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# Minimal graphs for hamiltonian extension

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## Abstract

For every  $n \geq 3$  we determine the minimum number of edges of graph with  $n$  vertices such that for any non edge  $xy$  there exists a hamiltonian cycle containing  $xy$ .

**Keywords:** 2-factor, hamiltonian cycle, hamiltonian path.

## 1 Introduction

For all graph theoretical terms and notations not defined here the reader is referred to [1]. We only consider simple finite loopless undirected graphs. For a graph  $G = (V, E)$  with  $|V| = n$  vertices, an edge is a pair of two connected vertices  $x, y$ , we denote it by  $xy, xy \in E$ ; when two vertices  $x, y$  are not connected this pair form the *non-edge*  $xy, xy \notin E$ . In  $G$  a 2-factor is a subset of edges  $F \subset E$  such that every vertex is incident to exactly two edges of  $F$ . Since  $G$  is finite a 2-factor consists of a collection of vertex disjoint cycles spanning the vertex set  $V$ . When the collection consists of an unique cycle the 2-factor is connected, so it is a hamiltonian cycle.

We intend to determine, for any integer  $n \geq 3$ , a graph  $G = (V, E), n = |V|$  with a minimum number of edges such that for every non-edge  $xy$  it is always possible to include the non-edge  $xy$  into a connected 2-factor, i.e., the graph  $G_{xy} = (V, E \cup \{xy\})$  has a hamiltonian cycle  $H, xy \in H$ . In other words for any non-edge  $xy$  of  $G$  there exists a hamiltonian path between  $x$  and  $y$ .

This problem is related to the minimal 2-factor extension studied in [3] in which the 2-factors are not necessary connected. It is also related to the problem of finding minimal graphs for non-edge extensions in the case of perfect matchings (1-factors) studied in [2].

**Definition 1.1** *Let  $G = (V, E)$  be a graph and  $xy \notin E$  an non-edge. If  $G_{xy} = (V, E \cup \{xy\})$  has a hamiltonian cycle that contains  $xy$  we shall say that  $xy$  has been extended (to a connected 2-factor, to an hamiltonian cycle).*

**Definition 1.2** *A graph  $G = (V, E)$  is connected 2-factor expandable or hamiltonian expandable (shortly expandable) if every non-edge  $xy \notin E$  can be extended.*

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**Definition 1.3** An expandable graph  $G = (V, E)$  with  $|V| = n$  and a minimum number of edges is a minimum expandable graph. The size  $|E|$  of its edge set is denoted by  $Exp_h(n)$ .

The case where the 2-factor is not constrained to be hamiltonian is studied in [3]. In this context  $Exp_2(n)$  denotes the size of a minimum expandable graph with  $n$  vertices. It follows that  $Exp_h(n) \geq Exp_2(n)$ .

We use the following notations. For  $G = (V, E)$ ,  $N(v)$  is the set of neighbors of a vertex  $v$ ,  $\delta(G)$  is the minimum degree of a vertex. A vertex with exactly  $k$  neighbors is a  $k$ -vertex. When  $P = v_i, \dots, v_j$  is a sequence of vertices that corresponds to a path in  $G$ , we denote by  $\bar{P} = v_j, \dots, v_i$  its mirror sequence (both sequences correspond to the same path).

We state our result.

**Theorem 1.1** The minimum size of a connected 2-factor expandable graph is:

$$Exp_h(3) = 2, Exp_h(4) = 4, Exp_h(5) = 6; Exp_h(n) = \lceil \frac{3}{2}n \rceil, n \geq 6$$

**Proof:** For  $n \geq 3$  we have  $Exp_h(n) \geq Exp_2(n)$ .

In [3] it is proved that the three graphs given by Fig. 1 are minimum for 2-factor extension. They are also minimum expandable for connected 2-factor extension.

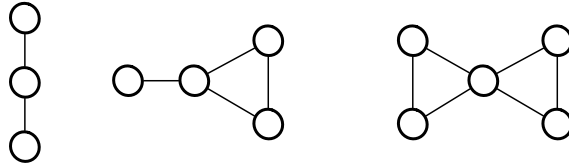


Figure 1:  $P_3$ , the paw, the butterfly.

Now let  $n \geq 6$ . From [3] we know the following when  $G$  a minimum expandable graph for the 2-factor extension:

- $G$  is connected;
- if  $\delta(G) = 1$  then  $Exp_2(n) \geq \frac{3}{2}n$ ;
- for  $n \geq 7$ , if  $u, v$  are two 2-vertices such that  $N(u) \cap N(v) \neq \emptyset$  then  $Exp_2(n) \geq \frac{3}{2}n$ ;

The graph given by Fig. 2 is minimum for 2-factor extension (see [3]). One can check that it is expandable for connected 2-factor extension. So we have  $Exp_h(6) = 9 = \frac{3}{2}n$ .

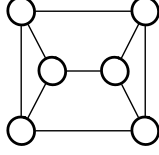


Figure 2: A minimum hamiltonian expandable graph with 6 vertices.

Suppose that  $G$  is a minimum expandable graph with  $n \geq 7$  and  $\delta(G) = 2$ . Let  $v \in V$  with  $d(v) = 2$ ,  $N(v) = \{u_1, u_2\}$ . If  $u_1u_2 \notin E$  then  $u_1u_2$  cannot be expanded into a hamiltonian cycle. So  $u_1u_2 \in E$ . If  $d(u_1) = 2$  then  $u_2 \in N(u_1) \cap N(v)$  and  $Exp_h(n) \geq \frac{3}{2}n$ . So from now on we may assume  $d(u_1), d(u_2) \geq 3$ . Suppose that  $d(u_1) = d(u_2) = 3$ . Let  $N(u_1) = \{v, u_2, v_1\}$ ,  $N(u_2) = \{v, u_1, v_2\}$ . If  $v_1 \neq v_2$  then  $u_1v_2$  is not expandable. If  $v_1 = v_2$  then  $vv_1$  is not expandable. From now we can suppose that  $d(u_1) \geq 3, d(u_2) \geq 4$ . Moreover  $v$  is the unique 2-vertex in  $N(u_2)$ . It follows that every 2-vertex  $u \in V$  can be matched with a distinct vertex  $u_2$  with  $d(u_2) \geq 4$ . Then  $\sum_{v \in V} d(v) \geq 3n$  and thus  $m \geq \frac{3}{2}n$ .

When  $\delta(G) \geq 3$  we have  $\sum_{v \in V} d(v) \geq 3n$ . Thus for any expandable graph we have  $|E| = m \geq \frac{3}{2}n, n \geq 7$ .

For any even integer  $n \geq 8$  we define the graph  $G_n = (V, E)$  as follows. Let  $n = 2p$ ,  $V = A \cup B$  where  $A = \{a_1, \dots, a_p\}$  and  $B = \{b_1, \dots, b_p\}$ .  $A$  (resp.  $B$ ) induces the cycle  $C_A = (A, E_A)$  with  $E_A = \{a_1a_2, a_2a_3, \dots, a_pa_1\}$  (resp.  $C_B = (B, E_B)$  with  $E_B = \{b_1b_2, b_2b_3, \dots, b_pb_1\}$ ). Now  $E = E_A \cup E_B \cup E_C$  with  $E_C = \{a_2b_2, a_3b_3, \dots, a_{p-1}b_{p-1}, a_1b_p, a_pb_1\}$ . Note that  $G_n$  is cubic so  $m = \frac{3}{2}n$ . (see  $G_{10}$  in Fig. 3)

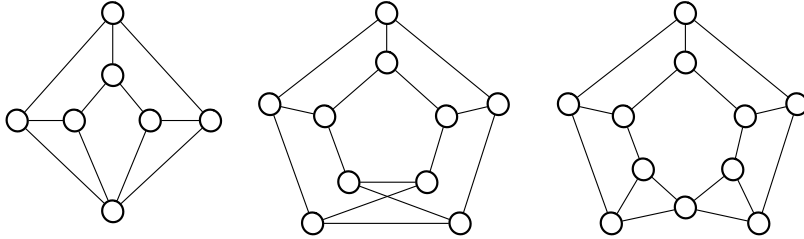


Figure 3: The graphs  $G_7, G_{10}, G_{11}$ , from the left to the right.

We show that  $G_n$  is expandable. First we consider a non-edge  $a_i a_j, p \geq j > i \geq 1$ . Note that the case of a non-edge  $b_i b_j$  is analogous. We have  $j \geq i + 2$  and since  $a_1 a_p \in E$  from symmetry we can suppose that  $j < p$ . Let  $P = a_j, a_{j-1}, \dots, a_{i+1}, b_{i+1}, b_{i+2}, \dots, b_{j+1}, a_{j+1}, a_{j+2}, b_{j+2}, \dots, c_j$  where  $c_j$  is either  $a_p$  or  $b_p$  and let  $Q = a_i, b_i, b_{i-1}, a_{i-1}, \dots, c_i$  where  $c_i$  is either  $a_1$  or  $b_1$ . From  $P$  and  $Q$  one can obtain an hamiltonian cycle containing  $a_i b_j$  whatever  $c_i$  and  $c_j$  are.

Now we consider a non-edge  $a_i b_j$ . Without loss of generality we assume  $j \geq i$ . Suppose first that  $j = i$ , so either  $i = 1$  or  $i = p$ . Without loss of generality we assume  $i = j = 1$ :  $a_1, b_p, b_{p-1}, \dots, b_2, a_2, a_3, \dots, a_p, b_1, a_1$  is a hamiltonian cycle. Now assume that  $j > i$ : Let  $P_j = b_j, b_{j-1}, \dots, b_{i+1}, a_{i+1}, a_{i+2}, \dots, a_{j+1}, b_{j+1}, b_{j+2}, a_{j+2}, \dots, c_p$  where either  $c_p = a_p$  or  $c_p = b_p$ ,  $P_i = a_i, b_i, b_{i-1}, a_{i-1}, a_{i-2}, \dots, c_1$  where either  $c_1 = a_1$  or  $c_1 = b_1$ . If  $c_p = a_p$  and  $c_1 = a_1$  then  $P_j, b_1, b_p, P_i, a_j$  is a hamiltonian cycle. If  $c_p = a_p$  and  $c_1 = b_1$  then  $P_j, a_1, b_p, P_i, a_j$  is a hamiltonian cycle. The two other cases are symmetric.

For any odd integer  $n = 2p + 1 \geq 7$  we define the graph  $G_n = (V, E)$  as follows. We set  $V = A \cup B \cup \{v_n\}$  where  $A = \{a_1, \dots, a_p\}$  and  $B = \{b_1, \dots, b_p\}$ .  $A \cup \{v_n\}$  (resp.  $B \cup \{v_n\}$ ) induces the cycle  $C_A = (A \cup \{v_n\}, E_A)$  with  $E_A = \{a_1 a_2, a_2 a_3, \dots, a_p v_n, v_n a_1\}$  (resp.  $C_B = (B \cup \{v_n\}, E_B)$  with  $E_B = \{b_1 b_2, b_2 b_3, \dots, b_p v_n, v_n b_1\}$ ). Now  $E = E_A \cup E_B \cup E_C$  with  $E_C = \{a_i b_i | 1 \leq i \leq p\} \cup \{a_1 v_n, b_1 v_n, a_p v_n, b_p v_n\}$ . Note that  $m = \lceil \frac{3}{2}n \rceil$ . (see  $G_7$  and  $G_{11}$  in Fig. 3)

We show that  $G_n$  is expandable. First, we consider a non-edge  $a_i a_j, p \geq j > i \geq 1$  (the case of a non-edge  $b_i b_j$  is analogous).  $a_i, a_{i+1}, \dots, a_{j-1}, b_{j-1}, b_{j-2}, b_{j-3}, \dots, b_i, b_{i-1}, a_{i-1}, a_{i-2}, b_{i-2}, \dots, v_n, c_p, d_p, d_{p-1}, c_{p-1}, \dots, c_j, d_j$ , where  $d_j = a_j$  and for any  $k, j \leq k \leq p$ , the ordered pairs  $c_k, d_k$  correspond to either  $a_k, b_k$  or  $b_k, a_k$ , is a hamiltonian cycle. Second, let a non-edge  $a_i b_j, p \geq j > i \geq 1$ . We use the same construction as above taking  $d_j = b_j$ .  $\square$

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