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A remark on triviality for the two-dimensional stochastic nonlinear wave equation

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Abstract

We consider the two-dimensional stochastic damped nonlinear wave equation (SdNLW) with the cubic nonlinearity, forced by a space-time white noise. In particular, we investigate the limiting behavior of solutions to SdNLW with regularized noises and establish triviality results in the spirit of the work by Hairer et al. (2012). More precisely, without renormalization of the nonlinearity, we establish the following two limiting behaviors; (i) in the strong noise regime, we show that solutions to SdNLW with regularized noises tend to 0 as the regularization is removed and (ii) in the weak noise regime, we show that solutions to SdNLW with regularized noises converge to a solution to a deterministic damped nonlinear wave equation with an additional mass term.

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1. Introduction

1.1. Stochastic damped nonlinear wave equation, renormalization, and triviality

We consider the Cauchy problem for the following stochastic damped nonlinear wave equation (SdNLW) with the cubic nonlinearity, posed on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + u^3 = \alpha\xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and $\xi(t, x)$ denotes a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$. The damped wave equation (without a stochastic forcing) appears as a model describing wave propagation with friction. It also appears as a modified heat conduction equation with the finite propagation speed property [6] and as stochastic models such as correlated random walk [19]. See [18] for further references. In the deterministic case, Eq. (1.1) has been studied extensively; see [16–18] and the references therein.

The stochastic nonlinear wave equations (SNLW) have been studied extensively in various settings; see [9, Chapter 13] for the references therein. In recent years, we have witnessed a rapid progress on the theoretical understanding of SNLW with singular stochastic forcing. In [12], Gubinelli, Koch, and the first author considered SNLW with an additive space-time white noise on $\mathbb{T}^2$:

$$\partial_t^2 u - \Delta u + u^k = \xi, \tag{1.2}$$

where $k \geq 2$ is an integer. The main difficulty of this problem already appears in the stochastic convolution Ψ, solving the linear equation:

$$\partial_t^2 \Psi - \Delta \Psi = \xi. \tag{1.3}$$

It is well known that for the spatial dimension $d \geq 2$, the stochastic convolution Ψ is not a classical function but is merely a Schwartz distribution. In particular, there is an issue in making sense of powers Ψ^k and a fortiori of the full nonlinearity u^k in (1.2). This requires us to modify the equation in order to take into account a proper *renormalization*.

In [12], by introducing appropriate time-dependent renormalization, the authors proved local well-posedness of (a renormalized version of) (1.2) on $\mathbb{T}^2$. In [13] with Tolomeo, they constructed global-in-time dynamics for (1.2) in the cubic case ($k = 3$). The local well-posedness argument in [12] essentially applies to SdNLW (1.1) with a general power-type nonlinearity u^k. When $\alpha = \sqrt{2}$, Eq. (1.1) corresponds to the so-called canonical stochastic quantization for the Φ_2^4-measure in Euclidean quantum field theory (see [28]), which formally preserves the Gibbs measure[3] for the deterministic nonlinear wave equation studied in [26].[4] By combining the local well-posedness argument with Bourgain's invariant measure argument [3,4], it was shown in [13] that SdNLW (1.1), with a general defocusing power-type nonlinearity u^{2k+1}, is almost surely globally well-posed with the Gibbs measure initial data and that the Gibbs measure is invariant under the dynamics. We also mention a recent extension [25] of these results to the case of two-dimensional compact Riemannian manifolds

[3] Namely, the Φ_2^4-measure on the u-component coupled with the white noise measure on the $\partial_t u$-component.

[4] Strictly speaking, the results mentioned here only apply to the nonlinear Klein–Gordon case, i.e. $-\Delta$ in (1.1) replaced by $1 - \Delta$.

without boundary and a recent work [11] in establishing local well-posedness of the quadratic SNLW on the three-dimensional torus $\mathbb{T}^3$.

In the works mentioned above, renormalization played an essential role, allowing us to give a precise meaning to the equations. Our main goal in this paper is to study the behavior of solutions to (1.1), in a suitable limiting sense, *without* renormalization. Namely, we consider Eq. (1.1) with a regularized noise, via frequency truncation, and study possible limiting behavior of solutions as we remove the regularization. In particular, we establish a triviality result in a certain regime; as we remove the regularization, solutions converge to 0 in the distributional sense. See Theorem 1.1.

Previously, Albeverio, Haba, and Russo [1] studied a triviality issue for the two-dimensional SNLW:

$$\partial_t^2 u - \Delta u + f(u) = \xi, \tag{1.4}$$

where f is a bounded smooth function. Roughly speaking, they showed that solutions to (1.4) with regularized noises tend to that to the stochastic linear wave equation (1.3). Let us point out several differences between [1] and our current work (besides considering the equations with/without damping). Our argument is strongly motivated by the solution theory recently developed in [12]. In particular, we carry out our analysis in a natural solution space $C([0,T]; H^{-\varepsilon}(\mathbb{T}^2))$, $\varepsilon > 0$. On the other hand, the analysis in [1] was carried out in the framework of Colombeau generalized functions, and as such, their solution does not a priori belong to $C([0,T]; H^{-\varepsilon}(\mathbb{T}^2))$. Furthermore, the cubic nonlinearity u^3 does not belong to the class of nonlinearities considered in [1]. Regarding a triviality result for Eq. (1.4), we also mention a recent work [24] on the stochastic wave equation with the sine nonlinearity.

In the parabolic setting, Hairer, Ryser, and Weber [15] studied the following stochastic Allen–Cahn equation on $\mathbb{T}^2$:

$$\partial_t u = \Delta u + u - u^3 + \alpha\xi. \tag{1.5}$$

By suitably adapting the strong solution theory due to Da Prato and Debussche [8], they established triviality for this equation; (i) in the strong noise regime, solutions to (1.5) with regularized noises tend to 0 as the regularization is removed and (ii) in the weak noise regime, solutions to (1.5) with regularized noises converge to a solution to a deterministic nonlinear heat equation. We will establish analogues of these results in the wave equation context; see Theorems 1.1 and 1.3.

We also mention a recent work [23] by Pocovnicu, Tzvetkov, and the first author on the cubic NLW on $\mathbb{T}^3$ with random initial data of negative regularity. As a byproduct of the well-posedness theory in this setting, they established a triviality result for the defocusing cubic NLW (without renormalization) with deterministic initial data perturbed by rough random data.

Lastly, we point out that, in the context of nonlinear Schrödinger-type equations, instability results in negative Sobolev spaces, analogous to triviality, are known even in the deterministic setting; see [14,27]. See also [7,20] for analogous results in the context of the modified KdV equation, showing the necessity of renormalization in the low regularity setting.

1.2. Main results

Given $N \in \mathbb{N}$, we denote by $\mathbf{P}_N$ the Dirichlet projection onto the spatial frequencies $\mathbb{Z}_N^2 \overset{\text{def}}{=} \{|n| \le N\}$. We study the following truncated equation:

$$\partial_t^2 u_N - \Delta u_N + \partial_t u_N + u_N^3 = \alpha_N \xi_N \tag{1.6}$$

with the truncated noise

$$\xi_N \stackrel{\text{def}}{=} \mathbf{P}_N \xi.$$

Here, $\{\alpha_N\}_{N\in\mathbb{N}}$ is a bounded sequence of non-zero real numbers, which reflects the strength of the noise. Our goal is to study the asymptotic behavior of u_N as $N \to \infty$ in the following two regimes:

$$\text{(i)} \lim_{N\to\infty} \alpha_N^2 \log N = \infty \qquad \text{and} \qquad \text{(ii)} \lim_{N\to\infty} \alpha_N^2 \log N \in [0, \infty).$$

We refer to the case (i) (and the case (ii), respectively) as the strong noise case (and the weak noise case, respectively).

Let us fix some notations. We write $e_n(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} e^{in\cdot x}$, $n \in \mathbb{Z}^2$, for the orthonormal Fourier basis in $L^2(\mathbb{T}^2)$. Given $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{T}^2)$ by the norm:

$$\|f\|_{H^s(\mathbb{T}^2)} = \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^2(\mathbb{Z}^2)},$$

where $\widehat{f}(n)$ is the Fourier coefficient of f and $\langle\cdot\rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We also set

$$\mathcal{H}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2).$$

When we work with space-time function spaces, we use short-hand notations such as $C_T H^s_x = C([0, T]; H^s(\mathbb{T}^2))$ and $L^p_\omega = L^p(\Omega)$. Given $A, B \geq 0$, we also set $A \wedge B = \min(A, B)$.

(i) Strong noise case: We first consider the strong noise case:

$$\lim_{N\to\infty} \alpha_N^2 \log N = \infty. \tag{1.7}$$

In this case, the noise remains singular (in the limit), which provides a strong cancellation property of the solution u_N to (1.6).

Given $N \in \mathbb{N}$ and $\alpha_N \in \mathbb{R}$, fix $\lambda_N = \lambda_N(\alpha_N) \geq 0$ (to be determined later; see (1.13)). We define a pair $(z^\omega_{0,N}, z^\omega_{1,N})$ of random functions by the following random Fourier series:

$$z^\omega_{0,N} = \frac{\alpha_N}{\sqrt{2}} \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle_N} e_n \qquad \text{and} \qquad z^\omega_{1,N} = \frac{\alpha_N}{\sqrt{2}} \sum_{|n|\leq N} h_n(\omega) e_n, \tag{1.8}$$

where $\langle n\rangle_N$ is defined by

$$\langle n\rangle_N \stackrel{\text{def}}{=} \sqrt{\lambda_N + |n|^2}$$

and $\{g_n\}_{n\in\mathbb{Z}^2}$ and $\{h_n\}_{n\in\mathbb{Z}^2}$ are sequences of mutually independent standard complex-valued[5] Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ conditioned so that $g_{-n} = \overline{g_n}$ and $h_{-n} = \overline{h_n}$, $n \in \mathbb{Z}^2$. We also assume that $\{g_n, h_n\}_{n\in\mathbb{Z}^2}$ is independent of the space-time white noise ξ in (1.1).

We now state our main result. Given $s, b \in \mathbb{R}$ and $T > 0$, we define the time restriction space

$$H^b([0, T]; H^s(\mathbb{T}^2))$$

by the norm

$$\|u\|_{H^b([0,T]; H^s(\mathbb{T}^2))} = \inf\big\{\|v\|_{H^b(\mathbb{R}; H^s(\mathbb{T}^2))} : v|_{[0,T]} = u\big\}. \tag{1.9}$$

[5] This means that $g_0, h_0 \sim \mathcal{N}_{\mathbb{R}}(0, 1)$ and $\operatorname{Re} g_n, \operatorname{Im} g_n, \operatorname{Re} h_n, \operatorname{Im} h_n \sim \mathcal{N}_{\mathbb{R}}(0, \frac{1}{2})$ for $n \neq 0$.

Here, the $H^b(\mathbb{R}; H^s(\mathbb{T}^2))$-norm is defined by

$$\|v\|_{H^b(\mathbb{R}; H^s(\mathbb{T}^2))} = \|\langle\tau\rangle^b \langle n\rangle^s \widehat{v}(\tau, n)\|_{L^2_\tau \ell^2_n},$$

where $\widehat{v}(\tau, n)$ denotes the space-time Fourier transform of v.

Theorem 1.1. *Let $\{\alpha_N\}_{N\in\mathbb{N}}$ be a bounded sequence of non-zero real numbers, satisfying (1.7). Then, there exists a divergent sequence $\{\lambda_N\}_{N\in\mathbb{N}}$ such that given any $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$, $T > 0$, $\varepsilon > 0$, and $N \in \mathbb{N}$, there exists almost surely a unique solution $u_N \in C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$ to (1.6) with initial data*

$$(u_N, \partial_t u_N)|_{t=0} = (v_0, v_1) + (z^\omega_{0,N}, z^\omega_{1,N}), \tag{1.10}$$

where $(z^\omega_{0,N}, z^\omega_{1,N})$ is as in (1.8). Furthermore, u_N converges in probability to the trivial solution $u \equiv 0$ in $H^{-\varepsilon}([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$ as $N \to \infty$.

Seeing the regularity of the stochastic term, one may think that the natural space for the convergence is $C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$. We need to work in a larger space $H^{-\varepsilon}([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$ in order to establish convergence of the deterministic (modified) linear solution (defined in (1.25)). See Lemma 2.7.

Our proof is strongly motivated by the arguments in [15,23]. The main idea can be summarized as follows; while we consider a model without renormalization, we artificially renormalize the nonlinearity at the expense of modifying the linear operator. More concretely, given a suitable choice of divergent constants λ_N, we first rewrite the truncated equation (1.6) as follows:

$$\mathcal{L}_N u_N + u_N^3 - \lambda_N u_N = \alpha_N \xi_N, \tag{1.11}$$

where $\mathcal{L}_N$ denotes the modified damped wave operator:

$$\mathcal{L}_N \overset{\text{def}}{=} \partial_t^2 - \Delta + \partial_t + \lambda_N. \tag{1.12}$$

As we see below, the constant λ_N will play a role of a renormalization constant. See (1.23).

We now set $\lambda_N = \lambda_N(\alpha_N)$ to be the unique solution to

$$\lambda_N = \frac{3\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \frac{1}{\langle n\rangle_N^2} = \frac{3\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \frac{1}{\lambda_N + |n|^2}. \tag{1.13}$$

See Lemma 2.2. With this choice of λ_N, it is easy to see that the corresponding linear dynamics:

$$\mathcal{L}_N u_N = \alpha_N \xi_N \tag{1.14}$$

possesses a unique invariant mean-zero Gaussian measure μ_N on $\mathcal{H}^0(\mathbb{T}^2)$ with the covariance operator

$$\frac{\alpha_N^2}{2} \begin{pmatrix} \mathbf{P}_N(\lambda_N - \Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.15}$$

See Lemma 2.1. Our choice of random functions $(z^\omega_{0,N}, z^\omega_{1,N})$ in (1.8) is such that the random part of the initial data (u_0, u_1) in (1.10) is distributed by the Gaussian measure μ_N. We point out that by setting σ_N by

$$\sigma_N \overset{\text{def}}{=} \mathbb{E}\big[(z^\omega_{0,N}(x))^2\big] = \frac{\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \frac{1}{\langle n\rangle_N^2}, \tag{1.16}$$

we have

$$\lambda_N = 3\sigma_N. \tag{1.17}$$

In Lemma 2.2, we show that

$$\lambda_N = \frac{3}{4\pi}\alpha_N^2 \log N + \text{lower order error}, \tag{1.18}$$

which allows us to show that the sequence $\{(z_{0,N}^\omega, z_{1,N}^\omega)\}_{N\in\mathbb{N}}$ is almost surely uniformly bounded in $\mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$ for any $\varepsilon > 0$.

In the following, we describe an outline of the proof of Theorem 1.1. The main idea is to apply the Da Prato–Debussche trick [8] and look for a solution to (1.6) (or equivalently to (1.11)) of the form $u_N = z_N + v_N$, where z_N denotes the singular stochastic part and v_N denotes a smoother residual part.

Given $N \in \mathbb{N}$, let z_N denote the solution to the linear equation (1.14) with $(z_N, \partial_t z_N)|_{t=0} = (z_{0,N}^\omega, z_{1,N}^\omega)$. It follows from the discussion above that z_N is a stationary process such that

$$\mathrm{Law}\big((z_N(t), \partial_t z_N(t))\big) = \mu_N$$

for any $t \in \mathbb{R}_+$. By expressing z_N in the Duhamel formulation (= mild formulation),[6] we have

$$z_N(t) = \partial_t \mathcal{D}_N(t) z_{0,N}^\omega + \mathcal{D}_N(t)(z_{0,N}^\omega + z_{1,N}^\omega) + \alpha_N \int_0^t \mathcal{D}_N(t-t')\mathbf{P}_N dW(t'), \tag{1.19}$$

where $\mathcal{D}_N(t)$ is given by

$$\mathcal{D}_N(t) \stackrel{\text{def}}{=} e^{-\frac{t}{2}} \frac{\sin\Big(t\sqrt{\lambda_N - \frac{1}{4} - \Delta}\Big)}{\sqrt{\lambda_N - \frac{1}{4} - \Delta}} \tag{1.20}$$

and W denotes a cylindrical Wiener process on $L^2(\mathbb{T}^2)$:

$$W(t) = \sum_{n\in\mathbb{Z}^2} \beta_n(t) e_n. \tag{1.21}$$

Here, $\{\beta_n\}_{n\in\mathbb{Z}^2}$ is a family of mutually independent complex-valued Brownian motions conditioned so that $\beta_{-n} = \overline{\beta_n}$, $n \in \mathbb{Z}^2$. Moreover, we assume that $\{\beta_n\}_{n\in\mathbb{Z}^2}$ is independent from $\{g_n, h_n\}_{n\in\mathbb{Z}^2}$ in (1.8). By convention, we normalize β_n such that $\mathrm{Var}(\beta_n(t)) = t$. Note that the space-time white noise ξ is given by $\xi = \frac{\partial W}{\partial t}$.

By setting $v_N = u_N - z_N$, it follows from (1.11) with (1.10) that v_N satisfies the following equation:

$$\begin{cases} \mathcal{L}_N v_N + (v_N + z_N)^3 - \lambda_N(v_N + z_N) = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (v_0, v_1). \end{cases} \tag{1.22}$$

By invariance of the Gaussian measure μ_N, we see that $z_N(t)$ has the same law as $z_{0,N}$ for any $t \in \mathbb{R}_+$. In particular, it follows from (1.8) that there is no uniform (in N) bound for $z_N(t)$, when measured in $L^2(\mathbb{T}^2)$. This causes an issue in studying the powers z_N^2 and z_N^3, uniformly in $N \in \mathbb{N}$.

[6] One can easily derive the propagator $\mathcal{D}_N(t)$ in (1.20) by writing the linear damped wave equation $\mathcal{L}_N u = 0$ on the Fourier side and solving it directly for each spatial frequency. See [17] for the case $\lambda_N = 0$ (on $\mathbb{R}^d$). Then, a standard variation-of-parameter argument yields the Duhamel formulation (1.19).

In [12,13], it is at this point that we introduced Wick renormalization and considered a renormalized equation to overcome this issue. Our goal is, however, to study the limiting behavior of the solution u_N to (1.6) *without* renormalization. In our current problem, we overcome this difficulty by following the idea in [15,23] and artificially introducing a renormalization constant λ_N in (1.11). By expanding the last two terms in (1.22), we have

$$(v_N + z_N)^3 - \lambda_N(v_N + z_N) = v_N^3 + 3v_N^2 z_N + 3v_N(z_N^2 - \sigma_N) + (z_N^3 - 3\sigma_N z_N), \tag{1.23}$$

where we used (1.17). Then, it follows from (1.16) that the last two terms precisely correspond to the renormalized powers of z_N^2 and z_N^3. See Section 2 for further details.

This artificial introduction of renormalization as in (1.23) allows us to study Eq. (1.22) for v_N. A standard contraction argument allows us to prove local well-posedness of (1.22), expressed in the Duhamel formulation:

$$v_N(t) = v_N^{\text{lin}}(t) + \int_0^t \mathcal{D}_N(t-t')\mathcal{N}(v_N + z_N)(t')dt', \tag{1.24}$$

where $\mathcal{N}(v_N + z_N) = (v_N + z_N)^3 - \lambda_N(v_N + z_N)$ and $v_N^{\text{lin}}(t)$ denotes the linear solution with deterministic initial data (v_0, v_1):

$$v_N^{\text{lin}}(t) = \partial_t \mathcal{D}_N(t)v_0 + \mathcal{D}_N(t)(v_0 + v_1). \tag{1.25}$$

On the one hand, the diverging behavior (1.18) of λ_N and (1.20) allow us to show that the second term on the right-hand side of (1.24) tends to 0 as $N \to \infty$. This explicit decay mechanism is analogous to that in the parabolic case studied in [15]. On the other hand, the linear solution v_N^{lin} does not enjoy such a decay property in an obvious manner. The crucial point here is that, in view of the asymptotics (1.18), the modified linear operator $\mathcal{L}_N$ in (1.11) introduces a rapid oscillation and, as a result, v_N^{lin} tends to 0 as a space-time distribution. This oscillatory nature of the problem is a distinctive feature of a dispersive problem, not present in the parabolic setting, and was also exploited in [23]. In this paper, we go one step further. By exploiting the rapid oscillation in the form of oscillatory integrals, we show that v_N^{lin} tends to 0 in $H^{-\varepsilon}([0,T]; H^{1-\varepsilon}(\mathbb{T}^2))$. See Lemma 2.7. This essentially explains the proof of Theorem 1.1 for short times.

In order to prove the claimed convergence on an arbitrary time interval $[0, T]$, we need to establish a global-in-time control of the solutions v_N. An energy bound in the spirit of Burq and Tzvetkov [5] allows us to prove global existence of v_N. Unfortunately, such an energy bound (at the level of $\mathcal{H}^1(\mathbb{T}^2)$) grows in N, which may cause a potential issue. In general, it may be a cumbersome task to obtain a global-in-time control on v_N, uniformly in $N \in \mathbb{N}$. One possible approach may be to adapt the I-method argument employed in [13]. In our case, however, the situation is much simpler since we know that the limiting solution is $u \equiv 0$, which allows us to reduce the problem to a small data regime.

Remark 1.2. (i) For simplicity, we only consider the regularization via the Fourier truncation operator $\mathbf{P}_N$ in (1.6). By a slight modification of the proof, we can also treat regularization by mollification with a mollifier ρ_ε, $\varepsilon \in (0, 1]$ and taking the limit $\varepsilon \to 0$.
(ii) We consider the stochastic NLW with damping. This allows us to have an invariant Gaussian measure μ_N for the linear dynamics (1.14), which in turn implies that the renormalization constant λ_N defined in (1.13) and (1.17) is time independent. If we consider the stochastic NLW without damping, then λ_N would be time dependent. This would then imply that the modified linear operator $\mathcal{L}_N$ in (1.12) is with a variable coefficient $\lambda_N(t)$, introducing an extra

complication to the problem. This is the reason we chose to study the stochastic NLW with damping.

(iii) In the parabolic setting [15], the triviality result was stated only with deterministic initial data. Namely, there was no need to add the random initial data as in (1.10). In [15], the residual part v_N satisfies an analogue of (1.22) with initial data essentially of the form (written in the wave context):

$$(v_N, \partial_t v_N)|_{t=0} = (v_0, v_1) - (z^{\omega}_{0,N}, z^{\omega}_{1,N}). \tag{1.26}$$

See the equation (Φ^{aux}_{ε}) on p. 6 in [15]. In the parabolic setting, this does not cause any difficulty since the strong parabolic smoothing allows us to handle rough initial data of the form (1.26) in the deterministic manner. On the other hand, in the current wave context, we cannot handle the random data in (1.26), unless we introduce a further renormalization (which would violate the point of this paper).

Let us now try to see what happens if we directly work with the non-stationary solution to the linear equation (1.14). Let $\widetilde{z}_N$ denote the solution to (1.14) with the trivial (= zero) initial data, given by

$$\widetilde{z}_N(t) = \alpha_N \int_0^t \mathcal{D}_N(t-t')\mathbf{P}_N dW(t') \tag{1.27}$$

Then, by setting

$$\widetilde{\lambda}_N(t) = 3\widetilde{\sigma}_N(t) = 3\mathbb{E}\big[(\widetilde{z}_N(t,x))^2\big],$$

we can rewrite (1.6) as

$$\mathcal{L}_N u_N + \big(u_N^3 - \widetilde{\lambda}_N(t)u_N\big) - (\lambda_N - \widetilde{\lambda}_N(t))u_N = 0, \tag{1.28}$$

where $\mathcal{L}_N$ is as in (1.12). By writing $u_N = v_N + \widetilde{z}_N$, we easily see that the expression $u_N^3 - \widetilde{\lambda}_N(t)u_N$ can be treated as in (1.23) without causing any difficulty. On the other hand, the weaker smoothing property of the damped wave equation (as compared to the heat equation) causes an issue in treating the last term $(\lambda_N - \widetilde{\lambda}_N(t))u_N$. Define z_N^{hom} by

$$z_N^{\text{hom}}(t) = \partial_t \mathcal{D}_N(t) z^{\omega}_{0,N} + \mathcal{D}_N(t)(z^{\omega}_{0,N} + z^{\omega}_{1,N}), \tag{1.29}$$

where $(z^{\omega}_{0,N}, z^{\omega}_{1,N})$ is as in (1.8). Then, it follows from (1.19), (1.27), and (1.29) together with independence of $\{g_n, h_n\}_{n\in\mathbb{Z}^2}$ in (1.8) and the cylindrical Wiener process W in (1.21) that

$$\begin{aligned}
\lambda_N - \widetilde{\lambda}_N(t) &= 3\mathbb{E}\big[(z_N^{\text{hom}}(t,x))^2\big] \\
&= \frac{3e^{-t}\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \bigg\{ \frac{1}{\langle n\rangle_N^2} \bigg(\cos\Big(t\sqrt{\lambda_N - \tfrac14 + |n|^2}\Big)\bigg)^2 \\
&\qquad + \frac{\Big(\sin\Big(t\sqrt{\lambda_N - \frac14 + |n|^2}\Big)\Big)^2}{\lambda_N - \frac14 + |n|^2} \bigg\} + O(1) \\
&= \frac{3e^{-t}\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \frac{1}{\lambda_N - \frac14 + |n|^2} + O(1),
\end{aligned}$$

which is logarithmically divergent for any $t \ge 0$. This shows that the last term in (1.28) (under the Duhamel integral) cannot be treated uniformly in $N \in \mathbb{N}$, thus exhibiting non-trivial difficulty in the non-stationary case. Compare this with the heat case, where the corresponding

expression for $\lambda_N - \widetilde{\lambda}_N(t)$ is uniformly bounded in $N \in \mathbb{N}$ for any $t > 0$ (and logarithmically divergent when $t = 0$) thanks to the strong smoothing property.
(iv) In Theorem 1.1, we treated the cubic case. It would be of interest to investigate the issue of triviality for a higher order nonlinearity. See also Remarks 1.5 and 4.4 on this issue in the weak noise case.

Our argument also makes use of the defocusing nature of the equation in an essential manner. In the focusing case, the modified linear operator $\mathcal{L}_N$ in (1.12) would be $\mathcal{L}_N = \partial_t^2 - \Delta + \partial_t - \lambda_N$. Namely, the diverging constant λ_N appears with a wrong sign and we do not know how to proceed at this point.

(ii) Weak noise case: Next, we consider the weak noise case:

$$\lim_{N\to\infty} \alpha_N^2 \log N = \kappa^2 \in [0, \infty). \tag{1.30}$$

In particular, we have $\alpha_N \to 0$ and thus we expect convergence to a deterministic damped NLW. In this case, we set

$$\mathcal{L} \stackrel{\text{def}}{=} \partial_t^2 - \Delta + \partial_t + 1. \tag{1.31}$$

Namely, we can simply set $\lambda_N \equiv 1$ in the previous discussion. With a slight abuse of notation, we then define μ_N to be the mean-zero Gaussian measure on $\mathcal{H}^0(\mathbb{T}^2)$ with the covariance operator

$$\frac{\alpha_N^2}{2} \begin{pmatrix} \mathbf{P}_N(1-\Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, it follows that μ_N is the unique invariant measure for the linear equation:

$$\mathcal{L} u_N = \alpha_N \xi_N. \tag{1.32}$$

With a slight abuse of notation, we use z_N to denote the solution to (1.32) with the random initial data $(z_N, \partial_t z_N)|_{t=0} = (z_{0,N}^\omega, z_{1,N}^\omega)$ distributed by μ_N as in the previous case. In particular, the random initial data in this case is given by (1.8) with $\lambda_N = 1$, namely

$$z_{0,N}^\omega = \frac{\alpha_N}{\sqrt{2}} \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle} e_n \quad \text{and} \quad z_{1,N}^\omega = \frac{\alpha_N}{\sqrt{2}} \sum_{|n|\le N} h_n(\omega) e_n. \tag{1.33}$$

We now state our second result.

Theorem 1.3. *Let $\{\alpha_N\}_{N\in\mathbb{N}}$ be a bounded sequence of real numbers, satisfying (1.30) for some $\kappa^2 \in [0, \infty)$. Then, given any $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$, $T > 0$, $\varepsilon > 0$, and $N \in \mathbb{N}$, there exists almost surely a unique solution $u_N \in C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$ to (1.6) with initial data*

$$(u_N, \partial_t u_N)|_{t=0} = (v_0, v_1) + (z_{0,N}^\omega, z_{1,N}^\omega), \tag{1.34}$$

where $(z_{0,N}^\omega, z_{1,N}^\omega)$ is as in (1.33). Furthermore, u_N converges in probability to w_κ in $C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$ as $N \to \infty$, where w_κ is the unique solution to the following deterministic damped NLW:

$$\begin{cases} \partial_t^2 w_\kappa - \Delta w_\kappa + \partial_t w_\kappa + \frac{3}{4\pi}\kappa^2 w_\kappa + w_\kappa^3 = 0 \\ (w_\kappa, \partial_t w_\kappa)|_{t=0} = (v_0, v_1). \end{cases} \tag{1.35}$$

Recall that, in Theorem 1.1, we needed to study the convergence in a space larger than $C([0,T];H^{-\varepsilon}(\mathbb{T}^2))$. This was due to the convergence property of the deterministic (modified) linear solution v_N^{lin} in (1.25). In Theorem 1.3, we estimate the difference of the solution u_N to (1.6) with initial data (1.34) and the limiting solution w_κ to (1.35). As such, the deterministic part (v_0, v_1) of the initial data cancels each other, allowing us to prove the convergence in a natural space $C([0,T];H^{-\varepsilon}(\mathbb{T}^2))$.

Remark 1.4. As mentioned in Remark 1.2, we consider the equation with damping so that the linear equation (1.32) preserves the Gaussian measure μ_N. This naturally yields the damped equation (1.35) as the limiting deterministic equation. In this weak noise regime, however, it is possible to introduce another parameter $\widetilde{\alpha}_N$ and tune the parameters such that the dynamics converges to that generated by a standard deterministic NLW without damping.

Consider the following SdNLW:

$$\partial_t^2 u_N - \Delta u_N + \widetilde{\alpha}_N \partial_t u_N + u_N^3 = \alpha_N \xi_N, \tag{1.36}$$

where $\widetilde{\alpha}_N$ is a positive number, tending to 0 as $N \to \infty$. For $N \in \mathbb{N}$, set $\gamma_N^2 = \frac{\alpha_N^2}{2\widetilde{\alpha}_N}$. We assume that $\{\gamma_N^2\}_{N\in\mathbb{N}}$ is bounded.[7] Then, by repeating the proof of Theorems 1.1 and 1.3, it is straightforward to see that the limiting behavior of the solution u_N to (1.36) is determined by

$$\lim_{N\to\infty} \gamma_N^2 \log N = \lim_{N\to\infty} \frac{\alpha_N^2}{2\widetilde{\alpha}_N} \log N = \gamma^2 \in [0,\infty].$$

We have the following two scenarios. (i) If $\gamma^2 = \infty$, then the solution u_N to (1.36) converges to 0. (ii) If $\gamma^2 \in [0,\infty)$, then the solution u_N to (1.36) converges to the solution w_γ, satisfying the following deterministic NLW (i.e. without damping):

$$\partial_t^2 w_\gamma - \Delta w_\gamma + \frac{3}{4\pi}\gamma^2 w_\gamma + w_\gamma^3 = 0.$$

The main point is that the tuning of the parameters, making the sequence $\{\gamma_N^2\}_{N\in\mathbb{N}}$ bounded, allows us to make use of certain invariant Gaussian measures for the (modified) linear dynamics.

Remark 1.5. In the weak noise case, it is possible to adapt our argument to a general defocusing power-type nonlinearity u^{2k+1}. See Remark 4.4 for further details.

2. Preliminary results for the strong noise case

In this section, we go over some preliminary materials for the strong noise case (Theorem 1.1), whose proof is presented in Section 3. In Section 2.1, we prove that the Gaussian measure μ_N with the covariance operator (1.15) is the (unique) invariant measure for the linear stochastic wave equation (1.14). In Section 2.2, we establish the asymptotic behavior (1.18) of the renormalization constant λ_N. In Section 2.3, we define the renormalized powers $:z_N^\ell:$ for the solution z_N to the linear equation (1.14) with $(z_N, \partial_t z_N)|_{t=0} = (z_{0,N}^\omega, z_{1,N}^\omega)$. Lastly, in Section 2.4, we study the decay property of the deterministic linear solution v_N^{lin} defined in (1.25).

[7] This in particular implies that α_N tends to 0 since our assumption states that $\widetilde{\alpha}_N$ tends to 0.

2.1. On the invariant measure for the linear equation

We begin by describing the invariant measure for the linear stochastic equation (1.14):

$$\mathcal{L}_N u_N = \alpha_N \xi_N.$$

We only sketch a proof since the argument is classical; see, for example, [13,25] for a more detailed discussion.

Lemma 2.1. *The linear stochastic wave equation (1.14) possesses a (unique) invariant mean-zero Gaussian measure μ_N on $\mathcal{H}^0(\mathbb{T}^2)$ with the covariance operator given in (1.15).*

Proof. We only present a sketch of the proof. For $|n| \le N$, let

$$X_n = \begin{pmatrix} \widehat{u}_N(n) \\ \partial_t \widehat{u}_N(n) \end{pmatrix}.$$

Then, in view of (1.21), we can rewrite the linear equation (1.14) as the following system of stochastic differential equations:

$$dX_n = \begin{pmatrix} 0 & 1 \\ -\langle n\rangle_N^2 & 0 \end{pmatrix} X_n dt + \left[\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} X_n dt + \begin{pmatrix} 0 \\ \alpha_N d\beta_n \end{pmatrix} \right]. \tag{2.1}$$

The first part on the right-hand side corresponds to the (modified) linear wave equation (without damping) whose semi-group acts as a rotation on each component of the vector X_n. Since the distribution of a complex-valued Gaussian random variable is invariant under a rotation, we see that the solution to this linear wave equation, starting from the random initial data $(z^\omega_{0,N}, z^\omega_{1,N})$ in (1.8), is stationary.

The second part on the right-hand side of (2.1) corresponds to the Langevin equation for the velocity $\partial_t \widehat{u}_N(n)$:

$$d(\partial_t \widehat{u}_N(n)) = -(\partial_t \widehat{u}_N(n))dt + \alpha_N d\beta_n,$$

whose solution is given by a complex-valued Ornstein–Uhlenbeck process. Namely, its real and imaginary parts are given by independent Ornstein–Uhlenbeck processes. Hence, it has a unique invariant measure given by the Gaussian distribution $\mathcal{N}_{\mathbb{C}}(0, \frac{\alpha_N^2}{2})$ (see, for example, [21, Theorem 7.4.7]), which is precisely the law of $\widehat{z}^\omega_{1,N} = \partial_t \widehat{z}_N(0)$ defined in (1.8).

For each $n \in \mathbb{Z}^2$ with $|n| \le N$, the generator of the dynamics (2.1) is given by the sum of the generators of the first and second parts on the right-hand side of (2.1). Hence, we conclude that the full linear stochastic wave equation (1.14), starting from $(z^\omega_{0,N}, z^\omega_{1,N})$ in (1.8), is also stationary. This means that the mean-zero Gaussian measure μ_N with the covariance operator (1.15) is invariant under (1.14). One can also prove that μ_N is actually the unique invariant measure for this equation; see Theorems 11.17 and 11.20 in [9]. □

Recall that z_N defined by (1.19) satisfies the linear stochastic wave equation (1.14). Then, due to the invariance of μ_N under the flow of (1.14), the variance of $z_N(t)$ is time independent and given by (1.16):

$$\sigma_N = \mathbb{E}\big[(z_N(t,x))^2\big] = \mathbb{E}\big[(z_N(0,x))^2\big] = \frac{\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \frac{1}{\langle n\rangle_N^2}. \tag{2.2}$$

2.2. On the renormalization constant

In this subsection, we study asymptotic properties of the renormalization constant λ_N implicitly defined by (1.13):

$$\lambda_N = \frac{3\alpha_N^2}{8\pi^2} \sum_{|n|\leq N} \frac{1}{\langle n\rangle_N^2} = \frac{3\alpha_N^2}{8\pi^2} \sum_{|n|\leq N} \frac{1}{\lambda_N + |n|^2}. \tag{2.3}$$

In particular, we prove the following lemma on the asymptotic behavior of λ_N as $N \to \infty$. See Lemma 3.1 in [15] and Lemma 6.1 in [23] for analogous results.

Lemma 2.2. *Given $N \in \mathbb{N}$, there exists a unique number $\lambda_N > 0$ satisfying Eq. (2.3). Moreover, if $\{\alpha_N\}_{N\in\mathbb{N}}$ is a bounded sequence of non-zero real numbers such that $\lim_{N\to\infty} \alpha_N^2 \log N = \infty$, then we have*

$$\lambda_N = \frac{3}{4\pi}\alpha_N^2 \log N + O(\alpha_N^2 \log\log N) \tag{2.4}$$

as $N \to \infty$.

Before proceeding to the proof of Lemma 2.2, we first recall the following bound. See Lemma 3.2 in [15].

Lemma 2.3. *Let a, $N \geq 1$. Then, we have*

$$\left| \sum_{|n|\leq N} \frac{1}{a + |n|^2} - \pi \log\left(1 + \frac{N^2}{a}\right)\right| \lesssim \frac{1}{\sqrt{a}} \min\left(1, \frac{N}{\sqrt{a}}\right).$$

We now present a proof of Lemma 2.2.

Proof of Lemma 2.2. Given $N \in \mathbb{N}$, let λ_N be as in (2.3). As λ_N increases from 0 to ∞, the right-hand side of (2.3) decreases from ∞ to 0. Hence, for each $N \in \mathbb{N}$, there exists a unique solution $\lambda_N > 0$ to (1.13).

From $\lambda_N > 0$, we obtain an upper bound $\lambda_N \lesssim \alpha_N^2 \log N$. From this upper bound and the uniform boundedness of α_N, we also obtain a lower bound $\lambda_N \gtrsim \alpha_N^2 \log N$ for any sufficiently large $N \gg 1$. Hence, we have

$$\lambda_N \sim \alpha_N^2 \log N \tag{2.5}$$

for any $N \gg 1$.

From Lemma 2.3, we have

$$\sum_{|n|\leq N} \frac{1}{\langle n\rangle^2} = 2\pi \log N + O(1).$$

Then, in view of the uniform boundedness of α_N, the error term R_N is given by

$$\begin{aligned} R_N &= \lambda_N - \frac{3}{4\pi}\alpha_N^2 \log N = \lambda_N - \frac{3}{8\pi^2}\alpha_N^2 \sum_{|n|\leq N} \frac{1}{\langle n\rangle^2} + O(\alpha_N^2) \\ &= \frac{3}{8\pi^2}\alpha_N^2 \sum_{|n|\leq N} \left(\frac{1}{\lambda_N + |n|^2} - \frac{1}{\langle n\rangle^2}\right) + O(\alpha_N^2) \\ &= \frac{3}{8\pi^2}\alpha_N^2 \sum_{|n|\leq N} \frac{1 - \lambda_N}{(\lambda_N + |n|^2)\langle n\rangle^2} + O(\alpha_N^2). \end{aligned} \tag{2.6}$$

Using (2.5), we can estimate the contribution to R_N in (2.6) from $\big\{|n| \gtrsim |\alpha_N|\sqrt{\log N}\big\}$ as $O(\alpha_N^2)$, while the contribution to R_N in (2.6) from $\big\{|n| \ll |\alpha_N|\sqrt{\log N}\big\}$ is $O(\alpha_N^2 \log\log N)$. Putting everything together, we obtain (2.4). $\square$

2.3. On the Wick powers

Given $N \in \mathbb{N}$, let z_N be the solution to the linear equation (1.14) with $(z_N, \partial_t z_N)|_{t=0} = (z_{0,N}^\omega, z_{1,N}^\omega)$. In the following, we define the renormalized powers of z_N and establish their regularity and decay properties.

Recall that the Hermite polynomials $H_k(x;\sigma)$ are defined via the generating function:

$$F(t,x;\sigma) = e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma).$$

In the following, we list the first few Hermite polynomials for readers' convenience:

$$H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x, \quad H_2(x;\sigma) = x^2 - \sigma, \quad H_3(x;\sigma) = x^3 - 3\sigma x. \tag{2.7}$$

Then, given $\ell \in \mathbb{Z}_{\geq 0} \overset{\text{def}}{=} \mathbb{N} \cup \{0\}$, we define the Wick powers $:z_N^\ell:$ by

$$:z_N^\ell(t,x): = H_\ell(z_N(t,x);\sigma_N) \tag{2.8}$$

in a pointwise manner, where σ_N is as in (2.2).

Before proceeding further, let us first state several lemmas. The first lemma states the orthogonality property of Wick products [29, Theorem I.3]. See also [22, Lemma 1.1.1].

Lemma 2.4. *Let f and g be Gaussian random variables with variances σ_f and σ_g. Then, we have*

$$\mathbb{E}\big[H_k(f;\sigma_f)H_m(g;\sigma_g)\big] = \delta_{km} k! \big\{\mathbb{E}[fg]\big\}^k.$$

Here, δ_{km} denotes the Kronecker delta function.

Next, we recall the Wiener chaos estimate. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-algebra generated by this sequence. Given $k \in \mathbb{Z}_{\geq 0}$, we define the homogeneous Wiener chaoses $\mathcal{H}_k$ to be the closure (under $L^2(\Omega)$) of the span of Fourier-Hermite polynomials $\prod_{n=1}^\infty H_{k_n}(g_n)$, where H_j is the Hermite polynomial of degree j and $k = \sum_{n=1}^\infty k_n$. (This implies that $k_n = 0$ except for finitely many n's.) Then, we have the following classical Wiener chaos estimate; see [29, Theorem I.22].

Lemma 2.5. *Let $\ell \in \mathbb{Z}_{\geq 0}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{\frac{\ell}{2}} \|X\|_{L^2(\Omega)}$$

for any random variable $X \in \mathcal{H}_\ell$ and any finite $p \geq 1$.

Our main goal here is to prove the following regularity and decay properties of the Wick powers $:z_N^\ell:$.

Proposition 2.6. (i) *Let $\ell \in \mathbb{N}$. Then, given any finite $p, q \geq 1$, $T > 0$, and $\varepsilon > 0$, we have*

$$\lim_{N\to\infty} \mathbb{E}\Big[\| :z_N^\ell(t): \|_{L_T^q W_x^{-\varepsilon,\infty}}^p\Big] = 0.$$

(ii) *Given any finite* $p \geq 1$, $T > 0$, *and* $\varepsilon > 0$, *we have*

$$\lim_{N\to\infty} \mathbb{E}\Big[\|z_N\|^p_{C_T H_x^{-\varepsilon}} \Big] = 0. \tag{2.9}$$

Proof. (i) By Sobolev's inequality, it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}\Big[\| :z_N^\ell(t): \|^p_{L_T^q W_x^{-\varepsilon,r}} \Big] = 0 \tag{2.10}$$

for any small $\varepsilon > 0$ and sufficiently large $r \gg 1$. We follow the argument in the proof of Proposition 2.1 in [12]. Fix $t \in \mathbb{R}_+$ and $x, y \in \mathbb{T}^2$. Then, by Lemma 2.4 with the invariance of the distribution of $z_N(t)$ and (1.8), we have

$$\begin{aligned}
\mathbb{E}\big[:z_N^\ell(t,x)::z_N^\ell(t,y):\big] &= \ell!\,\mathbb{E}[z_N(t,x)z_N(t,y)]^\ell = \frac{\ell!}{(8\pi^2)^\ell}\Big\{ \sum_{|n|\leq N} \frac{\alpha_N^2}{\langle n\rangle_N^2} e^{in\cdot(x-y)} \Big\}^\ell \\
&= \frac{\ell!}{(8\pi^2)^\ell} \sum_{n_1,\dots,n_\ell\in\mathbb{Z}_N^2} \Big(\prod_{j=1}^\ell \frac{\alpha_N^2}{\langle n_j\rangle_N^2} \Big) e^{i(n_1+\cdots+n_\ell)\cdot(x-y)}.
\end{aligned}$$

By applying the Bessel potentials $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ of order ε and then setting $x = y$, we obtain

$$\begin{aligned}
\mathbb{E}\Big[|\langle\nabla\rangle^{-\varepsilon} :z_N^\ell(t,x):|^2 \Big] &\sim \sum_{n_1,\dots,n_\ell\in\mathbb{Z}_N^2} \Big(\prod_{j=1}^\ell \frac{\alpha_N^2}{\langle n_j\rangle_N^2} \Big) \langle n_1+\cdots+n_\ell\rangle^{-2\varepsilon} \\
&\lesssim \lambda_N^{-\frac{\varepsilon}{2}} \sum_{n_1,\dots,n_\ell\in\mathbb{Z}_N^2} \frac{1}{\prod_{j=1}^{\ell-1}\langle n_j\rangle^2 \cdot \langle n_\ell\rangle^{2-\varepsilon}\langle n_1+\cdots+n_\ell\rangle^{2\varepsilon}} \\
&\lesssim \lambda_N^{-\frac{\varepsilon}{2}},
\end{aligned}$$

uniformly for all sufficiently large $N \gg 1$, where, in the first inequality, we used the uniform boundedness of α_N, Lemma 2.2, and the bound $\langle n\rangle_N \geq \lambda_N^{\frac{\varepsilon}{2}}\langle n\rangle^{1-\varepsilon}$ for $\varepsilon \in [0,1]$. Then, from Minkowski's integral inequality and the Wiener chaos estimate (Lemma 2.5), we obtain

$$\begin{aligned}
\Big\| \| :z_N^\ell: \|_{L_T^q W_x^{-\varepsilon,r}} \Big\|_{L^p(\Omega)} &\leq \Big\| \| \langle\nabla\rangle^{-\varepsilon} :z_N^\ell(t,x): \|_{L^p(\Omega)} \Big\|_{L_T^q L_x^r} \\
&\leq p^{\frac{\ell}{2}} \Big\| \| \langle\nabla\rangle^{-\varepsilon} :z_N^\ell(t,x): \|_{L^2(\Omega)} \Big\|_{L_T^q L_x^r} \\
&\leq C_\ell\, p^{\frac{\ell}{2}} T^{\frac{1}{q}} \lambda_N^{-\frac{\varepsilon}{4}}
\end{aligned} \tag{2.11}$$

for any finite $p \geq \max(q, r)$. The claim (2.10) follows from (2.11) and the asymptotic behavior (2.4) of λ_N proved in Lemma 2.2.

(ii) We prove (2.9) for any small $\varepsilon > 0$. Given $N \in \mathbb{N}$, define Ψ_N by

$$\Psi_N(t) = \int_0^t \mathcal{D}_N(t-t')\mathbf{P}_N dW(t'),$$

where $\mathcal{D}_N$ and W are as in (1.20) and (1.21). With a slight abuse of notation, define a Fourier multiplier operator D_N by the following symbol

$$D_N(n) = \sqrt{\lambda_N - \frac{1}{4} + |n|^2}. \tag{2.12}$$

Then, it follows from (1.19), the unitarity of $e^{\pm itD_N}$ on $H^s(\mathbb{T}^2)$, Minkowski's integral inequality, and Lemma 2.5 that

$$\begin{aligned}\Big\| \, \| z_N - \alpha_N \Psi_N \|_{C_T H_x^{-\varepsilon}} \Big\|_{L^p(\Omega)} &\lesssim \Big\| \|z_{0,N}^\omega\|_{H^{-\varepsilon}} \Big\|_{L^p(\Omega)} + \Big\| \|D_N^{-1} z_{1,N}^\omega\|_{H^{-\varepsilon}} \Big\|_{L^p(\Omega)} \\ &\lesssim \Big\| \|\langle\nabla\rangle^{-\varepsilon} z_{0,N}^\omega(x)\|_{L^p(\Omega)} \Big\|_{L^2_x} + \Big\| \|D_N^{-1}\langle\nabla\rangle^{-\varepsilon} z_{1,N}^\omega(x)\|_{L^p(\Omega)} \Big\|_{L^2_x} \\ &\lesssim_p \Big(\sum_{|n|\le N} \frac{\alpha_N^2}{\langle n\rangle^{2\varepsilon}\langle n\rangle_N^2} \Big)^{\frac12} + \Big(\sum_{|n|\le N} \frac{\alpha_N^2}{\langle n\rangle^{2\varepsilon}\big(D_N(n)\big)^2} \Big)^{\frac12} \lesssim \lambda_N^{-\frac{\varepsilon}{4}}.\end{aligned}$$

In the last step, we once again used the uniform boundedness of α_N and also the following bound:

$$D_N(n) \sim \langle n\rangle_N \gtrsim \lambda_N^{\frac{\varepsilon}{2}} \langle n\rangle^{1-\varepsilon} \tag{2.13}$$

uniformly for all sufficiently large $N \gg 1$, in view of (2.12) and Lemma 2.2.

Hence, it suffices to show that

$$\Big\| \sup_{0\le t\le T} \| \Psi_N(t)\|_{H^{-\varepsilon}} \Big\|_{L^p(\Omega)} \lesssim \lambda_N^{-\frac{\varepsilon}{4}}. \tag{2.14}$$

In view of (2.13), one can easily modify the proof of Proposition 2.1 in [12] to obtain (2.14). In the following, however, we apply the factorization method based on the elementary identity:

$$\int_{t_2}^t (t-t_1)^{\gamma-1}(t_1-t_2)^{-\gamma}\, dt_1 = \frac{\pi}{\sin \pi\gamma} \tag{2.15}$$

for any $\gamma \in (0,1)$ and $0 \le t_2 \le t$; see [9, Section 5.3].

Recall from (1.20) and (2.12) that

$$\mathcal{D}_N(t) = e^{-\frac{t}{2}} \frac{\sin(tD_N)}{D_N}. \tag{2.16}$$

Together with (2.15), we have

$$\begin{aligned}\Psi_N(t) &= \sum_{\sigma\in\{-1,1\}} \sigma \int_0^t \frac{e^{i\sigma(t-t')D_N}}{2iD_N} \mathbf{P}_N dW(t') \\ &= \frac{\sin\pi\gamma}{\pi} \sum_{\sigma\in\{-1,1\}} \sigma \int_0^t \frac{e^{i\sigma(t-t_1)D_N}}{2iD_N} (t-t_1)^{\gamma-1} Y_{\sigma,N}(t_1) dt_1,\end{aligned} \tag{2.17}$$

where

$$Y_{\sigma,N}(t_1) = \int_0^{t_1} e^{i\sigma(t_1-t_2)D_N}(t_1-t_2)^{-\gamma}\mathbf{P}_N dW(t_2).$$

Then, from (2.17) and the boundedness of $e^{i\sigma tD_N}$ on $H^s(\mathbb{T}^2)$, we have

$$\| \Psi_N\|_{L^p_\omega L^\infty_t([0,T])H_x^{-\varepsilon}} \lesssim \sum_{\sigma\in\{-1,1\}} \big\|(t-t_1)^{\gamma-1} D_N^{-1} Y_{\sigma,N}(t_1)\big\|_{L^p_\omega L^\infty_t([0,T])L^1_{t_1}([0,t])H_x^{-\varepsilon}}$$

By Hölder's inequality in t_1, we continue with

$$\lesssim \sum_{\sigma\in\{-1,1\}} \|D_N^{-1} Y_{\sigma,N}\|_{L^p_\omega L^p_T H_x^{-\varepsilon}}, \tag{2.18}$$

provided that $p > \frac{1}{\gamma}$.

By applying Fubini's theorem and Hölder's inequality, it suffices to estimate

$$\Big\| \int_0^{t_1} e^{i\sigma(t_1-t_2)D_N}(t_1-t_2)^{-\gamma} D_N^{-1}\mathbf{P}_N dW(t_2)\Big\|_{L^p_\omega H^{-\varepsilon}_x},$$

uniformly in $t_1 \in [0,T]$. From Minkowski's integral inequality (for $p \geq 2$), the Wiener chaos estimate (Lemma 2.5), and (2.13), we estimate this term by

$$\begin{aligned}\Big\| \int_0^{t_1} e^{i\sigma(t_1-t_2)D_N}(t_1-t_2)^{-\gamma} D_N^{-1}\mathbf{P}_N dW(t_2)\Big\|_{H^{-\varepsilon}_x L^2_\omega} \\ \sim \Big[\sum_{|n|\leq N}\langle n\rangle^{-2\varepsilon}\big(D_N(n)\big)^{-2}\int_0^{t_1}(t_1-t_2)^{-2\gamma}dt_2\Big]^{\frac{1}{2}} \\ \lesssim \lambda_N^{-\frac{\varepsilon}{4}}\end{aligned} \tag{2.19}$$

uniformly for all sufficiently large $N \gg 1$, provided that $\gamma < \frac{1}{2}$. The desired bound (2.14) follows from (2.18) and (2.19). This completes the proof of Part (ii). □

2.4. On the deterministic linear solution

In [23], the authors exploited a rapid oscillation to show that a deterministic linear solution tends to 0 as a space-time distribution. In the following lemma, by using a rapid oscillation to evaluate relevant oscillatory integrals, we show that the deterministic linear solution v_N^{lin} defined in (1.25) converges to 0 in $H^{-\varepsilon}([0,T]; H^{1-\varepsilon}(\mathbb{T}^2))$, $\varepsilon > 0$. This is the last ingredient for the proof of Theorem 1.1.

Lemma 2.7. *Given $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$, let v_N^{lin} be the solution to the linear wave equation with $(v, \partial_t v)|_{t=0} = (v_0, v_1)$ defined in (1.25). Then, given any $T > 0$ and $\varepsilon > 0$, v_N^{lin} converges to 0 in $H^{-\varepsilon}([0,T]; H^{1-\varepsilon}(\mathbb{T}^2))$ as $N \to \infty$.*

Proof. Fix $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $[0,1]$ and set $\chi_T(t) = \chi(T^{-1}t)$ for $T > 0$. By setting

$$V_0 = e^{-\frac{t}{2}}\cos\big(tD_N\big)v_0 \qquad \text{and} \qquad V_1 = \mathcal{D}_N(t)(v_1 + \tfrac{1}{2}v_0),$$

we have $v_N^{\text{lin}} = V_0 + V_1$. Then, from the definition (1.9) and Hölder's inequality in time, we have

$$\begin{aligned}\|v_N^{\text{lin}}\|_{H^{-\varepsilon}_T H^{1-\varepsilon}_x} &\leq \|\chi_T v_N^{\text{lin}}\|_{H^{-\varepsilon}_t H^{1-\varepsilon}_x} \\ &\leq \|\chi_T V_0\|_{H^{-\varepsilon}_t H^{1-\varepsilon}_x} + C_T\|V_1\|_{L^\infty_t H^{1-\varepsilon}_x}.\end{aligned} \tag{2.20}$$

In view of (2.16) with (2.13), the second term on the right-hand side of (2.20) can be estimated by

$$\|\langle\nabla\rangle^{1-\varepsilon}D_N^{-1}(v_0+v_1)\|_{L^2_x} \lesssim \lambda_N^{-\frac{\varepsilon}{2}}\|(v_0,v_1)\|_{\mathcal{H}^1} \longrightarrow 0 \tag{2.21}$$

as $N \to \infty$. As for the first term, we have

$$\begin{aligned}\mathcal{F}_{t,x}(\chi_T V_0)(\tau,n) &= \int_{\mathbb{R}} \chi_T(t)e^{-\frac{t}{2}}\cos\big(tD_N(n)\big)e^{-it\tau}\widehat{v_0}(n)dt \\ &= \frac{1}{2}\Big[\mathcal{F}_t(\chi_T e^{-\frac{t}{2}})(\tau - D_N(n)) + \mathcal{F}_t(\chi_T e^{-\frac{t}{2}})(\tau + D_N(n))\Big]\widehat{v_0}(n).\end{aligned}$$

Here, $\mathcal{F}_t$ and $\mathcal{F}_{t,x}$ denote the temporal and space-time Fourier transforms, respectively. Integrating by parts and using the properties of χ_T, we have

$$\begin{aligned}\big|\mathcal{F}_t(\chi_T e^{-\frac{t}{2}})(\tau)\big| &= \bigg|\int_{\mathbb{R}} \chi_T(t) e^{-\frac{t}{2}} e^{-it\tau} dt\bigg| = \langle\tau\rangle^{-2M}\bigg|\int_{\mathbb{R}} (1-\partial_t^2)^M\big[\chi_T(t)e^{-\frac{t}{2}}\big]e^{-it\tau}dt\bigg| \\ &\le C_{T,M}\langle\tau\rangle^{-2M}\end{aligned}$$

for $M \in \mathbb{Z}_{\ge 0}$ (and hence for any $M \ge 0$), uniformly in $\tau \in \mathbb{R}$. Therefore, we obtain

$$\begin{aligned}\|\chi_T V_0\|^2_{H_t^{-\varepsilon}H_x^{1-\varepsilon}} &\lesssim \sum_{|n|\le N}\langle n\rangle^{2(1-\varepsilon)}|\widehat{v_0}(n)|^2 \sum_{\sigma\in\{-1,1\}}\int_{\mathbb{R}}\langle\tau\rangle^{-2\varepsilon}\big|\mathcal{F}_t(\chi_T e^{-\frac{t}{2}})(\tau+\sigma D_N(n))\big|^2 d\tau \\ &\lesssim \sum_{|n|\le N}\langle n\rangle^{2(1-\varepsilon)}|\widehat{v_0}(n)|^2 \sum_{\sigma\in\{-1,1\}}\int_{\mathbb{R}}\langle\tau\rangle^{-2\varepsilon}\langle\tau+\sigma D_N(n)\rangle^{-1} d\tau \\ &\lesssim \sum_{|n|\le N}\langle n\rangle^{2(1-\varepsilon)}|\widehat{v_0}(n)|^2\langle D_N(n)\rangle^{-\varepsilon} \\ &\lesssim \lambda_N^{-\frac{\varepsilon}{2}}\|v_0\|^2_{H^1} \longrightarrow 0 \end{aligned} \tag{2.22}$$

as $N \to \infty$, where in the penultimate step we used the estimate

$$\int_{\mathbb{R}}\langle\tau\rangle^{-a}\langle\tau-\tau_0\rangle^{-b}d\tau \lesssim \langle\tau_0\rangle^{1-a-b}$$

for any $\tau_0 \in \mathbb{R}$ and any $a, b < 1$ with $a + b > 1$; see for example [10, Lemma 4.2]. Putting (2.20), (2.21), and (2.22) together, we conclude that v_N^{lin} converges to 0 in $H^{-\varepsilon}([0,T]; H^{1-\varepsilon}(\mathbb{T}^2))$ as $N \to \infty$. □

3. Trivial limit in the strong noise case

In this section, we prove triviality in the strong noise case (Theorem 1.1). In particular, we assume (1.7) in the following. As described in Section 1, we apply the Da Prato–Debussche trick and work in terms of the residual term $v_N = u_N - z_N$. From (1.22), (1.23), (2.7), and (2.8), we see that v_N satisfies

$$\begin{cases}\mathcal{L}_N v_N + v_N^3 + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3: = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (v_0, v_1).\end{cases} \tag{3.1}$$

The main idea is to use the decay properties of the Wick powers $:z_N^\ell:$ and the deterministic linear solution v_N^{lin} proved in Section 2.

We first establish almost sure global well-posedness of (3.1). Given $s \in \mathbb{R}$ and $T > 0$, define the solution space $X^s(T)$ by setting

$$X^s(T) \overset{\text{def}}{=} C([0,T]; H^s(\mathbb{T}^2)) \cap C^1([0,T]; H^{s-1}(\mathbb{T}^2)). \tag{3.2}$$

Proposition 3.1. *Let $N \in \mathbb{N}$. The Cauchy problem (3.1) is almost surely globally well-posed in $\mathcal{H}^1(\mathbb{T}^2)$. More precisely, given any $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$ and any $T > 0$, there exists a set $\Omega_T \subset \Omega$ of full probability such that, for any $\omega \in \Omega_T$ and $N \in \mathbb{N}$, there exists a unique solution $v_N \in X^1(T)$ to (3.1).*

We recall the following lemma from [12].

Lemma 3.2. *Let $0 \le s \le 1$.*

(i) *Suppose that* $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. *Then, we have*

$$\|\langle\nabla\rangle^s(fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|f\|_{L^{p_1}(\mathbb{T}^d)}\|\langle\nabla\rangle^s g\|_{L^{q_1}(\mathbb{T}^d)} + \|\langle\nabla\rangle^s f\|_{L^{p_2}(\mathbb{T}^d)}\|g\|_{L^{q_2}(\mathbb{T}^d)}.$$

(ii) *Suppose that* $1 < p, q, r < \infty$ *satisfy the scaling condition* $\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{s}{d}$. *Then, we have*

$$\|\langle\nabla\rangle^{-s}(fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle\nabla\rangle^{-s} f\|_{L^p(\mathbb{T}^d)}\|\langle\nabla\rangle^s g\|_{L^q(\mathbb{T}^d)}.$$

The first estimate is a consequence of the Coifman–Meyer theorem and the transference principle. See [12] for the references therein. Note that while the second estimate was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}$ in [12], the general case $\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{s}{d}$ follows from a straightforward modification.

We now present a proof of Proposition 3.1.

Proof of Proposition 3.1. Let $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$ and fix a target time $T > 0$ as in the statement. We first briefly go over local well-posedness of (3.1) with a control on $[0, T]$. By writing (3.1) in the Duhamel formulation, we have

$$\begin{aligned} v_N(t) &= \Gamma_N(v_N)(t) \\ &\overset{\text{def}}{=} \partial_t \mathcal{D}_N(t)v_0 + \mathcal{D}_N(t)(v_0 + v_1) \\ &\quad - \int_0^t \mathcal{D}_N(t-t')\big(v_N^3 + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3:\big)(t')dt'. \end{aligned} \tag{3.3}$$

Let $D_N(n)$ be as in (2.12). Recall from Lemma 2.2 that we have $\lambda_N > 0$. Then, by separately estimating the cases $D_N \gtrsim 1$ and $D_N \ll 1$, we have

$$\left| e^{-\frac{t}{2}} \frac{\sin t D_N(n)}{D_N(n)} \right| \lesssim \langle n\rangle^{-1} \tag{3.4}$$

for any $N \ge 1$, $n \in \mathbb{Z}^2$, and $t \ge 0$. Hence, in view of (2.16), we have

$$\|\Gamma_N(v_N)\|_{X^1(\delta)} \lesssim \|(v_0, v_1)\|_{\mathcal{H}^1} + \|v_N^3 + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3:\|_{L^1_\delta L^2_x} \tag{3.5}$$

for any $\delta > 0$.

Next, observe that from its definition (1.19), z_N satisfies

$$z_N = \mathbf{P}_N z_N,$$

which implies that we have

$$:z_N^\ell: = \mathbf{P}_{\ell N} :z_N^\ell:$$

for $\ell = 2, 3$. Hence, by Hölder's, Sobolev's and Bernstein's inequalities with the frequency support property of the Wick powers, we obtain

$$\begin{aligned} &\| v_N^3 + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3:\|_{L^1_\delta L^2_x} \\ &\lesssim \delta^{\frac{1}{2}} \Big(\|v_N\|^3_{L^\infty_\delta L^6_x} + \|v_N\|^2_{L^\infty_\delta L^4_x}\|z_N\|_{L^2_\delta L^\infty_x} \\ &\quad + \|v_N\|_{L^\infty_\delta L^2_x}\| :z_N^2: \|_{L^2_\delta L^\infty_x} + \| :z_N^3: \|_{L^2_\delta L^\infty_x} \Big) \\ &\lesssim \delta^{\frac{1}{2}} \Big(\|v_N\|^3_{X^1(\delta)} + N^\varepsilon \|v_N\|^2_{X^1(\delta)}\|z_N\|_{L^2_\delta W^{-\varepsilon,\infty}_x} \\ &\quad + N^\varepsilon \|v_N\|_{X^1(\delta)}\| :z_N^2: \|_{L^2_\delta W^{-\varepsilon,\infty}_x} + N^\varepsilon \| :z_N^3: \|_{L^2_\delta W^{-\varepsilon,\infty}_x} \Big) \end{aligned} \tag{3.6}$$

for $0 < \delta \le 1$.

Given a large target time $T > 0$, $M \geq 1$, and $N \in \mathbb{N}$, we set

$$\Omega_{N,T}^{M} = \Big\{\omega \in \Omega : \| :z_N^{\ell}: \|_{L^2_T W_x^{-\varepsilon,\infty}} \leq M, \ \ell = 1, 2, 3\Big\}.$$

Then, for any $\omega \in \Omega_{N,T}^{M}$, it follows from (3.5) and (3.6) that

$$\begin{aligned}\|\Gamma_N(v_N)\|_{X^1(\delta)} \leq C_0\|(v_0, v_1)\|_{\mathcal{H}^1} + C_1\delta^{\frac{1}{2}}\Big(&\|v_N\|^3_{X^1(\delta)} \\ &+ N^{\varepsilon}M\|v_N\|^2_{X^1(\delta)} + N^{\varepsilon}M\|v_N\|_{X^1(\delta)} + N^{\varepsilon}M\Big).\end{aligned}$$

In particular, if we set

$$R = 1 + 2C_0\|(v_0, v_1)\|_{\mathcal{H}^1} \qquad \text{and} \qquad \delta_{N,R} = (100C_1R^2N^{\varepsilon}M)^{-2},$$

then we see that Γ_N maps the ball $B_{N,R} = \{v_N : \ \|v_N\|_{X^1(\delta_{N,R})} \leq R\}$ into itself. Furthermore, by a similar computation, we can show that Γ_N is a contraction on $B_{N,R}$, establishing existence of a unique solution $v_N \in B_{N,R}$ to (3.1). A standard continuity argument allows us to extend the uniqueness to the whole space $X^1(\delta_{N,R})$.

It follows from (2.11) and Chebyshev's inequality (as in [2, Lemma 3][8]) that

$$P\Big(\| :z_N^{\ell}: \|_{L^2_T W_x^{-\varepsilon,\infty}} > M\Big) \leq Ce^{-cM^{\frac{2}{\ell}}T^{-\frac{1}{\ell}}\lambda_N^{\frac{\varepsilon}{2\ell}}}. \tag{3.7}$$

Then, defining Ω_T by

$$\Omega_T = \bigcap_{N\in\mathbb{N}} \Omega_{N,T} = \bigcap_{N\in\mathbb{N}} \bigcup_{M\in\mathbb{N}} \Omega_{N,T}^{M},$$

it follows from (3.7) that $\Omega_{N,T}$ has probability 1 and therefore Ω_T is a set of full probability. Furthermore, given $\omega \in \Omega_T$ and $N \in \mathbb{N}$, there exists $M = M(N) \in \mathbb{N}$ such that $\omega \in \Omega_{N,T}^{M}$ and thus the argument above shows local existence of a unique solution v_N to (3.1) on the time interval $[0, \delta_{N,R}(\omega)]$. This proves almost sure local well-posedness of (3.1). Note that we have the following blowup alternative for the maximal time $T^*_{N,R} = T^*_{N,R}(\omega)$ of existence; given $\omega \in \Omega_T$ and $N \in \mathbb{N}$, we have either

$$\lim_{t\nearrow T^*_{N,R}} \|v_N\|_{X^1(t)} = \infty \qquad \text{or} \qquad T^*_{N,R} \geq T. \tag{3.8}$$

Next, we prove almost sure well-posedness on the entire time interval $[0, T]$. We follow the argument introduced by Burq and Tzvetkov [5] in the context of random data global well-posedness of the cubic NLW on $\mathbb{T}^3$. In view of the blowup alternative (3.8), it suffices to show that, for each $\omega \in \Omega_T$, the $\mathcal{H}^1$-norm of $(v_N(t), \partial_t v_N(t))$ remains finite on $[0, T]$.

Define the energy $\mathcal{E}_N(v)$ by setting

$$\mathcal{E}_N(v)(t) = \frac{1}{2}\|\nabla v(t)\|^2_{L^2} + \frac{1}{2}\|\partial_t v(t)\|^2_{L^2} + \frac{1}{4}\|v(t)\|^4_{L^4} + \frac{1}{2}\lambda_N\|v(t)\|^2_{L^2}.$$

Then, for a solution v_N to (3.1), we have

$$\begin{aligned}\partial_t\mathcal{E}(v_N) &= -\int_{\mathbb{T}^2} \partial_t v_N\Big(\partial_t v_N + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3:\Big)dx \\ &\lesssim -\|\partial_t v_N\|^2_{L^2} + N^{\varepsilon}\|\partial_t v_N\|_{L^2}\Big(\|v_N\|^2_{L^4}\|z_N\|_{W^{-\varepsilon,\infty}} \\ &\qquad + \|v_N\|_{L^4}\| :z_N^2: \|_{W^{-\varepsilon,\infty}} + \| :z_N^3: \|_{W^{-\varepsilon,\infty}}\Big)\end{aligned}$$

[8] Lemma 2.2 in the arXiv version. See also Lemma 4.5 in [30].

By Young's inequality,

$$\lesssim \Big(1 + N^{\varepsilon}\|z_N\|_{W^{-\varepsilon,\infty}}\Big)\mathcal{E}(v_N) + N^{4\varepsilon}\|:z_N^2:\|^4_{W^{-\varepsilon,\infty}} + N^{2\varepsilon}\|:z_N^3:\|^2_{W_x^{-\varepsilon,\infty}}.$$

Then, it follows from Gronwall's inequality that given $T > 0$ and $N, M \in \mathbb{N}$, there exists a constant $C(N, T, M) > 0$ such that for any $\omega \in \Omega^M_{N,T}$, we have

$$\|v_N\|_{X^1(T^*_{N,R})} \lesssim \sup_{t\in[0,T^*_{N,R})} \mathcal{E}(v_N)(t) \le C(N, T, M)\mathcal{E}(v_N)(0) < \infty.$$

Since the choices of N and M are arbitrary, this implies that $T^*_{N,R}(\omega) \ge T$ for any $\omega \in \Omega_T$. This completes the proof of Proposition 3.1. □

We are now ready to present a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$. Fix $T > 0$. Given $N \in \mathbb{N}$, set

$$V_N = v_N - v_N^{\text{lin}}, \tag{3.9}$$

where v_N^{lin} is the linear solution defined in (1.25). Proposition 3.1 ensures that V_N exists almost surely on the time interval $[0, T]$, where it satisfies the Duhamel formulation. In the following, we show that V_N tends to 0 in $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$.

Fix $\varepsilon > 0$ sufficiently small. Then, from Lemma 3.2, we have

$$\begin{aligned}\|z_N(V_N + v_N^{\text{lin}})^2\|_{L^1_T H_x^{-\varepsilon}} &\lesssim T^{\frac12}\|z_N\|_{L^2_T W_x^{-\varepsilon,\frac1\varepsilon}}\Big\|\langle\nabla\rangle^{\varepsilon}\big[(V_N + v_N^{\text{lin}})^2\big]\Big\|_{L^\infty_T L_x^{\frac{2}{1-\varepsilon}}}\\ &\lesssim T^{\frac12}\|z_N\|_{L^2_T W_x^{-\varepsilon,\infty}}\|\langle\nabla\rangle^{\varepsilon}(V_N + v_N^{\text{lin}})\|^2_{L^\infty_T L_x^{\frac{4}{1-\varepsilon}}}\\ &\lesssim T^{\frac12}\|z_N\|_{L^2_T W_x^{-\varepsilon,\infty}}\|V_N + v_N^{\text{lin}}\|^2_{C_T H_x^{1-\varepsilon}}.\end{aligned} \tag{3.10}$$

Similarly, we have

$$\begin{aligned}\|:z_N^2:(V_N + v_N^{\text{lin}})\|_{L^1_T H_x^{-\varepsilon}} &\lesssim T^{\frac12}\|:z_N^2:\|_{L^2_T W_x^{-\varepsilon,\frac1\varepsilon}}\|\langle\nabla\rangle^{\varepsilon}(V_N + v_N^{\text{lin}})\|_{L^\infty_T L_x^{\frac{2}{1-\varepsilon}}}\\ &\lesssim T^{\frac12}\|:z_N^2:\|_{L^2_T W_x^{-\varepsilon,\infty}}\|V_N + v_N^{\text{lin}}\|_{C_T H_x^{1-\varepsilon}}.\end{aligned} \tag{3.11}$$

By Hölder's inequality, we have

$$\|:z_N^3:\|_{L^1_T H_x^{-\varepsilon}} \le T^{\frac12}\|:z_N^3:\|_{L^2_T H_x^{-\varepsilon}}. \tag{3.12}$$

In order to estimate the term $(V_N + v_N^{\text{lin}})^3$, we use (2.13) by assuming that $N \gg 1$ such that $\mathcal{D}_N$ is bounded from $L^2(\mathbb{T}^2)$ to $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$ with norm less than $\lambda_N^{-\frac{\varepsilon}{2}}$.[9] Then, Hölder's and Sobolev's inequalities yield

$$\begin{aligned}\Big\|\int_0^t \mathcal{D}_N(t-t')\big[(V_N + v_N^{\text{lin}})^3(t')\big]dt'\Big\|_{C_T H_x^{1-\varepsilon}} &\lesssim \lambda_N^{-\frac{\varepsilon}{2}}\|(V_N + v_N^{\text{lin}})^3\|_{L^1_T L^2_x}\\ &\le \lambda_N^{-\frac{\varepsilon}{2}} T\|V_N + v_N^{\text{lin}}\|^3_{L^\infty_T L^6_x}\\ &\lesssim \lambda_N^{-\frac{\varepsilon}{2}} T\|V_N + v_N^{\text{lin}}\|^3_{C_T H_x^{1-\varepsilon}}.\end{aligned} \tag{3.13}$$

[9] Note that this gain of $\lambda_N^{-\frac{\varepsilon}{2}}$ is not true for $\partial_t\mathcal{D}_N$. This is the reason we only prove convergence of V_N in $C([0, T]; H^{1-\varepsilon}(\mathbb{T}^2))$ instead of the smaller space $X^{1-\varepsilon}(T)$.

Moreover, from (3.4), we have

$$\|v_N^{\text{lin}}\|_{C_T H_x^{1-\varepsilon}} \lesssim \|(v_0, v_1)\|_{\mathcal{H}^{1-\varepsilon}}, \tag{3.14}$$

uniformly in N.

From (3.9), we have

$$V_N = \Gamma_N(V_N + v_N^{\text{lin}}) - v_N^{\text{lin}}$$

where Γ_N is as in (3.3). Then, putting (3.10)–(3.14) together along with the boundedness of $\mathcal{D}_N$ from $H^{-\varepsilon}(\mathbb{T}^2)$ to $C([0,T]; H^{1-\varepsilon}(\mathbb{T}^2))$, we obtain

$$\begin{aligned}\|V_N\|_{C_T H_x^{1-\varepsilon}} \lesssim{}& \lambda_N^{-\frac{\varepsilon}{2}} T\big(\|V_N\|_{C_T H_x^{1-\varepsilon}} + \|(v_0, v_1)\|_{\mathcal{H}^1}\big)^3 \\ &+ T^{\frac{1}{2}}\Big(\|z_N\|_{L_T^2 W_x^{-\varepsilon,\infty}}\big(\|V_N\|_{C_T H_x^{1-\varepsilon}} + \|(v_0, v_1)\|_{\mathcal{H}^1}\big)^2 \\ &+ \|:z_N^2:\|_{L_T^2 W_x^{-\varepsilon,\infty}}\big(\|V_N\|_{C_T H_x^{1-\varepsilon}} + \|(v_0, v_1)\|_{\mathcal{H}^1}\big) + \|:z_N^3:\|_{L_T^2 H_x^{-\varepsilon}}\Big).\end{aligned} \tag{3.15}$$

As in [15], we introduce a sequence of stopping times

$$\tau_N^\rho = T \wedge \inf\big\{\tau \geq 0 : \ \|V_N\|_{C_\tau H_x^{1-\varepsilon}} > \rho\big\} \tag{3.16}$$

for $\rho > 0$. Then, the bound (3.15) and the continuity in time of V_N (with values in $H^{1-\varepsilon}(\mathbb{T}^2)$) show that for any $\rho > 0$,

$$\begin{aligned}\|V_N\|_{C_{\tau_N^\rho} H_x^{1-\varepsilon}} \lesssim{}& \lambda_N^{-\frac{\varepsilon}{2}} T(\rho + \|(v_0, v_1)\|_{\mathcal{H}^1})^3 \\ &+ T^{\frac{1}{2}}\Big(\|z_N\|_{L_T^2 W_x^{-\varepsilon,\infty}}\big(\rho + \|(v_0, v_1)\|_{\mathcal{H}^1}\big)^2 \\ &+ \|:z_N^2:\|_{L_T^2 W_x^{-\varepsilon,\infty}}\big(\rho + \|(v_0, v_1)\|_{\mathcal{H}^1}\big) + \|:z_N^3:\|_{L_T^2 H_x^{-\varepsilon}}\Big).\end{aligned}$$

Taking an expectation, we conclude from Lemma 2.2 and Proposition 2.6 that

$$\lim_{N\to\infty} \mathbb{E}\big[\|V_N\|_{C_{\tau_N^\rho} H_x^{1-\varepsilon}}\big] = 0.$$

When $\tau_N^\rho < T$, it follows from the definition (3.16) of τ_N^ρ and the continuity in time of V_N that

$$\|V_N\|_{C_{\tau_N^\rho} H_x^{1-\varepsilon}} = \rho.$$

Hence, we obtain

$$P(\tau_N^\rho < T) \leq \frac{1}{\rho}\mathbb{E}\Big[\|V_N\|_{C_{\tau_N^\rho} H_x^{1-\varepsilon}} \mathbf{1}_{[0,T)}(\tau_N^\rho)\Big] \leq \frac{1}{\rho}\mathbb{E}\big[\|V_N\|_{C_{\tau_N^\rho} H_x^{1-\varepsilon}}\big] \longrightarrow 0$$

as $N \to \infty$. This in turn implies that, for any $\rho > 0$, we have

$$P(\|V_N\|_{C_T H_x^{1-\varepsilon}} > \rho) = P(\tau_N^\rho < T) \longrightarrow 0 \tag{3.17}$$

as $N \to \infty$.

Finally, recalling the decompositions $u_N = z_N + v_N$ and (3.9) and applying the embedding $C([0,T]; H^s(\mathbb{T}^2)) \subset H^{-\varepsilon}([0,T]; H^s(\mathbb{T}^2))$ for any $s \in \mathbb{R}$, we obtain

$$\begin{aligned}\|u_N\|_{H_T^{-\varepsilon} H_x^{-\varepsilon}} &= \|z_N + v_N^{\text{lin}} + V_N\|_{H_T^{-\varepsilon} H_x^{-\varepsilon}} \\ &\lesssim \|z_N\|_{C_T H_x^{-\varepsilon}} + \|v_N^{\text{lin}}\|_{H_T^{-\varepsilon} H_x^{1-\varepsilon}} + \|V_N\|_{C_T H_x^{1-\varepsilon}}.\end{aligned}$$

The first and third terms on the right-hand side converge to 0 in probability by Proposition 2.6 (ii) and (3.17), respectively, while the second term on the right-hand side converges to 0 by Lemma 2.7. This completes the proof of Theorem 1.1. □

4. Deterministic limit in the weak noise case

In this section, we work in the weak noise case:

$$\lim_{N\to\infty} \alpha_N^2 \log N = \kappa^2 \in [0, \infty) \tag{4.1}$$

and present a proof of Theorem 1.3. First, note that by setting $\lambda_N = 1$, the results in Section 2 hold in this case. In particular, the linear stochastic wave equation (1.32) admits a unique invariant measure, still denoted by μ_N.

Let $(z_{0,N}^\omega, z_{1,N}^\omega)$ be as in (1.33), distributed by the Gaussian measure μ_N. Denote by z_N the solution to (1.32) with $(z_N, \partial_t z_N)|_{t=0} = (z_{0,N}^\omega, z_{1,N}^\omega)$. Then, by invariance of μ_N, the variance of $z_N(t)$ is given by

$$\sigma_N \overset{\text{def}}{=} \frac{\alpha_N^2}{8\pi^2} \sum_{|n|\le N} \frac{1}{\langle n\rangle^2}. \tag{4.2}$$

We now define the Wick powers $:z_N^\ell:$ as in (2.8) with this new variance σ_N defined in (4.2). Note that from (4.2) with (4.1) and Lemma 2.3, we have

$$\lim_{N\to\infty} \sigma_N = \frac{1}{4\pi}\kappa^2. \tag{4.3}$$

As in the proof of Theorem 1.1, we proceed with the Da Prato–Debussche trick. Namely, write the solution u_N to (1.6) as $u_N = v_N + z_N$. Then, the residual term v_N satisfies

$$\begin{cases} \mathcal{L} v_N + (3\sigma_N - 1)(v_N + z_N) + v_N^3 + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3: = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (v_0, v_1), \end{cases} \tag{4.4}$$

where $\mathcal{L} = \partial_t^2 - \Delta + \partial_t + 1$ is as in (1.31).

Proceeding as in the proof of Proposition 2.6, we obtain the following lemma on the regularity and decay properties of the Wick powers $:z_N^\ell:$.

Lemma 4.1. *Let $\ell \in \mathbb{N}$. Given any finite $p, q \ge 1$, $T > 0$, and $\varepsilon > 0$, we have*[10]

$$\lim_{N\to\infty} \mathbb{E}\Big[\| :z_N^\ell(t): \|_{L_T^q W_x^{-\varepsilon,\infty}}^p\Big] = 0 \qquad \textit{and} \qquad \lim_{N\to\infty} \mathbb{E}\Big[\|z_N(t)\|_{X^{-\varepsilon}(T)}^p\Big] = 0,$$

where $X^s(T)$ is as in (3.2).

Lemma 4.1 follows as in Proposition 2.6 once we note the following; under (4.1), we have $\alpha_N \to 0$ as $N \to \infty$, which yields

$$\begin{aligned} \mathbb{E}\big[|\langle\nabla\rangle^{-\varepsilon} :z_N^\ell(t,x):|^2\big] &= \ell! \sum_{n_1,\dots,n_\ell\in\mathbb{Z}_N^2} \Big(\prod_{j=1}^\ell \frac{\alpha_N^2}{\langle n\rangle^2}\Big) \langle n_1 + \cdots + n_\ell\rangle^{-2\varepsilon} \lesssim_\ell \alpha_N^{2\ell} \\ &\longrightarrow 0, \end{aligned}$$

as $N \to \infty$.

[10] In this case, we also have convergence of $\partial_t z_N$ to 0 in $C([0,T]; H^{-1-\varepsilon}(\mathbb{T}^2))$ since the convergence to 0 comes from $\alpha_N \to 0$, not from a gain of a negative power of λ_N.

By arguing as in the proof of Proposition 3.1, we can show that Eq. (4.4) is almost surely globally well-posed in $\mathcal{H}^1(\mathbb{T}^2)$ in the sense that for any $T > 0$, there exists a set Ω_T of full probability such that for any $\omega \in \Omega_T$ and $N \in \mathbb{N}$, there exists a unique solution $v_N \in X^1(T)$ to (4.4), satisfying the bound

$$\|v_N\|_{X^1(T)} \leq C(N, T, \omega)\|(v_0, v_1)\|_{\mathcal{H}^1}.$$

Our main goal in this section is to prove the following proposition.

Proposition 4.2.
Let v_N be the solution to (4.4). Then, given any $T, \varepsilon > 0$, v_N converges in probability to the solution w_κ to (1.35) in $X^{1-\varepsilon}(T)$.

Once we have Proposition 4.2, Theorem 1.3 follows from the decomposition $u_N = z_N + v_N$ and the decay of z_N to 0 in $X^{-\varepsilon}(T)$ presented in Lemma 4.1. Hence, it remains to prove Proposition 4.2.

Proof of Proposition 4.2. Fix $T > 0$. By proceeding as in the proof of Proposition 3.1, we can show that the deterministic equation (1.35) admits a unique global solution $w_\kappa \in X^1(T)$, satisfying the energy bound

$$\|w_\kappa\|_{X^1(T)} \leq R_\kappa \overset{\text{def}}{=} C_\kappa(T)\|(v_0, v_1)\|_{\mathcal{H}^1}. \tag{4.5}$$

Define β_N by setting

$$\beta_N = 3\left(\sigma_N - \frac{\kappa^2}{4\pi}\right). \tag{4.6}$$

Then, we rewrite (4.4) as

$$\partial_t^2 v_N - \Delta v_N + \partial_t v_N + \frac{3}{4\pi}\kappa^2 v_N + v_N^3 + \mathcal{Q}_N(v_N) = 0,$$

where $\mathcal{Q}_N(v_N)$ is the "error" part given by

$$\mathcal{Q}_N(v_N) = \beta_N v_N + (3\sigma_N - 1)z_N + 3v_N^2 z_N + 3v_N :z_N^2: + :z_N^3: .$$

By setting $V_N = v_N - w_\kappa$, we see that V_N then solves

$$\begin{cases} \partial_t^2 V_N - \Delta V_N + \partial_t V_N + \frac{3}{4\pi}\kappa^2 V_N + V_N^3 \\ \qquad + 3V_N^2 w_\kappa + 3V_N w_\kappa^2 + \mathcal{Q}_N(V_N + w_\kappa) = 0 \\ (V_N, \partial_t V_N)\big|_{t=0} = (0, 0). \end{cases} \tag{4.7}$$

We first establish a good control on V_N on short time intervals. With a slight abuse of notations, we set

$$X^s(I) \overset{\text{def}}{=} C(I; H^s(\mathbb{T}^2)) \cap C^1(I; H^{s-1}(\mathbb{T}^2))$$

for an interval $I \subset \mathbb{R}_+$.

Lemma 4.3. *Given κ as in (4.1), let R_κ be as in (4.5). Then, for any $\rho > 0$ and small $\varepsilon > 0$, there exist $T_0 = T_0(\rho, R_\kappa)$ and $C_0 > 0$ such that if*

$$\|V_N\|_{X^{1-\varepsilon}([t_0, t_0+\tau])} \leq \rho \tag{4.8}$$

for some $t_0 \in [0, T)$ and $0 < \tau \leq T_0$ such that $t_0 + \tau \leq T$, then we have

$$\begin{aligned} \|V_N\|_{X^{1-\varepsilon}([t_0,t_0+\tau])} \leq C_0 \Big\{ & \|(V_N(t_0), \partial_t V_N(t_0))\|_{\mathcal{H}^{1-\varepsilon}} + \beta_N \tau(\rho + R_\kappa) \\ & + \tau \|z_N\|_{L^\infty_T H^{-\varepsilon}_x} + \tau^{\frac{1}{2}} \Big(\|z_N\|_{L^2_T W^{-\varepsilon,\infty}_x}(\rho^2 + R_\kappa^2) \\ & + \| :z_N^2: \|_{L^2_T W^{-\varepsilon,\infty}_x}(\rho + R_\kappa) + \| :z_N^3: \|_{L^2_T H^{-\varepsilon}_x} \Big) \Big\} . \end{aligned} \tag{4.9}$$

Proof. Given $t_0 \in [0, T)$ and $0 < \tau \leq T - t_0$, set $I = [t_0, t_0 + \tau]$. By estimating the Duhamel formulation of (4.7) on I as in the previous section, we have

$$\begin{aligned} \|V_N\|_{X^{1-\varepsilon}(I)} \lesssim & \|(V_N(t_0), \partial_t V_N(t_0))\|_{\mathcal{H}^{1-\varepsilon}} \\ & + \tau \|V_N\|_{X^{1-\varepsilon}(I)} \big(\|V_N\|^2_{X^{1-\varepsilon}(I)} + \|w_\kappa\|^2_{C_T H^1_x} \big) \\ & + \beta_N \tau \big(\|V_N\|_{X^{1-\varepsilon}(I)} + \|w_\kappa\|_{C_T H^{-\varepsilon}_x} \big) + (3\sigma_N - 1)\tau \|z_N\|_{L^\infty_T H^{-\varepsilon}_x} \\ & + \tau^{\frac{1}{2}} \Big(\|z_N\|_{L^2_T W^{-\varepsilon,\infty}_x}(\|V_N\|^2_{X^{1-\varepsilon}(I)} + \|w_\kappa\|^2_{C_T H^1_x}) \\ & + \| :z_N^2: \|_{L^2_T W^{-\varepsilon,\infty}_x}(\|V_N\|_{X^{1-\varepsilon}(I)} + \|w_\kappa\|_{C_T H^1_x}) + \| :z_N^3: \|_{L^2_T H^{-\varepsilon}_x} \Big), \end{aligned}$$

where the first term comes from the contribution of the linear evolution associated with the operator $\mathcal{L}^\kappa = \partial_t^2 - \Delta + \partial_t + \frac{3}{4\pi}\kappa^2$, starting from initial data $(V_N(t_0), \partial_t V_N(t_0))$. Hence, from (4.5) and (4.8), we obtain

$$\begin{aligned} \|V_N\|_{X^{1-\varepsilon}(I)} \lesssim & \|(V_N(t_0), \partial_t V_N(t_0))\|_{\mathcal{H}^{1-\varepsilon}} + \tau(\rho^2 + R_\kappa^2)\|V_N\|_{X^{1-\varepsilon}(I)} \\ & + \beta_N \tau(\rho + R_\kappa) + \tau \|z_N\|_{L^\infty_T H^{-\varepsilon}_x} + \tau^{\frac{1}{2}} \Big(\|z_N\|_{L^2_T W^{-\varepsilon,\infty}_x}(\rho^2 + R_\kappa^2) \\ & + \| :z_N^2: \|_{L^2_T W^{-\varepsilon,\infty}_x}(\rho + R_\kappa) + \| :z_N^3: \|_{L^2_T H^{-\varepsilon}_x} \Big), \end{aligned}$$

where we used the boundedness of $3\sigma_N - 1$ in view of (4.3).[11] Then, by choosing $T_0 = T_0(\rho, R_\kappa) > 0$ sufficiently small, we obtain the desired bound (4.9). □

We continue with the proof of Proposition 4.2. Fix small $\varepsilon > 0$. In the following, we proceed as in the previous section and introduce a sequence of stopping times

$$\tau_N^\rho = T \wedge \inf\big\{ \tau \geq 0 : \|V_N\|_{X^{1-\varepsilon}(\tau)} > \rho \big\} \tag{4.10}$$

for $\rho > 0$.

Let R_κ and T_0 be as in (4.5) and Lemma 4.3, respectively. Given $j = 0, \dots, \big[\frac{T}{T_0}\big] + 1$, set $t_j = jT_0$ for $0 \leq j \leq \big[\frac{T}{T_0}\big]$ and $t_{[\frac{T}{T_0}]+1} = T$.[12] Then, our goal is to apply Lemma 4.3 iteratively and show that

$$\lim_{N\to\infty} P(t_j \leq \tau_N^\rho < t_{j+1}) = 0, \tag{4.11}$$

[11] The bound on $3\sigma_N - 1$ depends on the entire sequence $\{\alpha_N\}_{N\in\mathbb{N}}$ but this does not cause an issue since we work with a *fixed* sequence $\{\alpha_N\}_{N\in\mathbb{N}}$.

[12] If T is a multiple of $T_0 > 0$, then we do not need to consider $j = \big[\frac{T}{T_0}\big] + 1$ and it suffices to prove (4.11) for all $j = 0, \dots, \big[\frac{T}{T_0}\big] - 1$.

for all $j = 0, \dots, \left[\frac{T}{T_0}\right]$. Once we prove (4.11), we obtain

$$P(\|V_N\|_{X^{1-\varepsilon}(T)} > \rho) = P(\tau_N^\rho < T) \le \sum_{j=0}^{\left[\frac{T}{T_0}\right]} P(t_j \le \tau_N^\rho < t_{j+1}) \longrightarrow 0$$

as $N \to \infty$.

From the definition (4.10) of τ_N^ρ, the continuity in time of $(V_N, \partial_t V_N)$ (with values in $\mathcal{H}^{1-\varepsilon}(\mathbb{T}^2)$), and applying Lemma 4.3 along with Lemma 4.1 and $\beta_N \to 0$ (which follows from (4.3) and (4.6)), we have

$$\begin{aligned} P(t_j \le \tau_N^\rho < t_{j+1}) &= \frac{1}{\rho}\mathbb{E}\Big[\|V_N\|_{X^{1-\varepsilon}([t_j,\tau_N^\rho])}\mathbf{1}_{[t_j,t_{j+1})}(\tau_N^\rho)\Big] \\ &\le \frac{C_0}{\rho}\mathbb{E}\Big[\|(V_N(t_j), \partial_t V_N(t_j))\|_{\mathcal{H}^{1-\varepsilon}}\mathbf{1}_{[t_j,t_{j+1})}(\tau_N^\rho)\Big] + o(1), \end{aligned} \tag{4.12}$$

as $N \to \infty$. When $j = 0$, we obtain (4.11) from (4.12) since $(V_N(0), \partial_t V_N(0)) = (0, 0)$. In general, by noting that

$$\|(V_N(t_j), \partial_t V_N(t_j))\|_{\mathcal{H}^{1-\varepsilon}} \le \|V_N\|_{X^{1-\varepsilon}([t_{j-1},t_j])},$$

we apply the bound (4.12) iteratively and obtain

$$\begin{aligned} P(t_j \le \tau_N^\rho < t_{j+1}) &\le \frac{C_0}{\rho}\mathbb{E}\Big[\|V_N\|_{X^{1-\varepsilon}([t_{j-1},t_j])}\Big] + o(1) \\ &\le \frac{C_0^2}{\rho}\mathbb{E}\Big[\|(V_N(t_{j-1}), \partial_t V_N(t_{j-1}))\|_{\mathcal{H}^{1-\varepsilon}}\Big] + o(1) \\ &\le \cdots \le \frac{C_0^j}{\rho}\mathbb{E}\Big[\|(V_N(0), \partial_t V_N(0))\|_{\mathcal{H}^{1-\varepsilon}}\Big] + o(1) \\ &\longrightarrow 0 \end{aligned}$$

as $N \to 0$ since $(V_N(0), \partial_t V_N(0)) = (0, 0)$. This proves (4.11). □

Remark 4.4. As mentioned in Remark 1.5 we can easily adapt the proof of Theorem 1.3 presented above to a general defocusing power-type nonlinearity u^{2k+1}, $k \in \mathbb{N}$, by using the following identity:

$$u_N^{2k+1} = \sum_{j=0}^{k} \binom{2k+1}{2j}(2j-1)!!\sigma_N^j \ :u_N^{2k+1-2j}:$$

in place of $u_N^3 = :u_N^3: + 3\sigma_N u_N$. Here, $(2j-1)!! = (2j-1)(2j-3)\cdots 3\cdot 1$ with the convention $(-1)!! = 1$. In this case, the solution u_N to

$$\begin{cases} (\partial_t^2 - \Delta + \partial_t)u_N + u_N^{2k+1} = \alpha_N \xi_N \\ (u_N, \partial_t u_N)\big|_{t=0} = (v_0, v_1) + (z_{0,N}^\omega, z_{1,N}^\omega) \end{cases}$$

converges to the solution w_κ to

$$\begin{cases} (\partial_t^2 - \Delta + \partial_t)w_\kappa + \sum_{j=0}^k \binom{2k+1}{2j}(2j-1)!!(\frac{\kappa^2}{4\pi})^j w_\kappa^{2k+1-2j} = 0 \\ (w_\kappa, \partial_t w_\kappa)\big|_{t=0} = (v_0, v_1) \end{cases}$$

as $N \to \infty$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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