eta + \phi)(1 + \epsilon + \theta)u + u((-\epsilon - \theta)^2 + (\phi + \eta - 1)^2) \quad (2)$$

$$4Z = (-\epsilon - \theta - 1)^2 + u^2(1 - \eta + \phi)^2 + (-(1 + \epsilon - \theta)(1 - \eta - \phi) + (1 - \eta + \phi)(1 + \epsilon + \theta))u \quad (3)$$

$$2T = \begin{pmatrix} 1 - \epsilon \\ -\eta - 1 \end{pmatrix} \wedge \begin{pmatrix} (1 + \epsilon - \theta + u^2(1 - \eta - \phi)) \\ u(-\epsilon - \eta - \theta + \phi) \end{pmatrix} \quad (4)$$

$$2V = \begin{pmatrix} 1 - \epsilon \\ -\eta - 1 \end{pmatrix} \wedge \begin{pmatrix} -u(-\epsilon - \eta - \theta + \phi) \\ 1 + \epsilon - \theta + u^2(1 - \eta - \phi) \end{pmatrix} \quad (5)$$

$$\epsilon^+ = \frac{-uX}{Y}, \quad \eta^+ = \frac{X}{Z}, \quad \theta^+ = \frac{T}{Y}, \quad \phi^+ = \frac{V}{Z} \quad (6)$$

The first (hopefully seminal) paper on this system gives insight into a grid of quadruplets of integer values, considered as initial values of the dynamic system, which happens to stay in a stable 2-or-4-cycle, or to diverge or to converge into another cycle not containing the initial quadruplet, and in some cases to run on a very long orbit. The grid with integer intervals precludes any detection of basins of attraction or transitions between these different behaviours. The study was for this reason oriented toward formal calculation with initial quadruplets depending continuously on a parameter. This paper is focused on a parameterized 2-cycle.

2 Analytical development

Several trajectories issued from unstable cycles, for instance 4-cycles beginning with(1, x, 0, -1) seem to converge onto the 2-cycle:

$$\begin{cases} E & -E & E + 1 & E + 1 \\ 1 & 1 & 0 & 2\frac{E-1}{E+1} \end{cases}$$

However, the application of equations 6 to these quadruplets encounters indetermination since all auxiliary variables X to V are equal to zero. Curiously, multiple applications of equations 6 to quadruplets close to $(E, -E, E + 1, E + 1)$ were never found to diverge. In order to go further, the system was initialised with $(E + e, -E, E + 1, E + 1)$ where e is very small. The calculation was done numerically but also formally, that is in function of variables E and e , at the first order in e (for instance substituting $1 - e$ for $\frac{1}{1+e}$).

This way, not only the cycle, but also the jacobian matrix, or gradient, can be calculated in function of the variables E and e defined for the first element of the cycle. Terms in $\frac{1}{e}$ appear in the gradient of the first function, which could

keep from considering limits on $e \rightarrow O$ but thanks to algebraic properties of the system, all these terms are cancelled on the second step by summing up (not multiplication of individual terms, but multiplication of matrices). The Jacobian matrices of first and second application are given in annex. Though it was calculated with a particular location of e in the quadruplet, the obtained gradient will be expected to be informative about the stability of the 2-cycle.

$$J_2 J_1 = \frac{1}{2} \begin{pmatrix} E(1-E) & E(E-3) & (2E-1)(E-1) & 1-E \\ -E(E+1) & E(E-1) & (2E+1)(E-1) & E-1 \\ -E(E+1) & (E-2)(E+1) & (2E-1)(E+1) & 1-E \\ (E+2)(E+1) & -E(E+1) & -(2E^2+3E-1) & -(E+1) \end{pmatrix}$$

The characteristic polynomial with eigenvalues λ writes:

$$P(\lambda, E) = \lambda^4 - (E^2 - 1)\lambda^3 - (E^2 + 1)\lambda^2 + (E^2 - 1)\lambda + E^2$$

which has real roots:

$$P(\lambda, E) = (\lambda - 1)(\lambda + 1)(\lambda^2 + (1 - E^2)\lambda - E^2)$$

$$P(\lambda, E) = (\lambda - 1)(\lambda + 1)^2(\lambda - E^2)$$

For $|E| \leq 1$, all $|\lambda| \leq 1$.

The eigenvectors associated with eigenvalues are:

$$\lambda_1 = 1, V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \lambda_2 = -1, V_2 = \begin{pmatrix} \frac{E^2+4E-1}{4(E+1)} \\ \frac{E+1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = -1, V_3 = \begin{pmatrix} -\frac{(E-1)^2}{2(E+1)} \\ -\frac{E-1}{2} \\ 0 \\ 1 \end{pmatrix}, \lambda_4 = E^2, V_4 = \begin{pmatrix} \frac{(1-E)}{(E+1)} \\ \frac{2E^2-E-4}{2(E+1)} \\ -1 \\ 1 \end{pmatrix}$$

V_1 is independent of E and represents a direction of perturbation which lets the subspace $(E, -E, E+1, E+1)$ invariant.

3 Simulation

The eigenvalues do not answer with certainty the question of stability. It is remarkable that 3 out of 4 eigenvalues are 1 or -1, more than enough for the standard theory to claim the stability undetermined, which makes necessary a study by simulation. For different values of E , figures 2 to 8 show the plotting of a projection of the system onto a 2-D basis. The basis is either (ϵ, η) , (V_1, V_4) or (V_3, V_4) , depending on which direction of eigenvector was thought conveying more information. The straight line where the system is undetermined, referred below as the “canyon” is drawn in red. On some figures, the second half of the trajectories is in dark gray, the 1000 first points are in green and the 1000 last ones in blue. The solid green line shows the initial perturbation equal to 10^{-7} ; it enables to see that the area explored by the system after a very large number of iterations remains very small compared to $\epsilon \dots \phi$. The neighbourhood of the canyon is generally empty on 2-D projections, but the canyon may be partially hidden behind points which are not in the same plane.

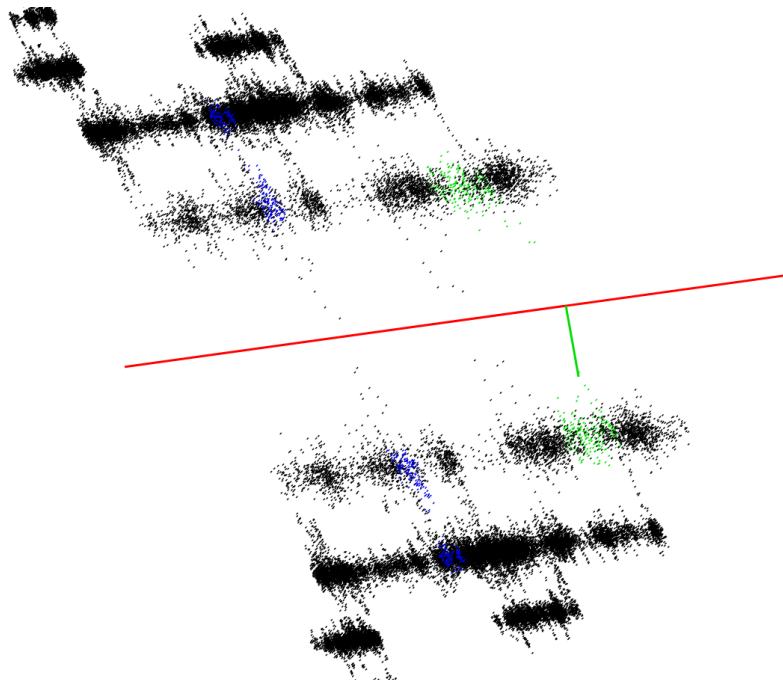


Figure 1: plotting of 100 000 points in basis (V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi = (1.75, -1.75, 2.75, 2.75)$. details top of section 3

Some comments follow for E in increasing order.

$E = -3$ departs from the canyon (not far away). *no Figure*

$E = -1$ perfectly stable. *no Figure*

$E = -0.25$ movement hardly visible in basis (ϵ, η) , oriented along V_4 . Figure 2

$E = 0$ moves parallel and very close to the canyon. *no Figure*

$E = 0.5$ moves parallel to the canyon. *no Figure*

$E = 1$ perfectly stable. *no Figure*

$E = 1.05$ Figure 3 and $E = 1.25$ Figure 4. corridor each side of the canyon.

$E = 2$ corridor, then escape. Figures 5

$E = 2.75$ Figure 6 transition with two preferential directions.

$E = 3$ Figures 7 to 9 Though the 4th eigenvalue has a large absolute value (equal to 9), which would trigger high unstability under linear theory, the system does not deviate so much from its origin after $5 \cdot 10^7$ iterations, whichever element $\epsilon \dots \phi$ receives the initial perturbation (Figure 7). For fear the proper eigenvector had not been excited in these simulations, simulations (Figure 8) were done with initial perturbation along V_4 and plotting in the basis (V_3, V_4) . The first points are aligned, but very soon, the trajectory goes aside and backward, though the general pattern keeps a preferential orientation. The maximum departure from the canyon is reached in the first millions iterations (checked in billions, Figure 9).

$E = 10$ departs from the canyon (not far away). *no Figure*

Two main behaviours may be identified:

- the system stay in a small area of irregular shape at some distance of the canyon
- after some wandering, the system follows two narrow corridors parallel to the canyon, sometimes going backwards.

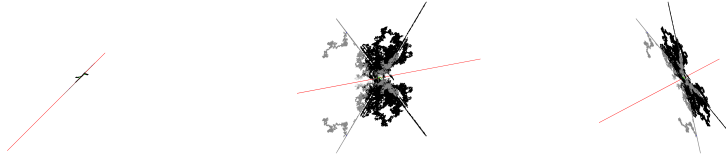


Figure 2: plotting of 10^7 points (left: ϵ, η ; centre: V_1, V_4 ; right: V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi = (-0.25, 0.25, 0.75, 0.75)$.

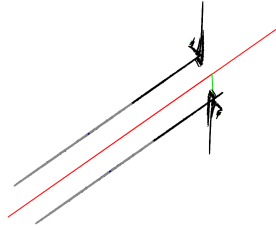


Figure 3: plotting of $4 \cdot 10^7$ points in basis (V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi = (1.05, -1.05, 2.05, 2.05)$

These observations stand for hundreds of millions of iterations, without giving evidence that results would be valid for a larger number or for other initial conditions. There is no clear limit between the two cases, rather a transition with numerous alignments of points. The non-linear stability literature, dealing with continuous time systems, does not seem helpful.

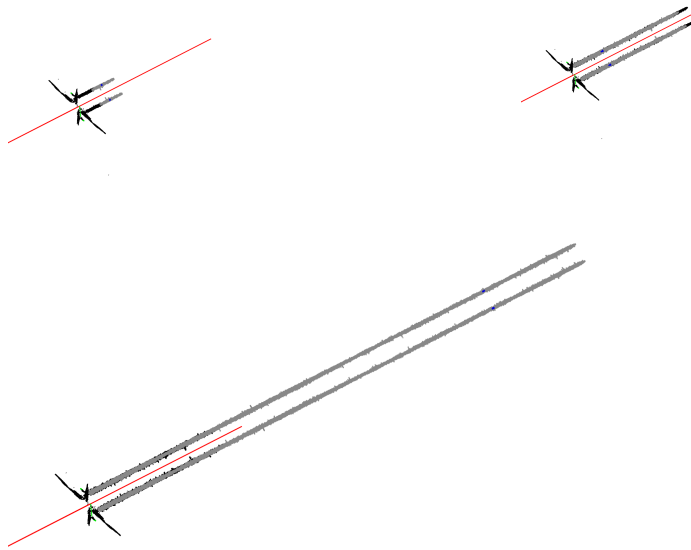


Figure 4: plotting of 10^6 , $5 \cdot 10^7$ and 10^8 points in basis (V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi = (1.25, -1.25, 2.25, 2.25)$

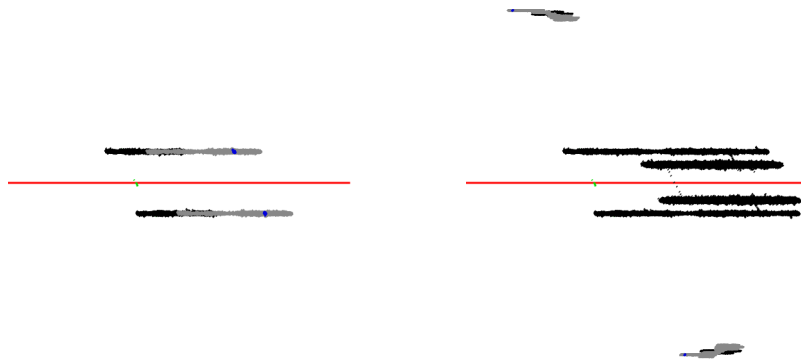


Figure 5: plotting of 10^6 and 10^7 points in basis (V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi = (2, -2, 3, 3)$. second half in gray.



Figure 6: plotting of 10^9 points in basis (V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi$,
 $= (2.75, -2.75, 3.75, 3.75)$.

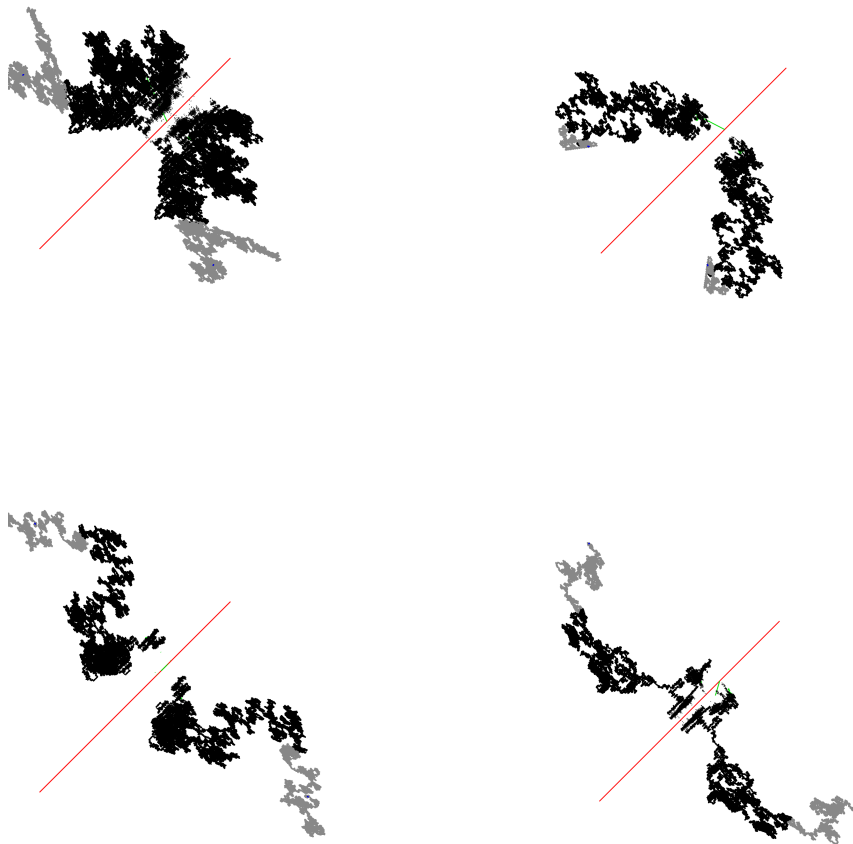


Figure 7: plotting of 10^7 points (ϵ, η) for 2-cycle beginning with $\epsilon \dots \phi = (3, -3, 4, 4)$. Initial perturbation on ϵ, η (top), θ, ϕ (bottom).

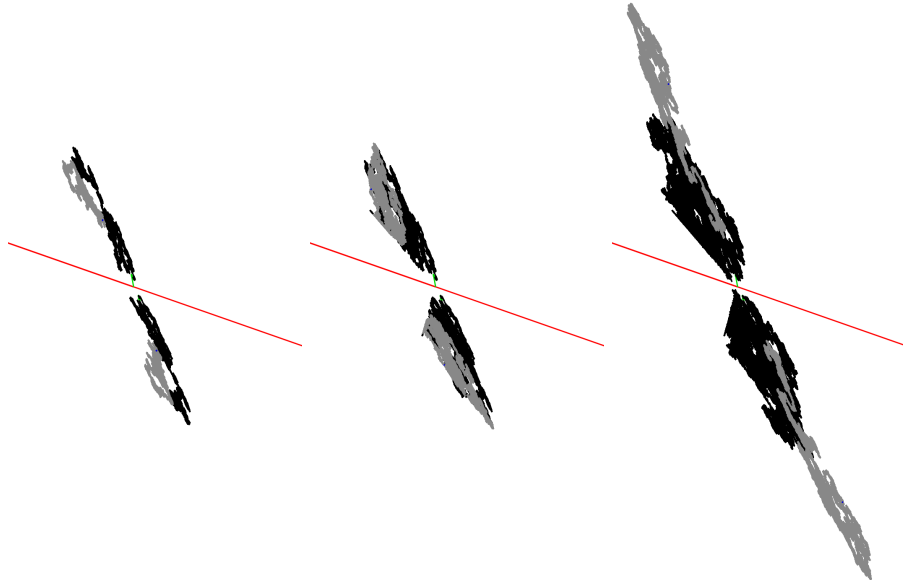


Figure 8: plotting of $5 \cdot 10^7$, $2 \cdot 10^8$ and 10^9 points in basis (V_3, V_4) for 2-cycle beginning with $\epsilon \dots \phi = (3, -3, 4, 4)$. second half in gray.



Figure 9: Maximum reached in the direction of V_4 versus iteration number ($2 \cdot 10^6$ and $2 \cdot 10^9$), for 2-cycle beginning with $\epsilon \dots \phi = (3, -3, 4, 4)$.

4 Discussion

Maybe the title of this paper should have been appended with a question mark. The stability issue generally arises about a point or a cycle. Here, we are interested in a subspace of dimension one composed of elements which are 2-cycles. Should we consider the stability of individual cycles, or the stability of the subset? The neighbourhood of a “canyon” of undetermined points, which does not capture the system, neither repels it far away, fostered the image of cliff edge stability. Most of the eigenvalues of the gradient are by construction exactly equal to ± 1 , so that the eigenvalues do not yield prediction of long-term behaviour. This is not linked to the existence of the canyon but a property of the system for a large set of initial conditions, which cannot be dismissed as highly unlikely. Part of the problem is to understand why the effect of an eigenvalue greater than 1 in a highly non linear system cannot be extrapolated from a linear system. Though the eigenvalues do not explain everything, the eigenvectors are useful for the analysis. The system tends to walk for a while in the direction of an eigenvector, then change (or not). A remarkable case is an orbit parallel to the canyon, which may split into several parallel segments. The reason why the system suddenly jumps to another pathway is probably not to be sought in the composed gradient of two iterations. It was shown in section 2 that the composition of two iterations eliminates an indetermination, but the system is computed separately at each iteration and may be subject to apparently random impulse due to limited precision.

This reduction of potentially infinite terms is part of an extraordinary simplification of complex calculation (see annex) ending in the amazingly short expression of the characteristic polynomial. What hidden fundamental reason behind?

Annex Jacobian matrices at first and second iteration

$$\begin{aligned}
 J_2 &= \frac{E+1}{4} \begin{pmatrix} 1-E & -E(E-1) & -E(E+1) & E+1 \\ E-1 & -E(E-1) & -E(E+1) & -(E+1) \\ 1-E & -(E+2)(E-1) & (E+2)(E+1) & E+1 \\ -(E+3) & -(E^2+E+2) & (E+2)(E+1) & E+1 \end{pmatrix} \\
 &+ \frac{e}{8} \begin{pmatrix} E(E-1)(3E+2) & 2E^4-3E^3-4E^2+1 & (E+1)(2E^3-E^2-3E-1) & -\frac{3}{2}E^2(E+1) \\ E(3E^2+3E-2) & (1-E)(2E^3+5E^2+E-1) & (E+1)(2E^3+5E^2-2E-1) & -E(E+1)(3E+4) \\ -(3E^3+3E^2+4E+2) & 2E^4+E^3-8E^2-10E-5 & (E+1)(-2E^3-3E^2+3E+3) & (E+1)(3E^2+4E+2) \\ -(3E^3+11E^2+10E) & 2E^4+7E^3+6E^2+2E-1 & (E+1)(2E^3+9E^2+9E+1) & (E+1)(E+2)(3E+2) \end{pmatrix} \\
 J_1 &= \begin{pmatrix} 0 & -\frac{2}{e} \left(1 - \frac{e}{2(E+1)}\right) & -\frac{2}{e} & \frac{1}{E+1} \left(1 - \frac{e}{2(E+1)}\right) \\ \frac{1}{E+1-\frac{e}{2}} & -\frac{2}{e} \left(1 + \frac{e}{(E+1)}\right) & -\frac{2}{e} \left(1 + \frac{3e}{2(E+1)}\right) & -e \left(1 + \frac{e}{2(E+1)}\right) \\ \frac{1}{E+1} \left(1 - \frac{e}{E+1}\right) & \frac{2(E-1)}{e(E+1)} \left(1 + \frac{e}{(E-1)(E+1)}\right) & \frac{2(E-1)}{e(E+1)} \left(1 - \frac{e}{2(E+1)}\right) & 0 \\ \frac{2E}{(E+1)(E+1-\frac{e}{2})} & -\frac{2(E-1)}{e(E+1)} \left(1 + \frac{e(5E+1)}{2(E-1)(E+1)}\right) & -\frac{2(E-1)}{e(E+1)} \left(1 + \frac{e(3E+1)}{(E-1)(E+1)}\right) & \frac{1-E}{1+E} \left(E+1 - \frac{e}{2}\right) \end{pmatrix}
 \end{aligned}$$

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