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Cycles in synchronous iterative voting: general robustness and examples in Approval Voting

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Abstract. We consider synchronous iterative voting, where voters are given the opportunity to strategically choose their ballots depending on the outcome deduced from the previous collective choices.

We propose two settings for synchronous iterative voting, one of classical flavor with a discrete space of states, and a more general continuous-space setting extending the first one. We give a general robustness result for cycles not relying on a tie-breaking rule, showing that they persist under small enough perturbations of the behavior of voters. Then we give examples in Approval Voting of electorates applying simple, sincere and consistent heuristics (namely Laslier's Leader Rule or a modification of it) leading to cycles with bad outcomes, either not electing an existing Condorcet winner, or possibly electing a candidate ranked last by a majority of voters. Using the robustness result, it follows that those "bad cycles" persist even if only a (large enough) fraction of the electorate updates its choice of ballot at each iteration.

We complete these results with examples in other voting methods, including ranking methods satisfying the Condorcet criterion; an *in silico* experimental study of the rarity of preference profiles exhibiting bad cycles; and an example exhibiting chaotic behavior.

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1 Introduction

Iterative Voting. In any voting system, the choice of a ballot by any voter is guided by her preferences between the candidates and the expected effect of each ballot she can cast. To estimate the effect of any ballot, one needs to know the ballots cast by the other voters, or at least to have some information on them. Assuming perfect information is neither realistic nor theoretically useful: in large electorates, changing one's ballot rarely changes the outcome of the election (hence, as is well-known, the notion of Nash equilibrium is ineffectively broad in the context of elections [Las09]). One thus often considers situations where information is imperfect, so that the ballot to be chosen by any given voter might change the outcome, with a tiny but non-zero probability. Another point of view is to consider the possibility for voters of aligned interests to form coalitions and decide together which ballots to cast. A third point of view, taking from both of the first two, is to consider that some (or all) information about voters' intents is common knowledge, and that voters choose their ballot under the assumption that other voters that have the same preferences as theirs will make the same reasoning as they will – thus overcoming the insufficient weight of one ballot to change the outcome.

In any case, these considerations introduce a *dynamical* component to voting: after all voters adjusted their intentions, the information under which this adjustment was made is outdated; the new intents result in new information that could be shared, and itself result in new strategic adjustments of voters' intents, etc. Natural questions are thus whether there is some dynamical equilibrium, i.e. intents of all voters that are preserved when all shared publicly; whether such equilibrium is unique; what properties equilibriums have, in particular which candidates can be elected at equilibrium; and which other sets of intents converge to an equilibrium after successive adjustments. The field of *iterative voting* is the study of these questions, under various modeling assumptions.

Synchronized and asynchronous iterative voting. By far the most studied version of iterative voting is *asynchronous*, i.e. only some voters (most commonly a single one) adjust their intents given the available information, and information is updated before some other voter(s) make their adjustments, etc. While this model has both a theoretical interest and practical applications to some collective decision processes, it is ill-suited to model large-scale political election, where the number of times when the information is updated (poll publications) is much lesser than the number of voters.

In contrast, we consider in this article *synchronous* iterative voting, where all voters are given the information at time t, and simultaneously adjust their intents to build the new situation at time $t+1$. It is important to observe that what makes iterative voting synchronous is not that the voters' intents change exactly at the same time, but that all voters have the opportunity to adjust their preferences before information is updated.

Synchronous iterative voting can be thought of as a model for political elections, where information is broadcasted through polls; we will thus name our dynamical model *Polling*

Dynamics; similar points of view have been notably taken in [CPP04], and [RE12] where the emphasis is on the effect of the amount of information given by the polls (ranks, or scores, etc).

However this is not our sole motivation: considering synchronized iterative voting is also interesting from a theoretical perspective, when trying to define a "best" ballot for each voter. Indeed, to be efficient in her choice a voter should not only vote strategically, but also anticipate that other voters will also do so, who will themselves try to anticipate all voters' strategies, etc. A model for this situation is to replace instantaneous anticipations by iterations: each voter is given a fixed heuristic, which dictates for each possible expected outcome which ballot to cast, and we inductively apply these heuristics and update the expected outcome. At a fixed point of the Polling Dynamics, which we call a *dynamical equilibrium*, anticipating each other's strategy result in the same situation, which is therefore stable. Conversely, the existence of a cycle means that in some circumstances there is no meaningful way for voters to inductively anticipate other's strategies. Note that even when limiting the number of steps of counter-strategies, the game-theoretic analysis of voting rules can be quite deep, see [GHRS19].

We shall be especially interested in Approval Voting – the voting system in which a ballot can contain the names of any subset of candidates, and the candidate whose name is present in the most ballots is elected. Approval Voting has the very interesting feature that several sincere ballots coexist for each order of preference, so that there exist heuristics that are both sincere, simple, and non-trivial; let us discuss this in more detail.

On sincerity, strategic voting and straightforwardness Durand notes in [Dur15] that in general the meaning of "sincerity" is open to interpretation, and that this word has often been used to argue against strategic voting. He makes a compelling point that strategic voting is to be expected, and even advised to voters, and that what causes a democratic problem is the *necessity* of resorting to strategies to get the best outcome rather than the fact that voters embrace this necessity ("*Manipuler c'est bien, la manipulabilité c'est mal*": "Manipulation is good, manipulability is bad"). Indeed strategies that either need a lot of information on others' plans or that are too contrived cannot be applied uniformly, creating asymmetries between voters. Even in the absence of manipulation or strategic voting, sincere ballots can be cast but afterward regretted in view of the outcome of the election, thus lowering trust and confidence in the democratic system. A most important property is thus *straightforwardness* [Gib73], i.e. the existence of strategies that yield an optimal outcome whatever choices are made by the other voters. Since straightforwardness cannot be hoped in general [Gib73, Sat75], one can try to determine whether certain systems often offer weak forms of straightforwardness (see again [Dur15]).

In Approval Voting, one can say that a ballot is *sincere* whenever all candidates present on the ballot are preferred to all candidates not present on it [BF07]. As soon as there are more than two candidates each voter has several sincere ballots, corresponding to the

various points in her order of preference where she can draw the line between acceptance and rejection. In particular, strategic voting (i.e. choosing one's ballot depending on the ballots expected to be cast by the other voters) can occur even when restricting to sincere ballots. In other words, sincerity in the above sense does not imply straightforwardness. One of our goals is to give examples showing how very far from straightforward Approval Voting can be even when restricting to sincere ballots.

Is Approval Voting a Condorcet system "in practice"? While Approval Voting is known not to be a Condorcet system, it has been argued that it might be close to Condorcet "in practice". We can e.g. read in [BF10]: "In particular, Condorcet candidates [...] almost always win under AV", and some elections where this assertion can be checked are mentioned as evidence.

Several theoretical results could also be seen as providing a basis for the above claim. First, even though many other voting systems also satisfy these properties, let us mention that Brams and Sanver [BS03] showed that when a Condorcet winner exists, her election is a strong Nash equilibrium; and in [SS04] it is shown that only Condorcet Winners are elected at strong Nash equilibriums. Strong Nash equilibrium are said by Brams and Sanver to be "globally stable", but one point not considered in either works above is whether they should be expected to be reached in practice.

A stronger argument originates in Laslier [Las09], where it is proved that under a large-electorate model with uncertainty in the recording of votes and perfect common information, the best course of action for voters results in a particular heuristic, the "Leader Rule" (LR). Additionally, he proved that if there is a Condorcet winner and all voters apply the Leader Rule, then there is at least one equilibrium, and any equilibrium elects the Condorcet winner.[1]

Description of the main results. To introduce the Polling Dynamics, we propose both a classical-flavored *discrete space* setting, and an enlarged *continuous space* setting that allows to continuously perturb a situation (i.e. a given electorate with fixed heuristics), for example in order to assume that at each step of the dynamics, a small fraction of voters fail to adjust their ballots. This makes it possible to discuss how dynamical phenomenons change under such perturbations.

Our first main result (Theorem A in Section 3) is a quite general robustness result, showing under a mild assumption that cycles or equilibriums in the discrete space setting persist under perturbation inside the continuous space setting. All the examples we will give benefit this robustness. Our main contribution here is in the framework for Polling Dynamics, the proof of Theorem A being rather straightforward once the definitions are set up.

[1] A similar argument has been raised by Warren D. Smith and is used by advocates of Approval Voting and Range Voting, see https://www.rangevoting.org/AppCW.html. The argument of Smith is less rigorous than Laslier's, since the equilibrium assumption is kept implicit.

The next main results are examples showing that in Approval Voting, the Polling Dynamics can exhibit a "bad cycle" where a disliked candidate can be elected, even in the presence of a Condorcet winner (Theorem B and C in Section 4). The main point shall not be the mere existence of cycles, a rather unsurprizing phenomenon,[2] but rather that such cycles can result in the election of a suboptimal candidate. As noted by Laslier, previous examples of cycles (notably in [BF07]) needed some voters to change their heuristic at some iteration of the process; in our examples the assumed strategic behaviors is consistent, i.e. constant in time; they are also sincere, simple, and strategically sound (the first example uses the Leader Rule, the second a slight modification where voters refuse to vote for several of their least-preferred candidates in any circumstance). Two obvious limitations to the relevance of these existence results are that such "bad cycles" could be rare, and that it is unrealistic to assume all voters to update their choice of ballots at each poll publication. The former limitation will be addressed by numerical experiments (which partly confirm it, and partly moderate it, see below); the latter is addressed by our robustness result. Indeed, theorem A shows that if instead of assuming that all voters update their choice of ballot at each iteration, we only assume that a large enough proportion of them do, then the same cycle of winners still occurs.

Regarding whether Approval Voting should often elects the Condorcet winner, our examples cast a doubt. Laslier's results mentioned above mean that in every situation where a Condorcet winner exists but is not elected, voters (individually) have a strategical interest in changing their ballots. Our examples expose that this does *not* mean that following this heuristic shall lead to the election of a Condorcet winner: it may also lead to a periodic pattern where the Condorcet winner is absent. More generally, even simple heuristics make the availability of information a possible impediment to reaching an equilibrium and/or to elect an existing Condorcet winner.

This new issue with the Approval Voting system complement in particular the examples provided in [SDL06].

We also provide a few examples in other voting systems in Section 5. Together with the Approval Voting examples, they show how extreme an impact polling can have on the election outcome: rigging *any one* poll can prevent the election of the Condorcet winner even if *all* subsequent polls are perfectly conducted and reported; and in fact, even if all polls are perfect but in the first one voters respond according to a pre-established expectation of the outcome of the election, this expectation can determine the outcome of the election even after arbitrary many polls: polling need not induce synchronization nor loss of memory. Even with perfect unrigged polls, the sheer *number* of polls (e.g. its parity) can decide the outcome of the election.

It could be expected that such bad cycles are in practice rare. In section 4.3 we explore numerically this question, for Approval Voting, under various cultures modeling

[2] An example of cycle under the Leader Rule is given as exercise 8.4.4 in [Mei18], but there are no Condorcet winner.

the construction of random voters' preferences. The main take-away is that bad cycles are rare when voters apply the Leader Rule, but can be more common, appearing in more than 15% of preference profiles, in its modification where voters give themselves a limit to the candidates they may approve of. Most striking is that the culture where bad cycles are rarest for the Leader Rule is the one where they are most common for its modification.

The continuous-state setting enables one to model a broad variety of voter's collective behavior. In Section 6, we consider an example where voters are reluctant to approve of a candidate who is ranked low in their preference order, but may do so if a very close-call makes this move susceptible to improve the outcome. This arbitration can surprisingly make the Continuous-State Polling Dynamics a *chaotic dynamical system*, making the prediction of the next winners from the observation of a sequence of previous expected winners practically impossible.

Brief overview of the literature. Let us end this section by describing some more previous works. With no pretense to exhaustivity, the aim is to describe some directions previously explored in iterative voting; a more detailed review can be found in [Mei17]. The present work distinguishes itself from previous ones by considering synchronous iterative voting, previously seldomly studied: most commonly *iterative voting* is used to mean what we named here *asynchronous iterative voting*. Another divide which mostly parallels the previous one is between works considering at each step a set of possible moves by voters (e.g. best-responses, quite commonly considered in asynchronous voting) and, as we do here, deterministic heuristics. The reader can find an overview on strategic voting in [Mei18].

In the case of Plurality Voting, discussed in slightly more details below, very general convergence results have been obtained in [MLR14, Mei15, MPRJ17]. Many other voting rules — not Approval — have been considered in [LR16, KSLR17]. Their theoretical results are negative (no guaranteed convergence) but empirical tests seem to indicate that cycles are rare. Restricted strategies, where voters may be constrained to some not necessarily best or better responses, have been studied in [OMP+15] and [GLR+13].

Non-myopic strategies have been considered for example in [BHY14, AGP17, OLP+15], showing notably that voters learning from past information produce relatively good outcomes. Specific behaviors of voters, assumed to be subject to either *truth bias* or *lazy bias*, are studied in [ROL+15].

Considering Plurality Voting, the situation is complicated by the rigidity of the single-name ballot, which forces voters to choose a trade-off between preferences and probability to improve the outcome of the election. The works [MLR14] and [Mei15] have studied in depth models taking into account the scores of the candidates and a level of uncertainty to define the possible voters' strategies. They obtained several results proving under some assumptions convergence to equilibrium (the result closest to our present setting is Theorem 5 in [Mei15], where at each iteration an arbitrary subset of voters adjust their

votes according to the current poll results, thus including the case studied here where all voters adjust their votes at each iteration). In the case when strategies are restricted, [OMP+15] gives sufficient conditions for convergence in many voting systems.

The question of the quality of equilibriums have been considered for Plurality, Veto and Borda in [BCMP13], where the voters can adjust their votes one at a time, and starting from the state where each voter casts her sincere ballot (which is unique in these voting systems). Concerning Plurality, they find that Nash equilibriums that can be attained in this way are all very good; but in their model, the individual updates are made greedily and any candidate that would start with two or more votes less than the starting winner will never receive any new vote. One could expect real voters to vote not for immediate improvement of the winner, but in order to give a better position to a contender that might receive more support from others. This is the kind of modeling assumptions that our framework is made to support.

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2 Polling dynamics with discrete or continuous space

This section mostly consists of notations and definitions; its main novelty is to propose a continuous space framework for iterative voting, which allows to model much more varied situations than the usual discrete space versions. It models a situation where voters are grouped by type, e.g. according to their preference order, but where not all voters of a given type will react in the same way to an information update. For example, we can assume that a fraction of voters of each type is not aware of an information update, and will thus keep their previous intended ballot for the next iteration. In another direction, if the win is a close call, some voter's type may strategically adjust their ballots accordingly; but how close the call should be for them to bother changing their intent may vary from individual to individual, and our continuous space framework will make it possible to model this.

Examples of uses of the framework developed here are given below, first for Approval voting and in Section 5 for several other voting systems.

2.1 Common notations

We start with the notation common to both the discrete space and continuous space frameworks.

The set of *voter types* is assumed to be finite, of cardinal n, and denoted by N. It will be convenient to denote voter types by positive integers ($N = \{1, \ldots, n\}$) in

the theoretical part, but by upper-case letters from the end of the alphabet ($N = \{Z, Y, X, \dots\}$) in examples. Some of these letters will be used for maps below, but the context will prevent any confusion. In many examples below, voter types will coincide with preferences; however distinguishing gives more flexibility in the modelling, e.g. we can handle the case of different types of voters having the same preferences but using different heuristics.

Let A be a finite set of *candidates* (or *alternatives*). Its cardinal is denoted by m and assumed to be at least 2 (most usually at least 3). Candidates will be denoted by lower-case letters from the beginning of the alphabet ($A = \{a, b, c, \dots\}$). The set of weak orderings on A, i.e. rankings of candidates allowing ties, is denoted by $\mathcal{R}(A)$. The set of linear ordering on A, i.e. rankings of candidates without ties, is denoted by $\mathcal{L}(A)$.

Preferences of each voter type will be modeled by an element of $\mathcal{R}(A)$; when disallowing ties, we will restrict to its subset $\mathcal{L}(A)$. An *preference profile* is a pair (P, w) where:

- $P = (\succsim_1, \dots, \succsim_n) \in \mathcal{R}(A)^n$ is a family of preferences, one for each voter type; the corresponding strict preferences are denoted by $\succ_i$,
- $w = (w_1, \dots, w_n) \in \mathbb{R}_+^n$ is a family of *weights* $w_i > 0$, which represents the number of voters of each type. A usual situation is to take $w_i = 1$ for all i (or a positive integer, when grouping several voters in each type), but we allow more general weights for the continuous space framework.

A preference profile is represented by an array, each column of which represents a voter type, the column head carrying the weight (we shall often indicate the voter type on top for clarity) and the column listing the candidates in decreasing order of preference. With 3 candidates a, b, c an example of preference profile is:

1	2	3
100	101	102
a	b	c
b	a	b
c	c	a

meaning that $n = 3$ and e.g. that $a \succ_1 b$ but $b \succ_2 a$, that there are 102 voters of type 3, etc. For concision, an element of $\mathcal{L}(A)$ is denoted by listing the candidates in decreasing order of preference, e.g. $\succsim := acb$ means $a \succ c \succ b$. Elements of $\mathcal{R}(A)$ are denoted similarly with tied elements between parentheses.

Let $\mathcal{B}$ be a finite set of *ballots* that can be cast by voters. For example $\mathcal{B}$ can be $\mathcal{L}(A)$, or for Approval Voting the set $\mathcal{P}(A)$ of subsets of A. An Approval Voting ballot $B \in \mathcal{P}(A)$ is said to be *sincere* for voters of type i whenever

$$\forall a, b \in A\colon (b \in B \text{ and } a \succ_i b) \implies a \in B$$

i.e. when every candidate strictly preferred to any approved candidate is also approved.

Let $\mathcal{O}$ be a set representing the possible *outcomes* of the election. We can let $\mathcal{O} = A$ if we are only interested in the winner, but it is convenient to take a larger set $\mathcal{O}$ to include all information that can be made available and used by voters to decide which ballot to cast. For example, for Approval Voting we can take $\mathcal{O} = [0,1]^A$, an element $r = (r_a, r_b, ...)$ of $\mathcal{O}$ giving the proportion of voters approving of each candidate. We let $W : \mathcal{O} \to A$ be the function mapping a result to the corresponding winner (we shall always assume a tie-breaking rule, e.g. breaking ties in favor of the earlier candidate in the alphabetical order).

2.2 Polling dynamics with discrete space

We start with the usual case where in any given circumstance all voters of the same type cast the same ballot, which we call here the *discrete space* framework. A *ballot profile* is thus an element of $\mathcal{B}^n$ and represents the ballots cast by each voter type.

We consider a *social choice function*, also called *voting rule* $f : \mathcal{B}^n \to A$ and assume the choice of $\mathcal{O}$ and W makes it possible to decompose it as $f = W \circ g$ where $g : \mathcal{B}^n \to \mathcal{O}$ is called the *information function*. Observe that by assuming a tie-breaking rule for W, we only consider *resolute* social choice functions. For example, with the above choice of outcomes, the information function is given by

$$g(B_1, \dots, B_n) = \left(\frac{1}{\sum_{i \in N} w_i} \sum_{i \in N \,|\, \alpha \in B_i} w_i \right)_{\alpha \in A}$$

and the social choice function maps $(B_1, \dots, B_n)$ to the candidates with the highest approval rating, ties broke by alphabetical order.

Each voter type i is assumed to have a *heuristic* $\sigma_i : \mathcal{O} \to \mathcal{B}$, representing the way voters of this type choose their next ballot given the information of the outcomes (which is itself determined from the previous ballot profile). Formally, the preferences of voter types do not appear in the Polling Dynamics to be defined in the next paragraph. Most usually, they do appear indirectly, through heuristics: more often than not, $\sigma_i(r)$ take the form $\sigma(\succsim_i, r)$ where σ is a map $\mathcal{R}(A) \times \mathcal{O} \to \mathcal{B}$ (or $\mathcal{L}(A) \times \mathcal{O} \to \mathcal{B}$ if ties in preferences are not allowed). An example of such a map σ is the Leader Rule of Laslier, which we mentioned earlier and recall below.

We obtain a *Polling Dynamics* on the set of ballot profiles

$$\begin{aligned} \varphi : \quad & \mathcal{B}^n \to \mathcal{B}^n \\ (B_1, \dots, B_n) &\mapsto \Big(\sigma_1(g(B_1, \dots, B_n)), \dots, \sigma_n(g(B_1, \dots, B_n))\Big) \end{aligned}$$

obtained by determining the outcome according to the ballot profile in argument and then applying each voter's heuristic to obtain a new ballot profile.

Alternatively, we can take as space the set of outcomes and consider the *shifted Polling Dynamics*

$$\begin{aligned} \psi : \mathcal{O} &\to \mathcal{O} \\ r &\mapsto g\Big(\sigma_1(r), \dots, \sigma_n(r)\Big) \end{aligned}$$

mapping an outcome to a new one, after applying each voter's heuristic then determining the resulting outcome.

These two maps are strongly related: $\psi \circ g = g \circ \varphi$ (in dynamical systems theory we say they are *semi-conjugated*); φ seems to carry more information, but their dynamical study are actually equivalent because we assumed in the model that heuristics only depend on the outcome, not on the full details of the ballots.

The Leader Rule and its modification. The Leader Rule is an example of heuristic for Approval Voting depending on tie-free preferences $\succsim \in \mathcal{L}(A)$; assuming the expected winner $\omega(r)$ and the expected runner-up $\rho(r)$ can both be deduced from the outcome r, it is defined by

$$\mathrm{LR}(\succsim, r) = \Big\{\alpha \in A \,\Big|\, \alpha \succ \omega(r) \text{ or } \alpha = \omega(r) \succ \rho(r)\Big\};$$

in words, all candidates preferred to the expected winner are approved, and the expected winner is approved if and only if she is preferred to the expected runner-up. When we say that voters (assumed to have tie-free preferences) apply the Leader Rule, we mean that their heuristics take the form $\sigma_i(r) = \mathrm{LR}(\succsim_i, r)$.

When ties are allowed, i.e. $\succsim \in \mathcal{R}(A)$, there are several possible ways to extend the Leader Rule. One could use the very same formula as above, but it would have the consequence that when the outcome is the ranking *abcd* and the preferences are $(ab)(cd)$, the resulting ballot is blank. But in this situation, casting the ballot $\{a, b\}$ may prevent a worst candidate to be elected, and is always preferable to a blank ballot. We will thus consider the following *Modified Leader Rule* for preferences with ties by

$$\mathrm{MLR}(\succsim, r) = \Big\{\alpha \in A \,\Big|\, \alpha \succ \omega(r) \text{ or } \alpha \simeq \omega(r) \succ \rho(r) \text{ or } (\alpha \simeq \omega(r) \text{ and } \nexists \beta \succ \alpha)\Big\}$$

in particular, compared to the above, when the expected winner is tied for top in the preferences, then she and all those tied with her are approved. Again, when we say that voters follow the Modified Leader Rule we mean that $\sigma_i(r) = \mathrm{MLR}(\succsim_i, r)$.

2.3 Polling dynamics with continuous space

We now turn to Polling Dynamics with continuous space (CS), the goal of which is to allow more flexibility in our assumption on voters' heuristics. In particular, we want to be able to consider a continuum of behaviors in each voter type, for example accounting for variable levels of bias, be it for example truth-bias (favoring sincere ballots) or lazy-bias (aversion to change one's ballot in view of new information). Example 4.8 illustrates this framework.

The discrete space of ballot profiles $\mathcal{B}^n$ thus has to be replaced by a more complicated object, encoding the proportion of each voter type casting each ballot. To this effect we use the *simplex* $\Delta(X)$ over a finite set X, defined as

$$\Delta(X) = \Big\{(p_x)_{x \in X} \in [0,1]^X \,\Big|\, \sum_{x \in X} p_x = 1\Big\}.$$

The *full continuous space* is then $\Delta(\mathcal{B})^n$; an element of this space is a doubly indexed family $(p^i_B)_{i\in N, B\in\mathcal{B}}$ where p^i_B is the proportion of ballots B cast among voters of type i, and is called a *CS ballot profile*.

We now define a *CS information function* as a map $G : \Delta(\mathcal{B})^n \to \mathcal{O}$, with the *associated CS social choice function* $F = W \circ G : \Delta(\mathcal{B})^n \to A$. For example, with the above choice of outcomes, for Approval Voting the CS information function is given by

$$G\Big((p^i_B)_{i,B}\Big) = \left(\frac{1}{\sum_{i\in N} w_i} \sum_{\substack{i\in N \\ B\in\mathcal{B}\,|\,\alpha\in B}} p^i_B w_i\right)_{\alpha\in A}$$

where $p^i_B w_i$ represents the total number of voters of type i casting the ballot B.

The full continuous space often has an unnecessarily large dimension, since under most heuristics voters will only cast ballots among a small subset $\mathcal{B}_i$ of $\mathcal{B}$ (one subset for each voter type; for example, for Approval Voting we could restrict to ballots that are sincere with respect to the type's preferences). We thus define a more convenient *continuous space* by

$$\mathcal{P} = \prod_{i\in N} \Delta(\mathcal{B}_i) \quad \subset \Delta(\mathcal{B})^n$$

the elements of which are *admissible CS ballot profiles*, or *states*. We still denote by G and F the restrictions of the CS information function and social choice function to this space. Observe that the full CS appears as a special case by taking $\mathcal{B}_i = \mathcal{B}$ for all i.

A *CS Polling Dynamics* is simply a map $\Phi : \mathcal{P} \to \mathcal{P}$. In general, it is not so much the dynamics of Φ that will be of interest, but the sequences of outcomes, which can be recovered as $G(\Phi^k(p_0))$ where k is the number of iterations and $p_0 = (p^i_B)_{i,B} \in \mathcal{P}$ is the initial ballots intended to be cast by the voters.

This very broad definition of a CS Polling Dynamics allows for much modeling flexibility, but the main interest is when such a map is deduced from some sort of heuristics of voters. We will not need this consideration in the robustness result which is very general, but let us give some definitions to show how this can be done.

We will assume that the choice of voters is based only on the predicted outcome and the ballots previously cast by voters of their type (this is enough to permit a fraction of voters not to adjust their choice, e.g. because they are not aware of the last poll). A *CS heuristic* for voters of type i is thus a map $\Sigma_i : \mathcal{O} \times \Delta(\mathcal{B}_i) \to \Delta(\mathcal{B}_i)$, and given a CS heuristic for each voter type the corresponding CS Polling Dynamics is given for all state $p = (p^i_B)_{i,B} \in \mathcal{P}$, setting $p^i = (p^i_B)_B$, by

$$\Phi(p) = \Big(\Sigma_1\Big(G(p), p^1\Big), \Sigma_2\Big(G(p), p^2\Big), \ldots, \Sigma_n\Big(G(p), p^n\Big)\Big).$$

3 Robustness of tie-free cycles

In this section we want to prove that cycles of a discrete space Polling Dynamics that do not rely on the tie-breaking rule are robust, i.e. they persist under small enough perturbation. "Small enough" is a void concept in the discrete space setting, and we

use CS Polling Dynamics to model the perturbations. The first step is to see that discrete space Polling Dynamics can always be realized as particular cases of CS Polling Dynamics.

3.1 Embedding the discrete space Polling Dynamics into the continuous space.

Observe that $\mathcal{B}^n$ embeds naturally into the full CS, by identifying $(B_1, \ldots, B_n)$ with the element $\pi(B_1, \ldots, B_n) = (p^i_B)_{i,B} \in \Delta(\mathcal{B})^n$ defined by

$$p^i_B = \begin{cases} 1 & \text{when } B = B_i \\ 0 & \text{otherwise} \end{cases}$$

Such an elements of the full CS, taking only the values 0 and 1, is called an *extreme state*. We shall say that the CS information function G *extends* the information function g whenever $G \circ \pi = g$, i.e. for all $(B_1, \ldots, B_n) \in \mathcal{B}^n$, $G(\pi(B_1, \ldots, B_n)) = g(B_1, \ldots, B_n)$. Without this assumption we would not be encoding the same voting method in both settings.

Consider fixed the (discrete space) heuristics σ_i and the corresponding Polling Dynamics φ, and define the *associated* CS Polling Dynamics by

$$\Phi_0(p) = \pi(\sigma_1(G(p)), \ldots, \sigma_n(G(p)),$$

i.e. starting from a CS ballot profile $p \in \Delta(\mathcal{B})^n$ we compute its outcome $G(p)$, then apply the heuristics to get new ballot choices for each voter type, then embed this into the full CS with the map π to obtain a new CS ballot profile. Note that Φ_0 is well-defined on the full CS, but all its values are extreme states.

The hypotheses we introduced in this section are precisely what needs to be assumed to ensure that the continuous space framework extends the discrete space situation: Φ_0 produces the same sequences of outcome than φ when starting at an extreme state.

Lemma 3.1. *Assume that G extends g and that the heuristics only pick admissible ballots, i.e. $\sigma_i(r) \in \mathcal{B}_i$ for all voter type $i \in N$ and all outcome $r \in \mathcal{O}$. Then the associated CS Polling Dynamics takes its values in $\mathcal{P}$, and for all $(B_1, \ldots, B_n)$ the sequences of outcomes*

$$g(\varphi^k(B_1, \ldots, B_n)) \qquad \textit{and} \qquad G(\Phi_0^k(\pi(B_1, \ldots, B_n)))$$

are the same.

Proof. Since heuristics only pick admissible ballots, for all $p \in \Delta(\mathcal{B})^n$ we have

$$\Big(\sigma_1(G(p)), \ldots, \sigma_n(G(p)\Big) \in \prod_{i \in N} \mathcal{B}^i$$

so that $\Phi_0(p) \in \mathcal{P}$.

Since G extends g, for all $(B_1, \dots, B_n)$ we have

$$\begin{aligned}\Phi_0(\pi(B_1, \dots, B_n)) &= \pi\Big(\sigma_1(G(\pi(B_1, \dots, B_n))), \dots, \sigma_n(G(\pi(B_1, \dots, B_n))\Big) \\ &= \pi\Big(\sigma_1(g(B_1, \dots, B_n)), \dots, \sigma_n(g(B_1, \dots, B_n)\Big) \\ &= \pi\Big(\varphi(B_1, \dots, B_n)\Big)\end{aligned}$$

By induction, we deduce that $\Phi_0^k \circ \pi = \pi \circ \varphi^k$ for all positive integer k. Using again that G extends g, we get $G \circ \Phi_0^k \circ \pi = G \circ \pi \circ \varphi^k = g \circ \varphi^k$, as desired. □

3.2 Perturbations and robustness.

The second step is to define what it means for two CS Polling Dynamics to be close one to the other. We consider the metric induced on the full CS and on $\mathcal{P}$ by the supremum norm $\|\cdot\|$, i.e. given two CS ballot profiles $p = (p_B^i)_{i,B}$ and $\bar{p} = (\bar{p}_B^i)_{i,B}$ we set

$$\|p - \bar{p}\| = \sup_{i \in N, B \in \mathcal{B}} \Big|p_B^i - \bar{p}_B^i\Big|.$$

This induces the usual topology on the full CS and, by restriction, on $\mathcal{P}$. We denote by $\bar{B}(p, \varepsilon)$ the closed ball of radius $\varepsilon \geq 0$ and center $p \in \mathcal{P}$ with respect to this metric.

We then consider the uniform metric, defined for every pair Φ_1, Φ_2 of CS Polling Dynamics by

$$D(\Phi_1, \Phi_2) := \sup_{p \in \mathcal{P}} \Big\|\Phi_1(p) - \Phi_2(p)\Big\|.$$

Definition 3.2. We say that a ballot profile $(B_1, \dots, B_n)$ is *tie-free* (implicitly, with respect to heuristics $(\sigma_i)_i$, an information function g and an extension G) whenever there exist a neighborhood U of $\pi(B_1, \dots, B_n)$ in $\mathcal{P}$ such that for all $p \in U$:

$$W \circ G(p) = W \circ g(B_1, \dots, B_n) \quad \text{and} \quad \sigma_i(G(p)) = \sigma_i(g(B_1, \dots, B_n)) \quad \forall i \in N.$$

A ballot profile $(B_1^0, \dots, B_n^0)$ *belongs to a tie-free cycle* for the Polling Dynamics φ whenever $\varphi^k(B_1^0, \dots, B_n^0) = (B_1^0, \dots, B_n^0)$ for some positive integer k (called a *period* of the cycle) and the successive ballot profiles $(B_1^j, \dots, B_n^j) = \varphi^j(B_1^0, \dots, B_n^0)$ are all tie-free.

Let us explain the rationale behind this definition. First, the condition $W \circ G(p) = W \circ g(B_1, \dots, B_n)$ means that the winner does not change if the considered CS ballot profile is close enough to $\pi(B_1, \dots, B_n)$. Second, all considered heuristics $\sigma : \mathcal{O} \to \mathcal{B}^n$ and their composition with the CS information function $\sigma \circ G : \mathcal{P} \to \mathcal{B}^n$ will be piecewise continuous, and thus piecewise constant (since they take value in a finite set). A state p at which $\sigma \circ G$ is not continuous means that either a small change in the state can change the outcomes radically, or a small change in the outcome can change the resulting ballot profile for this heuristic. This correspond to breaking a tie, either in the information function or in the heuristic. For example in Approval Voting, when applying

the Leader Rule with the above continuous choice of outcomes $\mathcal{O} = [0,1]^A$: when the first two candidates are tied, the tie-breaking rule embedded in W is taken into account to determine the expected winner; and when several candidates are tied for runner-up, a tie-breaking rule must implicitly be written in the heuristic itself.

Theorem A. *Assume that G extends g and that the heuristics $\sigma_1, \dots, \sigma_n$ only pick admissible ballots. If the Polling Dynamics φ has a tie-free cycle of period k, an element of which is denoted by $(B_1^0, \dots, B_n^0)$, then there exist $\varepsilon > 0$ with the following property: for all CS Polling Dynamics Φ such that $D(\Phi, \Phi_0) \leq \varepsilon$, for all $p^0 \in \mathcal{P}$ such that $\|p^0 - \pi(B_1^0, \dots, B_n^0)\| \leq \varepsilon$, for all positive integer j,*

$$W \circ G\Big(\Phi^j(p^0)\Big) = W \circ g\Big(\varphi^j(B_1^0, \dots, B_n^0)\Big).$$

If moreover Φ is continuous, then there exist $p^0 \in \mathcal{P}$ such that $\|p^0 - \pi(B_1^0, \dots, B_n^0)\| \leq \varepsilon$ and $\Phi^k(p^0) = p^0$.

In words, if the discrete space dynamics has a tie-free cycle, then perturbing the dynamics and starting point in the CS setting does not change the (periodic) sequence of winners; and if the perturbed CS Polling Dynamics is continuous, then this sequence of winners is furthermore realized by a cycle in $\mathcal{P}$, which is a perturbation of the original cycle. We shall see that for some natural examples, the cycle moreover attracts a neighborhood of p^0 (see Example 4.8, which is easily generalized).

Proof. For each positive integer j, set $B^j := (B_1^j, \dots, B_n^j) := \varphi^j(B_1^0, \dots, B_n^0)$. Since B^j is tie-free, $\pi(B^j)$ admits a neighborhood U_j (and we can choose the sequence $(U_j)_j$ to be k-periodic) such that for all $p \in U_j$,

$$W \circ G(p) = W \circ g(B^j) \quad \text{and} \quad \sigma_i(G(p)) = \sigma_i(g(B^j)) \quad \forall i \in N.$$

By definition of a neighborhood, for each j there exist $\varepsilon_j > 0$ such that $\bar{B}(\pi(B^j), \varepsilon_j) \subset U_j$; let $\varepsilon = \min(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-1})$.

The second tie-free condition $\sigma_i(G(p)) = \sigma_i(g(B^j))$ for all $i \in N$ and the definition of Φ_0 ensures that for all $p \in U_j$, $\Phi_0(p) = \pi(B^{j+1})$. Let Φ be a CS Polling Dynamics such that $D(\Phi, \Phi_0) \leq \varepsilon$ and let $p^0 \in \mathcal{P}$ such that $\|p^0 - \pi(B_1^0, \dots, B_n^0)\| \leq \varepsilon$. Then $\Phi_0(p^0) = \pi(B^1)$ and $\Phi(p^0) \in \bar{B}(\pi(B^1), \varepsilon) \subset U_1$. The same reasoning and an induction ensures that for all j, the state $p^j := \Phi^j(p^0)$ lies in $\bar{B}(\pi(B^j), \varepsilon) \subset U_j$. The first tie-free condition $W \circ G(p^j) = W \circ g(B^j)$ then gives the desired conclusion.

Assume further that Φ is continuous. Observe that $\mathcal{P}$ is a polyhedron of $\mathbb{R}^d$ for some d, and that balls $\bar{B}(s, \varepsilon)$ are thus convex sets, in particular homeomorphic to closed balls. For all $p \in \bar{B}(\pi(B^0), \varepsilon)$, we have $\Phi^{k-1}(p) \in U_{k-1}$ and thus $\Phi^k(p) \in \bar{B}(\pi(B^0), \varepsilon)$: Φ^k is a continuous map sending the topological ball $\bar{B}(\pi(B^0), \varepsilon)$ into itself. Brouwer's fixed point theorem ensures that there exist $p^0 \in \bar{B}(\pi(B^0), \varepsilon)$ such that $\Phi^k(p^0) = p^0$. $\square$

Remark 3.3. Theorem A applies to dynamical equilibriums, by taking $k = 1$.

4 Bad cycles in Approval Voting

Let us recall some definitions relative to what can be considered "good" or "bad" collective choices among the candidates.

Given a preference profile, a candidate α is said to *dominate* a candidate β (sometimes written $\alpha > \beta$, but beware that this is not a transitive relation) whenever there are strictly more voters that strictly prefer α to β than voters that strictly prefer β to α (i.e. in the *majority graph*, assuming indifferent voters abstain, there is an arrow from α to β). A candidate α is then said to be a *Condorcet winner* whenever she dominates every other candidate; a Condorcet winner may or may not exist, but if she exists she is unique. When preferences have no ties, this is the usual definition of a (strong) Condorcet winner; when there are ties, a stronger definition could be possible: to dominate, one could ask for a majority of all voters, including abstainers. Similarly a candidate β is said to be a *Condorcet loser* whenever she is dominated by every other candidate; again, a Condorcet loser may or may not exist and is unique if she exist.

Last, we will use a stronger notion than Condorcet loser: a candidate is said to be an *absolute majority loser* whenever she is a Condorcet loser and there is a strict majority of the electorate that ranks her last (possibly tied with others) in their preferences. An absolute majority loser rarely exists.

4.1 First example: non-convergence of the Leader Rule

In this Section all heuristics will be given by the Leader Rule, which only relies on the identity of the expected winner and runner-up, not on the expected scores of candidates (its definition is recalled in Section 2.2). We can thus consider a simple set of outcomes, $\mathcal{O} = \{\omega\rho \colon \omega \neq \rho \in A\}$ the set of order pairs of distinct candidates, and the winner map is defined by $W(\omega\rho) = \omega$; ρ represents the runner-up. Preferences of voter types are without tie (i.e. in $\mathcal{L}(A)$) as needed for the Leader Rule. The information function g is obtained by ranking candidates by decreasing order of approval numbers ($r_\alpha = \sum_{i|\alpha\in B_i} w_i$), breaking ties by alphabetical order, and then selecting the two first candidates, keeping their order. The CS information function G is defined similarly, with approval numbers

$$r_\alpha = \sum_{\substack{i\in N \\ B|\alpha\in B}} p^i_B w_i,$$

making it an extension of g. The admissible ballots of type i are the sincere ballots with respect to the preference order $\succsim_i$; the LR ensures that heuristics only pick admissible ballots. Our goal is to prove the following result.

Theorem B. *Using Approval Voting, there exists a preference profile for* 4 *candidates such that:*

- *there is a Condorcet winner,*
- *each voter has preferences without ties,*

- *assuming voters follow the Leader Rule, the Polling Dynamics has a cycle where no state elects the Condorcet winner. Moreover, more than two thirds of the outcomes lead to this cycle.*

Proof. We set $A = \{a, b, c, d\}$ and consider the following preference profile with 7 types of voters:

T	U	V	W	X	Y	Z
100	1000	1001	1002	1004	1008	1016
a	b	c	d	b	c	d
b	a	a	a	c	d	b
c	c	d	b	a	a	a
d	d	b	c	d	b	c

and heuristics given by the Leader Rule.

The voters of type U, V, W like a but each prefers one of b, c, d better, while the voters of type X, Y, Z do not like a too much but distaste one of b, c, d even more, creating a cycle in the majority graph $b \succ c \succ d \succ b$ with a close to tie with each of b, c, d. Meanwhile, voters of type T prefers a to any other candidate, and their moderate number suffice to make a a Condorcet winner while maintaining the cycle b, c, d in the majority graph. The precise numbers of types U to Z are chosen, for the sake of fanciness, to exclude any perfect tie (different sums of distinct powers of 2 never agree).

Figure 1 represents the shifted Polling Dynamics ψ in the form of a graph. We only give the details of the computations along the cycle, others are similar. Consider the

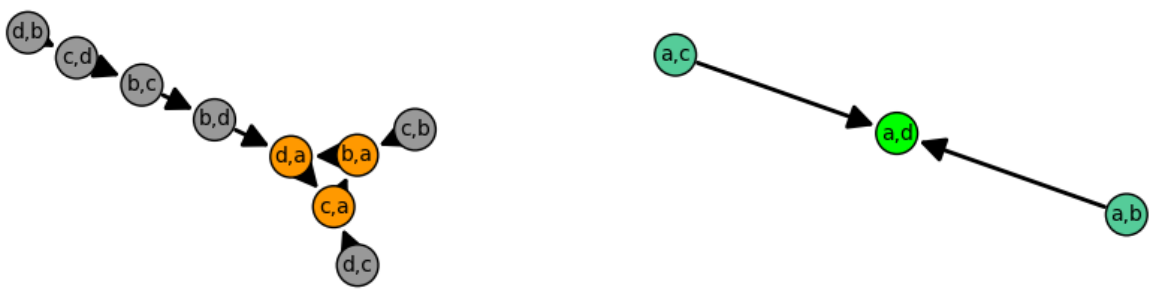

Figure 1: The shifted Polling Dynamics of the first example. Outcomes where the Condorcet winner is elected are shown in green, light green for the dynamical equilibrium. The periodic outcomes not electing the Condorcet winner are shown in orange.

outcome ba. Under the Leader Rule, it leads to the following ballot profile and results:

$$B_T = \{a\} \qquad B_U = \{b\} \qquad B_X = \{b\} \qquad r_a = 3111 \qquad r_b = 3020$$
$$B_V = \{c, a, d\} \qquad B_Y = \{c, d, a\} \qquad r_c = 2009$$
$$B_W = \{d, a\} \qquad B_Z = \{d, b\} \qquad r_d = 4027$$

so that $\psi(ba) = da$ – i.e. a stays second, while the previously unthreatening d comes in first position. The strategic adjustments triggered by the outcome da are as follows:

$$\begin{array}{lllll} B_T = \{a,b,c\} & B_U = \{b,a,c\} & B_X = \{b,c,a\} & r_a = 3105 & r_b = 2104 \\ & B_V = \{c,a\} & B_Y = \{c,d\} & & r_c = 4113 \\ & B_W = \{d\} & B_Z = \{d\} & & r_d = 3026 \end{array}$$

so that $\psi^2(ba) = \psi(da) = ca$. The corresponding strategic adjustments are then:

$$\begin{array}{lllll} B_T = \{ab\} & B_U = \{b,a\} & B_X = \{b,c\} & r_a : 3118 & r_b : 4122 \\ & B_V = \{c\} & B_Y = \{c\} & & r_c : 3013 \\ & B_W = \{d,a,b\} & B_Z = \{d,b,a\} & & r_d : 2018 \end{array}$$

so that $\psi^3(ba) = ba$. □

Remark 4.1. It is worthwhile to observe (although a bit tedious to check) that in this example, *all* outcomes can be obtained from a certain sincere ballot profile. For example, dc is obtained if all voters cast the ballot with only their preferred candidates. That all outcomes are covered is hardly surprising: there are only 12 of them, but $3^7 = 2187$ combinations of sincere ballots (excluding abstentions).

Remark 4.2. In the cases of $m > 4$ candidates, we can take the above example and add $n - 4$ dummy candidates that appear at the end of all voters preferences. The only property that may not be preserved in this operation is the size of the basin of attraction of the 3-cycle: for example the states starting with one of the dummy candidates will all be sent by φ to a state starting with a, since voters would vote for all of a, b, c, d. However this is easily fixed by adding a voter type, for example in the case of a fifth candidate e one could take 50 voters of type S with preference order $becda$. Indeed, this voters will break the tie between a, b, c, d whenever e is the expected winner, in favor of b, thus leading to the basin of attraction of the 3-cycle.

Remark 4.3. As pointed out by one of the anonymous referees, Theorem B does not hold for 3 candidates. Assume there is a Condorcet Winner a and two other candidates b, c. Since preferences have no ties, there are 6 possible preferences, and we can reduce to six voter types; let $T_{\alpha\beta}$ denotes the voters ranking α first and β second, and let it also denotes the number of such voters. We assume there is no equality in any partition into two groups of three voter types, which is a generic condition, to avoid dealing with ties. Since a is a Condorcet winner,

$$T_{ab} + T_{ac} + T_{ca} > \frac{1}{2} > T_{bc} + T_{ba} + T_{cb}$$
$$T_{ab} + T_{ac} + T_{ba} > \frac{1}{2} > T_{bc} + T_{ca} + T_{cb}$$

One of b and c would win a duel against the other, and without loss of generality we assume b does, i.e.

$$T_{ba} + T_{bc} + T_{ab} > \frac{1}{2} > T_{ca} + T_{cb} + T_{ac}.$$

We assume that all voters follow the Leader Rule and prove that the Polling Dynamic converges to one of the outcomes ab or ac.

First, whatever the expected outcome is, a receives either $T_{ab}+T_{ac}+T_{ca}$ or $T_{ab}+T_{ac}+T_{ba}$ votes, depending on whether b is above c or not, in any case more than half.

When a is expected winner, b and c receives $T_{bc}+T_{ba}+T_{cb} < \frac{1}{2}$ and $T_{bc}+T_{ca}+T_{cb} < \frac{1}{2}$ votes respectively, so that a stays ahead and either ab or ac is an equilibrium, attracting the other one of these two states.

Whenever a is second, none of the voters for a vote for the expected winner, and it follows that in the next round, the contender will be behind a. The third candidate will receive the votes of voters preferring her to the expected winner; in the case of c, it means either $T_{ca}+T_{cb}+T_{ac}$ or $T_{ca}+T_{cb}+T_{bc}$ votes, according to whether b or a is expected to win, and both are less than $\frac{1}{2}$. It remains to consider the outcome cab, where b receives the votes $T_{ba}+T_{bc}+T_{ab}$. In this case, we can have a sequence of outcomes ca, ba but then a wins in the next outcome. Whenever a gets at least second, she will therefore become first and stay first ever after.

Now, in every case where a is third, she will receive more than half the votes and c will receive less than half, so that a will get at least the second position in the next round, after which she will take the lead. This ends the proof of convergence to equilibrium.

Remark 4.4. If we do not ask for a Condorcet winner to exist, a cycle can be produced with $m = 3$ candidates, e.g. with the preference profile

X	Y	Z
10	11	12
a	b	c
b	c	a
c	a	b

where starting with voters approving only their top candidate, we obtain a cycle of outcomes cb, bc, ab.

From Theorem A we deduce that the cycle persists when we perturb the Polling Dynamics in the CS setting (to be meaningful, here we change back the set of outcomes to $\mathcal{O} = [0,1]^A$).

Corollary 4.5. *Consider a preference profile as given by Theorem B. Let Φ_0 be the induced CS Polling Dynamics on the continuous space $\mathcal{P}$ defined by making admissible exactly the sincere ballots. Then there exist $\varepsilon > 0$ and an open subset U of $\mathcal{P}$ such that for* all *CS Polling Dynamics Φ with $D(\Phi, \Phi_0) \leq \varepsilon$ and all $p^0 \in U$, the sequence of winners*

$$W \circ G(\Phi^j(p^0)) \qquad j = 0, 1, 2, \ldots$$

is periodic and does not contain the Condorcet winner.

4.2 Second example: the possible election of an absolute majority loser

Our second example, at the small cost of introducing a modification of the LR accounting for ties, improves on the previous one on two accounts: it necessitates only 3 candidates, and it exhibits a cycle where an absolute majority loser could get elected. We conserve most the setting of the previous Section: $\mathcal{O}$, W, g and G are as above. The only change is that we allow for ties in preferences and more varied heuristics.

Theorem C. *Using Approval Voting, there exists a preference profile on* 3 *candidates with ties in preferences allowed and sincere heuristics such that:*

- *there are a Condorcet winner and an absolute majority loser,*
- *the Polling Dynamics has a* 2*-cycle, one of whose state elects the absolute majority loser; moreover the cycle attracts two thirds of the outcomes,*
- *there is an equilibrium not electing the Condorcet winner.*

Proof. We consider the following preference profile:

Z	Y	X	W
3	1	3	5
a	a	b	c
b	bc	a	ab
c		c	

with each voter type $i \in N$ using the Modified Leader Rule. In particular voters of type W will not choose between a and b, thus always casting the ballot $\{c\}$, no matter which outcome is expected. Similarly, voters of type Y always cast the ballot $\{a\}$ (this last type is only introduced here for tie-breaking).

Note that a is a Condorcet winner, beating b with a score of 4 to 3 (voters of type W abstaining) and c with a score of 7 to 5. Moreover c is a Condorcet loser, loosing to b by 6 to 5; actually c is a worst candidate of 7 out of the 12 voters, making it an absolute majority loser.

Assume as starting expected outcome the result obtained if each voter votes for every candidates she does not rank last:

$$B_Z = \{a, b\} \quad B_Y = \{a\} \quad B_X = \{b, a\} \quad B_W = \{c\}$$
$$r_a = 7 \quad r_b = 6 \quad r_c = 5$$

leading to a being expected winner and b expected runner-up (corresponding to the Condorcet order). This leads voters of type Z and X to adjust their votes: their favorite candidate is either threatened by their second-favorite (for Z) or have a shot at winning

the election from a current runner-up position (for X). Consistently with their heuristics they choose to vote only for their favorite candidate:

$$B_Z = \{a\} \qquad B_Y = \{a\} \qquad B_X = \{b\} \qquad B_W = \{c\}$$
$$r_a = 4 \qquad r_b = 3 \qquad r_c = 5.$$

The second poll thus results in a win of c with a as runner-up. This result induces voters of type Z and X to resume approving both a and b, in order not to let c be elected (again, this is a consistent application of the modified Leader Rule). This results in the same ballots being cast as in the first poll, so we get a 2-cycle, one of the states electing the worst candidate.

Figure 2 represents the shifted Polling Dynamics ψ in graph form.

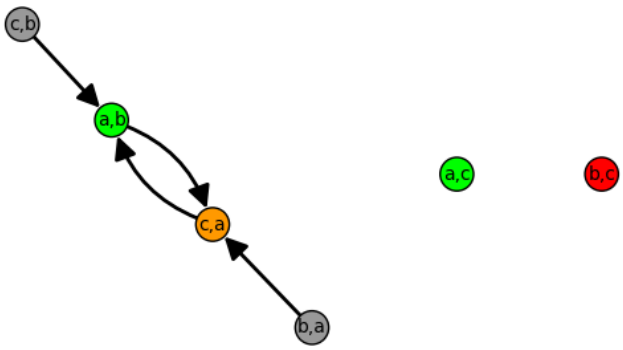

Figure 2: The shifted Polling Dynamics of the second example. Light green states are accessible after an arbitrary large number of iterations and elect the Condorcet winner; orange (periodic) and red (fixed point) states are accessible after an arbitrarily large number of iterations but do not elect the Condorcet winner. The orange state is arguably the most problematic, as it elects the absolute majority loser c.

We see that of 6 outcomes, 4 lead to the cycle that can elect either the Condorcet winner a or the absolute majority loser c depending on whether the number of polls conducted before the election is odd or even, while the other 2 are equilibriums, one electing the Condorcet winner a and the other the Condorcet runner-up b. □

Remark 4.6. If we accept to depart further from the LR, we could avoid ties in preferences and preserve the features of the examples by splitting W into two types of voters of equal size, with respective preferences *cab* and *cba* and heuristic to always vote $\{c\}$ (with the interpretation that these voters prefer c to the other two by far, but still have a slight preference between a and b). This shows that Theorem C also holds with the stronger definition of Condorcet winner (see beginning of Section 4).

Remark 4.7. An argument that could be raised against this example is that it needs that a large proportion of voters having c as favorite candidate would never vote for any other candidate. While this is indeed a crucial feature of the voters preferences in

this example, there are two counter-arguments. First, this situation seems not all that unlikely: far-right candidates with a strong anti-establishment discourse can have many supporters who would consider all other candidates (or at least those with a chance of being elected) as part of the very same "establishment" and thus would only approve of c. Second, this can be a textbook case of manipulation by a coalition: if the minority of all voters who prefer c (with preferences cb or ca say) gather in a coalition and decide to vote only for c, they get a good chance to have c elected against the will of a two-third majority! Actually, these counter-arguments feed on each other: an anti-establishment discourse can serve the purpose of forming a coalition-in-practice of voters who will not express their preferences between a and b in order to favor c.

Again, Theorem A ensures that this bad cycle is robust. Let us make this more concrete by considering an explicit perturbation.

Example 4.8. We now give a CS setting extending the discrete-space example in the proof of Theorem C. Let the sets of admissible ballots be

$$\mathcal{B}_Z = \Big\{\{a\},\{a,b\}\Big\}, \quad \mathcal{B}_Y = \Big\{\{a\}\Big\}, \quad \mathcal{B}_X = \Big\{\{b\},\{a,b\}\Big\} \quad \text{and} \quad \mathcal{B}_W = \Big\{\{c\}\Big\}$$

and consider outcomes giving the proportion of votes received by each candidates: $\mathcal{O} = [0,1]^A$.

Observe that since the simplex over a singleton is a singleton, and the simplex over a two-element set is an interval, $\mathcal{P}$ can be identified with the square $[0,1]^2$, with coordinates (x,z) where x (respectively z) represents the proportion of voters of type X (respectively Z) casting the ballot $\{a,b\}$. The leftmost part of Figure 3 represents $\mathcal{P}$. The three lines corresponding to ties (of equation $(z = x + \frac{1}{3})$ for a and b; $(x = \frac{1}{3})$ for a and c; $(z = \frac{2}{3})$ for b and c); they are concurrent at the point where all three candidates are tied, and delimit six areas, one for each possible outcome (all of which are possible under the chosen preference profile and restriction of ballots, as we can see in the figure). We denote by A_{abc} the region where the outcome is abc (boundary segments are attributed according to the tie-breaking rule), and similarly for the other five regions.

Given $\delta \in [0,1]$, we let $\mathcal{T}_\delta$ be the set of $(x,z) \in \mathcal{P}$ such that $(r_a, r_b, r_c) := V(x,z)$ avoids ties by a margin at least δ, i.e. $|r_\alpha - r_\beta| \geq \delta$ for all $\alpha \neq \beta \in A$. We consider any perturbation $(\Phi_\varepsilon)_{\varepsilon\in[0,1]}$ of the embedded discrete space Polling Dynamics such that:

$$\Phi_\varepsilon(x,z) = (1-\varepsilon)\Phi_0(x,z) + \varepsilon(x,z) \qquad \forall (x,z) \in \mathcal{T}_\delta$$

which models the situation where after each poll where no two candidates are δ-close to be tied, a proportion ε of voters keep their ballots unchanged and the remaining $(1-\varepsilon)$ apply the heuristics of their types defined in the discrete setting (here the LR for types X and Z, types Y and W always voting only for their top candidate).

We will show that taking $\delta = .04$ and $\varepsilon = .15$ is small enough for the conclusion of Theorem A to apply. Figure 4 shows the first five iterations of such a perturbation for $\delta = .04$ and $\varepsilon = .15$, and we can observe a bad 2-cycle with points in A_{abc} and A_{cab}.

The region $A_1 \subset A_{abc}$ delimited by the lines of equations $(z < x + \frac{1}{6})$ and $(z > \frac{5}{6})$ results in the outcome abc with margins of $\frac{1}{24}$th of the electorate, i.e. slightly over 4%. Similarly,

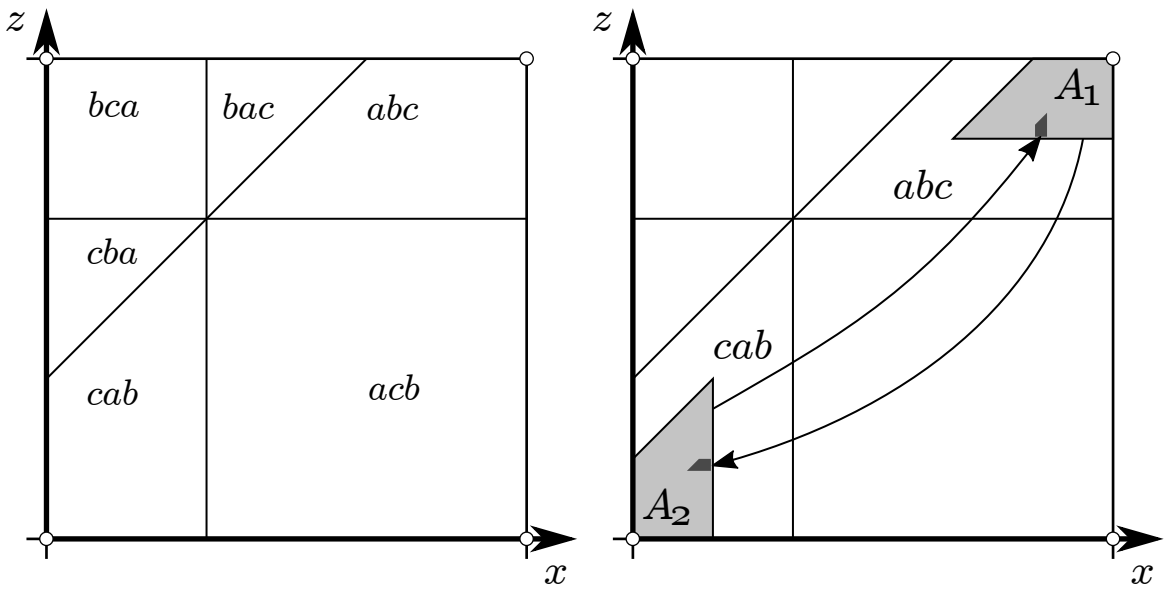

Figure 3: A continuous-space example. *Left:* the state space $\mathcal{P}$. The corners are the extreme states, corresponding to the four outcomes that are attainable from a ballot profile under the discrete-space Polling Dynamics. *Right:* on the same state space, any CS Polling Dynamics where 85% of voters adjust their ballot according to the modified Leader Rule when candidates are separated by 4% margins will send the light-grey regions A_1 and A_2 into each other (images shown in dark grey), thus ensuring a 2-cycle with one state near the upper-rigth corner, one near the lower-left corner.

the region $A_2 \subset A_{cab}$ delimited by the lines of equations $(z < x + \frac{1}{6})$ and $(x < \frac{1}{6})$ result in the outcome *cab* with the same margins. We have $\Phi_{.15}(x, z) = (.15x, .15z)$ whenever $(x, z) \in A_1$ and $\Phi_{.15}(x, z) = (.85 + .15x, .85 + .15z)$ whenever $(x, z) \in A_2$. One easily checks that $\Phi_{.15}(A_1) \subset A_2$ and $\Phi_{.15}(A_2) \subset A_1$. It follows that $\Phi^2_{.15}(A_1) \subset A_1$, and since $\Phi^2_{.15}$ is a contraction on A_1 (of ratio $.15^2$, much small than 1), the Contraction Mapping Theorem ensures it has a fixed point $(x_1, z_1) \in A_1$. Then the orbit of (x_1, z_1) is a 2-cycle with one state inducing the outcome *abc* and the other inducing *cab*. Moreover any element of the open set $A_1 \cup A_2$ is attracted to this cycle exponentially fast.

4.3 Numerical study

The examples of the previous section are interesting from a theoretical point of view, showing that the existence of equilibriums and the quality of candidates they elect do not by themselves suffice to ensure a good outcome. But from a practical point of view, it could be that the bad cycles we exhibited are so rare that they do not matter too much. In this section, we present experimental estimates of the frequency of bad cycle in preference profiles where a Condorcet winner exists. We continue to follow the setup of Sections 4.1 and 4.2, using either the Leader Rule or the modified Leader Rule. This leaves us to model the preference profile, and for this we compare several "cultures".

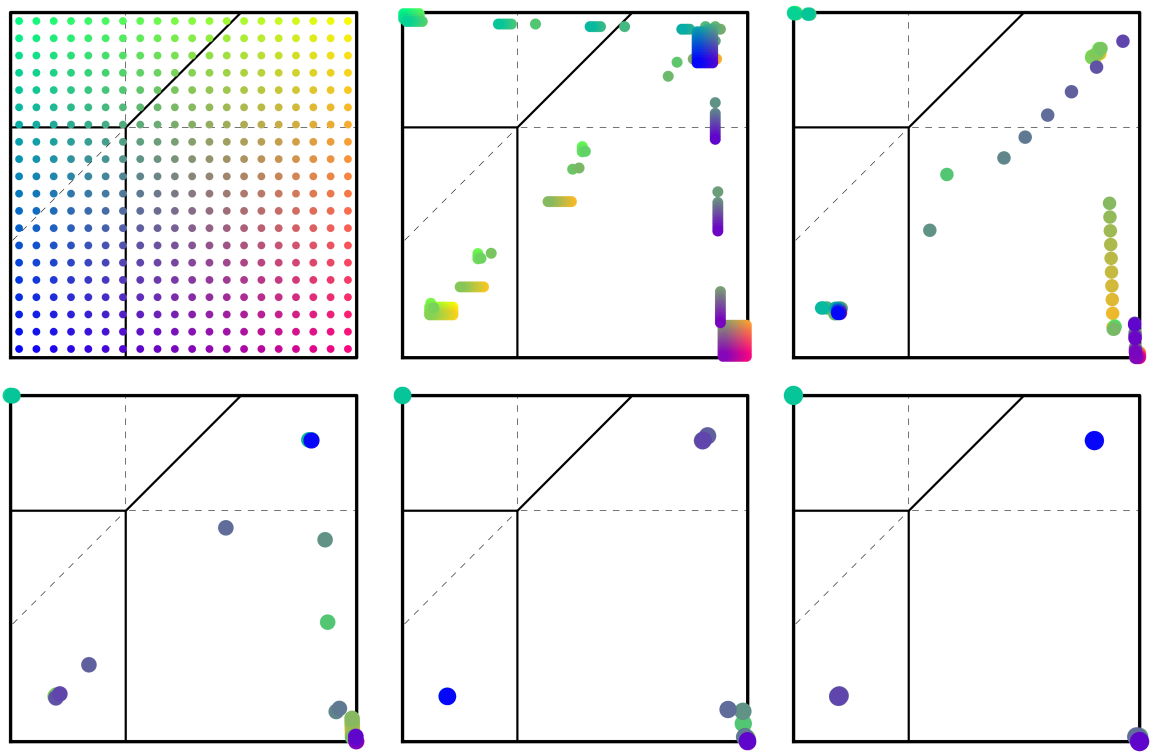

Figure 4: An example of a CS Polling Dynamics obtained by perturbation of the Polling Dynamics used in the proof of Theorem C, showing robustness of the bad 2-cycle. Top-left: initial points of the plane; then from left to right then top to bottom, the first five iterations of a continuous CPD where, whenever margins are above 4 %, 85 % of voters apply the Leader Rule and 15 % keep their ballot unchanged. Points are drawn with the color corresponding to initial position (darker above for better readability). After the first iteration, already most points are attracted to the periodic points near the four corners. The top-left and bottom-right corners are attracting fixed points, and there is an attractive orbit of period 2 near the other corners.

Our results complement for example the findings of [LLV00] and [Las10]. Compared to the latter, for each preference profile we determine the existence of a bad cycle anywhere in the space of outcomes, and of arbitrary length.

Main experimental setup We fix the number of candidates to $m = 6$, and the number of voter types to $n = 20$, $N = \{1, \ldots, 20\}$, a compromise between complexity of the preference profile and computational power limitations. The influence of these numbers is tested at the end of the section.

Each voter type is assigned a number of voters uniformly drawn in $[0, 1]$ and is assigned a preference modeled by a linear order on $\bar{A} = A \cup \{\mathsf{limit}\}$; when we use the Leader Rule, limit is simply ignored, but when we use its modified version, the candidates ranked beyond limit are considered tied for the last rank and thus never put on this voters'

ballots. For example, the order $a \succ b \succ \mathsf{limit} \succ c \succ d \succ e \succ f$ corresponds to the preference $abcdef$ when the LR is used, and to $ab(cdef)$ when its modified version is used.

We consider the following ways of constructing preferences:

- *Impartial Culture*: each voter type draws uniformly an element $\mathcal{L}(\bar{A})$,
- *d-dimensional culture*: each candidate and each voter types is given a uniform random position in $[0,1]^d$ where each coordinate represents a "political axis"; the preferences of a voter type is obtained by sorting the candidates by increasing distance to the voter type, using the ℓ^1 metric

$$d((x_1,\dots,x_d),(y_1,\dots,y_d)) = \sum_{i=1}^{d} |x_i - y_i|$$

  that sums the levels of disagreement along the various axes; limit is then inserted by drawing a random "distance" to the voter type, with the same law than the distance between two uniform points in $[0,1]^d$

We ran experiments for each combination of a heuristic among the Leader Rule and its modification and each political culture among the Impartial Culture and the d-dimensional cultures with $d \in \{1,2,3,400\}$. In each case, we generated 100 000 independent preference profiles and recorded whether a Condorcet winner exists, and if yes whether the Polling Dynamics has a "bad cycle" (or bad equilibrium), i.e. where some of whose states do not elect the Condorcet winner. Each run took a couple of hours on a single core of a modern CPU. The results are provided in Table 1. The first take-away is that in practice

The Leader Rule is very effective in electing the Condorcet winner.

Only the Impartial Culture witnesses slightly larger odds of not electing her while structured cultures very rarely produce bad cycles, the linear ($d = 1$) culture having produced none in 100 000 attempts (this is no accident, see Section 4.4).

On the contrary, when voters apply the Modified Leader Rule and thus, with the kind of preferences used here, refuse to approve of candidates below their personal threshold no matter what, bad cycles are more common. They never dominate, but they culminate *precisely* in the case of a linear culture, with 15% of preference profiles leading to a bad cycle in the Polling Dynamics. The situation that is most favorable to the Leader Rule, is also the worst one for its modification!

Imagine that voters of a given type are aware of a Condorcet winner quite low on their preferences, and that they know that when everyone uses the Leader Rule, the election of the Condorcet winner is very likely. By not applying the Leader Rule and instead refuse to approve candidates below a threshold, they can get significant odds to obtain the election of a candidate they prefer to the Condorcet winner:

> *When some voters expect a candidate they do not like to be a Condorcet winner, their best interest can be not to apply the Leader Rule, at least not below a certain threshold.*

Heuristic \ Culture	Impartial	$d = 1$	$d = 2$	$d = 3$	$d = 400$
Leader Rule	(70%) 1.3%	(100%) 0.0%	(90%) 0.2%	(87%) 0.2%	(85%) 0.3%
Modified Leader Rule	(75%) 6.3%	(92%) 15%	(89%) 7.8%	(88%) 6.0%	(87%) 3.0%

Table 1: Experimental results with 6 candidates and 20 voter types. In small, the proportion of preference profiles where a Condorcet winner exists; in normal size, the proportion of preference profiles with a bad cycle or equilibrium, among preference profiles having a Condorcet winner (both rounded). The 95% confidence Wilson score interval gives a deviation of less than 0.23 percentage point for all values; the deviation is even less than 0.04 percentage point for the value 0.3% and below.

At first sight, this may seem difficult to reconcile with the optimality of the Leader Rule proven by Laslier [Las09]. The point is that Laslier considers that each voter assumes imperfect recording of ballots and think as if she would be the only one to change her ballot, trying to optimize the expected outcome. Our numerical results show that whenever voters expect other voters of the same type to also change their votes strategically, the Leader Rule might not be their best heuristic.

This experiment also gave the opportunity to find especially problematic Polling Dynamics, see Figures 5 and 6 where green states correspond to the Condorcet winner being elected, orange states are part of a cycle but do not elect the Condorcet winner, and red states are equilibriums not electing the Condorcet winner.

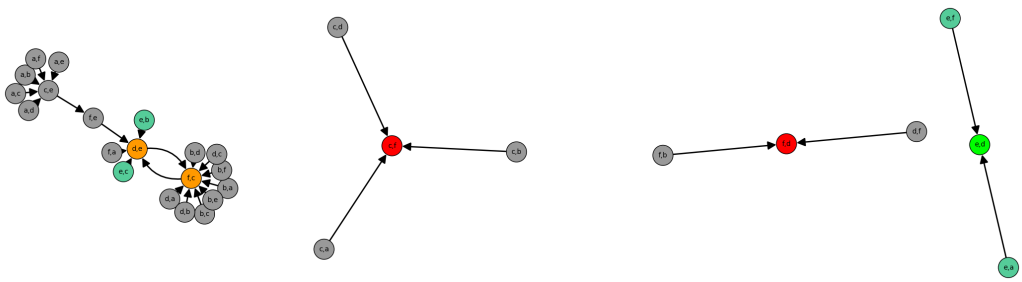

Figure 5: A bad cycle and two equilibriums not electing the Condorcet winner under the modified Leader Rule, produced with the Impartial Culture. Only 3 outcomes out of 30 end up electing the Condorcet winner, and there are outcomes where the Condorcet winner is elected, but that lie in the basin of attraction of the bad cycle (this never happens with the Leader Rule).

Effect of the numbers of candidates and voter types We report in Table 2 more modest numerical experiment to assert the influence of the number of candidates and the

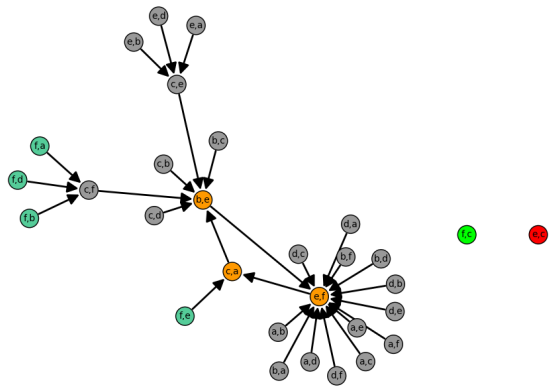

Figure 6: An example in the one-dimensional culture with the modified Leader Rule: the bad cycle attracts all outcomes but two, including most outcomes electing the Condorcet winner.

number of voters. We see that the number of voter types has relatively small influence, at least in the range tested, with the exception of the 1-dimensional culture with the modified LR, where a larger voter type makes bad cycles and equilibriums more likely. The main take away, valid in all considered conditions except the 1-dimensional culture with the LR, is that:

> *A larger number of candidates seems to make bad cycles and equilibriums more common among situations where a Condorcet Winner exists.*

Computational power prevented us to explore larger number of candidates with samples large enough to draw conclusions.

4.4 Convergence of the Leader Rule in one-dimensional cultures

In view of the above experiments, it seems that the one-dimensional culture combines well with the Leader Rule. Black's Median Voter Theorem [Bla86] implies the existence of a Condorcet winner, and from [Las09] it follows that there exist equilibriums and that all equilibriums elect ω. Here we prove that in this particular culture, we moreover have convergence of the Polling Dynamics.

Definition 4.9. We say that a preference profile *can be modeled by a one-dimensional culture* when there exist a *positional mapping*

$$x : A \cup N \to \mathbb{R}$$

with the following property: the preferences $\succsim$ of any voter type $i \in N$ is given by

$$\forall \alpha \neq \beta \in A, \qquad \alpha \succsim_i \beta \quad \text{if and only if} \quad |x(\alpha) - x(i)| < |x(\beta) - x(i)|.$$

For simplicity, we assume further that $|x(\alpha) - x(i)| \neq |x(\beta) - x(i)|$ for all $\alpha \neq \beta$ and all i, so that all preferences are linear orders (without ties); and we assume that there is

Cul. & Strat. \ $\lvert\mathcal{C}\rvert$ & $\lvert\mathcal{T}\rvert$	3, 10	4, 10	6, 10	**6, 20**	6, 30	8, 20
Impartial, LR	(92%) 0	(84%) 0.3%	(71%) 1.0%	(**70%**) **1.3%**	(69%) 1.3%	(60%) 2.0%
1-dim., LR	(100%) 0	(100%) 0	(100%) 0	(**100%**) **0**	(100%) 0	(100%) 0
2-dim., LR	(97%) 0	(94%) 0.05%	(90%) 0.3%	(**90%**) **0.2%**	(91%) 0.05%	(87%) 0.5%
3-dim., LR	(96%) 0	(92%) 0.1%	(85%) 0.4%	(**87%**) **0.2**%	(89%) 0.1%	(84%) 0.4%
400-dim., LR	(95%) 0	(90%) 0.1%	(81%) 0.3%	(**85%**) **0.3%**	(89%) 0.3%	(81%) 0.6%
Impartial, modified LR	(94%) 2, 1%	(88%) 3.5%	(77%) 5, 9%	(**75%**) **6.3%**	(75%) 6.4%	(66%) 8.0%
1-dim., modified LR	(98%) 2.6%	(96%) 5.4%	(90%) 10%	(**92%**) **15%**	(92%) 19%	(88%) 20%
2-dim., modified LR	(97%) 1.5%	(95%) 3.4%	(89%) 6.2%	(**89%**) **7.8%**	(91%) 7.8%	(84%) 11%
3-dim., modified LR	(97%) 1.4%	(93%) 2.7%	(86%) 5.3%	(**88%**) **6%**	(90%) 5.9%	(84%) 8.2%
400-dim., modified LR	(96%) 0.9%	(92%) 2.1%	(84%) 3.4%	(**87%**) **3.0%**	(89%) 2.6%	(82%) 4.1%

Table 2: Complementary experimental results to check robustness. Samples have 6 250 pseudo-random preference profiles (except for 6 candidates and 20 voter types, where we reported the previous results with larger samples), yielding typically 4 times larger confidence intervals than samples of size 100, 000 and leaving the expected deviations under the percentage point. In small, the proportion of preference profiles where a Condorcet winner exists; in normal size, the proportion of preference profiles with a bad cycle or equilibrium, among profiles having a Condorcet winner (both rounded).

no partition of the voter types in two groups that have equal total number of voters, so that no duel between two candidates would end up in a tie (these are *generic* conditions: they are stable under small perturbation of the data and any data can be approximated arbitrarily close by data satisfying those conditions).

The goal of this section is to prove the following result.

Theorem 4.10. *Under Approval Voting, if the preference profile can be modeled by a one-dimensional culture and voters apply the Leader Rule, then the Polling Dynamics converges to an equilibrium (which elects the Condorcet winner, as prescribed by Laslier's Theorem).*

The end of the section is dedicated to the proof of Theorem 4.10. We use outcomes reduced to the winner and runner-up, we assume that the preference profile can be

modeled by a one-dimensional culture and we fix a positional mapping x. Denoting by $w = \sum_{i \in N} w_i$ the total number of voters, a *median* is a value $m \in \mathbb{R}$ such that

$$\sum_{i\colon x(i) \leq m} w_i \geq \frac{w}{2} \qquad \text{and} \qquad \sum_{i\colon x(i) \geq m} w_i \geq \frac{w}{2}$$

i.e., at least half the voters lie on the left of m, and at least half have lie on the right of m. The assumption that there is no partition of voter types in two groups of equal size ensures that there is a unique median m, coinciding with the position of some voter type: $m = x(i_m)$, where the voter type i_m is called the *median type*. By assumption, there is a single candidate μ whose position is closest to $x(i_m)$.

We start with the following particular instance of Black's Median Voter Theorem, which we prove for the sake of completeness.

Lemma 4.11. *The candidate μ is a Condorcet winner.*

Proof. Let α be any other candidate. Voters preferring μ to α are those positioned on the half line L of endpoint $\frac{1}{2}(x(\alpha)+x(\mu))$ and containing $x(\mu)$. Since $x(\mu)$ is closer from $x(X_m)$ than $x(\alpha)$, L contains $x(i_m)$ and thus contains at least half the voters. Since there are no possible ties, μ dominates α. □

Lemma 4.12. *Let $\alpha\beta$ be an outcome such that $\alpha \neq \mu$ and let $\alpha'\beta' = \psi(\alpha\beta)$ be the next outcome in the (shifted) Polling Dynamics. Then either $|x(\alpha') - x(\mu)| < |x(\alpha) - x(\mu)|$, or $\alpha' = \alpha$ and $|x(\beta') - x(\mu)| < |x(\beta) - x(\mu)|$.*

Proof. The positions $x(\alpha)$ and $x(\beta)$ divide the real line in three component: the open bounded interval I between them, the open half-line H_α with extremity $x(\alpha)$ avoiding I, and the open half-line H_β with extremity $x(\beta)$ avoiding I.

If $x(\mu) \in H_\alpha$, α will receive the votes of all voters positioned in a half line starting at $(x(\alpha) + x(\beta))/2$ and containing $x(\mu)$, while other candidates receive votes only from voters positioned in H_α or in $I \cup H_\beta$. It follows that $\alpha' = \alpha$ and β' is the candidate positioned next to α, in the direction of $x(\mu)$ (possibly $\beta' = \mu$). In this case, we thus have $\alpha' = \alpha$ and $|x(\beta') - x(\mu)| < |x(\beta) - x(\mu)|$.

In all other cases, α' is the candidate positioned next to α, in the direction of μ (possibly $\alpha' = \mu$), therefore $|x(\alpha') - x(\mu)| < |x(\alpha) - x(\mu)|$. □

Let $>$ denote the "geo-lexicographic" order on outcomes, defined by $\alpha\beta > \alpha'\beta'$ whenever

$$|x(\alpha') - x(\mu)| < |x(\alpha) - x(\mu)|, \text{ or } \alpha' = \alpha \text{ and } |x(\beta') - x(\mu)| < |x(\beta) - x(\mu)|.$$

The previous lemma shows that along an orbit of ψ, the outcome can only decrease in this order until μ becomes winner. Since there cannot be an infinite decreasing sequence of outcomes, eventually μ becomes winner. Now, as is well-known, this is a stable situation: for all β, $\psi(\mu\beta) = \mu\beta_0$ where β_0 is the candidate getting the better score in a duel against μ. Therefore, every outcome converges under ψ to the outcome $\mu\beta_0$ (and every ballot profile of B^n converges under φ to the ballot profile induced by the outcome $\mu\beta_0$ and the Leader Rule). The proof of Theorem 4.10 is complete.

5 Other Voting rules

In order to show how our framework applies to general voting rule, let us consider three more examples: one for Plurality voting, where heuristics use quantitative information in the outcome rather than only the ranking; one for Instant Run-Off voting (IRV); one for the *condorcified* version of IRV (or, for that matter, of any ranked voting).

5.1 A plurality cycle

We consider Plurality voting, with outcomes giving the total number of votes of each candidate, with $n = 4$ voter types and $m = 3$ candidates, $A = \{a, b, c\}$. We thus set $\mathcal{O} = [0, w]^A$ where $w = \sum_{i \in N} w_i$ is the total number of voters, an element of which is written as a triple $r = (r_a, r_b, r_c)$. Ballots bear a single name, i.e. $\mathcal{B} = A$, and the information function g sends a family $(B_i)_{i \in N} \in A^N$ of ballots to the outcome

$$g((B_i)_{i \in N}) = \Big(\sum_{i \in N, B_i = \alpha} w_i \Big)_{\alpha \in A.}$$

The winner map sends (r_a, r_b, r_c) to the candidate with the most votes, with ties broken in favor of the earlier candidate in the alphabetical order.

We consider the preference profile

X	Y	Z
10	13	14
a	b	c
b	a	ab
c	c	

and assume the following heuristics:

- voters of type Z always vote for c,
- voters of type X vote as follows. If their least-preferred candidate c is in a close contest for winning with their second-preferred candidate b, more precisely if

  $$|r_c - r_b| \leq .03(r_a + r_b + r_c) \quad \text{and} \quad \min(r_b, r_c) > r_a + .03(r_a + r_b + r_c)$$

  then they vote for b; in all other cases they vote sincerely, for a,
- voters of type Y vote as voters of type X, with a and b switched.

These heuristics allow voters of type X and Y to vote strategically whenever their preferred choice is far from winning, but their second-preferred choice needs support to avoid the win of their least-preferred candidate. The factor .03 is chosen because it is the typical uncertainty in polls on about a thousand voters. With the above preference profile, $.03(r_a + r_b + r_c) = 1.11$.

If voters first report their preferred candidates, we obtain the outcome

$$r_0 = (10, 13, 14)$$

and we are in the situation where c wins in close contest with b. The above heuristics lead voters of type X to vote for b, while others remain sincere, leading to the new outcome $r_1 = \psi(r_0) = (0, 23, 14)$ where b wins by a large margin. The above heuristics lead every voter to vote sincerely, so that $\psi(r_1) = r_0$, and we have a two-cycle (an outcome of which elects the absolute majority loser). By theorem A, this bad cycle persists in a small enough CS perturbation.

5.2 An IRV cycle

We now consider IRV and give an example with $n = 3$ and $m = 4$. Ballots and outcomes are linear rankings of candidates: $\mathcal{B} = \mathcal{O} = \mathcal{L}(A)$. The information function g is defined as follows. Given an argument $(B_i)_{i \in N} \in \mathcal{B}^N$, we first rank candidates according to the total weight of ballots where they are ranked top (again, ties broke in alphabetical order). The last ranked candidate is eliminated, and we rank the remaining candidates according to the total weight of ballots where they are ranked top *among candidates that have not been eliminated*. The last ranked candidate is again eliminated. The resulting outcome is denoted with the last standing candidate first, then the other by reversed order of elimination. The winner map sends a ranking $r \in \mathcal{O} = \mathcal{L}(A)$ to its top candidate.

We consider the preference profile

X	Y	Z
10	11	12
b	c	d
a	a	a
c	b	b
d	d	c

so that the Condorcet order coincides with the alphabetical order, in particular a is the Condorcet winner. Let us define simple heuristics that make use of an expected outcome to try to improve it. Denote by $\alpha\beta\gamma\delta$ the preferences of the considered type i of voters (e.g. for $i = X$, $\alpha = b$, $\beta = a$, $\gamma = c$ and $\delta = d$). Given an outcome r, if the expected winner is neither α nor β and β was eliminated before α then $\sigma_i(r) = \beta\alpha\gamma\delta$; for every other outcome r, $\sigma_i(r) = \alpha\beta\gamma\delta$. In other words, voters strategically invert their two first preferred candidates when the top one had a better run but still could not prevent a worse candidate to be elected, giving their second choice a chance to do better.

If voters start voting sincerely, we obtain the outcome *cdba* with the Condorcet winner eliminated with no votes at all in the first round, and ballots of type X voters ultimately transferred to candidate c against d. The above heuristics have voters of type X and Z invert their two most preferred candidates and vote *abcd*, *adbc* respectively. The

outcome is then *acbd*. Then all voters resume voting sincerely, and we have a 2-cycle one of whose states elects the third candidate in the Condorcet order. Observe that every outcome where the Condorcet winner is elected leads voters to cast their sincere ballot, thus leading to this bad cycle.

According to Theorem A, this cycle is again robust under perturbation in a CS setting.

5.3 A cycle in condorcified ranked voting

Consider a ranked voting rule, i.e. one with $\mathcal{B} = \mathcal{L}(A)$. It has been proved by Durand, Mathieu and Noirie [DMN14, DMN16] and independently by Green-Armytage, Tideman and Cosman [GATC16] that under mild conditions *condorcification* of a voting system (i.e. electing the Condorcet winner if she exist and applying the given rule otherwise) cannot increase manipulability and often reduces it. We can therefore ask whether a condorcified voting rule is less susceptible to the presence of bad cycles when a Condorcet winner is present. Unsurprisingly, the answer is negative when voters who do not like the Condorcet winner strategically choose to rank her lower in their ballot

Let us consider a condorcified voting rule, i.e. for all $(B_i)_{i\in N} \in \mathcal{B}^N$ for which their is a Condorcet winner γ, $W \circ g((B_i)_{i\in N}) = \gamma$. Assume moreover that $\mathcal{O}$ contains the information whether or not the winner was a Condorcet winner. Consider the preference profile

Z	Y	X	W
150	102	101	100
a	b	c	b
b	c	a	a
c	a	b	c

for which a is the sincere Condorcet winner. Assume the following heuristics for voter of any type i, whose preferences are denoted as above by $\alpha\beta\gamma$: if β is elected as Condorcet winner, then they cast the ballot $\alpha\gamma\beta$, in an attempt to give α a shot. Otherwise, they vote sincerely.

If at first all voters vote sincerely, in the first outcome r_0 the winner is a, declared a Condorcet winner. This leads voters of type X to next cast the ballot *cba* and voters of type W to cast the ballot *bca*. This makes b the Condorcet winner of the new ballot profile. The above heuristics then lead all voters to vote sincerely, except voters of type Z who cast the ballot *acb*; this makes again a the Condorcet winner, and we have a 2-cycle, one of whose ballot profiles elects a candidate that is not the Condorcet winner of the preference profile. Moreover this cycle attracts all outcomes where a is a Condorcet winner of the ballot profile.

Again, Theorem A ensures that this cycle is robust under perturbation in a CS setting.

6 Chaos

We finish with an example illustrating the flexibility of the CS setting for the Polling Dynamics. The starting point is to observe that the embedding Φ_0 of a discrete space Polling Dynamics in a CS setting cannot be continuous (unless it is a constant map). Indeed, it takes a finite number of values, and the state space $\mathcal{P}$ is connected. There must exist some CS ballot profiles near which arbitrarily close CS ballot profiles are sent to very different images by Φ_0. In practice, assuming the outcomes carry continuous information (e.g. shares of votes for each candidate), a heuristic like the Leader Rule would probably not be applied blindly by all voters when the expected winner and runner-off are in a close-call contest, or when the runner-up is in a close-call contest with the next candidate. Different voters will have different confidence in the available information and a small change in the expected outcome should result in a small change in the ballots cast, i.e. the CS Polling Dynamics should be continuous.

We now give a relatively simple example with $m = 3$, based on a reluctance to add one's second-preferred candidate unless it seems likely to improve the outcome, and we observe that a complicated dynamics emerges. The take-away is that

> *Even simple CS Polling Dynamics can exhibit a chaotic behavior, where the sequence of winners is impossible to predict reliably from the observation of arbitrarily many of its first terms.*

By looking at the influence of a small number of core parameters of the model, we shall see that chaos is neither universal nor restricted to exceptional parameters.

A simple continuous CS Polling Dynamics We consider the four-types preference profile used in the proof of Theorem C:

Z	Y	X	W
3	1	3	5
a	a	b	c
b	bc	a	ab
c		c	

for which a is a Condorcet winner and c an absolute majority loser. As above, the corresponding sets of admissible ballots makes $\mathcal{P}$ a square with coordinates (x, z) where x, z are the proportions of voters of type X , Z respectively voting $\{a, b\}$. This choice of ballot can be seen either as a form of cooperation between the two types of voters, or as the result of risk aversion (which are complementary views, not opposed ones).

An outcome is a triple $r = (r_a, r_b, r_c) \in [0, 1]^3$ giving the shares of votes obtained by each candidate, and we choose the following CS heuristics (here the resulting distribution of ballots does not depend upon the value x or z in $\Delta(\mathcal{B}_i)$ that is by definition part of their argument):

$$\sigma_Z((r_a, r_b, r_c), z) = C_Z \circ S(r_a, r_b, r_c) \qquad \sigma_X(x, (r_a, r_b, r_c), x) = C_X \circ S(r_b, r_a, r_c)$$

where C_Z, C_X, S will be defined below, to be interpreted as follows:

- S is a "safety function", quantifying how unlikely it seems that cooperating would be useful to counter a threat by c (notice how r_a, r_b are exchanged in σ_X, to take into account the preferences of this type). S will be in particular very small when both c is expected winner or close to be expected winner, *and* collaborating has a good chance to prevent her to win,
- C is a "collaboration function", translating a level of safety into a proportion of collaborations (high safety resulting in low collaboration).

We take here a very simple safety function:

$$S(r_1, r_2, r_3) = \begin{cases} |r_2 - r_3| & \text{when } r_2 > r_1 \\ \frac{1}{2}|r_2 - r_3| + \frac{1}{2}|r_1 - r_3| & \text{otherwise.} \end{cases}$$

i.e. when the second-preferred candidate is ranked higher than the preferred one, the safety is the margin in her race with the least-preferred one; otherwise, safety is an average of the margins in races of the two preferred candidates against the last one. For example, if $r_1 \gg r_3 \simeq r_2$ then the safety is not too small (the least preferred candidate has little chance of winning) but not maximal, meaning some voters will prefer to collaborate in order to ensure c finishes at the last rank. When $r_1 \simeq r_2$, the safety is close to both $|r_1 - r_3|$ and $|r_2 - r_3|$; when c is far above or far below, one's vote seems unlikely to change the outcome and the incentive to collaborate is small.

Finally, we choose a simple collaboration function extrapolating linearly between the value 1 (all voters collaborate) when the safety vanishes, and the value 0 (no voter collaborates) when the safety is large enough:

$$C_i(t) = (1 - c_i t)_+ := \max(0, 1 - c_i t)$$

where as indicated $(\cdot)_+$ is the positive part, and where the coefficient c_i $(i = Z, X)$ quantifies the risk tolerance: the higher, the less likely X and Z voters will collaborate (low values of c_i thus correspond to high aversion to risk). For now, we take $c_Z = c_X = 5$, a rather moderate value: $1 - 5t$ reaches 0 only at $t = 0.2$, i.e. when the safety margins reaches a staggeringly high 20% of voters.

Since a state (x, z) results in the outcome $(r_a, r_b, r_c) = (3x+4, 3z+x, 5)$, these choices yield the CS Polling Dynamics Φ defined by:

$$\Phi(x, z) = \Big(\big(1 - 5|3x - 1|\big)_+, \big(1 - 2.5|3x - 1| - 2.5|3z + x - 5|\big)_+\Big)$$

when $2x + 4 \geq 3z$ and

$$\Phi(x, z) = \Big(\big(1 - 2.5|3z + x - 5| - 2.5|3x - 1|\big)_+, \big(1 - 5|3z + x - 5|\big)_+\Big)$$

when $2x + 4 \leq 3z$. While Φ may look like an innocent map, drawing an orbit of this map reveals an interesting pattern (Figure 7). It turns out *all* orbits yield pretty much the same image, typical of a *chaotic attractor*.

In dynamical systems, *chaos* is not a formally defined word, but refers to a number of properties all seeking to translate the idea that orbits behave in an unpredictable fashion. The most prominent one is *entropy*, which comes in a number of versions, of which we shall only give one flavor (more information is available in numerous books, e.g. [KH95]). It is to be understood that "chaotic" situations are those of positive entropy, zero entropy being the sign for a relatively "tame" system.

While we will not *prove* chaos here, we will show compelling numerical evidence that some entropy is positive, meaning that an orbit have the same level of complexity as a random sequence of (skewed) coin tosses. Note that the fact that the map Φ is chaotic is not the most important point: what matters is that the sequence of winners along many orbits are chaotic, which here results from the attractor intersecting regions electing different candidates.

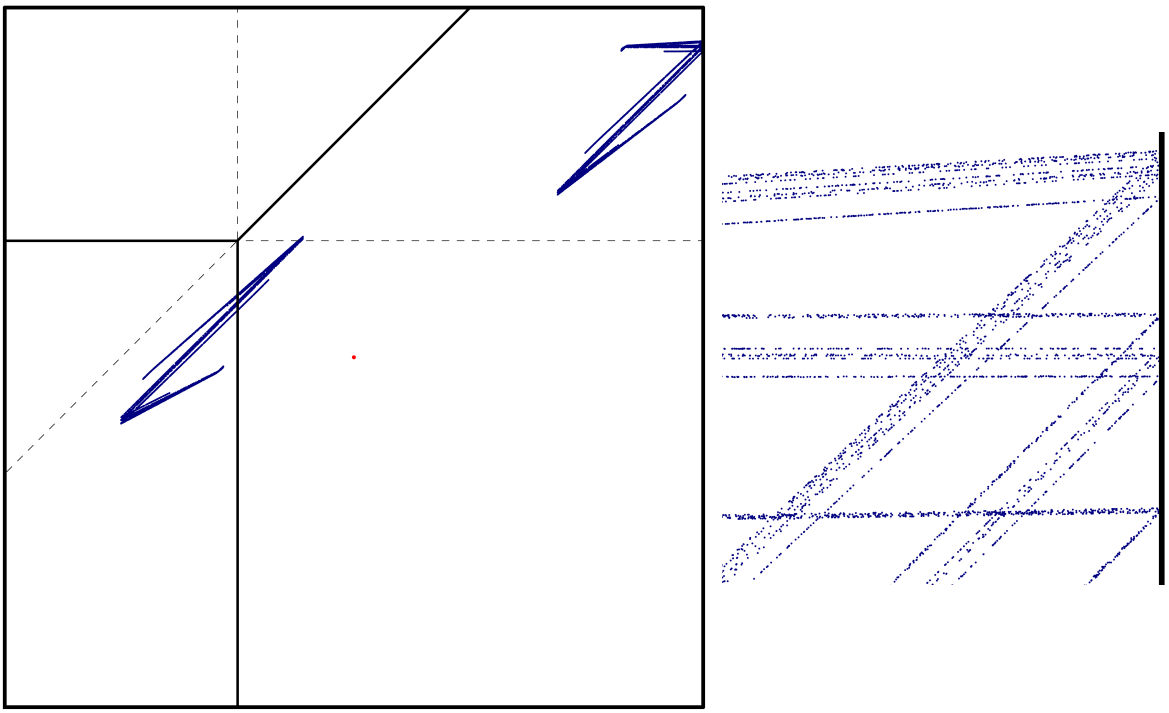

Figure 7: One orbit of the CS Polling Dynamics Φ. Left: full picture. Right: a zoom to the upper-right part showing a linear/fractal struture.

Numerical quantification of the chaos Given any state $s \in \mathcal{P}$, its *winners word* is the word $\omega(s)$ whose k-th letter is $W(G(\Phi^k(s)))$, the winner after k iterations. In the example above the word of the successive winners starting from the center of the space $\mathcal{P}$ is (32 first letters shown, capital c for better readability):

$$\omega(.5, .5) = aaaCaCaaaCaCaCaaaCaCaCaCaCaaaCaC \cdots$$

(the choice of $(.5, .5)$ is arbitrary, but experimentally one checks that it has no bearing on the results exposed here). There is some visible structure in this word, for example

there are never two successive c no four a in a row; apart from the first three letters, the word seems constructed only with the blocs $\beta = ca$ and $\gamma = caaa$. However, the succession of these blocks looks somewhat random:

$$\omega(.5, .5) = aaa\gamma\beta\gamma\gamma\beta\gamma\gamma\gamma\gamma\beta\gamma\gamma\beta\gamma\gamma\gamma\gamma\beta\beta\beta\beta\gamma\ldots$$

To quantify the randomness of $\omega(.5, .5)$, we use the following definition.

Definition 6.1. Let $\omega = \alpha_1\alpha_2\cdots$ be an infinite word on the alphabet A, i.e. a sequence of elements of A (written without parentheses or comma). For all $k < j \in \mathbb{N}$ we set $\omega_{k:j} = \alpha_k\alpha_{k+1}\alpha_{k+2}\cdots\alpha_j$; each $\omega_{k:j}$ is called a *subword* of ω (of length $j - k + 1$).

For all $k, \ell \in \mathbb{N}$ with $\ell \leq k$ and all finite word θ of length ℓ, we denote by $S_k^\theta(\omega)$ the number of times the subword θ appears in $\omega_{1:k}$, we set

$$P_k^\theta(\omega) := \frac{S_k^\theta(\omega)}{k - \ell + 1}$$

the proportion of length ℓ subwords of $\omega_{1:k}$ equal to θ, and by $P_k^\ell(\omega)$ the "probability vector" $(P_k^\theta(\omega))_{\theta\in A^\ell}$. Finally the *Kolmogorov-Sinai entropy* of the word ω is defined as

$$h_{\mathrm{KS}}(\omega) = \lim_{\ell\to\infty} \frac{1}{\ell} \limsup_{k\to\infty} H(P_k^\ell(\omega))$$

where

$$H(p_1, p_2, \ldots, p_k) = \sum_{i=1}^{k} -p_i \log p_i \qquad \text{with the convention } 0 \log 0 = 0.$$

Remark 6.2. The limit in the definition exists by Fekete's lemma, the needed subadditivity being of the perks of the function H. The entropy $h_{\mathrm{KS}}(\omega)$ is never larger than $\log m$, with equality when all possible subwords of length ℓ appear in ω with the same asymptotic frequency.

The value of $h_{\mathrm{KS}}(\omega)$ is to be interpreted as the "uncertainty" of a random guess. The reference case is that of an object uniformly drawn among N, having uncertainty $\log N$; Observe that this makes uncertainty linear in the number of objects to be guessed in the sense that guessing ℓ independent objects each uniformly drawn among N has uncertainty $\log(N^\ell) = \ell \log N$. More generally, one defines the uncertainty of a choice made according to a probability vector $(p_1, \ldots, p_N)$ as $H(p_1, \ldots, p_N)$, and this definition enjoys many natural properties (see e.g. [Wal82] Theorem 4.1). The uncertainty of guessing a length-ℓ subword of ω, when ℓ is large and we look far on the right of the word, is asymptotically of the order of $\ell \cdot h_{\mathrm{KS}}(\omega)$.

The sequence $\ell \mapsto H(P_{2^{20}}^\ell(\omega))$ is plotted in Figure 8. We observe an extremely good alignment from $\ell = 4$ onward, with slope $\simeq 0.229$, a strong indication that $h_{\mathrm{KS}}(\omega(.5, .5))$ is close to this value. Every starting state s yields very similar results, a strong numerical indication that the map Φ is chaotic, with chaotic sequences of winners. To check whether all states have orbits accumulating on the attractor pictured in Figure 4, we

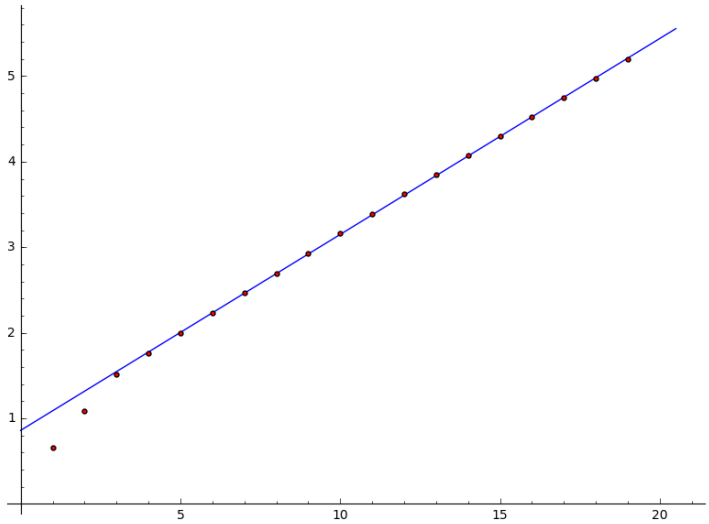

Figure 8: Plot of the entropy-estimating sequence $\ell \mapsto H(P^{\ell}_{2^{20}}(\omega))$. The slope of the marked line is $\simeq 0.2291$

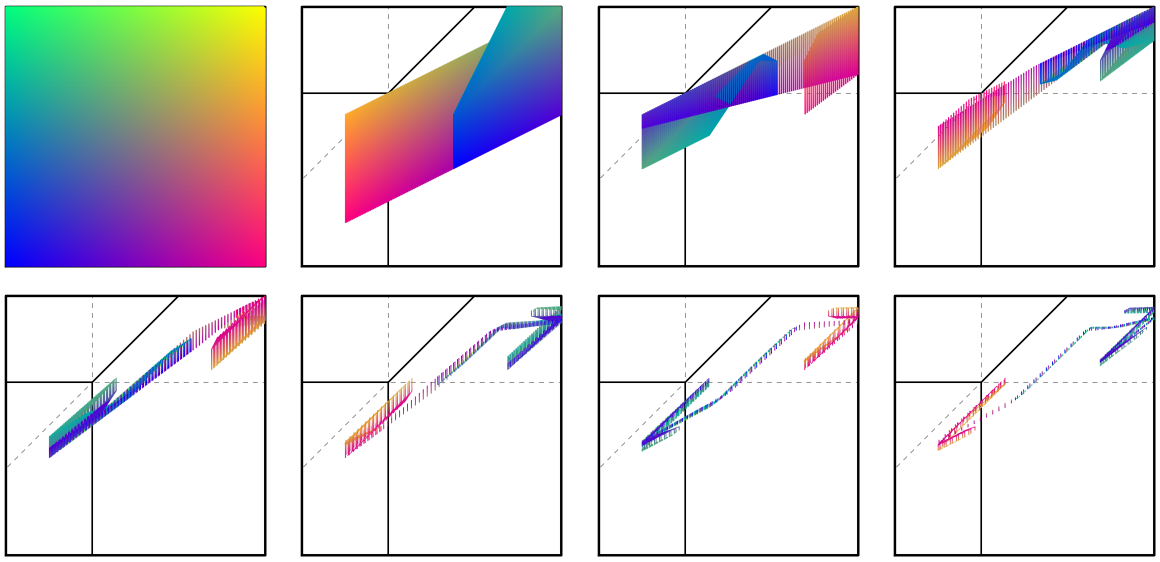

Figure 9: From left to right then top to bottom, 200×200 points of the state space regularly spaced and their 7 first iterates by Φ (colors preserved along each orbit).

have drawn the first iterates of Φ in Figure 9. All orbits seem to accumulate to the attractor, which appears in fact connected (a fact that can be confirmed by topological arguments), made of a very thin curve between two arrow-shaped parts, which are exchanged by Φ.

While this is only a specific example, it was constructed from basic principles and it is possible that chaos be a quite common feature of CS Polling Dynamics. In order to start testing this hypothesis, we look at what happens when we let c_X and c_Z vary, see Figure 10. We observe that entropy is mostly concentrated in an L shaped region, where both voter types have not too high a risk aversion, and either type has not too high risk tolerance. However when Z has high risk-aversion, moderate-high values of c_X still result in positive entropy. This asymmetry must comes from the asymmetry of the preference profile introduced by voters of type Y.

Looking at sequences of winners with various parameters show that in the upper-right square where entropy vanishes (high risk tolerance for both voter types), several different dynamics can happen: c can be constantly elected or we can observe a periodic pattern involving a and c as winners. When X has high risk aversion, we only observed constant election of a. When Z has high risk aversion, for increasing values of c_X, we observe successively: constant election of a, periodic patterns involving a and b, chaotic patterns involving a and b, constant election of b.

Let us end with the influence of the preference profile. We look at the basic example ($c_X = c_Z = 5.$) with randomly chosen weights for all four voter types. The following examples complements the main one to form a representative sample of what a quick experimental exploration revealed for this model.

Example 6.3. With $w_X = 0.05$, $w_Y = 0.02$, $w_Z = 0.8$ and $w_W = 0.8$, a is constantly elected. Observe that it always gets at least .82 votes while c receives only .8, so the only possible contender would be b, but she would need almost all of Z voters to cooperate and most X voters not to cooperate, a behavior prevented in our model. This is a non-chaotic, actually very stable example.

Example 6.4. Take now $w_X = 0.6$, $w_Y = 0.08$, $w_Z = 0.56$ and $w_W = 0.82$. Then applying the same method as above, we get a higher entropy estimation:

$$h_{\mathrm{KS}}(\omega(.5, .5)) \simeq 0.36$$

but the word itself looks a bit different, with long sequences of consecutive a:

$$\omega(.5, .5) = aaaCaaaCaCaaaaaCaCaaaaaaaaaCaaaaaaaaaaaaaCaCaCaCaCa \cdots$$

This could be example of an "intermittent" behavior, with slow regions where Φ is relatively tame but from which all orbits eventually escape to enter strongly chaotic regions, constantly alternating between two behaviors: predictable and chaotic. See one orbit in Figure 11 (left); other orbits produce very similar pictures.

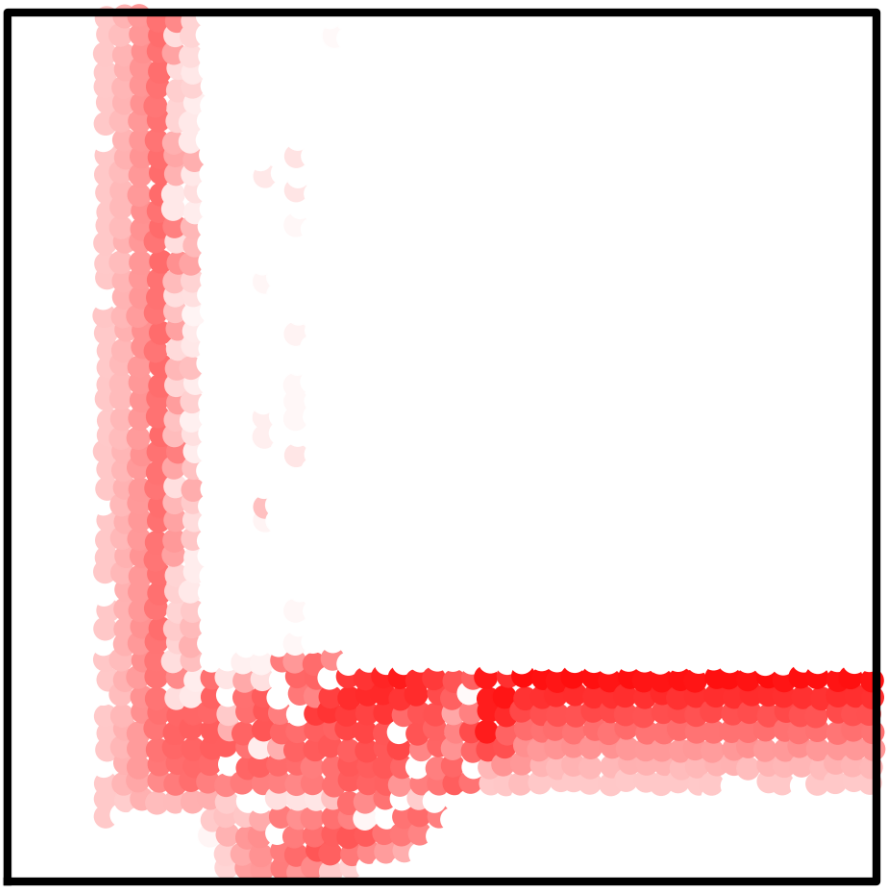

Figure 10: Entropy estimated for various values of c_X (abscissa) and c_Z (ordinate), from 1. to 30., mostly regularly spaced with a random perturbation. Each dot is one value of the pair (c_X, c_Z), color indicating the estimated entropy (linear regression made on subwords of length 4 to 11 in $\omega(.5, .5)$): white is entropy 0 while pure red is entropy $\ln(2)$ or above.

Example 6.5. Changing only slightly the previous example with $w_X = 0.6$, $w_Y = 0.08$, $w_Z = 0.56$ and $w_W = 0.81$, a radical change in long-term behavior appears: entropy seems to vanish, with $H(P^{\ell}_{2^{20}})$ plateauing abruptly from $\ell = 10$ onward. When looking at subwords of $\omega(.5, .5)$ of length 10 and more, one observes that there are exactly 22 of them, no matter the length. Closer observation then reveals that $\omega(.5, .5)$ is the concatenation of copies of the length 22 word *aaacacaaacacacacaaaaaa*. Other starting states yield similar results, indicating that Φ has here an attracting periodic orbit, of period 22 (see Figure 11, right). At the time scale of an election, it would appear chaotic, but in the longer run it is not.

This examples incites us to plot very long orbits on top of our entropy estimation, to rule out an attracting periodic orbit of length significantly larger than the maximal length of subwords used in the entropy estimation. Rigorously proving positive entropy

would need more sophisticated mathematical tools (one would search for a “horseshoe”).

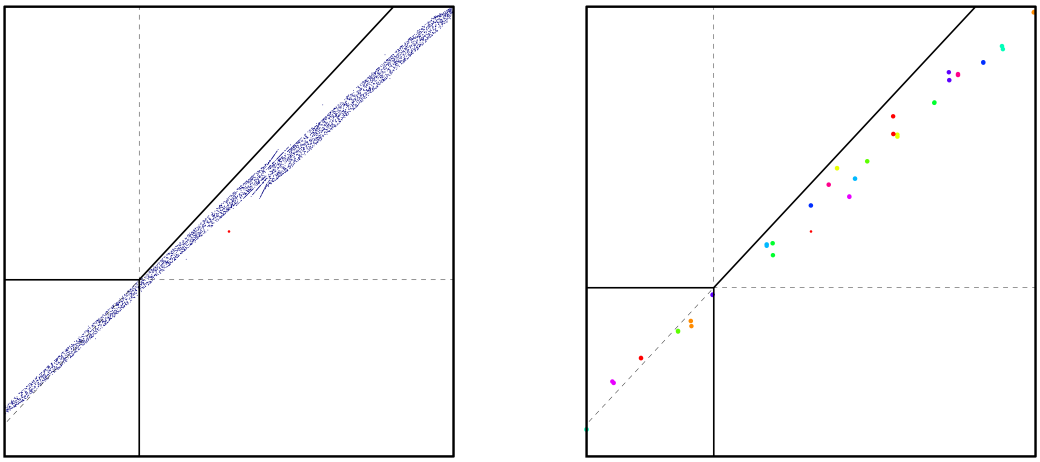

Figure 11: Comparison of the orbits of $s = (.5, .5)$ for two close populations: example 6.4 on the left, example 6.5 on the right, with colors of period 22. Both runs have 5000 points (larger points are used on the right for readability).

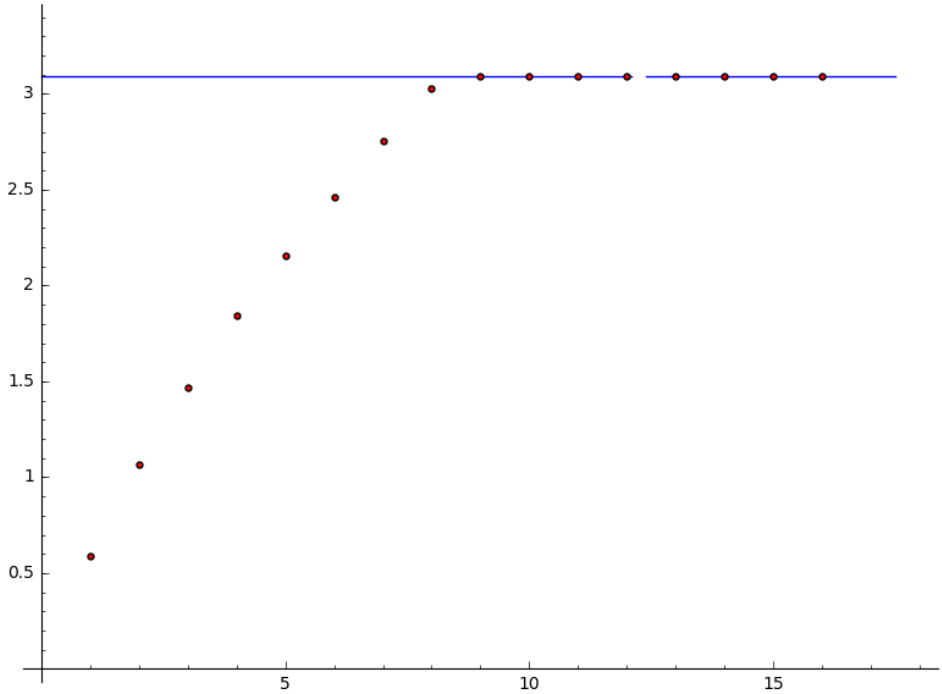

Figure 12: Plot of the entropy-estimating sequence $\ell \mapsto H(P^{\ell}_{2^{20}}(\omega))$ for example 6.5

7 Conclusion

We have explained why *synchronized* iterated voting is relevant both to model political elections preceded by polls, and to iterative anticipation of other voters' strategies. We proposed a framework for synchronized iterated voting, producing a "Polling Dynamics", in two flavors: one with a discrete space of states, the other with a continuous space allowing more flexible modeling. We showed that the former can be seen as a particular case of the latter.

Our first main result is that cycles of outcomes are robust under perturbation, so that finding a cycle in the discrete-space setting ensures any small enough variation of the model in the continuous-space setting still exhibit the same cycle.

Then we produced two examples in Approval voting, where voters apply *simple, consistent, sincere* heuristics but cycles with sub-optimal or outright bad outcomes appear. The first example somewhat mitigate an important result of Laslier: while the equilibriums of the Polling Dynamics following from his Leader Rule, as he proved, elect the Condorcet winner when she exist, they may fail to attract most of the possible outcomes.

We performed *in silico* experiments to assert the prevalence of these electoral conundrums. They are very rare for the Leader Rule, and even impossible when the culture is unidimensional. However a slight relaxation of the Leader Rule makes precisely the unidimensional culture the worst one among those tested, with up to more than 15% of electorate producing a bad cycle.

Under Approval Voting, not only convergence to equilibrium may not happen, but

cycles can lead individually sound heuristics to result collectively in the worst possible outcome.

We then gave example of bad cycles for other voting systems, in particular Condorcet systems, showing that these issues are not at all specific to Approval Voting.

Last, we considered a simplistic example of continuous-space Polling Dynamics ensuring continuity, i.e. small changes in the expected outcome leads to small changes in the ballots cast. It turned out this model has a chaotic behavior, and we conjectured that chaos is not uncommon at all for continuous-space Polling Dynamics. We supported this conjecture by looking at the influence of some parameters of the model, showing that chaos can be observed in a non-negligible range. The way in which a particular model of voters behavior and a particular preference profile result in either constant, periodic or chaotic patterns is only illustrated here, and would deserve a full study that is way beyond the scope of the present work.

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