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► **To cite this version:**

| Michele Ancona. Random real branched coverings of the projective line. 2020. hal-02434851

HAL Id: hal-02434851

<https://hal.science/hal-02434851>

Preprint submitted on 10 Jan 2020

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Random real branched coverings of the projective line.

Michele Ancona*

Abstract

In this paper, we construct a natural probability measure on the space of real branched coverings from a real projective algebraic curve (X, c_X) to the projective line $(\mathbb{CP}^1, conj)$. We prove that the space of degree d real branched coverings having "many" real branched points (for example more than $\sqrt{d}^{1+\alpha}$, for any $\alpha > 0$) has exponentially small measure. In particular, maximal real branched coverings, that is real branched coverings such that all the branched points are real, are exponentially rare.

Introduction

Let (X, c_X) be a real algebraic curve, that is a smooth complex complex curve equipped with an anti-holomorphic involution c_X , called the real structure. We denote by $\mathbb{R}X$ the real locus of X , that is the set $\text{Fix}(c_X)$ of fixed points of c_X . For example the projective line $(\mathbb{CP}^1, conj)$ is a real algebraic curve whose real locus equals \mathbb{RP}^1 .

The central objects of this paper are real branched coverings from X to \mathbb{CP}^1 , that is, the branched coverings $u : X \rightarrow \mathbb{CP}^1$ such that $u \circ c_X = conj \circ u$. Let us denote by $\mathcal{M}_d^{\mathbb{R}}(X)$ the set of degree d real branched coverings from X to \mathbb{CP}^1 . The first purpose of the paper is to show that $\mathcal{M}_d^{\mathbb{R}}(X)$ has a natural probability measure μ_d induced by a compatible volume form ω of X (that is $c_X^* \omega = -\omega$), which we fix once for all. Later in the introduction we will sketch the construction of the measure μ_d , which we will give in details in Section 1.3. By Riemann-Hurwitz formula, the number of critical points, counted with multiplicity, of a degree d branched covering $u : X \rightarrow \mathbb{CP}^1$ equals $2d + 2g - 2$, where g is the genus of X . The probability measure μ_d allows us to ask the following question.

What is the probability that all the critical points of a real branched covering $u \in \mathcal{M}_d^{\mathbb{R}}(X)$ are real?

In [2], it is proved that the expected number of real critical points is equivalent to $c\sqrt{d}$ as the degree d of the random branched covering goes to infinity. The constant c is explicit, given by $c = \sqrt{\frac{\pi}{2}} \text{Vol}(\mathbb{R}X)$, where $\text{Vol}(\mathbb{R}X)$ is the length of the real locus of X with respect to the Riemannian metric induced by ω . The main theorem of the paper is the following exponential rarefaction result for real branched coverings having "many" real critical points.

Theorem 0.1. *Let X be a real algebraic curve. Let $\ell(d)$ be a sequence of positive real numbers such that $\ell(d) \geq B \log d$, for some $B > 0$. Then there exist positive constants c_1 and c_2 such that the following holds*

$$\mu_d \{ u \in \mathcal{M}_d^{\mathbb{R}}(X), \#(\text{Crit}(u) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d} \} \leq c_1 e^{-c_2 \ell(d)^2}.$$

For example, for any fixed $\alpha > 0$, we can consider the sequence $\ell(d) = \sqrt{d}^\alpha$. Theorem 0.1 says that the space of real branched coverings having more than $\sqrt{d}^{1+\alpha}$ real critical points has exponential small measure. In particular, maximal real branched coverings (i.e. branched coverings such that all the critical points are real) are exponentially rare.

The probability measure on $\mathcal{M}_d^{\mathbb{R}}(X)$. The construction of the probability measure on $\mathcal{M}_d^{\mathbb{R}}(X)$ uses the fact that there is a natural map from $\mathcal{M}_d^{\mathbb{R}}(X)$ to the space of degree d real holomorphic line bundle $\text{Pic}_{\mathbb{R}}^d(X)$, see Proposition 1.5. This map sends a degree d morphism u to the degree d line bundle $u^* \mathcal{O}(1)$. The fiber of this map over $L \in \text{Pic}_{\mathbb{R}}^d(X)$ is the open dense subset of $\mathbb{P}(\mathbb{R}H^0(X, L)^2)$ given by (the class of) pairs of global sections without common zeros. In order to construct a

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probability measure on $\mathcal{M}_d^{\mathbb{R}}(X)$, we produce a family of probability measures $\{\mu_L\}_{L \in \text{Pic}_{\mathbb{R}}^d(X)}$ on each space $\mathbb{P}(\mathbb{R}H^0(X, L)^2)$. The probability measure μ_L on $\mathbb{P}(\mathbb{R}H^0(X, L)^2)$ is the measure induced by the Fubini-Study metric associated with a real Hermitian product on $\mathbb{R}H^0(X, L)^2$. This Hermitian product is the natural \mathcal{L}^2 -product induced by ω , see Section 1.1. This family of measures, together with the Haar probability measure on the base $\text{Pic}_{\mathbb{R}}^d(X)$, gives rise to the probability measure μ_d on $\mathcal{M}_d^{\mathbb{R}}(X)$.

An example: the projective line. Let us consider the case $X = \mathbb{C}\mathbb{P}^1$ equipped with the conjugaison $\text{conj}([x_0 : x_1]) = [\bar{x}_0 : \bar{x}_1]$. Given two degree d real polynomials $P, Q \in \mathbb{R}_d^{\text{hom}}[X_0, X_1]$ without common zeros, we produce a degree d real branched covering $u_{PQ} : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ by sending $[x_0 : x_1]$ to $[P(x_0, x_1) : Q(x_0, x_1)]$. We also remark that the pair $(\lambda P, \lambda Q)$ defines the same branched covering. Conversely, one can prove that any degree d real branched covering $u : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ is of the form $u = u_{PQ}$ for some (class of) pair of polynomials (P, Q) without common zeros. This means that $\mathcal{M}_d^{\mathbb{R}}(\mathbb{C}\mathbb{P}^1) = \mathbb{P}(\mathbb{R}_d^{\text{hom}}[X_0, X_1]^2 \setminus \Lambda_d)$, where Λ_d is the set of polynomials with at least one common zero. Consider the affine chart $\{x_1 \neq 0\}$, the corresponding coordinate $x = \frac{x_0}{x_1}$ and the polynomials $p(X) = P(X_0, 1)$ and $q(X) = Q(X_0, 1)$. Then, one can see that a point $x \in \{x_1 \neq 0\}$ is a critical point of u_{PQ} if and only $p'(x)q(x) - q'(x)p(x) = 0$ (see Proposition 2.14).

In the the previous paragraph, we constructed a probability measure on this space by fixing a compatible volume form on source space, in this case $\mathbb{C}\mathbb{P}^1$. Indeed, a compatible volume form induces a \mathcal{L}^2 -scalar product on $\mathbb{R}_d^{\text{hom}}[X_0, X_1]$ which will induce a Fubini-Study volume form on $\mathbb{P}(\mathbb{R}_d^{\text{hom}}[X_0, X_1]^2)$ and then a probability on $\mathcal{M}_d^{\mathbb{R}}(\mathbb{C}\mathbb{P}^1)$. If we equip $\mathbb{C}\mathbb{P}^1$ with the Fubini-Study form, then the induced scalar product on $\mathbb{R}_d^{\text{hom}}[X_0, X_1]$ is the one which makes $\{\sqrt{\binom{d}{k}}X_0^kX_1^{d-k}\}_{0 \leq k \leq d}$ an orthonormal basis. This scalar product was considered by Kostlan in [8] (see also [11]). It is the only scalar product invariant under the action of the orthogonal group $O(2)$ (which acts on the variables X_0 and X_1) and such that the standard monomials are orthogonal to each other.

About the proof. There are two main steps in the proof of our main theorem. First, we reduce our problem into the problem of the computation of the *Gaussian* measure of a cone $\mathcal{C}_{\ell(d)}$ which lies inside the space of pairs of global sections of a real holomorphic line bundle over X . This cone is defined by using the Wronskian of a pair of global sections, which plays a key role. Then, we use peak sections theory to estimate some Markov moments related to this Wronskian. These moments, together with Poincaré-Lelong formula, allow us to estimate the measure of the cone $\mathcal{C}_{\ell(d)}$.

Let sketch the proof in more details. We fix a degree 1 real holomorphic line bundle F over X , so that, for any $L \in \text{Pic}_{\mathbb{R}}^d(X)$ there exists a unique $E \in \text{Pic}_{\mathbb{R}}^0(X)$ such that $L = F^d \otimes E$. Recall that any class of pairs of real global sections without common zeros $[\alpha : \beta] \in \mathbb{P}(\mathbb{R}H^0(X, F^d \otimes E)^2)$ defines a real branched covering $u_{\alpha\beta}$ by sending a point $x \in X$ to $[\alpha(x) : \beta(x)] \in \mathbb{C}\mathbb{P}^1$. Theorem 0.1 will follow from the estimate

$$\mu_{F^d \otimes E} \{[\alpha : \beta] \in \mathbb{P}(\mathbb{R}H^0(X, F^d \otimes E)^2), \#(\text{Crit}(u_{\alpha\beta}) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d}\} \leq c_1 e^{-c_2 \ell(d)^2} \quad (1)$$

where $\mu_{F^d \otimes E}$ is the probability measure induced by the Fubini-Study metric on $\mathbb{P}(\mathbb{R}H^0(X, F^d \otimes E)^2)$. Indeed, if we integrate the inequality (1) along $\text{Pic}_{\mathbb{R}}^0(X)$ we exactly obtain Theorem 0.1. To prove the estimate (1), we will use the following two facts. First, a point x is a critical point of $u_{\alpha\beta}$ if and only if it is a zero of the Wronskian $W_{\alpha\beta} \doteq \alpha \otimes \nabla \beta - \beta \otimes \nabla \alpha$. Second, the pushforward (with respect to the projectivization) of the Gaussian measure on $\mathbb{R}H^0(X, F^d \otimes E)^2$ is exactly the probability measure $\mu_{F^d \otimes E}$. These two facts imply that the estimate (1) is equivalent to the fact that the Gaussian measure of the cone

$$\mathcal{C}_{\ell(d)} \doteq \{(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \#(\text{real zeros of } W_{\alpha\beta}) \geq \ell(d)\sqrt{d}\} \quad (2)$$

is bounded from above by $c_1 e^{-c_2 \ell(d)^2}$.

In order to estimate the Gaussian measure of $\mathcal{C}_{\ell(d)}$, inspired by [5], we bound from above the moments of the random variable $(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2 \mapsto \log \|W_{\alpha\beta}(x)\|$, where x is a point in X such that $\text{dist}(x, \mathbb{R}X)$ is bigger than $\frac{\log d}{\sqrt{d}}$, see Proposition 2.15. This condition on the distance is natural, it is strictly related to peak section's theory (see [12, 6]) and it is the reason why we need the hypothesis on the growth of the sequence $\ell(d)$ in Theorem 0.1. The estimate of these moments uses two ingredients: the theory of peak sections and the comparison between the norms of two different evaluation maps (and more generally jet maps). Once these moments are estimated, Markov inequality and Poincaré-Lelong formula gives us the exponential rarefaction of the Gaussian measure of the cone (2).

Organization of the paper. The paper is organized as follows. In Section 1.1 we introduce the main objects and notations of this paper. In Sections 1.2 and 1.3 we study the geometry of the manifold $\mathcal{M}_d^{\mathbb{R}}(X)$ and we construct the probability measure μ_d on it.

The purpose of Section 2 is to prove Proposition 2.15, that is, to estimate of the moments of the random variable $(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2 \mapsto \log \|(\alpha \otimes \nabla\beta - \beta \otimes \nabla\alpha)(x)\|$, for F and E are respectively a degree 1 and 0 real holomorphic line bundles. In order to do this, in Section 2.1 we introduce Gaussian measures on $\mathbb{R}H^0(X, F^d \otimes E)^2$ and in Section 2.2 we study jet maps at points $x \in X$ which are far from the real locus. Finally, in Section 3, we deduce Theorem 0.1 from the estimates established in Section 2.

1 Random real branched coverings

1.1 Background

Let (X, c_X) be a real algebraic curve, that is a complex, projective, smooth curve equipped with an anti-holomorphic involution c_X , called the real structure. We assume that the real locus $\mathbb{R}X \doteq \text{Fix}(c_X)$ is non empty. For example $(\mathbb{C}\mathbb{P}^1, \text{conj})$, where $\text{conj}([x_0 : x_1]) = [\bar{x}_0 : \bar{x}_1]$, is a real algebraic curve whose real locus is $\mathbb{R}\mathbb{P}^1$. A real holomorphic line bundle $p : L \rightarrow X$ is a line bundle equipped with an anti-holomorphic involution c_L such that $p \circ c_X = c_L \circ p$ and c_L is complex-antilinear in the fibers. We denote by $\mathbb{R}H^0(X; L)$ the real vector space of real holomorphic global sections of L , i.e. sections $s \in H^0(X; L)$ such that $s \circ c_X = c_L \circ s$. Let $\text{Pic}_{\mathbb{R}}^d(X)$ be the set of degree d real line bundles. It is a principal space under the action of the compact topological abelian group $\text{Pic}_{\mathbb{R}}^0(X)$ and so it inherits a normalized Haar measure that we denote by $d\mathbb{H}$ (see, for example, [7]). Finally, recall that a real Hermitian metric h on L is a Hermitian metric on L such that $c_L^* h = \bar{h}$.

Proposition 1.1. *Let (X, c_X) be a real algebraic curve and let ω be a compatible volume form of mass 1, that is $c_X^* \omega = -\omega$ and $\int_X \omega = 1$. Let $L \in \text{Pic}_{\mathbb{R}}^d(X)$ be a degree d real holomorphic line bundle over X , then there exists a unique real Hermitian metric h (up to multiplication by a positive real constant) such that $c_1(L, h) = d \cdot \omega$.*

Proof. For the existence and unicity of such metric, see [1, Proposition 1.4]. The fact that the metric h is real follows from the following argument. Let us consider the Hermitian metric $\bar{c}_L^* h$ on L . Claim: its curvature equals $-d \cdot c_X^* \omega$. Indeed, for any $x \in X$ we consider a real meromorphic section s of L such that x and $c_X(x)$ are neither zero nor pole of s (such section exists by Riemann-Roch Theorem). Then, the curvature of $(L, \bar{c}_L^* h)$ around x is $\partial\bar{\partial} \log(\bar{c}_L^* h)_x(s(x), s(x)) = \partial\bar{\partial} \log h_{c_X(x)}(c_L(s(x)), c_L(s(x))) = \partial\bar{\partial} \log h_{c_X(x)}(s(c_X(x)), s(c_X(x))) = \partial\bar{\partial} c_X^* \log h(s, s) = -c_X^* \partial\bar{\partial} \log h(s, s)$, where the last equality is due to the anti-holomorphicity of c_X . Then, the claim follows from the fact that $\partial\bar{\partial} \log h(s, s) = d \cdot \omega$. Now, consider the real Hermitian metric $(h \cdot \bar{c}_L^* h)^{1/2}$. Its curvature equals

$$\frac{1}{2}(\partial\bar{\partial} \log h(s, s) + \partial\bar{\partial} \log(\bar{c}_L^* h)(s, s)) = \frac{1}{2}(d \cdot \omega - d \cdot c_X^* \omega) = d \cdot \omega,$$

where the last equality follows from the fact that ω is compatible with the real structure. By unicity of the metric with curvature $d \cdot \omega$, this implies that $(h \cdot \bar{c}_L^* h)^{1/2}$ is a multiple of h . We actually have the equality $(h \cdot \bar{c}_L^* h)^{1/2} = h$, because for a real point $x \in \mathbb{R}X$ and a real vector $v \in \mathbb{R}L_x$ we get $(h_x(v, v) \cdot (\bar{c}_L^* h)_x(v, v))^{1/2} = (h_x(v, v) \cdot h_x(v, v))^{1/2} = h_x(v, v)$. \square

Definition 1.2. Let ω be a compatible volume form of mass 1, let $L \in \text{Pic}_{\mathbb{R}}^d(X)$ be a degree d line bundle over X and h be the real Hermitian metric given by the previous proposition. We define the \mathcal{L}^2 -scalar product on $\mathbb{R}H^0(X; L)$ by

$$\langle \alpha, \beta \rangle_{\mathcal{L}^2} = \int_{x \in X} h_x(\alpha(x), \beta(x)) \omega$$

for any pair of real holomorphic sections $\alpha, \beta \in \mathbb{R}H^0(X; L)$.

1.2 The space of real branched coverings

In this section we introduce and study the space of real branched coverings from a real algebraic curve (X, c_X) to $(\mathbb{C}\mathbb{P}^1, \text{conj})$.

Definition 1.3. We denote by $\mathcal{M}_d^{\mathbb{R}}(X)$ the space of all degree d real branched coverings $u : X \rightarrow \mathbb{C}\mathbb{P}^1$, that are the branched coverings such that $u \circ c_X = \text{conj} \circ u$.

A natural way to define a degree d real branched covering is the following one. Consider a degree d real holomorphic line bundle $L \in \text{Pic}_{\mathbb{R}}^d(X)$ and two real holomorphic sections $\alpha, \beta \in \mathbb{R}H^0(X, L)$ without common zeros. Then, we can define the degree d real branched covering $u_{\alpha\beta}$ defined by

$$u_{\alpha\beta} : x \in X \mapsto [\alpha(x) : \beta(x)] \in \mathbb{C}\mathbb{P}^1.$$

Proposition 1.4. *Two pairs $(\alpha, \beta), (\alpha', \beta')$ of real holomorphic sections of L define the same real branched covering if and only if $(\alpha', \beta') = (\lambda\alpha, \lambda\beta)$ for some $\lambda \in \mathbb{R}^*$.*

Proof. The proof follows the lines of [1, Proposition 1.1]. □

Proposition 1.5. *There exists a natural map from $\mathcal{M}_d^{\mathbb{R}}(X)$ to the space $\text{Pic}_{\mathbb{R}}^d(X)$ of degree d real line bundles over X . This natural map is given by $u \in \mathcal{M}_d^{\mathbb{R}}(X) \mapsto u^*\mathcal{O}(1) \in \text{Pic}_{\mathbb{R}}^d(X)$. The fiber over $L \in \text{Pic}_{\mathbb{R}}^d(X)$ is the open subset of $\mathbb{P}(\mathbb{R}H^0(X; L)^2)$ given by (the class of) pair of sections (α, β) without common zeros.*

Proof. Given a degree d real branched covering $u : X \rightarrow \mathbb{C}\mathbb{P}^1$, we get a degree d real line bundle $u^*\mathcal{O}(1)$ over X and the class of two real holomorphic global holomorphic sections without common zeros $[u^*x_0 : u^*x_1] \in \mathbb{P}(\mathbb{R}H^0(X; u^*\mathcal{O}(1))^2)$. On the other hand, given a degree d real line bundle $L \rightarrow X$ and two real holomorphic global sections without common zeros $(\alpha, \beta) \in \mathbb{R}H^0(X; L)^2$, then the map $u_{\alpha\beta} : X \rightarrow \mathbb{C}\mathbb{P}^1$ defined by $x \mapsto [\alpha(x) : \beta(x)]$ is a degree d real branched covering. Moreover, by Proposition 1.4, two pairs (α, β) and (α', β') of real holomorphic sections of L define the same real branched covering if and only if $(\alpha', \beta') = (\lambda\alpha, \lambda\beta)$ for some $\lambda \in \mathbb{R}^*$, hence the result. □

1.3 Probability on $\mathcal{M}_d^{\mathbb{R}}(X)$

Let X be a real algebraic curve equipped with a compatible volume form ω of total mass 1. In this section, we construct a natural probability measure on the space $\mathcal{M}_d^{\mathbb{R}}(X)$ of degree d real branched coverings from X to $\mathbb{C}\mathbb{P}^1$.

Let $L \in \text{Pic}_{\mathbb{R}}^d(X)$ be a degree d real line bundle equipped with the real Hermitian metric h given by Proposition 1.1. We recall that in Definition 1.2 we defined the \mathcal{L}^2 -scalar product on the space $\mathbb{R}H^0(X; L)$ induced by the Hermitian metric h . This \mathcal{L}^2 -scalar product induces a scalar product on the Cartesian product $\mathbb{R}H^0(X; L)^2$ and then a Fubini-Study metric on $\mathbb{P}(\mathbb{R}H^0(X; L)^2)$. We recall that the Fubini-Study metric is constructed as follows. First, we restrict the scalar product to the unit sphere of $\mathbb{R}H^0(X; L)^2$. The obtained metric is invariant under the action of $\mathbb{Z}/2\mathbb{Z}$ and the Fubini-Study metric is then the quotient metric on $\mathbb{P}(\mathbb{R}H^0(X; L)^2)$.

Definition 1.6. Let L be a real holomorphic line bundle over X . We denote by μ_L the probability measure on $\mathbb{P}(\mathbb{R}H^0(X; L)^2)$ induced by the normalized Fubini-Study volume form. Here, the Fubini-Study metric on $\mathbb{P}(\mathbb{R}H^0(X; L)^2)$ is the one induced by the Hermitian metric on L given by Proposition 1.1.

Proposition 1.7. *The probability measure μ_L over $\mathbb{P}(\mathbb{R}H^0(X; L)^2)$ does not depend on the choice of the multiplicative constant in front of the metric h given by Proposition 1.1.*

Proof. The proof follows the line of [1, Proposition 1.7] □

Remark 1.8. *For a real holomorphic line bundle L , we denote by Λ_L the space of pair of sections $(s_0, s_1) \in \mathbb{R}H^0(X; L)^2$ with at least a common zeros. By [2, Proposition 2.11], the set Λ_L has zero measure (it is an hypersurface), at least if the degree of L is large enough. This implies that μ_L induces a probability measure on $\mathbb{P}(\mathbb{R}H^0(X; L)^2 \setminus \Lambda_L)$, still denoted by μ_L .*

Definition 1.9. We define the probability measure μ_d on $\mathcal{M}_d^{\mathbb{R}}(X)$ by the following equality:

$$\int_{\mathcal{M}_d^{\mathbb{R}}(X)} f d\mu_d = \int_{L \in \text{Pic}_{\mathbb{R}}^d(X)} \left(\int_{\mathcal{M}_d^{\mathbb{R}}(X, L)} f d\mu_L \right) d\mathbb{H}(L)$$

for any $f \in \mathcal{M}_d^{\mathbb{R}}(X)$ measurable function. Here:

- $\mathcal{M}_d^{\mathbb{R}}(X, L)$ is the fiber of the natural morphism $\mathcal{M}_d^{\mathbb{R}}(X) \rightarrow \text{Pic}_{\mathbb{R}}^d(X)$ defined in Proposition 1.5.
- μ_L denotes (by a slight abuse of notation) the restriction to $\mathcal{M}_d^{\mathbb{R}}(X, L)$ of the probability measure on $\mathbb{P}(\mathbb{R}H^0(X, L)^2)$ defined in Definition 1.6.
- dH denotes the normalized Haar measure on $\text{Pic}_{\mathbb{R}}^d(X)$.

Remark 1.10. *The probability measure μ_d of Definition 1.9 is the real analogue of the one constructed in the complex setting in [1] for the study of random branched coverings from a fixed Riemann surface to $\mathbb{C}\mathbb{P}^1$. Also, in the complex setting, a similar construction has been considered by Zelditch in [14] in order to study large deviations of empirical measures of zeros on a Riemann surface.*

Example 1.11. *Let us consider the case $(X, c_X) = (\mathbb{C}\mathbb{P}^1, \text{conj})$, where $\mathbb{C}\mathbb{P}^1$ is equipped with the Fubini-Study form ω_{FS} . For the projective line $\mathbb{C}\mathbb{P}^1$, the unique degree d real line bundle is the line bundle $\mathcal{O}(d)$, which is naturally equipped with a real Hermitian metric h_d whose curvature equals $d \cdot \omega$. The space of real holomorphic global sections $\mathbb{R}H^0(\mathbb{C}\mathbb{P}^1; \mathcal{O}(d))$ is isomorphic to the space of degree d homogeneous polynomials $\mathbb{R}_d^{\text{hom}}[X_0, X_1]$ and the \mathcal{L}^2 -scalar product coincides with the Kostlan scalar product (i.e. the scalar product which makes $\{\sqrt{\binom{d}{k}} X_0^k X_1^{d-k}\}_{0 \leq k \leq d}$ an orthonormal basis, see [8, 11]). Then, a random real branched covering $u : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ is given by the class of a pairs of independent Kostlan polynomials.*

2 Gaussian measures and estimates of higher moments

In this section, we introduce some Gaussian measures on the spaces $\mathbb{R}H^0(X; L)^2$ and $H^0(X; L)^2$, as in [6, 5, 10, 2]. We follow the notations of Section 1. In particular, (X, c_X) is a real algebraic curve whose real locus $\mathbb{R}X$ is not empty.

2.1 Gaussian measures

In this section, given any degree d real line $L \in \text{Pic}_{\mathbb{R}}^d(X)$, we equip the cartesian product $\mathbb{R}H^0(X; L)^2$ of the space of real holomorphic section with a Gaussian measure γ_L . In order to do this, we fix a compatible volume form ω of total volume 1 (i.e. $c_X^* \omega = -\omega$ and $\int_X \omega = 1$). Given $L \in \text{Pic}_{\mathbb{R}}^d(X)$, we equip L by the real Hermitian metric h with curvature $d \cdot \omega$ (the metric h is unique up to a multiplicative constant, see Proposition 1.1).

In Definition 1.2, we defined a \mathcal{L}^2 -Hermitian product on the space $\mathbb{R}H^0(X; L)$ of real holomorphic global holomorphic sections of L denoted by $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$ and defined by

$$\langle \alpha, \beta \rangle_{\mathcal{L}^2} = \int_{x \in X} h_x(\alpha(x), \beta(x)) \omega$$

for all α, β in $\mathbb{R}H^0(X; L)$.

Definition 2.1. The \mathcal{L}^2 -scalar product on $\mathbb{R}H^0(X; L)^2$ induces a Gaussian measure γ_L on $\mathbb{R}H^0(X; L)^2$ defined by

$$\gamma_L(A) = \frac{1}{\pi^{N_d}} \int_{(\alpha, \beta) \in A} e^{-\|\alpha\|_{\mathcal{L}^2}^2 - \|\beta\|_{\mathcal{L}^2}^2} d\alpha d\beta$$

for any open subset $A \subset \mathbb{R}H^0(X; L)^2$. Here $d\alpha d\beta$ is the Lebesgue measure on $(\mathbb{R}H^0(X; L)^2; \langle \cdot, \cdot \rangle_{\mathcal{L}^2})$ and N_d denotes the dimension of $\mathbb{R}H^0(X; L)$, which equals the complex dimension of $H^0(X; L)$.

Remark 2.2. *If $d > 2g - 2$, where g is the genus of X , then $H^1(X; L) = 0$ and then, by Riemann-Roch theorem, we have $N_d = d + 1 - g$.*

Proposition 2.3. *[1, Proposition 1.12] Let f be a function on an Euclidian space $(V, \langle \cdot, \cdot \rangle)$ which is constant over the lines, i.e. $f(v) = f(\lambda v)$ for all $v \in V$ and all $\lambda \in \mathbb{R}^*$. Denote by $d\gamma$ the Gaussian measure on V induced by $\langle \cdot, \cdot \rangle$ and by $d\mu$ the normalized Fubini-Study measure on the projectivized $\mathbb{P}(V)$. Then, for all cones $A \subset V$, we have*

$$\int_A f d\gamma = \int_{\mathbb{P}(A)} [f] d\mu$$

where $\mathbb{P}(A)$ is the projectivized of A and $[f]$ is the function on $\mathbb{P}(V)$ induced by f .

We will be also interested in the *complex* Gaussian measure on the space $H^0(X, L)^2$. Indeed, the Hermitian metric h on L defines a \mathcal{L}^2 -Hermitian product on $H^0(X, L)$ by the formula

$$\langle \alpha, \beta \rangle_{\mathcal{L}^2} = \int_{x \in \Sigma} h_x(\alpha(x), \beta(x)) \omega$$

for all α, β in $H^0(\Sigma; L)$.

Definition 2.4. The complex Gaussian measure $\gamma_L^{\mathbb{C}}$ on $H^0(\Sigma; L)^2$ is defined by

$$\gamma_L^{\mathbb{C}}(A) = \frac{1}{\pi^{2N_d}} \int_{(\alpha, \beta) \in A} e^{-\|\alpha\|_{\mathcal{L}^2}^2 - \|\beta\|_{\mathcal{L}^2}^2} d\alpha d\beta$$

for any open subset $A \subset H^0(\Sigma; L)^2$. Here $d\alpha d\beta$ is the Lebesgue measure on $(H^0(\Sigma; L)^2; \langle \cdot, \cdot \rangle_{\mathcal{L}^2})$ and N_d denotes the complex dimension of $H^0(\Sigma; L)$.

2.2 Jet maps and peak sections

Let F and E be respectively degree 1 and 0 real holomorphic line bundles over X . We equip F and E by the real Hermitian metrics given by Proposition 1.1 which we denote by h_F and h_E . In particular the real Hermitian metric $h_d \doteq h_F^d \otimes h_E$ on $F^d \otimes E$ is such that its curvature equals $d \cdot \omega$. Finally, recall that the space $H^0(X, F^d \otimes E)$ is endowed with the \mathcal{L}^2 -Hermitian product

$$\langle \alpha, \beta \rangle_{\mathcal{L}^2} = \int_{x \in X} h_d(\alpha(x), \beta(x)) \omega$$

defined by for any α, β in $H^0(X; F^d \otimes E)$.

Definition 2.5. For any $x \in X$, let H_x be the kernel of the map $s \in H^0(X, F^d \otimes E) \mapsto s(x) \in (F^d \otimes E)_x$. Similarly, we denote by H_{2x} the kernel of the map $s \in H_x \mapsto \nabla s(x) \in (F^d \otimes E)_x \otimes T_{X,x}^*$. We define the following jet maps:

$$\begin{aligned} ev_x &: s \in H^0(X, F^d \otimes E)/H_x \mapsto s(x) \in (F^d \otimes E)_x, \\ ev_{2x} &: s \in H_x/H_{2x} \mapsto \nabla s(x) \in (F^d \otimes E)_x \otimes T_{X,x}^*. \end{aligned}$$

The previous definition has the following real analogue:

Definition 2.6. For any point $x \in X$, we define the real vector spaces $\mathbb{R}H_x^0 = H_x^0 \cap \mathbb{R}H^0(X, F^d \otimes E)$ and $\mathbb{R}H_{2x}^0 = H_{2x}^0 \cap \mathbb{R}H^0(X, F^d \otimes E)$ and the real jet maps by

$$\begin{aligned} ev_x^{\mathbb{R}} &: s \in \mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x^0 \mapsto s(x) \in (F^d \otimes E)_x, \\ ev_{2x}^{\mathbb{R}} &: s \in \mathbb{R}H_x/\mathbb{R}H_{2x} \mapsto \nabla s(x) \in (F^d \otimes E)_x \otimes T_{X,x}^*. \end{aligned}$$

By the fact that F is ample (recall that $\deg F = 1$), we get that for d large enough the maps $ev_x^{\mathbb{R}}$, ev_x , $ev_{2x}^{\mathbb{R}}$ and ev_{2x} are invertible. The following proposition estimates the norms of this maps and of their inverses.

Proposition 2.7. [6, Propositions 4 and 6] For any $B > 0$, then there exists an integer d_B and a positive constant c_B such that, for any $d \geq d_B$ and any point $x \in X$ with $\text{dist}(x, \mathbb{R}X) \geq B \frac{\log d}{\sqrt{d}}$, the maps $d^{-\frac{1}{2}} ev_x^{\mathbb{R}}$, $d^{-\frac{1}{2}} ev_x$, $d^{-1} ev_{2x}^{\mathbb{R}}$ and $d^{-1} ev_{2x}$ as well as their inverse have norms and determinants bounded from above by c_B .

Remark 2.8. In [6, Propositions 4 and 6], the constant B equals 1, and the line bundle E is trivial. The same proof actually holds for any fixed $B > 0$ and any $E \in \text{Pic}_{\mathbb{R}}^0(X)$. Indeed, the proof is based on the theory peak sections and Bergman kernels and this theory holds in this more general setting (see for example [4] or [9, Theorem 4.2.1]).

Using the \mathcal{L}^2 -Hermitian product on $H^0(X, F^d \otimes E)$, we can identify $H^0(X, F^d \otimes E)/H_x$ with the orthogonal complement of H_x in $H^0(X, F^d \otimes E)$. Similarly, we identify the quotient H_x/H_{2x} with the orthogonal complement of H_{2x} in H_x . We then have an orthogonal decomposition

$$H^0(X, F^d \otimes E) = H^0(X, F^d \otimes E)/H_x \oplus H_x/H_{2x} \oplus H_{2x}.$$

Similarly, using the \mathcal{L}^2 -scalar product on $\mathbb{R}H^0(X, F^d \otimes E)$, we have the orthogonal decomposition

$$\mathbb{R}H^0(X, F^d \otimes E) = \mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x \oplus \mathbb{R}H_x/\mathbb{R}H_{2x} \oplus \mathbb{R}H_{2x}.$$

The map $ev_x \times ev_{2x}$ (resp. $ev_x^{\mathbb{R}} \times ev_{2x}^{\mathbb{R}}$) gives an isomorphism between $H^0(X, F^d \otimes E)/H_x \oplus H_x/H_{2x}$ (resp. $\mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x \oplus \mathbb{R}H_x/\mathbb{R}H_{2x}$) and the fiber $(F^d \otimes E)_x \oplus (F^d \otimes E)_x \otimes T_{X,x}^*$.

Moreover, remark that we have natural identifications $H^0(X, F^d \otimes E)/H_x \oplus H_x/H_{2x} = H_{2x}^\perp$ and $\mathbb{R}H^0(X, F^d \otimes E)/\mathbb{R}H_x \oplus \mathbb{R}H_x/\mathbb{R}H_{2x} = \mathbb{R}H_{2x}^\perp$. A direct consequence of Proposition 2.7 is the following

Corollary 2.9. *For any $B > 0$, there exist an integer d_B and a positive constant c_B such that, for any $d \geq d_B$ and any x with $\text{dist}(x, \mathbb{R}X) \geq B \frac{\log d}{\sqrt{d}}$, the map $(ev_x^{\mathbb{R}} \times ev_{2x}^{\mathbb{R}})^{-1} \circ (ev_x \times ev_{2x}) : H_{2x}^\perp \rightarrow \mathbb{R}H_{2x}^\perp$ has determinant bounded from above by c_B and from below by $1/c_B$.*

Definition 2.10. We denote by s_0 and s_1 the global holomorphic sections of $L^d \otimes E$ of unit \mathcal{L}^2 -norm which generates respectively the orthogonal of H_x in $H^0(X, F^d \otimes E)$ and the orthogonal of H_{2x} in H_x . We call these sections the *peak sections* at x .

The pointwise estimate of the norms (with respect to the Hermitian metric h_d of curvature $d \cdot \omega$) of the peak sections are well known and strictly related to the estimates of the Bergman kernel along the diagonal (see [12, 13, 3, 9]). With a slight abuse of notation, we will denote by $\|\cdot\|$ any norm induced by h_d .

Lemma 2.11. ([1, Proposition 1.5]) *For any $x \in X$, let s_0 and s_1 be the peak sections defined in Definition 2.10. Then, as $d \rightarrow +\infty$, we have the estimates $\|s_0(x)\| = \frac{\sqrt{d}}{\sqrt{\pi}}(1 + O(d^{-1}))$ and $\|\nabla s_1(x)\| = \frac{d}{\sqrt{\pi}}(1 + O(d^{-1}))$, where the error terms are uniform in $x \in X$.*

2.3 Wronskian and higher moments

Let F and E be respectively degree 1 and 0 real holomorphic line bundles over X . The purpose of this section is to prove Proposition 2.15, which gives key estimates of the higher moments of the random variable $(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2 \mapsto \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\|$, where $W_{\alpha\beta}$ is the Wronskian, given by the following

Definition 2.12. Let ∇ be a connection on $F^d \otimes E$. For any pair of real holomorphic global sections $(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2$, we denote by $W_{\alpha\beta}$ the Wronskian $\alpha \otimes \nabla \beta - \beta \otimes \nabla \alpha$, which is a real holomorphic global section of $F^{2d} \otimes E^2 \otimes T_X^*$.

Remark 2.13. *The Wronskian $W_{\alpha\beta}$ does not depend on the choice of a connection on $F^d \otimes E$. Indeed, two connections ∇ and ∇' on $F^d \otimes E$ differ by a 1-form θ , and then we have*

$$(\alpha \otimes \nabla \beta - \beta \otimes \nabla \alpha) - (\alpha \nabla' \beta - \beta \nabla' \alpha) = \alpha \otimes (\nabla - \nabla') \beta - \beta \otimes (\nabla - \nabla') \alpha = \alpha \otimes \beta \otimes \theta - \beta \otimes \alpha \otimes \theta = 0.$$

Proposition 2.14. [1, Proposition 2.3] *Let F and E be respectively degree 1 and 0 real line bundles over X and $(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2$ be a pair of sections without common zeros. A point $x \in X$ is a critical point of the map $u_{\alpha\beta} : x \in X \mapsto [\alpha(x) : \beta(x)] \in \mathbb{C}\mathbb{P}^1$ if and only if it is a zero of the Wronskian $W_{\alpha\beta}$ defined in Definition 2.12.*

Proposition 2.15. *Let X be a real algebraic curve equipped with a compatible volume form ω of total volume 1 and let $F \in \text{Pic}_{\mathbb{R}}^1(X)$. For any $B > 0$ there exists an integer d_B and a constant c_B such that for any $E \in \text{Pic}_{\mathbb{R}}^0(X)$, any $m \in \mathbb{N}$, any $d \geq d_B$ and any point $x \in X$ with $\text{dist}(x, \mathbb{R}X) \geq B \frac{\log d}{\sqrt{d}}$, we have*

$$\int_{(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2} \left| \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\| \right|^m d\gamma_d(\alpha, \beta) \leq c_B(m+1)!.$$

Here, $\text{dist}(\cdot, \cdot)$ is the distance in X induced by ω , γ_d is the Gaussian measure on $\mathbb{R}H^0(X, F^d \otimes E)^2$ constructed in Section 2.1 and $\|\cdot\|$ denote the norm induced by the Hermitian metrics on F and E given by Proposition 1.1.

Proof. Let us consider the integral we want to estimate:

$$\int_{(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2} \left| \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\| \right|^m d\gamma_d(\alpha, \beta). \quad (3)$$

First, remark that the function in the integral (3) only depends on the 1-jet of the sections α and β . We will then write the orthogonal decomposition $\mathbb{R}H^0(X, F^d \otimes E) = \mathbb{R}H_{2x} \oplus \mathbb{R}H_{2x}^\perp$, where $\mathbb{R}H_{2x}$ is the space of real sections s such that $s(x) = 0$ and $\nabla s(x) = 0$. As the Gaussian measure is a product measure, after the integration over the orthogonal of $\mathbb{R}H_{2x}^\perp \times \mathbb{R}H_{2x}^\perp$, we get that the integral (3) is equal to

$$\int_{(\alpha, \beta) \in \mathbb{R}H_{2x}^\perp \times \mathbb{R}H_{2x}^\perp} \left| \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\| \right|^m d\gamma_d \big|_{\mathbb{R}H_{2x}^\perp \times \mathbb{R}H_{2x}^\perp}(\alpha, \beta). \quad (4)$$

Using the notations of Section 2.2, and in particular Definitions 2.5 and 2.6, let $J_d : H_{2x}^\perp \rightarrow \mathbb{R}H_{2x}^\perp$ be the map $(ev_x^\mathbb{R} \times ev_{2x}^\mathbb{R})^{-1} \circ (ev_x \times ev_{2x})$ and denote by

$$I_d = J_d \times J_d : H_{2x}^\perp \times H_{2x}^\perp \rightarrow \mathbb{R}H_{2x}^\perp \times \mathbb{R}H_{2x}^\perp.$$

By changing of variables given by the isomorphism I_d , we get

$$(4) = \int_{(\alpha, \beta) \in H_{2x}^\perp \times H_{2x}^\perp} \left| \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\| \right|^m (I_d^{-1})_* (d\gamma_d \big|_{\mathbb{R}H_{2x}^\perp \times \mathbb{R}H_{2x}^\perp})(\alpha, \beta). \quad (5)$$

By Corollary 2.9, the maps I_d and I_d^{-1} have determinants bounded from above by a constant which only depends on B . In particular, there exists a constant c_1 , depending only on B , such that

$$(5) \leq c_1 \int_{(\alpha, \beta) \in H_{2x}^\perp \times H_{2x}^\perp} \left| \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\| \right|^m d\gamma_d^\mathbb{C} \big|_{H_{2x}^\perp \times H_{2x}^\perp}(\alpha, \beta) \quad (6)$$

where $\gamma_d^\mathbb{C}$ is the complex Gaussian measure defined in Definition 2.4. In order to prove the result, we have to bound from above the quantity

$$\int_{(\alpha, \beta) \in H_{2x}^\perp \times H_{2x}^\perp} \left| \log \left\| \frac{\pi}{d^{3/2}} W_{\alpha\beta}(x) \right\| \right|^m d\gamma_d^\mathbb{C} \big|_{H_{2x}^\perp \times H_{2x}^\perp}(\alpha, \beta) \quad (7)$$

Let s_0 and s_1 be the peak sections at x introduced in Definition 2.10 and we write $\alpha = a_0\sigma_0 + a_1\sigma_1$ and $\beta = b_0\sigma_0 + b_1\sigma_1$. We then have

$$\|W_{\alpha\beta}(x)\| = |a_0b_1 - a_1b_0| \|(s_0 \otimes \nabla s_1 - s_1 \otimes \nabla s_0)(x)\| = |a_0b_1 - a_1b_0| \frac{d^{3/2}}{\pi} (1 + O(d^{-c_2(B)})),$$

where the last equality follows from Proposition 2.11. This implies that the integral in (7) equals

$$\begin{aligned} & \int_{\substack{a=(a_0, a_1) \in \mathbb{C}^2 \\ b=(b_0, b_1) \in \mathbb{C}^2}} \left| \log \left(|a_0b_1 - a_1b_0| \left\| \frac{\pi}{d^{3/2}} (s_0 \otimes \nabla s_1 - s_1 \otimes \nabla s_0)(x) \right\| \right) \right|^m \frac{e^{-|a|^2 - |b|^2}}{\pi^4} da db \\ &= \int_{\substack{a=(a_0, a_1) \in \mathbb{C}^2 \\ b=(b_0, b_1) \in \mathbb{C}^2}} \left| \log \left(|a_0b_1 - a_1b_0| \right) \right|^m \frac{e^{-|a|^2 - |b|^2}}{\pi^4} (1 + O(d^{-c_3(B)})) da db \\ &\leq 2 \int_{\substack{a \in \mathbb{C}^2 \\ b \in \mathbb{C}^2}} |\log |a_0b_1 - a_1b_0||^m \frac{e^{-|a|^2 - |b|^2}}{\pi^4} da db \end{aligned} \quad (8)$$

where the last inequality holds for $d \geq d_B$, for some d_B large enough.

In the remaining part of the proof, we will estimate the last integral appearing in (8). In order to do this, for any $a = (a_0, a_1)$ we make an unitary transformation of \mathbb{C}^2 (of coordinates b_0, b_1) by sending the vector $(1, 0)$ to $v_a = \frac{1}{\sqrt{|a_0|^2 + |a_1|^2}}(a_0, a_1)$ and the vector $(0, 1)$ to $w_a = \frac{1}{\sqrt{|a_0|^2 + |a_1|^2}}(-\bar{a}_1, \bar{a}_0)$. We will write any vector of \mathbb{C}^2 as a sum $tv_a + sw_a$ with $s, t \in \mathbb{C}$. Under this change of variables, the integral appearing in (8) becomes

$$\leq 2 \int_{\substack{a \in \mathbb{C}^2 \\ (s, t) \in \mathbb{C}^2}} |\log |s|| |a| \left\| \frac{e^{-|a|^2 - |s|^2 - |t|^2}}{\pi^4} \right\| da ds dt = 2 \int_{\substack{a \in \mathbb{C}^2 \\ s \in \mathbb{C}}} |\log |s|| |a| \left\| \frac{e^{-|a|^2 - |s|^2}}{\pi^3} \right\| da ds. \quad (9)$$

We pass to polar coordinates $a = re^{i\theta}$, for $\theta \in S^1$ and $r \in \mathbb{R}_+$, and $s = \rho e^{i\phi}$, for $\phi \in S^1$ and $\rho \in \mathbb{R}_+$, and we obtain

$$2 \int_{\substack{a \in \mathbb{C}^2 \\ s \in \mathbb{C}}} |\log |s| | |a| |^m \frac{e^{-|a|^2 - |s|^2}}{\pi^3} da ds = 8 \int_{\substack{r \in \mathbb{R}_+ \\ \rho \in \mathbb{R}_+}} |\log \rho r|^m e^{-r^2 - \rho^2} r^3 \rho dr d\rho. \quad (10)$$

Writing $\log \rho r = \log \rho + \log r$, developing the binomial and using the triangular inequality, we obtain

$$(10) \leq 8 \int_{\substack{r \in \mathbb{R}_+ \\ \rho \in \mathbb{R}_+}} \sum_{k=0}^m \binom{m}{k} |\log \rho|^k |\log r|^{m-k} e^{-r^2 - \rho^2} r^3 \rho dr d\rho. \quad (11)$$

Let us study the integrals $\int_{\rho \in \mathbb{R}_+} |\log \rho|^n e^{-\rho^2} \rho d\rho$ and $\int_{r \in \mathbb{R}_+} |\log r|^n e^{-r^2} r^3 dr$. To compute these two integrals, we will use the following formula obtained by integration by part:

$$\int (\log x)^n dx = x \log x - n \int (\log x)^{n-1} dx, \quad n > 0. \quad (12)$$

- Computation of the integral $\int_{\rho \in \mathbb{R}_+} |\log \rho|^n e^{-\rho^2} \rho d\rho$. We write

$$\int_{\rho \in \mathbb{R}_+} |\log \rho|^n e^{-\rho^2} \rho d\rho = \int_{\rho=0}^1 (-\log \rho)^n e^{-\rho^2} \rho d\rho + \int_{\rho=1}^{\infty} (\log \rho)^n e^{-\rho^2} \rho d\rho. \quad (13)$$

For the first term of this sum we have

$$\int_{\rho=0}^1 (-\log \rho)^n e^{-\rho^2} \rho d\rho \leq \frac{\sqrt{2}}{2} \int_{\rho=0}^1 (-\log \rho)^n d\rho = \frac{\sqrt{2}}{2} n! \quad (14)$$

where we used first that $e^{-\rho^2} \rho \leq \frac{\sqrt{2}}{2}$ for $\rho \in [0, 1]$ and then we used n times the formula (12).

For the second term of the sum in (13), we use first the fact that $e^{-\rho^2} \rho \leq \frac{e^{-\frac{1}{\rho^2}}}{\rho^3}$ for any $\rho \geq 1$ and then the change $t = 1/\rho$, to have

$$\int_{\rho=1}^{\infty} (\log \rho)^n e^{-\rho^2} \rho d\rho \leq \int_{\rho=1}^{\infty} (\log \rho)^n \frac{e^{-\frac{1}{\rho^2}}}{\rho^3} d\rho \stackrel{t=1/\rho}{=} - \int_1^0 (\log(1/t))^n t e^{-t} dt = \int_0^1 (-\log(t))^n t e^{-t} dt. \quad (15)$$

The last integral is the same as in (14), so from (14) and (15) we obtain

$$\int_{\rho=1}^{\infty} (\log \rho)^n e^{-\rho^2} \rho d\rho \leq \frac{\sqrt{2}}{2} n! \quad (16)$$

Putting (14) and (16) in (13), we obtain

$$\int_{\rho \in \mathbb{R}_+} |\log \rho|^n e^{-\rho^2} \rho d\rho \leq \sqrt{2} n!. \quad (17)$$

- Computation of the integral $\int_{r \in \mathbb{R}_+} |\log r|^n e^{-r^2} r^3 dr$. As before, we write

$$\int_{r \in \mathbb{R}_+} |\log r|^n e^{-r^2} r^3 dr = \int_{r=0}^1 (-\log r)^n e^{-r^2} r^3 dr + \int_{r=1}^{\infty} (\log r)^n e^{-r^2} r^3 dr. \quad (18)$$

For the first term of the sum, we get

$$\int_{r=0}^1 (-\log r)^n e^{-r^2} r^3 dr \leq (-1)^n \frac{\sqrt{2}}{\sqrt{3}} \int_{r=0}^1 (\log r)^n dr = \frac{\sqrt{2}}{\sqrt{3}} n! \quad (19)$$

where the first inequality follows from $e^{-r^2} r^3 \leq \frac{\sqrt{2}}{\sqrt{3}}$, for $r \in [0, 1]$, and the last equality is obtained using n times the formula (12).

For the second term of the sum in the right-hand side of (18), we use integration by parts with respect to the functions $-\frac{1}{2}(\log r)^n r^2$ and $-2re^{-r^2}$ to obtain

$$\int_{s=1}^{\infty} (\log r)^n e^{-r^2} r^3 dr = \left[-\frac{1}{2}(\log r)^n r^2 e^{-r^2}\right]_{r=1}^{\infty} + \frac{n}{2} \int_{r=1}^{\infty} (\log r)^{n-1} r e^{-r^2} dr + \int_{r=1}^{\infty} (\log r)^n r e^{-r^2} dr. \quad (20)$$

As $\left[-\frac{1}{2}(\log r)^n r^2 e^{-r^2}\right]_{r=1}^{\infty} = 0$ we obtain, by using (16) in (20), that

$$\int_{s=1}^{\infty} (\log r)^n e^{-r^2} r^3 dr \leq \frac{3\sqrt{2}}{4} n!. \quad (21)$$

Putting (19) and (21) in (18), we get

$$\int_{r \in \mathbb{R}_+} |\log r|^n e^{-r^2} r^3 ds \leq \frac{4\sqrt{6} + 9\sqrt{2}}{12} n!. \quad (22)$$

Now, we use (17) and (22) and we obtain the following estimate:

$$\int_{\substack{r \in \mathbb{R}_+ \\ \rho \in \mathbb{R}_+}} \sum_{k=0}^m \binom{m}{k} |\log \rho|^k |\log r|^{m-k} e^{-r^2 - s^2} r^3 \rho dr d\rho \leq \frac{4\sqrt{3} + 9}{6} \sum_{k=0}^m \binom{m}{k} k!(m-k)! \leq \frac{4\sqrt{3} + 9}{6} (m+1)!. \quad (23)$$

Putting (23) in (11) and using (10), (9) and (8), we obtain the desired estimate for (7), hence the result. \square

3 Proof of Theorem 0.1

In this section, we prove our main result. We follow the notations of Sections 1 and 2.

Proposition 3.1. *Let X be a real algebraic curve equipped with a compatible volume form ω of total volume 1 and let $F \in \text{Pic}_{\mathbb{R}}^1(X)$. Fix a sequence of positive real numbers $(a_d)_d$. Then, for any $B > 0$ there exists $d_B \in \mathbb{N}$ and a constant c_B such that, for any $E \in \text{Pic}_{\mathbb{R}}^0(X)$, any $d \geq d_B$ and any sequence of smooth functions $(\varphi_d)_d$ with $\text{dist}(\text{supp}(\varphi_d), \mathbb{R}X) \geq B \frac{\log d}{\sqrt{d}}$, the following holds*

$$\begin{aligned} & \gamma_{F^d \otimes E} \left\{ (\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial \bar{\partial} \varphi_d \right| \geq a_d \right\} \\ & \leq c_B \exp \left(- \frac{a_d}{2 \|\partial \bar{\partial} \varphi_d\|_{\infty} \text{Vol}(\text{Supp}(\partial \bar{\partial} \varphi_d))} \right). \end{aligned}$$

Here, $\text{dist}(\cdot, \cdot)$ is the distance in X induced by ω , $\gamma_{F^d \otimes E}$ is the Gaussian measure on $\mathbb{R}H^0(X, F^d \otimes E)^2$ constructed in Section 2.1 and $\|\cdot\|$ denote the pointwise norm induced by the Hermitian metrics on F and E given by Proposition 1.1.

Proof. For any $t_d > 0$, let us denote

$$\exp \left(t_d \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial \bar{\partial} \varphi_d \right| \right) = \sum_{m=0}^{\infty} \frac{t_d^m}{m!} \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial \bar{\partial} \varphi_d \right|^m. \quad (24)$$

Remark that

$$\left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial \bar{\partial} \varphi_d \right| \geq a_d d \Leftrightarrow \exp \left(t_d \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial \bar{\partial} \varphi_d \right| \right) \geq e^{t_d a_d} \quad (25)$$

so that, by Markov inequality, we have

$$\begin{aligned} & \gamma_{F^d \otimes E} \left\{ (\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \left| \int_X \log \frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \partial \bar{\partial} \varphi_d \right| \geq a_d \right\} \leq \\ & e^{-t_d a_d} \int_{\mathbb{R}H^0(X, F^d \otimes E)^2} \exp \left(t_d \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial \bar{\partial} \varphi_d \right| \right) d\gamma_{F^d \otimes E}. \end{aligned} \quad (26)$$

Now, we have

$$\left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \partial\bar{\partial}\varphi_d \right|^m \leq \|\partial\bar{\partial}\varphi_d\|_\infty^m \left| \int_{\text{Supp}(\partial\bar{\partial}\varphi_d)} \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \omega \right|^m. \quad (27)$$

We then apply Hölder inequality with m and $m/(m-1)$ for the functions $\log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right)$ and 1, so that

$$(27) \leq \|\partial\bar{\partial}\varphi_d\|_\infty^m \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d))^{m-1} \int_{\text{Supp}(\partial\bar{\partial}\varphi_d)} \left| \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}(x)\| \right) \right|^m \omega. \quad (28)$$

By Proposition 2.15, there exists $d_B \in \mathbb{N}$ and a positive constant c_B such that for any $d \geq d_B$ we get

$$\text{right-hand side of (28)} \leq \|\partial\bar{\partial}\varphi_d\|_\infty^m \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d))^m c_B (m+1)!. \quad (29)$$

Then, by (26), (24) and (29), we have

$$\begin{aligned} & \gamma_{F^d \otimes E} \left\{ (\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \left| \int_X \log \frac{\|W_{\alpha\beta}(x)\|}{d^{3/2}} \partial\bar{\partial}\varphi_d \right| \geq a_d \right\} \leq \\ & e^{-t_d a_d} c_B \sum_{m=0}^{\infty} (m+1) \left(\|\partial\bar{\partial}\varphi_d\|_\infty \cdot \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d)) \right)^m t_d^m. \end{aligned} \quad (30)$$

Now, we have the identity $\sum_{m=0}^{\infty} (m+1)x^m = \frac{d}{dx} \sum_{m=1}^{\infty} x^m = \frac{d}{dx} \left(\frac{1}{(1-x)} - 1 \right) = \frac{1}{(1-x)^2}$, so that the right hand side in (30) equals

$$\frac{c_B \exp(-t_d a_d)}{(1-t_d \|\partial\bar{\partial}\varphi_d\|_\infty \cdot \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d)))^2} \quad (31)$$

Putting $t_d = (2\|\partial\bar{\partial}\varphi_d\|_\infty \cdot \text{Vol}(\text{Supp}(\partial\bar{\partial}\varphi_d)))^{-1}$, we get the result. \square

Lemma 3.2 (Lemma 2 of [5]). *There exist positive constants C_i , $i \in \{1, \dots, 4\}$, and a family of cutoff functions $\chi_t : X \rightarrow [0, 1]$, defined for $t \in (0, t_0]$, for some $t_0 > 0$, such that*

1. $\text{Vol}(\text{supp}(\partial\bar{\partial}\chi_t)) \leq C_1 t$;
2. $\text{Vol}(X \setminus \chi_t^{-1}(1)) \leq C_2 t$;
3. $\|\partial\bar{\partial}\chi_t\|_{L^\infty} \leq C_3 t^{-2}$;
4. $\text{dist}(\text{supp}(\chi_t), \mathbb{R}X) \geq C_4 t$.

We now prove the following fiberwise version of Theorem 0.1.

Theorem 3.3. *Let $\ell(d)$ be a sequence of positive real numbers such that $\ell(d) \geq B(\log d)$ for some $B > 0$. Then there exist positive constants c_1 and c_2 such that*

$$\mu_{F^d \otimes E} \{ u \in \mathcal{M}_d^{\mathbb{R}}(X, F^d \otimes E), \#(\text{Crit}(u) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d} \} \leq c_1 e^{-c_2 \ell(d)^2}.$$

Here, $\mu_{F^d \otimes E}$ is the probability measure defined in Definition 1.6 and $\mathcal{M}_d^{\mathbb{R}}(X, F^d \otimes E)$ is defined in Definition 1.9.

Proof. For any pair of real global sections $(\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2$ without common zeros, let $u_{\alpha\beta}$ be the real branched covering defined by $x \mapsto [\alpha(x) : \beta(x)]$. Consider the set

$$\mathcal{C}_{\ell(d)} \doteq \{ (\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \#(\text{Crit}(u_{\alpha\beta}) \cap \mathbb{R}X) \geq \ell(d)\sqrt{d} \}. \quad (32)$$

Remark that this set is a cone in $\mathbb{R}H^0(X, F^d \otimes E)^2$. By Proposition 2.3, this implies that the Gaussian measure of $\mathcal{C}_{\ell(d)}$ equals the Fubini-Study measure of its projectivization, which is exactly the measure we want to estimate. In order to obtain the result, we will then compute the Gaussian measure of the cone (32). Moreover, by Proposition 2.14, we have that $x \in \text{Crit}(u_{\alpha\beta})$ if and only if $W_{\alpha\beta}(x) = 0$,

so that, in order to compute $\#\text{Crit}(u_{\alpha\beta})$, we can compute the number of zeros of $W_{\alpha\beta}$. To do this, we will use Poincaré-Lelong formula, that is the following equality between currents

$$\omega_d - \sum_{x \in \{W_{\alpha\beta}=0\}} \delta_x = \frac{1}{2\pi i} \partial\bar{\partial} \log \|W_{\alpha\beta}\|, \quad (33)$$

where $\|\cdot\|$ is the (induced) metric on $F^{2d} \otimes E^2 \otimes T_X^*$ given by Proposition 1.1 and ω_d is the corresponding curvature form. Remark that ω_d equals $2d \cdot \omega + O(1)$ (the term $2d \cdot \omega$ comes from the curvature form of $F^{2d} \otimes E^2$ and the term $O(1)$ from the curvature form of T_X^*). Moreover, remark that the Hermitian metric $\frac{\pi}{d^{3/2}} \|\cdot\|$ has the same curvature of the Hermitian metric $\|\cdot\|$, because the curvature form is not affected by a multiplicative constant. Then, Poincaré-Lelong formula (33), can also be read

$$2d \cdot \omega + O(1) - \sum_{x \in \{W_{\alpha\beta}=0\}} \delta_x = \frac{1}{2\pi i} \partial\bar{\partial} \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}\| \right) \quad (34)$$

where the equality is in the sense of currents. We will apply (34) for the functions χ_{t_d} given by Lemma 3.2, for $t_d = \frac{\ell(d)}{4C_2\sqrt{d}}$, where C_2 is the constant appearing in Lemma 3.2. By (34), we then get

$$\frac{1}{2\pi} \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}\| \right) \partial\bar{\partial} \chi_{t_d} \right| \geq \left| 2d \left(1 - \frac{\ell(d)}{4\sqrt{d}} \right) + O(1) - \sum_{x \in \{W_{\alpha\beta}=0\}} \chi_{\frac{\ell(d)}{\sqrt{d}}}(x) \right|. \quad (35)$$

Remark that, for any pair of real global sections (α, β) in the cone $\mathcal{C}_{\ell(d)}$ defined in (32), we have

$$\sum_{x \in \{W_{\alpha\beta}=0\}} \chi_{\frac{\ell(d)}{\sqrt{d}}}(x) \leq 2d + 2g - 2 - \ell(d)\sqrt{d}, \quad (36)$$

where g is the genus of X . Then, putting (36) in (35), we get

$$\frac{1}{2\pi} \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}\| \right) \partial\bar{\partial} \chi_{\frac{\ell(d)}{\sqrt{d}}} \right| \geq \frac{1}{2} \ell(d)\sqrt{d} + O(1),$$

for any $(\alpha, \beta) \in \mathcal{C}_{\ell(d)}$. Then, for d large enough, the cone (32) is included in the set

$$\left\{ (\alpha, \beta) \in \mathbb{R}H^0(X, F^d \otimes E)^2, \left| \int_X \log \left(\frac{\pi}{d^{3/2}} \|W_{\alpha\beta}\| \right) \partial\bar{\partial} \chi_{\frac{\ell(d)}{\sqrt{d}}} \right| \geq \ell(d)\sqrt{d} \right\}.$$

The result then follows from Proposition 3.1 and Lemma 3.2. \square

Proof of Theorem 0.1. We fix a degree 1 real holomorphic line bundle F over X , so that for any $L \in \text{Pic}_{\mathbb{R}}^d(X)$ there exists an unique degree 0 real holomorphic line bundle $E \in \text{Pic}_{\mathbb{R}}^0(X)$ such that $L = F^d \otimes E$. The result then follows by integrating the inequality appearing in Theorem 3.3 along the compact base $\text{Pic}_{\mathbb{R}}^0(X) \simeq \text{Pic}_{\mathbb{R}}^d(X)$ (the last isomorphism is given by the choice of the degree 1 real line bundle F). \square

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