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# Effective choices of representations in problem solving

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*This paper investigates the types of representations used by high-achieving eleven-year-old pupils in their argumentation in the process of mathematical problem solving. The literature review addresses pedagogical and psychological aspects of knowing representations and using them in the process of problem solving. In our empirical study we examine a sample of 109 eleven-year-old pupils who took part in the final round of a national mathematical competition in Serbia. A total of 656 problem solutions was analyzed. The results show that these pupils used and often combined inventive iconic and conventional numerical/symbolical representations in their argumentation. We discuss pupils' choice of representations which lead to correct solutions and adjusted representations corresponding to the type of problem (algebraic or logical-combinatorial). The aim of the paper is to draw attention to the importance of studying representational approaches pupils take when solving problems.*

*Keywords: Representations, problem solving, argumentation, solution representation.*

## Introduction

There has been rising interest among mathematics educators in the usefulness of representations in the process of learning and problem solving. Competent use of representations in problem solving appears to be a significant indicator of level of mathematics literacy (De Lange, 2003). Then, it is a plausible assumption that investigations on how high-achieving pupils use representations in problem solving could lead to the development of strategies for choosing adequate representations in problem solving process.

Some of the principal objectives of mathematics education are to enable pupils to be good problem solvers (Schoenfeld, 1992). Solving problems in various ways on one hand and “decoding, encoding, translating, distinguishing between, and interpreting different forms of representations of mathematical objects and situation” on the other hand are recognized as some of the key competences needed for mathematical literacy as it is defined by national curriculums worldwide (De Lange, 2003, p.77). Representations, as tools for presenting problems, are considered to be means for understanding. It is also advocated that using multiple representations in problems may contribute to higher achievement (Cai, 2013). Role of representations in problem solving is recognized as important issue (Janvier, 1987; Cuoaco & Curcio, 2003; Shoenfeld, 1992). Two important issues are 1) which representations pupils invent when solving non-routine tasks and 2) which representations more often assure successful argumentation.

## Theoretical background

In numerous conceptualizations of mathematical competences and mathematical thinking, ability to use representations (symbolic, numerical, visual, or verbal expressions) is identified as important

parameter of mathematical abilities. Tchoshanov (2002) discusses types of representations on the concrete to abstract continuum as they reflect different modes of representation: concrete (real object, physical model, manipulative), pictorial (photograph, picture, drawing, sketch, graph), abstract (sign, symbol, written, verbal language). Matteson (2007) identified five categories of mathematical representations in problem solving: numerical, iconic, verbal, symbolic, and dual. Numerical representation focuses on specific numerical values in a variety of formats, such as decimal, fraction, percentage, or a numerical list (such as a list of numbers appearing as outcomes of probability). According to Matteson, iconic representations encompass different visual representations, from pictorial, (realistic) models, horizontal charts, vertical charts to graphs and coordinates graphs. Pictorial representations are pictures of real-world objects such as toys, dice, etc. As we slightly depart from Matteson's classification, we will consider iconic representation as distinct from pictorial (which closely resembles real world objects). For example, an apple may be "iconically" represented as a dot. Verbal representations require application of written (or spoken) language to express understanding, to describe, analyze, explain, or reflect upon numerical, algebraic, or graphic representation (which does not include brief phrases such as directions for solving the problem). Symbolic representations involve symbolic notation and include usage of variables and formulas such as: equation, expression, algebraic equation, algebraic expression, and formula (Matteson, 2007). Yet pupils may not use only one representation in problem solving. Under the term "dual representation" Matteson considers dealing with two categories of representations in problem solving (e.g. numerical and verbal). Matteson claims that "problems' (solutions) incorporating multiple representations generally result in more incorrect solutions" (Matteson, 2007, p. 60).

Some researchers pointed out that visual representations play an important role in supporting reflection and as a tool of communicating mathematical ideas (Arcavi, 2003; Gagatsis et. all, 2010; Sfard, 1991). The main functions of visual representation are to illustrate symbolic representation and to resolve conflict between intuition and symbolic solution. Successfully arriving at solutions to mathematical problems utilizes a combination of problem representation skills and symbol manipulation skills (Brenner et al., 1997). The first mentioned function involves skills which "include constructing and using mathematical representations in words, graphs, tables, and equations" (ibid). Lesh and colleagues argue that besides the significance of each representational system, attention should be directed toward translation among representational systems as well as to transformation within them (Lesh et al, 1987). They point their research shows that the act of representing tends to be "plural, unstable and evolving". Along the line, Duval discusses changes of register of representation in terms of cognitive operations: 1) transformations within the same register, like for example in case of algebraic operations or numerical computation, 2) translation of representation into different register, which is more cognitively challenging (Duval, 1999, 2003).

Several researchers attended to multiple representations in problem solving (Lowrie, 2001; Hegarty and Kozhevnikov, 1999; Matteson, 2007). Lowrie (2001), in a study of the relationship between different forms of problem representation and pupils' performance in problem solving, has found that high-achieving middle school pupils tend to use nonvisual methods when solving problems. He pointed out that most studies examining representational preferences along a visual-nonvisual continuum have considered preference alone and did not attend to effectiveness of method (ibid).

Lowrie found that those “pupils’ who were able to use visual representations regularly and in an efficient manner, were able to solve mathematics problems in a more effective manner than pupils’ who were more inclined to use nonvisual aids on a regular basis” (Lowrie, 2001, p. 360).

Stylianou and Silver (2004) examined differences in using representations between mathematicians and undergraduate students (as novices). They found that experts construct visual representations more frequently and use qualitatively better way to understand problem situation. School teachers are traditionally inclined to rely on symbolic, analytical representations. Consequently, pupils tend to do the same. The role of different types of representation in problem solving, particularly the effects of the usage of multiple representations is not fully explored.

## Methodology

We investigated types of representations used by high-achieving eleven-year-old pupils in solving problems. The main research questions were: 1. what representations pupils most frequently use when solving problems; 2. what representations are associated with correct solutions; and 3. how many representations used at the same time in argumentation are most often associated correct argumentation. Our questions were proposed based on previous research from Matteson, Lowrie, and Brenner and colleagues.

The data was collected in the final round of a national mathematics competition “Mislisa”, similar to “Kangaroo without borders” In this competition, mathematics relevant for solving problems is not strictly limited to the national curriculum. Successful problem solving in this competition requires critical thinking and applying knowledge in non-standardized situations. The problems range from the simplest computation problems, to algebraic, logical, and combinatorial problems most often set in realistic context. The competition promotes the value of mathematics literacy in pupils.

The study sample consists of 109 pupils who qualified for the finals of the competition by achieving the maximum score in the preliminary round of competition. We examined problem solutions from all finalists. The subset of 8 problems in which competitors were explicitly asked to provide explanation was analyzed in our study. Descriptive statistics was done for those eight problems.

Solutions for two of the problems are analyzed qualitatively in greater details. Our preliminary analysis of all solutions, resulted in selection of *The Bottle problem* and *The Castle problem* as exemplars for the points we make. They are the following:

*The Bottle problem* There is three times more milk in the first bottle than in the second one. If 3l of milk is added in the first bottle, and 5l of milk in the second, than there would be two times more milk in the first bottle than in the second one. How many litres of milk has been in the bottles at the beginning? Explain your answer.

*The Castle problem* A Castle has 6 towers. Each tower has a room. Each room has one door. Each door has a lock. All rooms are locked and keys are mixed up. What is the maximum number of trials a guard needs to check, in order to open the doors?

In the Bottle problem, finding way(s) to represent relationships between amount of milk in the first and the second bottles is the key for finding the solution. Problems similar to this are commonly

solved in schools with the strategy of forming an algebraic expression (symbolic representation). But there are alternative strategies for solving this problem such as 1) the “method of segments” which relies on expressing relationships between quantities on line segments (iconic representation) or 2) the “method of trials” (involving number representation) or 3) simply explaining the solution in words (verbal representation) or 4) some other creative approach.

The Castle problem is logical-combinatorial. The maximum number of trials in the first cycle to get to the door that can be unlocked is obviously equal to the number of rooms. In the following cycle of trials, one door is already unlocked so five doors remained to be checked in. There is no one “school” strategy for solving this problem.

## Results and Discussion

Here we present descriptive analysis of pupils’ solutions. A total of 656 problem solutions are taken as units of analyses. Out of them, 355 items were solutions to algebraic problems and 301 items solutions to logical-combinatorial problems.

Table 1 shows that choice of representations for algebraic problems did not match representations used in combinatorial problems. Since pupils often used more than one representation, the percentages in each column do not add to 100.

Type of representation	Algebraic problems	Logical-combinatorial problems
Numerical	31,9%	57,8%
Iconic	22%	20,8%
Verbal	21,8%	61,2%
Symbolical	45,9%	8,3%

**Table 1: Percentages of problem solutions with particular type of representation**

In accordance to Duval’s claim, pupils tended to use non-visual representations. To explain algebraic problem solutions, pupils most frequently used symbolical representation. It is not surprising since traditionally solving algebraic problems in a way other than writing down equivalent equations is undervalued in classrooms. In contrast, the logical-combinatorial problem solutions pupils primarily relayed on verbal and numerical representations. Particularly, verbal representations were often used for argumentation in solving logical combinatorial tasks, rarely for algebraic problems.

We examined whether there is an association between number of used representations (1, 2 or 3) and correctness of solution (False/Partially correct/Correct). The percentages are presented in the Table 2. Percentage of correct solutions including more representation, were bigger than of correct solutions with smaller number of representations used. We performed the  $\chi^2$  test of independence for variables Number of used representations (1, 2 or 3) and Correctness of solution (False/Partially correct/Correct) on problems. There is a significant relationship between “number of used representations (1,2,3)” and “Correctness of solution (false, partially correct, correct)”. ( $\chi^2=67,478$ ,  $df=4$ ,  $N=656$ ,  $p<0,001$ ). Cramers’s V coefficient of 0,227,  $p<0,001$  suggests moderate correlations.

Thus, successful problem solvers constructed significantly more representations than unsuccessful problem solvers. Our finding confirm Lesh’s observation about plural and evolving nature of the act of representing. The finding does not reinforce Matteson’s statement that “multiple representations result in more incorrect solutions” (Matteson, 2007)

Number of representations	False solution	Partially correct solution	Correct solution
1	35,4%	23,7%	40,9%
2	20,7%	10,4%	68,9%
3	4,5%	9,1%	86,4%
Total	27,3%	17,2%	55,5%

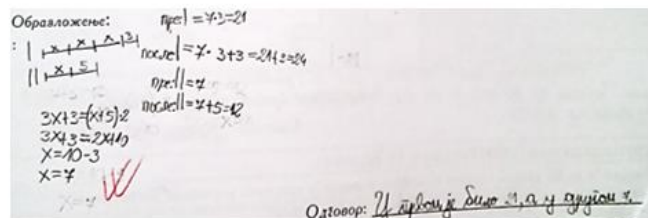
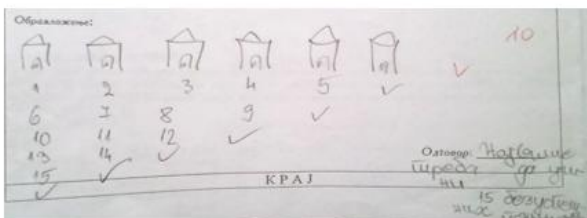
**Table 2: Percentages of number of used representations according to correctness of solution**

Iconic representations are considered to be more intuitively comprehensible and closer to reality than symbolic representations which involve abstraction and generalization in mathematics domain (Liu, 2012). Yet sufficient experience with such representations is a prerequisite for their’ appreciation. The high-achieving pupils in our study had a tendency to use symbolic representations in algebraic problems. Almost half of the solutions involved using of symbolic representation in those problems. The test of independence showed that there is significant association between correctness of problem solution and use of symbolic representation ( $\chi^2=62,417$ ,  $df=2$ ,  $p<0,001$ ). The strength of relationship is medium (Cramer’s  $V=0,419$ ,  $p<0,001$ ).

Many pupils obviously assumed that they were expected to use pictures when explaining their solutions. But often pictures were neither sufficient nor effective aid in argumentation. Some pupils combined pictorial with numerical representation (e.g. counting up number of attempts in the Castle problem). The others combined pictorial with symbolic and verbal representations by creating number sentences and written explanations of steps.

### Prototypical representations in pupil’s solutions

Pupils created diverse representations for their argumentation. In the Castle problem a significant number of pupils used numerical representation. Yet, others supported argumentation with pictorial representation in combination with other representations. Figure 1 shows an example of argumentation based on dual representation (pictorial and numerical).



**Figure 1: Dual representation - pictorial and numerical (left)**

**Figure 2: Dual representation - number line and algebraic expressions) (right)**

When using iconic representation, most commonly the pupils relied on number line and the method of segments. This method is commonly taught in Serbian schools. Figure 2 illustrates an example of argumentation based on number line representation in solving the Bottle problem. In her problem solution, Ana used the method of segments (number line representation) to show the relationship between elements in the initial state and the final state (so this pupil represented how the situation changed by drawing two pairs of number line segments). Ana switched from the number line representation to symbolic ones as she wrote a chain of equivalent equations.

An example of the solution of the Castle problem, solely based on iconic representation is presented in Figure 3. Marko put a key for his iconic presented argumentation on the right, where “x” meant unsuccessful attempt, “ ”, successful attempt, and “-” that the door cannot be opened.

Кључ	1	2	3	4	5	6
1	x	x	x	x	x	✓
2	x	x	x	x	✓	-
3	x	x	x	✓	-	-
4	x	x	✓	-	-	-
5	x	✓	-	-	-	-
6	✓	-	-	-	-	-

Безусловне покуше: 15

Кључи: 1, 2, 3, 4, 5, 6

Легенда:  
 x - безуспешна покуша  
 ✓ - успешна покуша  
 - - врата која не могу да се отворе

**Figure 3: Iconic representation**

Figure 4 shows Sara’s solution, assessed to be a creative argumentation involving Roman and Arabic numerals in solving the Castle problem. The solution consisted of five equalities. Sara counted the cycle of attempts by Roman numerals. She used Arabic numerals to show number of attempts in the particular cycle. Formally, without (absent) supplementary explanation the record was not mathematically correct as it stated e.g. I = 6 and V = 1. Sara used different numeration to distinguish between number of attempts and number of trials in the cycle, overlooking meaning of the equality sign. Her solution was assessed to be too sketchy and although the answer was correct, she did not earn points for argumentation.

Pupils often combined different representations in solving problems. In those cases they usually started with pictorial representation, than they turned to verbal or symbolic representation. Using pictorial representation, enabled them to provide simpler explanation. An effective presentation of thought process with no verbal explanation is presented in Figure 1.

15. I = 5  
 II = 4  
 III = 3  
 IV = 2  
 V = 1  
 VI = 1  
 15

Образложење:

Пошто има 6 кључева и 6 врата, првим кључем највише може да буде 10 5 промоваја, средњим 4 по средњем 3 и тако сваком средњем са 1 мере ✓

Рачун: 5 + 4 + 3 + 2 + 1 + 0 = 9 + 5 + 1 + 0 = 14 + 1 + 0 = 15 + 0 = 15 ✓

Одговор: 15

**Figure 4: Combined numerical representations (left)**

**Figure 5: Multiple representations - pictorial, verbal and numerical (right)**

In Figure 5 is Luka's solution of The Castle problem with multiple representations. He initially used pictorial and numerical representations along with verbal explanation and finalized the solution with numerical calculation. The realistic picture of doors itself hardly helped in solving the problem. Nevertheless Luka used it as a tool in argumentation. He probably meant to indicate with that picture that all 6 doors were checked with one key. The other cycles of tries were not visually presented in Luka's solution. But his verbal explanation presented on the right side of the picture was accurate. Luka wrote: "Since there were 6 keys and 6 doors, with the first key there could be no more than five failed attempts, in the next step 4, than 3, and so on, in each successive step 1 less".

## Conclusion

Successful problem solvers construct significantly more representations than unsuccessful problem solvers. Our results do not endorse previous arguments that multiple representations are associated with more incorrect solutions. Using combined representations often results in correct solution. Typically, pupil's argumentation starts with pictures, than turns to verbal or symbolic representation. Using pictures frequently enables pupils to provide sound explanation.

We found that talented pupils had a tendency to use symbolic representations. Pupils' choice of representation could be a consequence of limited experience in using different representations in the past. Ultimately teachers need to help pupils to develop flexibility in using different representations and ability to combine representations (or move in argumentation from one to another one).

In particular, future research should test whether pupils should be encouraged to provide argumentation by 1) combining numerical and verbal representations in the case of logical combinatorial problems and 2) numerical pictorial and iconic representations in the case of algebraic problems. More generally, we need to focus on developing methods for improving pupils' abilities to visualize and verbalize thought process and to be flexible in combining representations in the process of problem solving.

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