



**HAL**  
open science

# Geometric prediction: proposing a theoretical construct to analyze students' thinking in geometrical problem-solving

Elisa Miragliotta

## ► To cite this version:

Elisa Miragliotta. Geometric prediction: proposing a theoretical construct to analyze students' thinking in geometrical problem-solving. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02435297

**HAL Id: hal-02435297**

**<https://hal.science/hal-02435297v1>**

Submitted on 10 Jan 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Geometric prediction: proposing a theoretical construct to analyze students' thinking in geometrical problem-solving

Elisa Miragliotta

University of Modena and Reggio Emilia, Department of Physics, Computer Science and Mathematics, Modena, Italy; [elisa.miragliotta@unimore.it](mailto:elisa.miragliotta@unimore.it)

*We consider geometric prediction (GP) as a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant. In this paper, we explain our interest in this process and how we define it. Furthermore, we describe a tool for analyzing students' productions. In particular, we present brief analyses of students' interviews while they are solving a geometric open problem explicitly designed to elicit processes of GP. Finally, we summarize some preliminary results about the features of this construct. These results are part of a doctoral project aimed at gaining insight into the process of GP.*

*Keywords: Spatial reasoning, visualization, intuition, figural concepts.*

## Introduction

It is well known that spatial reasoning and visualization are topics of interest for mathematics educators (e.g. Duval, 1995; Presmeg, 2006) to the extent that Cuoco, Goldenberg and Mark (1996) have proposed to organize curriculum around *mathematical habits of mind*, which include visualization and tinkering. These are described as being at the heart of mathematical research and, therefore, crucial for helping students learn to think like mathematicians. These habits are particularly involved in geometrical problem-solving. Indeed, when solvers approach a geometrical task they can interact with visual or mental images in several ways. Cognitive Psychology explains this interaction by the intervention of several *visuo-spatial abilities* such as *imagery generation ability* and *imagery manipulation ability* (Cornoldi & Vecchi, 2004).

In a previous study (Miragliotta & Baccaglini-Frank, 2017) we tried to analyze students' solution processes when solving a geometrical task, but found shortcomings in using the visuo-spatial abilities for at least two reasons: much effort goes into establishing which abilities are used during each analyzed process, and interpretation plays a large role and is not consistent across researchers; none of the abilities explicitly deal with the Theory of Euclidean Geometry within which figures and their properties are defined, when the reasoning is carried out in this context. This is because as mathematicians we know that, when we solve a geometrical task, we can imagine consequences of (mental) transformations on a geometrical object which are consistent with theoretical constraints, given or induced by a particular construction. Indeed, mathematical objects have a particular nature, the investigation of which usually requires theoretical elements which refer to a domain different from the one of perception. In line with Neisser (1989), we consider perceiving and thinking as different cognitive activities. *Perception* "is immediate, effortless, and veridical" in one word "direct"; *thinking* is "indirect", because it "may not depend on the immediate environment at all, it is often anything but effortless or immediate, and frequently goes astray" (Neisser, 1989, p. 11).

In the specific domain of 2D Euclidean Geometry, in order to cope with the difficulties listed above, we defined a new theoretical construct called *geometric prediction (GP)*. Although at the moment, we are working towards a finer operative definition, it seems to be an interesting construct to use in

order to shed light onto problem solving processes in geometry. Furthermore we expect it to be trainable, that is to ameliorate through appropriate educational practices. The aim of our research is to gain insight into the process of GP. So we set out to observe and analyze what happens during the resolution of particular open problems, focusing on instances of *geometric prediction* in solvers' discourse, gestures, drawings, and dragging modalities used in a second phase of the resolution of the given problem in a Dynamic Geometry Environment (DGE). Because of what we present in this paper, in which the role of the DGE is marginal, and since the space limited, we will not discuss this second phase. In this paper, we present the construct of GP and describe a tool used for analyzing its emerging features in students' data that are collected in task-based interviews.

## **Theoretical framework**

In the specific domain of Mathematics Education, under the umbrella of *spatial reasoning* literature collects a large set of definitions, which share the reference to the activity of imagining objects and interact with them through mental transformations (rotation, stretch, reflection, etc.). Among these definitions, we have considered *spatial reasoning* as defined by the Spatial Reasoning Study Group: "the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 146). We considered *visualization*, according to Presmeg's (2006) description, as "taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics" (p. 206). Here visualization is considered to be explicitly linked to mathematical activity, and it could involve mental images.

## **Geometric prediction**

Previous research on visualization informs us that when solvers face a geometrical task, they can interact with mental images; they can imagine the consequence of (mental) manipulation of the geometrical figure. Moreover, it seems that in this process the theoretical elements play an important role. With "theoretical elements" we mean those related to the Theory of Euclidean Geometry (TEG). The process of *geometric prediction* is "a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant" (Mariotti & Baccaglioni-Frank, 2018, p. 157).

Products of GP may or may not be coherent with theoretical constraints. Indeed, in order to take into account all theoretical constraints, the solver must *theoretically control* the figure, that is "mentally impose on a figure theoretical elements that are coherent in the Theory of Euclidean Geometry" (Mariotti & Baccaglioni-Frank, 2018, p. 156).

## **Fischbein's Theory of Figural Concepts and Intuitive knowledge**

In order to explain the nature of geometrical objects, we refer to the *Theory of Figural Concepts* (Fischbein, 1993), according to which they have a dual nature. Geometrical objects are completely described and controlled by an axiomatic system of definition and theorems, but at the same time they maintain certain figural aspects of images. Fischbein (1993) thinks of a *figural concept* as a complex mental entity (different from pure concepts and pure images) "which simultaneously possesses both conceptual and figural properties" (p. 144), realizing a fusion between the conceptual and the figural components.

A construct which seems to be involved in GP is *intuition* (Fischbein, 1987). It is a kind of cognition, characterized by self-evidence and immediacy, different from perception, and implying an extrapolation of information beyond that directly accessible. Intuitions can have an anticipatory role. Indeed, anticipatory is a kind of intuition, which belongs explicitly to problem-solving activity; it provides a global view of a solution, which precedes the analytical one.

## Research questions and methodology

The data we present are part of a doctoral research project on GP for which 6 geometrical problems were designed and proposed to 37 Italian high school students (ages 14-18), undergraduates, graduate and Ph.D. students in mathematics (ages 19-33), during the months of February and April 2018. The problems were designed to elicit processes of GP and they were used within task-based interviews (Goldin, 2000). We decided to use a semi-structured type of interview: the first question is always the same; then there is a sequence of questions defined a priori and a set of stimuli in order to elicit students' comments.

Consistently with our purposes, we designed a particular kind of open problems (Arsac, Germain, Mante, 1998; Silver, 1995), *prediction open problems*, in which the solver is asked to describe possible alternative arrangements of a geometric configuration (imagined, given by a drawing and/or by a step-by-step construction) maintaining given properties. Predictions could be asked for explicitly or not.

Although all data have not yet been thoroughly analyzed, we use some of the data to present a preliminary report on the following questions: (a) How can we characterize GP processes (in the case of the prediction open problems given)? (b) In solving a prediction open problem, what kind of gestures, words and drawings are performed by solvers? What do these allow us to infer about the GP processes?

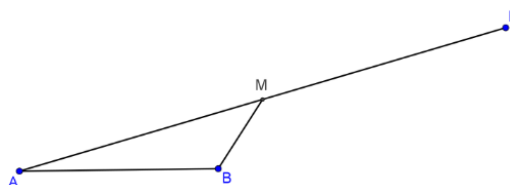
In the following section we report on an example of the task used during interviews.

### The “locus of P” problem

The task used in this study is composed of two parts. The first one is:

Read and perform the following step-by-step construction: fix two points  $A$  and  $B$ ; connect them with a segment  $AB$ ; choose a point  $P$  on the plane; connect  $A$  and  $P$  with a segment  $AP$ ; construct  $M$  as the midpoint of  $AP$ ; construct the segment  $MB$  and name its length  $d$ .  $A$  and  $B$  are fixed, and the length of  $MB$  has to always be  $d$ .

Then the interviewer asks: “What can you say about the point  $P$ ?”



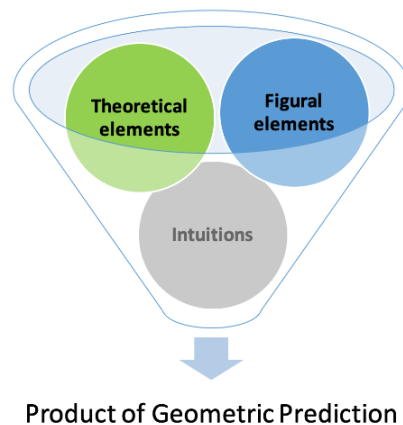
**Figure 1: Instance of figure obtained by step-by-step construction**

The step-by-step construction could be accomplished with paper and pencil (obtaining a construction like the one in Figure 1). Once the solver had proposed a solution or stated that s/he

was not able to find one, part two of the task was given: the interviewer opened a dynamic geometry sketch and she asked the solver to move  $P$  in the figure, consistently with her/his prediction, or else to explore the figure to help reach a solution. As highlighted before, in this paper we will not focus on this second phase.

### Construction of a new tool for data analyses

In our data analyses, we have been highlighting instances of GP by spotting and analyzing students' gestures, drawings and discourse. In particular, we are trying to identify elements which belong to the students' conceptual component and their figural component of the geometrical objects in focus. In order to analyze solvers' GPs, we collect these elements in a sort of "funnel" (Figure 2) which shows a product of GP in gestural and/or discursive form.



**Figure 2: Elements collected in the funnel and its product**

The tool, as shown in Table 1 and sketched out in Figure 2, is composed of 3 columns. In the first and in the third there are, respectively, *theoretical elements* and *figural elements*, referred by solvers in gestural or discursive way or inferred by us (in square brackets). These two constructs are defined and identified according to Fischbein's (1993) distinction between conceptual and figural components of geometrical objects. In the center column, we used two different colors to highlight which element of the two columns is expressed by the solver at a specific moment. Indeed, the vertical order of boxes follows the chronological sequence in which solvers made elements explicit. Furthermore, we added an "X" when the element is mathematically incorrect or incoherent with respect to the given geometrical construction. At the end of the funnel, there is the product of the GP. Each funnel represents the sequence of the observable steps of a new GP.

As an example, here we show the construction of the funnel referred to an excerpt of Student A's interview. She is solving the "locus of  $P$ " problem. This excerpt begins right after the interviewer asked the first question.

Student A: The point  $P$ ...meanwhile **is part...of the straight line** in which...on which there is also... $AM$  and is **outside the triangle  $AMB$** .

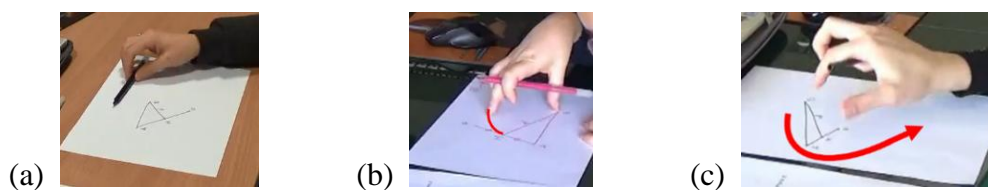
Student A: In this case – but I think it is a particular case – **perpendicular** to...not the point  $P$ ! It is a **straight line** on which there is the point  $P$  that is **perpendicular to  $MB$** , but it is a particular case, I think!

Interviewer: Ok. Make a prediction. Do you think that the point  $P$  can have other positions?

Student A: If I draw again...if **the length of  $MB$  must always be  $d$** , it could be **like a mirror**, so it could take the place...the same position, only **on the other side of the segment**.

In this excerpt, we recognize some figural elements (and underlined them), like segment  $AM$  and triangle  $AMB$ . We use bold type to highlight the identifiable theoretical elements, like perpendicularity and (line) symmetry. We recognize a product of a GP in the last student's statement:  $P$  takes a symmetric position with respect to  $AB$ . The GP is both discursive and gestural (the student shows the expected position of  $P$  using a pen (Figure 3a)). Then, she undertakes another GP, strictly connected with the figural elements observed before. Indeed, she claims:

Student: Yes, it [point  $P$ ] moves along the segment  $M$ ...along the **half-line  $MP$**  and then etcetera, etcetera... along the half-line that would continue, because it does not interfere with the **triangle**.



**Figure 3: Gestures performed by (a) Student A (b) Student C (c) Student D**

We collected all elements in two funnels, as shown in Table 1 (see Student A). The blue arrow highlights how the product of the second GP (named GP2) is directly connected with and derives from the product of the first GP (named GP1).

## Preliminary findings

### Funnels as a tool to gain insight into solvers' processes of GP

At the moment we have analyzed 15 interviews during which the "locus of  $P$ " problem was proposed. Here we show only four examples of funnels (Table 1): Student A is a 9th grade student (15 years old); Student B is a 13th grade student (18 years old); Student C is an undergraduate student and Student D had just completed a master's degree in mathematics. The chosen funnels summarize some of the solvers' common behaviors during the resolution of the first part of the task and highlight how this tool can be used to gain insight into the process of GP.

The first funnel (Student A) represents an example of a mathematically incorrect GP during which the student focuses on particular figural elements of the given configuration. Indeed, Student A explicitly mentioned the triangle  $ABM$  and then the segment  $MP$ . Our analyses reveal that in all cases in which the triangle is mentioned (5 interviewees), the solvers seem to have considered the configuration as made up of two completely independent figural elements: a triangle and a segment. This hypothesis is consistent with the solvers' stated predictions about the behavior of the configuration and it is confirmed by instances of surprise observed during the subsequent exploration in a DGE. The relation between figural elements induces Student A to speak of moving  $P$  along a half-line, maintaining  $MB$  not only constant in length, but also in the same position. Possibly, it is the lack of theoretical control that induces the solver to infer the necessity of maintaining invariant so many properties. In this case, the invariance of the length is maintained by

the invariance of the whole triangle. Furthermore, we notice that our analytical tool allows us to notice that during the second GP the student mostly referred to figural elements.

On the contrary, Student C and Student D provide examples of GPs that explicitly make use of theoretical elements. In particular, in the funnel of Student C, we observe an ongoing dialectic between figural and theoretical elements. In this case, we notice another common phenomenon: observing the development of the fourth GP (whose outcome is GP4), Student C recalls the prototype of circle (defined as the locus of points at a given distance from a given point); she reproduces its construction using her fingers as a compass (Figure 3b), making the circle (or some parts of it) into a figural element, even if it was not actually drawn on paper.

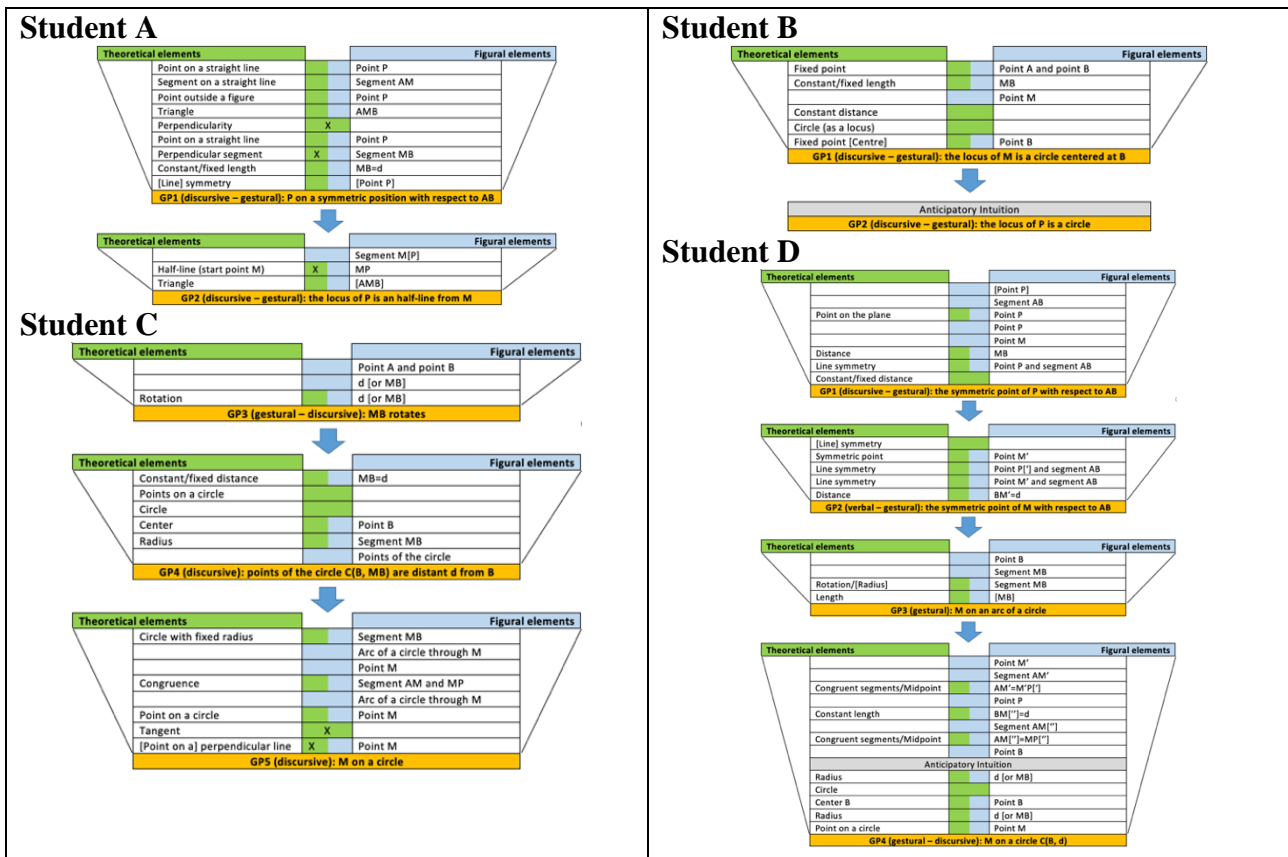


Table 1: Four examples of solvers' funnels

Such a phenomenon is evident also in Student D's production. First of all, she produces a GP about  $P$ . Suddenly, the point constructed as a symmetric point of  $P$  becomes in its own right one of the figural elements involved in the subsequent GP. We can see this effect also in the last funnel. Student D imagines to construct some points by rotating  $MB$  (Figure 3c). These points become part of the configuration, even if they are not visible on the drawing.

These findings allow us to highlight two features of GP processes. They are independent of each other, but they come in a sort of flow or chain of predictions. Furthermore, often their products seem to "freeze" for the student (see Student C and Student D), becoming integral parts of the configuration even if they are not drawn, and they become starting points for new GPs.

We stress how important is the role of the theoretical elements. Although Student D and Student A start from a similar product of GP regarding point  $P$ , Student D's GP is accompanied by more

detailed and coherent (with the TEG) theoretical elements. Moreover, she does not mention triangles, but she focuses only on segments and points. This seems to allow her to refine the first product of her first GP and, finally, to reach the last one (point  $M$  on a circle).

Student D's processes of GP highlight another emergent feature of GPs: often they are accompanied by intuitions. In particular, we observe a kind of intuition which appears during a solution process and that expresses an immediate and global view of a solution: *anticipatory intuition*. The occurrence of this kind of intuition could explain what happens in short chains of GP, like the one performed by Student B. His funnel shows a chain of GP in which most rows contain theoretical elements and no other elements could be recognized between the two products of GP: GP1 and GP2. This could be explained by his resorting to intuitive knowledge.

## Conclusion

The analyses of students' videos and transcripts reveal that GP is a process involving an interplay between perception and reasoning processes. Indeed, GP seems strictly related to figural components. In the case of the problem analyzed, it seems that students who see the configuration as a triangle and an independent segment do not reach a mathematically correct (or coherent with the TEG) product of GP. The notion seems to be in line with Duval's (1995) findings related to perceptual apprehensions of a geometric configuration and dimensional change. Nevertheless, GP also involves theoretical elements which belong to the TEG, and it has a strong relation with the idea of students' theoretical control over a figure. Indeed, theoretical control that is coherent with the TEG should lead to correct predictions (see Student C and D). Moreover, GP is related to conceptual components to the extent that in the case of Student B the discursive element "the length of  $MB$  must always be constant" fostered recollection of the definition of circle, leading to immediate recognition of the locus of  $M$ . So, it seems that theoretical elements play an important role in the immediacy and correctness of GPs. Student B reveals how for an "expert" solver GP can become an immediate and automatic process. This finding is promising for the trainability of GP through appropriate educational practices.

Furthermore, we notice that often GP is coupled with *anticipatory intuition* (see Student B and Student D). Nevertheless, the two constructs are not identifiable. Indeed, in other cases (see Student C) correct products of GPs are produced without any recognizable intuitive knowledge.

In conclusion, processes of GP seem to be observable through the solver's productions; funnels are useful tools for shedding light onto this process, but they must be accompanied by qualitative analyses of transcriptions for gaining deeper insight into the GP. Moreover, GP seems to be an interesting construct in order to shed light onto problem solving processes in geometry.

We believe that this kind of research can provide new insights into students' difficulties in learning geometry. Moreover, awareness of prediction processes could be very helpful for teachers. Outcomes of GPs seem to be windows onto students' mental images and processes generating them. Such knowledge could help in guiding mathematical teaching and learning, because teachers could know which properties students are imposing on a given configuration by asking them to make explicit their GPs. If students are imposing on a figure too many or incorrect properties, the teacher can intervene appropriately. Moreover, geometrical activities to strengthen students' theoretical control by fostering awareness of their conceived invariants can be designed.



## References

- Arsac, G., Germain, G., & Mante, M. (1991). *Problème ouvert et situation-problème*. Université Claude Bernard Lyon I.
- Bruce, C. D., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, M., ... & Okamoto, Y. (2017). Understanding gaps in research networks: using “spatial reasoning” as a window into the importance of networked educational research. *Educational Studies in Mathematics*, 95(2), 143–161.
- Cornoldi, C., & Vecchi, T. (2004). *Visuo-spatial working memory and individual differences*. Hove (UK): Psychology Press.
- Cuoco, A., Goldenberg, P. E., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, 15(4), 375–402.
- Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processing. In R. Sutherland, & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142–157). Berlin: Springer.
- Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Reidel, Dordrecht, The Netherlands: Kluwer Academic.
- Fischbein, E. (1993). The theory of figural concepts. *Educational studies in mathematics*, 24(2), 139–162.
- Goldin, G. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In R. Lesh, & A.E. Kelly (Eds.), *Research design in mathematics and science education*. Erlbaum, Hillsdale, pp. 517–545.
- Mariotti, M. A., & Baccaglini-Frank, A. (2018). Developing geometrical exploration skills through dynamic geometry. In S. Carreira, N. Amado, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving: A focus on technology, creativity and affect* (pp. 153–176). Springer.
- Miragliotta, E., & Baccaglini-Frank, A. (2017). Visuo-spatial abilities and Geometry: a first proposal of a theoretical framework for interpreting processes of visualization. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 3952–3959). Dublin, Ireland: DCU Institute of Education and ERME.
- Neisser, U. (Ed.). (1989). *Concepts and conceptual development: Ecological and intellectual factors in categorization*. Cambridge University Press.
- Presmeg, N.C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrez, & P. Boero (Eds.) *Handbook of research on psychology of mathematics education: Past, present and future*, (pp. 205–235). Rotterdam/Taipei: Sense Publishers.
- Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *Zentralblatt fur Didaktik der Mathematik/International Reviews on Mathematical Education*, 27(2), 67–72.