Robust output feedback stabilization for two heterodirectional linear coupled hyperbolic PDEs

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Abstract

We solve in this article the problem of robust output feedback regulation for a system composed of two hyperbolic equations with collocated input and output in presence of a general class of disturbances and noise. Importantly, the robustness of the controller is considered with respect to delays in the actuation and in the measurements but also with respect to uncertainties on parameters, most importantly transport velocities. The proposed control law introduces three degrees of freedom (by means of tuning parameters) on which we give general conditions to guarantee the existence of robustness margins. We show that to tune these degrees of freedom and allow potential robustness trade-offs, it is necessary to consider all the different types of uncertainties simultaneously as it is the only way to ensure the existence of non-zero robustness margins. Provided that these conditions are satisfied, these tuning parameters enable a trade-off between performance and robustness, between disturbance rejection and sensitivity to noise. The existence of robustness margins and the Input To State Stability of the system are proved combining backstepping transformations and classical complex analysis techniques.

Key words: Hyperbolic Partial Differential Equations; stabilization; backstepping; neutral system; complex analysis.

1 Introduction

In this paper, we solve the problem of robust output feedback regulation for a system of two linear hyperbolic Partial Differential Equations (PDEs) with collocated boundary input and output in presence of disturbances and measurement noise. The robustness of this control law is considered with respect to delays in the actuation and in the measurements but also with respect to uncertainties on parameters, most importantly transport velocities. Inspired by the results of [4] and [24], the proposed design combines a backstepping approach with an integral action, which is used to ensure Input-to-State Stability (ISS) and convergence of the output to zero for constant disturbance. The resulting output feedback controller presents three tuning parameters: the amount of reflection to be cancelled at the boundary by the actuator, the gain of the integral action and the amount of boundary reflection cancelled in the observer. We give general conditions on these degrees of freedom that guarantee robustness. Provided that these conditions are satisfied, these tunable parameters enable various trade-offs between performance and robustness, e.g. between disturbance rejection and noise sensitivity.

Most physical systems involving a transport phenomenon can be modelled using hyperbolic partial differential equations (PDEs): heat exchangers [34], open channel flow [16], multiphase flow [19] or power systems [32]. The backstepping approach [13,22] has enabled the design of stabilizing full-state feedback laws for these systems. The generalization of these stabilization results for a large number of systems has been a focus point in the recent literature (details in [6,10,13,22]). The main objective of these controllers is to ensure convergence in the minimum achievable time (as defined in [11,27]), thereby omitting the robustness aspects that are known to be the major limitation for practical applications. It has been for instance observed (see [15,28]) that for many feedback systems, the introduction of arbitrarily small time delays in the loop may cause instability for any feedback. In particular, in [28], a systematic frequency domain treatment of this phenomenon for distributed parameter systems is presented. This has induced the notion of *delay-robust stabilization.* For linear first order hyperbolic PDEs, considering uncertainties in different parameters parameters, the notion of w-stability has been introduced in [14]. These robustness aspects have been the purpose of recent investigations: in presence of uncertainties in the system, the design of adaptive control laws using filter or swapping design is the purpose of [1,2]. Considering scalar linear hyperbolic systems, recent contributions [4] have stressed the necessity of a change of strat-

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egy to guarantee the existence of robustness margins for the closed loop system. In particular the authors have proved the necessity to preserve some reflection terms in the control law to ensure delay-robustness. This has been done by means of a *tuning parameter* introduced in the design of the control law. The robustness analysis has been done rewriting the hyperbolic system as a difference system, as these two classes of problems have been proved to be equivalent [7].
The control law proposed in [4] has been modified in [24], combining it with an integral action (to ensure the output regulation in presence of noise and disturbances) and a state-observer (to ensure the output-feedback stabilization). The resulting closed loop system has been proved to be Input-to-State Stable (ISS) and to ensure the stabilization of the output in presence of constant disturbances. Besides, the class of disturbances considered in [24], namely bounded signals, is more general than the one proposed in [17,18] in which the disturbance signal is generated by an exosystem of finite dimension, or than the smooth disturbances considered in [26,25]. However, the control law proposed in [24] only proves the ISS of the closed loop system without assessing any robustness of the closed loop system with respect to delays and uncertainties. This control law introduces three degrees of freedom but the choice of these tuning parameters and the underlying trade-offs still have to be qualitatively and quantitatively analyzed.

The main contribution of this article is to give a set of sufficient conditions for robustness on the degrees of freedom of the control law derived in [24]. These three tuning parameters are the amount of reflection to be cancelled by the actuator at the boundary, the gain of the integral action and the amount of boundary reflection cancelled in the observer. Importantly, we show that is necessary to consider uncertainties on the transport velocities, delay on the actuation and delay on the measurements simultaneously while tuning these parameters as it is the only way to ensure the existence of non-zero robustness margins. The introduced tuning parameters enable multiple trade-offs (performance-robustness, noise sensitivity-disturbance rejection) that are qualitatively analyzed on a toy example. Our approach is the following: considering the output-feedback control law proposed in [24] we prove, by means of backstepping transformations and using the characteristics method, that the resulting closed loop system can be transformed into a Neutral Differential System. Under some conditions on the introduced degrees of freedom, this later system is proved to be robust to delays and uncertainties. This is done using classical Laplace analysis techniques [20]. The ISS property can then be obtained, adjusting the techniques developed in [24].

The paper is organized as follows. In Section 2, we introduce the system under consideration (with delays, uncertainties or disturbances) and recall the results obtained in [24] in which a stabilizing output feedback law has been designed for the nominal system (i.e. the one without any uncertainty). This control law introduces three degree of freedom that can be tuned to ensure robust stabilization and potential robustness trade-offs. We then give some general definitions regarding the stability and robustness properties of the considered system. We conclude this section giving the main result of this paper that is a set general conditions on the tuning parameters to guarantee the robustness of the uncertain real system. As the robustness analysis requires technical and long computations, we introduce in Section 3 an operator framework to simplify to make the proof easier to follow. The robustness of the closed loop system with respect to uncertainties and delays in proved in Section 4. This is done combining backstepping transformations to rewrite the system as a Neutral Differential System, which is then proved to be exponentially stable using classical complex analysis techniques. Adjusting the techniques developed in [24] the ISS property of the feedback law is stated in Section 5. Finally, some simulations results on a toy problem are proposed in Section 6. These simulations highlight the different trade-offs that can be considered, using the previously introduced degrees of freedom. They constitutes a first step towards a quantitative analysis of these tuning parameters. Some concluding remarks are given in Section VII.

2 Problem under consideration and main results

2.1 Problem under consideration

In this paper, we consider the following uncertain linear hyperbolic system

$$\partial_t u(t,x) + \bar{\lambda}\partial_x u(t,x) = \bar{\sigma}^{+-}(x)v(t,x) + d_1(t,x), \quad (1)$$
$$\partial_t v(t,x) - \bar{\mu}\partial_x v(t,x) = \bar{\sigma}^{-+}(x)u(t,x) + d_2(t,x), \quad (2)$$

evolving in $\{(t,x)| \quad t>0,\ x\in[0,1]\}$, with the following linear boundary conditions

$$u(t,0) = \bar{q}v(t,0) + d_3(t), \quad (3)$$
$$v(t,1) = \bar{\rho}u(t,1) + (1+\delta_V)V(t-\delta_0) + d_4(t). \quad (4)$$

Such systems model, e.g. open channel flows of traffic flows [8]. The uncertain in-domain coupling terms $\bar{\sigma}^{+-}(x)$ and $\bar{\sigma}^{-+}(x)$ are assumed to belong to $\mathcal{C}([0,1],\mathbb{R})^2$. More precisely, they are defined by

$$\bar{\sigma}^{+-}(x) = \sigma^{+-}(x) + \delta_\sigma^{+-}(x), \quad (5)$$
$$\bar{\sigma}^{-+}(x) = \sigma^{-+}(x) + \delta_\sigma^{-+}(x), \quad (6)$$

where the functions $\sigma^{+-}(x)$ and $\sigma^{-+}(x)$ belong to $\mathcal{C}([0,1],\mathbb{R})^2$ and represent known in-domain coupling terms, while the continuous functions $\delta_\sigma^{+-}(x)$ and $\delta_\sigma^{-+}(x)$ represent uncertainties acting on these coupling terms. The velocities $\bar{\lambda}>0$ and $\bar{\mu}>0$ are defined as $\bar{\lambda}=\lambda+\delta_\lambda,\ \bar{\mu}=\mu+\delta_\mu$, where $\lambda>0$ and $\mu>0$ are known constant velocities, while the terms δ_λ and δ_μ represent constant uncertainties on these velocities. We assume that $\mu-|\delta_\mu| \leq \mu+|\delta_\mu| < 0 < \lambda-|\delta_\lambda| \leq \lambda+|\delta_\lambda|$. Note that the velocities are assumed to be constant, but the results of this paper can extended (with some technical adjustments) to spatially varying velocities (and spatially varying uncertainties). The boundary couplings $\bar{q} \neq 0$ (distal reflection) and $\bar{\rho}$ (proximal reflection) are defined as $\bar{q} = q + \delta_q$, $\bar{\rho} = \rho + \delta_\rho$, where q and ρ are known boundary couplings, while the terms δ_q and δ_ρ represent constant uncertainties on the distal and proximal reflections. We assume that $\bar{q} \neq 0$ and $|\bar{q}\bar{\rho}| < 1$ and $|\rho q| < 1$, in order to guarantee the existence of a delay-robust linear feedback control law (see [4,28]). The function V is an input function (control law) that has values in $\mathbb{R}$, while the term $\delta_V \neq -1$ is a constant uncertainty acting on the actuation. We assume that there is a delay, denoted δ_0, acting on the actuation. The functions d_1 and d_2 correspond to disturbances acting on the right-hand side of (1) and (2). The functions d_3 and d_4 correspond to disturbances acting on the right-hand side of (3) and (4), respectively. Finally, we consider the case of delayed noisy collocated measurements, i.e.

$$y_m(t) = u(t - \delta_1, 1) + n(t)\,, \tag{7}$$

where we have denoted δ_1 the delay acting on the measurements. The initial conditions denoted u_0 and v_0 are assumed to be bounded and therefore belong to $L^\infty([0,1])^2$. In the following, we define the characteristic time τ and $\bar{\tau}$ as

$$\tau = \frac{1}{\lambda} + \frac{1}{\mu},\ \bar{\tau} = \frac{1}{\bar{\lambda}} + \frac{1}{\bar{\mu}}. \tag{8}$$

We denote κ the maximal bound for the uncertainties:

$$\begin{aligned}\kappa = \max\{ &\max_{x\in[0,1]}(\delta_\sigma^{+-}(x)), \max_{x\in[0,1]}(\delta_\sigma^{-+}(x)), \delta_\lambda, \\ &\delta_\mu, \delta_\rho, \delta_q, \delta_V\}.\end{aligned} \tag{9}$$

We make the following assumption on the disturbances and on the noise.

Assumption 1 *The disturbances $d_1(\cdot,x), d_2(\cdot,x), d_3, d_4$, are in $W^{2,\infty}((0,\infty);\mathbb{R})$, the noise n is assumed to be in $L^\infty((0,\infty);\mathbb{R})$. We also have $d_1(t,\cdot)$ and $d_2(t,\cdot)$ that are in $\mathcal{C}([0,1];\mathbb{R}^+)$.*

With this assumption, using the method of characteristics and classical fixed point arguments we have the following result (see e.g. [9]).

Theorem 2 *[9] The open loop system (1)-(4) with bounded initial condition $(u_0, v_0)^\top$ admits a unique solution in $\mathcal{C}\left([0,\infty); L^\infty\left((0,1);\mathbb{R}^2\right) \cap L^1\left((0,1);\mathbb{R}^2\right)\right)$.*

2.2 Observer design

In this section, we consider the nominal system associated to (1)-(4), i.e. we assume $\kappa = 0$, $d_i(t) \equiv 0$, $n(t) \equiv 0$ and $\delta_i = 0$ (absence of uncertainties, disturbances, noise and delays). For such a nominal system, a stabilizing output feedback law has been designed in [24]. Let us consider the following observer (defined in [24])

$$\partial_t \hat{u} + \lambda \partial_x \hat{u} = \sigma^{+-}(x)\hat{v} - P^+(x)(\hat{u}(t,1) - y_m(t)), \tag{10}$$

$$\partial_t \hat{v} - \mu \partial_x \hat{v} = \sigma^{-+}(x)\hat{u} - P^-(x)(\hat{u}(t,1) - y_m(t)), \tag{11}$$

with the boundary conditions

$$\hat{u}(t,0) = q\hat{v}(t,0), \tag{12}$$
$$\hat{v}(t,1) = \rho(1-\epsilon)\hat{u}(t,1) + \rho\epsilon y_m(t) + V(t). \tag{13}$$

The gains $P^+(\cdot)$ and $P^-(\cdot)$ are defined as

$$P^+(x) = -\lambda P^{uu}(x,1) + \mu\rho(1-\epsilon)P^{uv}(x,1), \tag{14}$$
$$P^-(x) = -\lambda P^{vu}(x,1) + \mu\rho(1-\epsilon)P^{vv}(x,1), \tag{15}$$

where the kernels P^{uu}, P^{uv}, P^{vu}, and P^{vv} belong to $L^\infty(\mathcal{T}_u)$ (where $\mathcal{T}_u = \{(x,\xi) \in [0,1]^2 |\quad \xi \geq x\}$) and are defined in [24]. The initial conditions $\hat{u}_0$ and $\hat{v}_0$ are assumed to be bounded. The degree of freedom $\epsilon \in [0,1]$ that appears in (13) can be seen as a measure of trust in the measurements relative to the model where $\epsilon = 1$ results in relying more on the measurements and $\epsilon = 0$ relying more on the model. We now consider the output feedback law given in [24] and defined by

$$V(t) = V_{BS}(t) + k_I\eta(t) + k_I V_I(t), \tag{16}$$
$$\dot{\eta}(t) = y_m(t), \tag{17}$$

where the initial condition of η is denoted η_0 and where

$$\begin{aligned}V_{BS}(t) &= -\tilde{\rho}(1-\epsilon)\hat{u}(t,1) - \tilde{\rho}\epsilon y_m(t) \\ &- (\rho - \tilde{\rho})\int_0^1 (K^{uu}(1,\xi)\hat{u}(t,\xi) + K^{uv}(1,\xi)\hat{v}(t,\xi))d\xi \\ &+ \int_0^1 (K^{vu}(1,\xi)\hat{u}(t,\xi) + K^{vv}(1,\xi)\hat{v}(t,\xi))\, d\xi,\end{aligned} \tag{18}$$

$$\begin{aligned}V_I(t) &= -\int_0^1 l_1(\xi)(\hat{u}(t,\xi) - \int_0^\xi K^{uu}(\xi,\nu)\hat{u}(t,\nu)d\nu \\ &- \int_0^\xi K^{vu}(\xi,\nu)\hat{v}(t,\nu)d\nu)d\xi - \int_0^1 l_2(\xi)(\hat{v}(t,\xi) - \\ &\int_0^\xi K^{vu}(\xi,\nu)\hat{u}(t,\nu) + K^{vv}(\xi,\nu)\hat{v}(t,\nu)d\nu)d\xi.\end{aligned} \tag{19}$$

The kernels K^{uu}, K^{uv}, K^{vu}, K^{vv} belong to $L^\infty(\mathcal{T}_b)$ (where $\mathcal{T}_b = \{(x,\xi) \in [0,1]^2 |\quad \xi \leq x\}$) and are defined by a set of PDEs given in [33]. The corresponding inverse kernels are denoted $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$ and $L^{\beta\beta}$. They also belong to $L^\infty(\mathcal{T}_b)$ and are also defined by a set of PDEs given in [33]. The functions l_1 and l_2 are defined on the interval $[0,1]$ as the solutions of the system

$$\lambda l_1'(x) = L^{\alpha\alpha}(1,x), \quad \mu l_2'(x) = -L^{\alpha\beta}(1,x)\,, \tag{20}$$

with the boundary conditions

$$l_2(1) = 0\,, \quad l_1(0) = \frac{\mu}{q\lambda} l_2(0). \tag{21}$$

The control law V defined in (16) has three components: $V_{BS}(t)$, $V_I(t)$ and $k_I\eta$. The component V_{BS} corresponds to the control law derived in [4]. Its purpose is to *cancel the effect of the potentially destabilizing in-domain coupling terms.* It stabilizes the system (1)-(4) in the absence of uncertainties, disturbance, noise in the measurements or delays in the loop. Note that the purpose of the term $\tilde{\rho}\hat{u}(t,1)$ is to avoid a complete cancellation of the proximal reflection and thus to guarantee some delay-robustness [4]. The purpose of the integral term $k_I\eta$ is to *enable rejection of constant disturbance* [24]. Finally, the last term of the control law $(k_I V_I(t))$ is related to the presence of the integrator $k_I\eta$. More precisely, the term $k_I\eta$ which is used to enable disturbance rejection may have an effect on the stability of the system. This has to be *compensated* by the term $(k_I V_I(t))$ (see [24] for details). The stability of the closed loop system has been assessed in [24] under some conditions. The first one is given by the following assumption.

Assumption 3

$$1+\int_0^1 L^{\alpha\alpha}(1,\xi)d\xi+\frac{1}{q}\int_0^1 L^{\alpha\beta}(1,\xi)d\xi \neq 0\,. \qquad (22)$$

Unfortunately, no physical interpretation has been found for this assumption, which is necessary to guarantee the effect on the integral action. The second condition is a condition on the tuning parameter k_I and explicitly depends on Assumption 3.

Condition 4 *Let us define $k_1=(\rho-\tilde{\rho})q$ and $k_2=q(1+l_1(1)\lambda)$. We have assumed that $|k_1|<1$ and we impose $k_I k_2<0$. Moreover k_I is chosen such that*

$$\begin{aligned} |k_I| &< -\frac{\sqrt{1-k_1^2}}{|k_2|\,\tau}\arctan\left(\frac{\sqrt{1-k_1^2}}{|k_1|}\right) \\ &\quad+\frac{\pi\sqrt{1-k_1^2}}{|k_2|\,\tau}\,, \qquad \textit{if}\; k_1\in(-1,0)\,, && (23)\\ |k_I| &< \frac{\pi}{2\,|k_2|\,\tau}\,, \qquad \textit{if}\; k_1=0\,, && (24)\\ |k_I| &< \frac{\sqrt{1-k_1^2}}{|k_2|\,\tau}\arctan\left(\frac{\sqrt{1-k_1^2}}{k_1}\right),\; \textit{else}\,. && (25)\end{aligned}$$

This condition implies that the magnitude of k_I is directly related to the one of τ. Note that this condition is directly related to Assumption 3 (since $1+l_1(1)\lambda = 1+\int_0^1 L^{\alpha\alpha}(1,\xi)d\xi+\frac{1}{q}\int_0^1 L^{\alpha\beta}(1,\xi)d\xi$). Thus, Assumption 3 is required to guarantee $k_2\neq 0$ and the possibility to have an integral action. With this choice of k_I and $\tilde{\rho}$, the complex equation $s-(k_1 s+k_2 k_I)\mathrm{e}^{-s\tau}=0$ (where s denotes the Laplace variable) has all its solutions in the complex left-half plane [12]. We can finally write the following theorem that assesses the stability of the closed loop system (1)-(4) in the absence of uncertainties, disturbance, noise in the measurements or delays in the loop

Theorem 5 *Nominal Stabilization [24, Theorem 5]. Consider the nominal system composed of* (1)-(4) *(in the absence of uncertainties, disturbance, noise in the measurements or delays in the loop) and of the observer system* (10)-(13) *along with the control law* (16). *If Assumption 3, and Condition 4 are satisfied, then, for any bounded initial condition (u_0^{nom}, v_0^{nom}), for any bounded observer initial condition $(\hat{u}_0,\hat{v}_0)$, the solution $(u^{nom}, v^{nom}, \hat{u}, \hat{v})$ converges (in the sense of the L^∞-norm) to zero.*

2.3 Robustness aspects

A controller ensuring stability of a PDE system may exhibit poor closed-loop behavior and even instability in practice due to vanishing delay margins. For this reason, several concepts of robust stability have been introduced to ensure that the stability holds even in the presence of (possibly small) uncertainties on the delays. We recall here several definitions relevant to our control problem.

Definition 6 (Delay-robust stabilization [28]) *Consider a plant transfer function G and a feedback controller K such that GK is regular*[1] *and K stabilizes G. The closed-loop system is robustly stable with respect to delays if and only if there exists ϵ_0 such that, for all $\epsilon\in[0,\epsilon_0]$ the closed-loop transfer function in the presence of a delay ϵ in the actuation (i.e. $GK\left(I+\mathrm{e}^{-\epsilon s}GK\right)^{-1}$) is stable. Let us denote $H=GK$ and $\mathfrak{P}_H$ the (discrete) set of its poles in $\mathbb{C}^+=\{s\in\mathbb{C}\mid Re(s)>0\}$. Let us define $\gamma=\lim_{\substack{|s|\to\infty\\ s\in\mathbb{C}^+\setminus\mathfrak{P}_H}} Sp(H(s))$, where Sp stands for the spectral radius. If $\gamma<1$, then the closed-loop system is robustly stable with respect to delays. If $\gamma>1$, then the closed-loop system is not robustly stable with respect to delays.*

The more general concept of w-stability proves more useful in this context.

Definition 7 (w-stability [14]) *Consider a plant transfer function G and a feedback controller K such that GK is regular. The closed-loop system is* w-stable *if and only if for any approximate identity I_δ, the closed-loop transfer function $GK\left(I+I_\delta GK\right)^{-1}$ is stable. An approximate identity is a family of transfer functions I_δ such that*

(1) $\|I_\delta\|_\infty<1$, $I_0=I$;
(2) On every compact set of the open Right-Half Plane, I_δ converges to I when δ goes to zero.

Suppose that (G,K) is input-output stable. Then (G,K) is w-stable *if there exists a $\rho>0$ such that*

$$\lim_{\{s\in\,\mathbb{C}^+|\ |s|>\rho\}} ||G(s)K(s)||<1. \qquad (26)$$

[1] i.e. GK is bounded on some Right-Half plane has a limit at $+\infty$ along the real axis, see [28, Section 2] for details.

Approximate identities may include more general transfer functions than the ones stemming from uncertainties on the delays. Thus, w-stability implies delay-robust stability. Moreover it is easy to show that the condition $\gamma < 1$ implies (26). Hence the conditions for delay-robustness and w-stability are the same, possibly except for the case $\gamma = 1$. Uncertainties in the transport velocities result in additional terms in the closed-loop transfer functions that do not take the form of approximate identities. This is illustrated by the following example.

Example 1 *Consider the following linear hyperbolic system*

$$u_t + (1+\delta)u_x = 0, \quad v_t - v_x = 0 \tag{27}$$
$$u(0,t) = qv(0,t), \qquad v(1,t) = \rho u(1,t) + V(t) \tag{28}$$

where ρ and q are positive, $|\rho q| < 1$ and δ is the uncertainty on the transport velocity of u. Then, the transfer function between $v(1,\cdot)$ and V reads

$$v(1,s) = \underbrace{\frac{1-\rho q e^{-2s}}{1-\rho q e^{-\left(1+\frac{1}{1+\delta}\right)s}}}_{I_\delta} \frac{1}{1-\rho q e^{-2s}} V(s) \tag{29}$$

The function I_δ is not an approximate identity since $||I_\delta||_\infty = \frac{1+\rho q}{1-\rho q} > 1$.

It has been proved in [7] that linear hyperbolic PDEs system have equivalent stability properties to those of a specific class of neutral systems. For this class of system, the notion of strong stability has been considered in the literature [21,20,29,30].

Definition 8 (Strong Stability [20]) . *The following difference system*

$$x(t) = \sum_{k=1}^{n} x(t-\tau_k) + \int_0^{\tau_n} g(s)x(t-s)ds \tag{30}$$

is strongly stable if it is stable when subjected to small variations in the delays τ_k.

In what follows, we show strong stability of the closed-loop system, in the presence of additional uncertainties on the coupling coefficients of the PDE. Regarding the effect of the noise and disturbances, we show in this paper the Input-to-State Stability of the closed loop system in the sense of the following definition.

Definition 9 *Input-to-State Stability (ISS). The output of the closed loop system* (1)-(4) *along with the control law* (16) *and the observer* (10)-(13) *is ISS with respect to n and d_i, $i = 1, \ldots, 4$ if there exist a $\mathcal{KL}$ function h_1 and a $\mathcal{K}$ function h_2 such that for any bounded initial condition $(u_0, v_0, \hat{u}_0, \hat{v}_0)^\top$ and any measurable locally essentially bounded input $K(t)$ (that depends on $n(t)$ and $d_i(t)$), the following holds*

$$||(u,v)||_{L^2} \leq h_1\left((u_0, v_0, \hat{u}_0, \hat{v}_0)^\top, t\right)$$
$$+ h_2\left(\|K(t)\|_{L^\infty((0,t);\mathbb{R})}\right). \tag{31}$$

2.4 Main result

We now give the main result of the paper. Under some conditions on the **three degrees of freedom**: $\tilde{\rho}$, ϵ and k_I, the control law $V(t)$ defined by (16) ensures the robust-stabilization of the closed loop system (1)-(4) with the observer (10)-(13), in the absence of noise and disturbance. These degrees of freedom can then be tuned inside the obtained stability domain to allow multiple trade-offs. To conditions (23)-(25) on k_I, we add the following one on (ρ, ϵ).

Condition 10 *Let us consider three integers i, j, k and denote E^k_{ij} the square matrix of dimension k whose all components are equal to zero except the one located at the intersection of the i^{th} row and the j^{th} column. Let us consider the following matrices*

$$A_1 = \rho q E^4_{11} - \tilde{\rho} q(1-\epsilon)E^4_{12} + (q\epsilon(\rho-\tilde{\rho}) + q\tilde{\rho})E^4_{31} - \tilde{\rho}q(1-\epsilon)E^4_{32}, \tag{32}$$
$$A_2 = \tilde{\rho}q(1-\epsilon)E^4_{14} + (\rho-\tilde{\rho})q(1-\epsilon)E^4_{33} + \tilde{\rho}q(1-\epsilon)E^4_{34}, \tag{33}$$
$$A_3 = E^4_{21} + E^4_{43}, \tag{34}$$
$$A_4 = -\tilde{\rho}q\epsilon E^4_{12} + (\tilde{\rho}-\rho)q\epsilon E^4_{31} - \tilde{\rho}q\epsilon E^4_{32}. \tag{35}$$

The parameters $\tilde{\rho}$ and ϵ are chosen such that

$$\sup_{\theta_k \in [0,2\pi]^4} Sp\left(\sum_{k=1}^{4} A_k \exp(i\theta_k)\right) < 1, \tag{36}$$

where Sp stands for the spectral radius.

Note that since $|\rho q| < 1$, Condition 10 is always satisfied for e.g. $\epsilon = 1$ and $\tilde{\rho} = 0$. We have the following theorem

Theorem 11 *Suppose that Assumption 3, and Conditions 4 and 10 are satisfied. There exist $\delta_{marg} > 0$ and $\kappa_0 > 0$ such that if $\delta_0 < \delta_{marg}$, $\delta_1 < \delta_{marg}$ and $\kappa < \kappa_0$ and if $d_1 \equiv d_2 \equiv d_3 \equiv d_4 \equiv n \equiv 0$, then, the closed loop system* (1)-(4) *along with the control law* (16) *and the observer* (10)-(13) *is exponentially stable.*

This theorem assesses the robustness of the closed loop system with respect to delays and uncertainties on the parameters. Its proof is based on a rewriting of the closed loop system as a neutral system and analyzing the root location for the associated characteristic equation. This theorem is stronger (and consequently more restrictive in its requirements) than [14, Theorem 9.5.4]. It implies w-stability as we will have $||G(s)K(s)|| < 1$. Regarding the effect of noise and disturbances, we have the following theorem.

Theorem 12 *Suppose that Assumption* (3)*, and Conditions* (4) *and* (10) *are satisfied. There exist* $\delta_{marg} > 0$ *and* $\kappa_0 > 0$ *such that if* $\delta_0 < \delta_{marg}$*,* $\delta_1 < \delta_{marg}$ *and* $\kappa < \kappa_0$ *then, the output of the closed loop system* (1)-(4) *along with the control law* (16) *and the observer* (10)-(13) *is ISS. Moreover, for any bounded initial conditions* $(u_0, v_0, \hat{u}_0, \hat{v}_0, \eta_0)$*, there exists a positive constant* M *such that the controlled output* $y(t)$ *satisfies*

$$|y(t)| \leq M. \tag{37}$$

Furthermore, if $\dot{d}_1(t) = \dot{d}_2(t) = \dot{d}_3(t,\cdot) = \dot{d}_4(t,\cdot) = n(t) = 0$*, then the controlled output satisfies*

$$\lim_{t\to\infty} |y(t)| = 0. \tag{38}$$

3 Operator framework and preliminary results

In this section, we introduce some important preliminary results that make the proof of Theorem 11 simpler. We recall that we denote $\mathcal{C}([a,b],\mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of the uniform convergence. For a function $\phi : [-\tau,\infty) \to \mathbb{R}$, we define its partial trajectory $\phi_{[t]} \in L^2([-\tau,0],\mathbb{R})$ by

$$\phi_{[t]}(\theta) \dot{=} \phi(t+\theta), \ -\tau \leq \theta \leq 0. \tag{39}$$

Consider a strictly positive integer p, two collections of strictly positive constants $\mathfrak{R} : (\tau_1, \cdots, \tau_p)$ and $\mathfrak{E} : (\epsilon_1, \cdots, \epsilon_p)$, and a collection of non-negative constants $\mathfrak{U} : (\mathfrak{u}_1, \cdots, \mathfrak{u}_p)$. We define $\tau_{\max} = \max_{1\leq i\leq p}(\tau_i)$. The sequence $\mathfrak{E}$ represents a sequence of delays (namely δ_0 and δ_1). The sequence $\mathfrak{R}$ represents a sequence of transport times that appear in the robustness analysis. They are linear combinations of the characteristic transport times of the system (1)-(4) and (10)-(13) and of the delays δ_0 and δ_1. Finally, the sequence $\mathfrak{U}$ represents a sequence of small uncertainties. In order to have the same number of elements in every collection $\mathfrak{R}$, $\mathfrak{E}$ and $\mathfrak{U}$, some elements can be repeated. We assume that all the elements of $\mathfrak{E}$ and $\mathfrak{U}$ can be considered as small as wanted if κ (defined by (9)) and $\max(\delta_0, \delta_1)$ tend to zero. In the presence of delays and uncertainties, the closed loop system features operators with specific properties. We classify these operators in the following three categories.

Definition 13 *An operator* $\mathcal{I}$ *belongs to* $\mathfrak{I}$ *if there exists a real* $\tau_{\mathcal{I}} \in \mathfrak{R}$*, a compact support function* $f_{\mathcal{I}} \in L^1([0,\tau_{max}])$ *whose support is* $[0,\tau_{\mathcal{I}}]$ *such that*

$$\begin{aligned}\mathcal{I} : L^2([-\tau_{max},0],\mathbb{R}) &\to \mathbb{R}\\ \phi_t &\mapsto \int_{-\tau_{max}}^{0} f_{\mathcal{I}}(-\nu)\phi_t(\nu)d\nu.\end{aligned} \tag{40}$$

For all $n \in \mathbb{N}^*$*, we denote* $\mathfrak{M}_n(\mathfrak{I})$ *the set of square matrix operators such that for any* $\mathcal{M} \in \mathfrak{M}_n(\mathfrak{I})$*, for all* $(i,j) \in [1,n]^2$*,* $\mathcal{M}_{i,j} \in \mathfrak{I}$*.*

The class $\mathfrak{I}$ corresponds to integral terms appearing through backstepping transformations and posing no threat to delay-robustness.

Definition 14 *An operator* $\mathcal{D}$ *belongs to* $\mathfrak{D}$ *if there exists a real* $\epsilon_{\mathcal{D}} \in \mathfrak{E}$ *such that*

$$\begin{aligned}\mathcal{D} : L^2([-\tau_{max},0],\mathbb{R}) &\to \mathbb{R}\\ \phi_t &\mapsto \phi_t(-\epsilon_{\mathcal{D}}).\end{aligned} \tag{41}$$

The class $\mathfrak{D}$ corresponds to delay operators appearing due to the delays in the measurements and in the actuation.

Definition 15 *An operator* $\mathcal{W}$ *belongs to* $\mathfrak{W}$ *if one of the three following conditions is satisfied*

(1) there exist $\mathcal{I} \in \mathfrak{I}$ *and* $\mathcal{D} \in \mathfrak{D}$ *such that* $\mathcal{I}\circ\mathcal{D} \in \mathcal{I}$ *and*

$$\begin{aligned}\mathcal{W} : L^2([-\tau_{max},0],\mathbb{R}) &\to \mathbb{R}\\ \phi_t &\mapsto \mathcal{I}(\phi_t - \mathcal{D}(\phi_t)),\end{aligned} \tag{42}$$

(2) there exist $\mathcal{D} \in \mathfrak{D}$ *and* $\mathfrak{u} \in \mathfrak{U}$ *such that*

$$\begin{aligned}\mathcal{W} : L^2([-\tau_{max},0],\mathbb{R}) &\to \mathbb{R}\\ \phi_t &\mapsto \mathfrak{u}\mathcal{D}(\phi_t),\end{aligned} \tag{43}$$

(3) there exist $\mathcal{I} \in \mathfrak{I}$*,and* $\mathfrak{u} \in \mathfrak{U}$ *such that*

$$\begin{aligned}\mathcal{W} : L^2([-\tau_{max},0],\mathbb{R}) &\to \mathbb{R}\\ \phi_t &\mapsto \mathfrak{u}\mathcal{I}(\phi_t).\end{aligned} \tag{44}$$

For all $n \in \mathbb{N}^*$*, we denote* $\mathfrak{M}_n(\mathfrak{W})$ *the set of square matrix operators such that for any* $\mathcal{M} \in \mathfrak{M}_n(\mathfrak{W})$*, for all* $(i,j) \in [1,n]^2$*,* $\mathcal{M}_{i,j} \in \mathfrak{W}$*.*

The class $\mathfrak{W}$ corresponds in the first case to the difference between integral terms and the same delayed terms. In the second case, it corresponds to delayed terms multiplied by arbitrarily small terms; while in the third case are considered integral delayed terms multiplied by an arbitrarily small term. These terms naturally appear in the computations while using the method of characteristics. These terms do not pose any threat for delay-robustness provided the delays and uncertainties are small enough. In what follows we denote $\hat{\mathcal{I}}$ the Laplace transform of the operator $\mathcal{I}$ (provided it is well-defined).

Lemma 16 *Consider a positive constant* $\eta > 0$*. There exist* $\epsilon_0 > 0$ *and* $\kappa_0 > 0$ *such that if for all* $i \in [1,p]$*,* $\epsilon_i < \epsilon_0$ *and* $\mathfrak{u}_i < \kappa_0$*, then for any* $\mathcal{W} \in \mathfrak{W}$*, its Laplace transform* $\hat{\mathcal{W}}$ *satisfies* $|\hat{\mathcal{W}}(s)| < \eta$ *for all* $s \in \mathbb{C}^+$*.*

PROOF. If the operator $\mathcal{W}$ satisfies (42), the proof is a consequence of Riemann-Lebesgues lemma (see e.g. [31, p.103] or [14, Property A.6.2, p.636]). Otherwise, it is a consequence of the boundedness of the operators.

The following theorem proves that the operators that belong to $\mathfrak{M}_n(\mathfrak{W})$ do not have any major impact on the stability properties, assuming that the ϵ_i and $\mathfrak{u}_i$ can be chosen as small as we want.

Theorem 17 *Consider $n \in \mathbb{N}^*$ and an operator $\mathcal{F} : L^2([-\tau_{max}, 0], \mathbb{R}^n) \to \mathfrak{M}_{n,n}(\mathbb{R})$ such that there exists p matrices $A_i \in (\mathfrak{M}_{n,n}(\mathbb{R}))^p$ and $\mathcal{I} \in \mathfrak{M}_n(\mathfrak{I})$, such that for all $\phi \in L^2([-\tau_{max}, 0], \mathbb{R}^n)$*

$$\mathcal{F}(\phi_t) = \phi_t - \sum_{i=1}^{p} A_i \phi_t(-\tau_i) - \mathcal{I}(\phi_t). \tag{45}$$

Consider an operator $\mathcal{W} \in \mathfrak{M}_n(\mathfrak{W})$. If the semigroup associated to the operator $\mathcal{F}$ is exponentially stable, then there exists $\epsilon_0 > 0$ and $\kappa_0 > 0$ such that the semigroup associated to the operator $\mathcal{F} + \mathcal{W}$ is exponentially stable for all $i \in [1, p]$, $\epsilon_i < \epsilon_0$ and $\mathfrak{u}_i < \kappa_0$.

PROOF. Let us denote $\hat{\mathcal{F}}(s)$ (resp. $\hat{\mathcal{W}}(s)$) the Laplace transform of the operator $\mathcal{F}$ (resp $\mathcal{W}$) (see [21]). The proof can be done by contradiction, adjusting the proof of [5, Theorem 2].

The following theorem guarantees that systems with a strongly unstable principal part are necessarily unstable.

Theorem 18 *Consider $n \in \mathbb{N}^*$ and a differential delay matrix operator $\mathcal{F} : L^2([-\tau_{max}, 0], \mathbb{R}^n) \to \mathfrak{M}_{n,n}(\mathbb{R})$ such that there exists p matrices $A_i \in (\mathfrak{M}_{n,n}(\mathbb{R}))^p$ such that for all $\phi \in L^2([-\tau_{max}, 0], \mathbb{R}^n)$*

$$\mathcal{F}(\phi_t) = \phi_t - \sum_{i=1}^{p} A_i \phi_t(-\tau_i). \tag{46}$$

If the characteristic equation associated to the operator $\mathcal{F}$ has a non finite number of zeros in the open-right half plane, then, for any set of $\epsilon_i > 0$ and $\mathfrak{u}_i > 0$, for any $\mathcal{I} \in \mathfrak{M}_n(\mathfrak{I})$ and $\mathcal{W} \in \mathfrak{M}_n(\mathfrak{W})$, the operator $\mathcal{F}+\mathcal{I}+\mathcal{W}$ generates an unstable semigroup.

PROOF. Let us consider an arbitrary set of strictly positive coefficients $\mathfrak{E}$ and positive coefficients $\mathfrak{U}$. Consider $\mathcal{I} \in \mathfrak{M}_q(\mathfrak{I})$ and $\mathcal{W} \in \mathfrak{M}_q(\mathfrak{W})$. The operator $\mathcal{W}$ can be rewritten as $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ where the components of $\mathcal{W}_1$ are defined either by (42) or (44) and the components of $\mathcal{W}_2$ are defined by (43). Due to [21, Theorem 6.1], the operator $\mathcal{F} + \mathcal{W}_2$ has an infinite number of zeros in the right half-plane. Using Riemann-Lebesgues' lemma we have that the holomorphic function $|\hat{\mathcal{I}}(s) + \hat{\mathcal{W}}_1(s)|$ converges to 0 for $|s|$ large enough (with $\Re(s) \geq 0$). It implies (see [4, Lemma 3] for details) that the characteristic equation associated to the operator $\mathcal{F} + \mathcal{I} + \mathcal{W}$ has an infinite number of zeros in the right half-plane and consequently generates an unstable semigroup.

Note that due to the vector space structure of $\mathfrak{M}_n(\mathfrak{W})$, similar results hold if the operator $\mathcal{W}$ is replaced by a linear combination of operators that belong to $\mathfrak{M}_n(\mathfrak{W})$. In the next section, to prove Theorem 11, we express the observer-controller equations as delay-differential equations (with potential integral terms). During the derivations, multiple terms that belong to $\mathfrak{M}_n(\mathfrak{W})$ appear. Since they can be neglected for the stability analysis, for the sake of simplicity and brevity, every time one of this term appears we write it as $\mathcal{O}(X_t)$ (where X_t is related to the state of the system). In other words, all the terms included in $\mathcal{O}(X_t)$ are terms that do not have any influence on the stability if the delays and uncertainties are small enough. This approach is consistent with the one proposed in [14, Chapter 9]. For convenience, the Laplace transform of such terms is denoted $\hat{\mathcal{O}}(X_t)(s)$.

4 Robustness aspects: proof of Theorem 11

In this section we prove Theorem 11. In what follows we assume that $d_i \equiv 0$ and $n(t) \equiv 0$. Using the backstepping method, we first rewrite the closed loop (1)-(4) along with the observer (10)-(13) and the control law (16) in a simpler set of coordinates in which the in-domain coupling terms have been removed. In this new set of coordinates, it becomes possible to rewrite the corresponding PDEs as delay differential equations. We finally analyse the stability properties of this resulting neutral system. In all this section we consider that Assumption 3, and Conditions 4 and 10 are satisfied. For sake of clarity, most of the proofs are given in Appendix. The global strategy of the proof is summarized as follows

(1) Using backstepping transformations, we transform the original and error systems (u, v) and $(\tilde{u} = u - \hat{u}, \tilde{v} = v - \hat{v})$ into simpler target systems (α, β), $(\tilde{\alpha}, \tilde{\beta})$ from which the in-domain coupling terms have been removed (Sections 4.1, 4.2).
(2) We can then express the state $(\beta(t, 1), \tilde{\beta}(t, 1), V(t))$ as the solution of a delay differential system (Section 4.3).
(3) By means of an analysis of the associated characteristic equation, we prove that the conditions given in Theorem 11 on the tuning parameters guarantee the exponential stability of $\beta(t, 1)$ (Section 4.4).
(4) Using the invertbility of the backstepping transformations, it implies the exponential stability of the original system (u, v).

4.1 Backstepping transformation of the original system (1)-(4)

Consider system (1)-(4) and the invertible Volterra change of coordinates

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \begin{pmatrix} \alpha(t,x) \\ \beta(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} \bar{L}^{\alpha\alpha}(x,\xi) & \bar{L}^{\beta\alpha}(x,\xi) \\ \bar{L}^{\alpha\beta}(x,\xi) & \bar{L}^{\beta\beta}(x,\xi) \end{pmatrix} \begin{pmatrix} \alpha(t,\xi) \\ \beta(t,\xi) \end{pmatrix} d\xi, \tag{47}$$

where the kernels $\bar{L}^{\alpha\alpha}$, $\bar{L}^{\alpha\beta}$, $\bar{L}^{\beta\alpha}$ and $\bar{L}^{\beta\beta}$ belong to $L^\infty(\mathcal{T}_b)$ and are defined a set of PDEs analogous to the ones that satisfied the kernels $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$ and $L^{\beta\beta}$ which can be found in [33] (changing the nominal parameters by the uncertain ones). The dynamics of the system (1)-(4) in the new coordinates are given by

$$\partial_t\alpha(t,x) + \bar{\lambda}\partial_x\alpha(t,x) = 0, \quad (48)$$
$$\partial_t\beta(t,x) - \bar{\mu}\partial_x\beta(t,x) = 0, \quad (49)$$

with the following linear boundary conditions

$$\alpha(t,0) = \bar{q}\beta(t,0), \quad (50)$$
$$\beta(t,1) = \bar{\rho}\alpha(t,1) + (1+\delta_V)V(t-\delta_0) - \int_0^1 \bar{N}^\alpha(\xi)\alpha(t,\xi) + \bar{N}^\beta(\xi)\beta(t,\xi)d\xi, \quad (51)$$

with $\bar{N}^\alpha(\xi) = \bar{L}^{\beta\alpha}(1,\xi) - \bar{\rho}\bar{L}^{\alpha\alpha}(1,\xi)$ and $\bar{N}^\beta(\xi) = \bar{L}^{\beta\beta}(1,\xi) - \bar{\rho}\bar{L}^{\alpha\beta}(1,\xi)$. Moreover, we have

$$\dot{\eta} = \alpha(t-\delta_1,1) + \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\alpha(t-\delta_1,\xi)d\xi + \int_0^1 \bar{L}^{\alpha\beta}(1,\xi)\beta(t-\delta_1,\xi)d\xi. \quad (52)$$

Due to the cascade structure of system (48)-(51) and due to the invertibility of the Volterra transformation (47), if $\beta(t,1)$ exponentially converges to zero then it implies that the system (1)-(4) is exponentially stable.

The transformation (47) is invertible and the inverse transformation can be expressed as

$$\begin{pmatrix}\alpha(t,\xi)\\ \beta(t,\xi)\end{pmatrix} = \begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} - \int_0^x \begin{pmatrix}\bar{K}^{uu}(x,\xi) & \bar{K}^{uv}(x,\xi)\\ \bar{K}^{vu}(x,\xi) & \bar{K}^{vv}(x,\xi)\end{pmatrix}\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} d\xi, \quad (53)$$

where the kernels $\bar{K}^{uu}, \bar{K}^{vu}, \bar{K}^{uv}, \bar{K}^{vv}$ belongs to $L^\infty(\mathcal{T}_b)$ and satisfy a set of PDEs analogous to the ones satisfied by the kernels $K^{uu}, K^{vu}, K^{uv}, K^{vv}$ and which are given in [33]. Due to the continuity of the kernels with respect to the kernel PDEs parameters, the kernels $\bar{K}^{uu}, \bar{K}^{vu}, \bar{K}^{uv}$, and $\bar{K}^{vv}$ respectively converge to K^{uu}, K^{vu}, K^{uv}, and K^{vv} and the kernels $\bar{L}^{\alpha\alpha}, \bar{L}^{\beta\alpha}, \bar{L}^{\alpha\beta}$, and $\bar{L}^{\beta\beta}$ respectively converge to $L^{\alpha\alpha}, L^{\beta\alpha}, L^{\alpha\beta}$, and $L^{\beta\beta}$ when κ goes to zero. Finally, we define the function $\bar{l}_1$ and $\bar{l}_2$ on the interval $[0,1]$ as the solution of the system

$$\bar{\lambda}\bar{l}_1'(x) = \bar{L}^{\alpha\alpha}(1,x), \quad \bar{\mu}\bar{l}_2'(x) = -\bar{L}^{\alpha\beta}(1,x), \quad (54)$$

with the boundary conditions

$$\bar{l}_2(1) = 0, \quad \bar{l}_1(0) = \frac{\bar{\mu}}{\bar{q}\bar{\lambda}}\bar{l}_2(0). \quad (55)$$

4.2 Backstepping transformation of the error system

Combining the observer (10)-(13) and the real system (1)-(4) yields the following error system (denoting $\tilde{u}(t,x) = u(t,x) - \hat{u}(t,x)$ and $\tilde{v}(t,x) = v(t,x) - \hat{v}(t,x)$):

$$\begin{aligned}\partial_t\tilde{u}(t,x) + \lambda\partial_x\tilde{u}(t,x) &= \sigma^{+-}(x)\tilde{v}(t,x) - P^+(x)\tilde{u}(t,1)\\ &\quad - \delta_\lambda\partial_x u(t,x) + \delta_{\sigma^{+-}}(x)v(t,x)\\ &\quad + P^+(x)(u(t,1) - u(t-\delta_1,1)), \quad (56)\\ \partial_t\tilde{v}(t,x) - \mu\partial_x\tilde{v}(t,x) &= \sigma^{-+}(x)\tilde{u}(t,x) - P^-(x)\tilde{u}(t,1)\\ &\quad + \delta_\mu\partial_x v(t,x) + \delta_{\sigma^{-+}}(x)u(t,x)\\ &\quad + P^-(x)(u(t,1) - u(t-\delta_1,1)), \quad (57)\end{aligned}$$

with the boundary conditions

$$\tilde{u}(t,0) = q\tilde{v}(t,0) + \delta_q v(t,0), \quad (58)$$
$$\tilde{v}(t,1) = \rho(1-\epsilon)\tilde{u}(t,1) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) + (1+\delta_V)V(t-\delta_0) - V(t). \quad (59)$$

Once again, the objective is to find a suitable transformation to remove the in-domain couplings $\sigma^{+-}(x)\tilde{v}(t,x)$ and $\sigma^{-+}(x)\tilde{u}(t,x)$. Let us consider the transformation defined by

$$\tilde{\alpha}(t,x) = \tilde{u}(t,x) + \int_x^1 R^{\alpha\alpha}\tilde{u}(t,\xi) + R^{\beta\alpha}\tilde{v}(t,\xi)d\xi, \quad (60)$$
$$\tilde{\beta}(t,x) = \tilde{v}(t,x) + \int_x^1 R^{\beta\alpha}\tilde{u}(t,\xi) + R^{\beta\beta}\tilde{v}(t,\xi)d\xi, \quad (61)$$

where the kernels $R^{\alpha\alpha}, R^{\alpha\beta}, R^{\beta\alpha}$ and $R^{\beta\beta}$ are L^∞ functions defined on $\mathcal{T}_t$ that are the inverse kernels of the kernels $P^{\cdot\cdot}$ defined in (14)-(15). We have the following lemma whose proof is done in Appendix A.

Lemma 19 *There exist two L^∞ functions f and g such that the states $\tilde{\alpha}$ and $\tilde{\beta}$ defined by (60)-(61) satisfy the following set of PDEs*

$$\begin{aligned}&\partial_t\tilde{\alpha}(t,x) + \lambda\partial_x\tilde{\alpha}(t,x) = \delta_{\sigma^{+-}}(x)v(t,x) - f(x)(u(t,1)\\ &- u(t-\delta_1,1)) + \mu R^{\alpha\beta}(x,1)\delta_\rho u(t,1) - \delta_\lambda\partial_x u(t,x)\\ &+ \int_x^1 R^{\alpha\beta}(x,\xi)(\delta_{\sigma^{-+}}(\xi)u(t,\xi) + \delta_\mu\partial_x v(t,\xi))d\xi\\ &+ \mu R^{\alpha\beta}(x,1)((1+\delta_V)V(t-\delta_0) - V(t))\\ &+ \int_x^1 R^{\alpha\alpha}(x,\xi)(\delta_{\sigma^{+-}}(\xi)v(t,\xi) - \delta_\lambda\partial_x u(t,\xi))d\xi, \quad (62)\\ &\partial_t\tilde{\beta}(t,x) - \mu\partial_x\tilde{\beta}(t,x) = \delta_{\sigma^{-+}}(x)u(t,x) - g(x)(u(t,1)\\ &- u(t-\delta_1,1)) + \mu R^{\alpha\beta}(x,1)\delta_\rho u(t,1) + \delta_\mu\partial_x v(t,x)\\ &+ \int_x^1 R^{\beta\beta}(x,\xi)(\delta_{\sigma^{-+}}(\xi)u(t,\xi) + \delta_\mu\partial_x v(t,\xi))d\xi\\ &+ \mu R^{\beta\beta}(x,1)((1+\delta_V)V(t-\delta_0) - V(t))\\ &+ \int_x^1 R^{\beta\alpha}(x,\xi)(\delta_{\sigma^{+-}}(\xi)v(t,\xi) - \delta_\lambda\partial_x u(t,\xi))d\xi, \quad (63)\end{aligned}$$

along with the boundary conditions

$$\tilde{\alpha}(t,0) = q\tilde{\beta}(t,0) + \delta_q \beta(t,0), \quad (64)$$

$$\tilde{\beta}(t,1) = \rho(1-\epsilon)\tilde{\alpha}(t,1) + \rho\epsilon(u(t,1) - u(t-\delta_1,1)) + \delta_\rho u(t,1) + (1+\delta_V)V(t-\delta_0) - V(t). \quad (65)$$

4.3 Neutral delay-differential system

We are now able to express the two states $\beta(t,1)$ and $\tilde{\beta}(t,1)$ as the solutions of a neutral delay-differential system. We define the extended state $X(t)$ as

$$X(t) = \begin{pmatrix}\beta(t,1) & \tilde{\beta}(t,1) & V(t)\end{pmatrix}^T. \quad (66)$$

We define the collections $\mathfrak{R}$, $\mathfrak{E}$ and $\mathfrak{U}$ as

$$\mathfrak{R} :(\tau, \bar{\tau}, \frac{1}{\lambda} + \frac{1}{\bar{\mu}}, \tau+\delta_0, \bar{\tau}+\delta_0, \bar{\tau}+\delta_1, \bar{\tau}+\delta_0+\delta_1, \frac{1}{\lambda}+\frac{1}{\bar{\mu}}+\delta_0),\ \mathfrak{E} : (0, \delta_0, \delta_1),$$
$$\mathfrak{U} :(\delta_\lambda, \delta_\mu, \delta_q, \delta_\rho, \delta_V, \max_{x\in[0,1]}(\delta_\sigma^{+-}(x)), \max_{x\in[0,1]}(\delta_\sigma^{-+}(x))).$$

In what follows, we heavily rely on the definitions of Section 3 to ease the notations, grouping terms that correspond to operators in $\mathfrak{I}, \mathfrak{D}$ or $\mathfrak{W}$ and explicitly retaining only the terms that are critical for the robustness analysis. The strategy is the following

(1) Using equations (48)-(51), we start expressing $\beta(t,1)$ as the solution of a neutral equation whose terms only depend on the state X_t.
(2) Using the observer equations (62)-(65), we express the state $\tilde{\beta}(t,1)$ as the solution of a neutral equation that depends on the state X_t.
(3) Finally, we can simplify the expression of the control law (16) and express it as a function of X_t.
(4) We obtain a neutral system satisfied by X_t whose stability analysis is easier. This is done in Section 4.4 by means of an analysis of the associated characteristic equation.

4.3.1 Expression of the state $\beta(t,1)$

Considering equations (48)-(51), using the method of characteristics, they simply rewrite for any $t \geq \bar{\tau}+\delta_0+\delta_1$ as

$$\beta(t,1) = \bar{q}\bar{\rho}\beta(t-\bar{\tau},1) - \int_0^{\bar{\tau}} \tilde{N}(\xi)\beta(t-\xi,1)d\xi + (1+\delta_V)V(t-\delta_0), \quad (67)$$

where $\tilde{N}$ is defined by

$$\tilde{N}(\xi) = \begin{cases} \bar{\mu}\bar{N}^\beta(1-\bar{\mu}\xi) & \text{for} \quad \xi \in [0, \frac{1}{\bar{\mu}}) \\ \bar{\lambda}\bar{q}\bar{N}^\alpha(\bar{\lambda}\xi - \frac{\bar{\lambda}}{\bar{\mu}}) & \text{for} \quad \xi \in (\frac{1}{\bar{\mu}}, \bar{\tau}]. \end{cases},$$

4.3.2 Expression of the observer state $\tilde{\beta}(t,1)$

Let us consider equations (62)-(65). We have the following lemma, whose proof is given in Appendix B

Lemma 20 *For any $t \geq \tau + \delta_0 + \delta_1$*

$$\tilde{\beta}(t,1) = \rho q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \rho q\epsilon(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + V(t-\delta_0) - V(t) + \mathcal{O}(X_t). \quad (68)$$

4.3.3 Expression of the control law $V(t)$

We now express the control law $V(t) = V_{BS}(t) + k_I V_I(t) + k_I\eta(t)$ defined in (16) as a function of X_t. Regarding equation (52), there are some integral terms in the expression of $\dot{\eta}$. We choose to incorporate the term $-k_I\int_0^1 \bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi)d\xi$ into the integral action. To do so, let us consider the invertible transformation

$$\gamma(t) = \bar{\eta}(t) - \int_0^1 (\bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi)\beta(t,\xi))d\xi. \quad (69)$$

We then have the following lemma, whose proof is given in Appendix C.

Lemma 21 *There exists a Lipschitz function $\tilde{F}$ such that the control law $V(t)$ rewrites*

$$V(t) = \tilde{\rho}\epsilon q(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\tilde{\beta}(t-\tau,1) - \tilde{\rho}q\beta(t-\bar{\tau},1) + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu + \int_0^{\bar{\tau}} \tilde{N}(\xi)\beta(t-\xi,1) + k_I\gamma(t) + \mathcal{O}(X_t). \quad (70)$$

where γ satisfies

$$\dot{\gamma}(t) = q(\beta(t-\bar{\tau}-\delta_1,1) + \lambda l_1(1)\beta(t-\bar{\tau},1)) + \mathcal{O}(\beta(\cdot,1)_t). \quad (71)$$

4.3.4 Neutral system

Using the previous computations and simplifications we are now able to give the neutral system satisfied by X_t. Injecting (70) inside (67), (68), we obtain the following system

$$\beta(t,1) = \rho q\beta(t-\bar{\tau},1) - \tilde{\rho}q(1-\epsilon)\beta(t-\bar{\tau}-\delta_0,1) - \tilde{\rho}q\epsilon\beta(t-\bar{\tau}-\delta_0-\delta_1,1) + k_I\gamma(t-\delta_0) + \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau-\delta_0,1) + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu + \mathcal{O}(X_t), \quad (72)$$

$$\tilde{\beta}(t,1) = (\rho-\tilde{\rho})q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \tilde{\rho}q(1-\epsilon)\tilde{\beta}(t-\tau-\delta_0,1) + q((\rho-\tilde{\rho})\epsilon+\tilde{\rho})\beta(t-\bar{\tau},1) + (\tilde{\rho}-\rho)q\epsilon\beta(t-\bar{\tau}$$

$$- \delta_1, 1) - (1-\epsilon)\tilde{\rho} q \beta(t - \bar{\tau} - \delta_0, 1) + k_I \gamma(t - \delta_0)$$
$$- k_I \gamma(t) - \tilde{\rho}\epsilon q \beta(t - \bar{\tau} - \delta_0 - \delta_1, 1) + \mathcal{O}(X_t), \qquad (73)$$

where γ satisfies (71). The equation satisfied by $V(t)$ is given in (C.3) and not rewritten here. As equations (72)-(73) require the expression of $\gamma(t)$ and since only its derivative is available, we choose to differentiate (72)-(73) with respect to time.

$$\begin{aligned}
&\dot{\beta}(t,1) = \rho q \dot{\beta}(t-\bar{\tau},1) - \tilde{\rho} q(1-\epsilon)\dot{\beta}(t-\bar{\tau}-\delta_0,1) \\
&\quad - \tilde{\rho} q\epsilon \dot{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) + k_I q \beta(t-\bar{\tau}-\delta_1-\delta_0,1) \\
&\quad + \tilde{\rho} q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau-\delta_0) + k_I q\lambda l_1(1)\beta(t-\bar{\tau}-\delta_1) + \\
&\int_0^\tau \tilde{F}(\nu)\dot{\tilde{\beta}}(t-\nu-\delta_0,1)d\nu + \mathcal{O}(\dot{X}_t) + \mathcal{O}(X_t), \qquad (74) \\
&\dot{\tilde{\beta}}(t,1) = (\rho-\tilde{\rho})q(1-\epsilon)\dot{\tilde{\beta}}(t-\tau,1) + \tilde{\rho} q(1-\epsilon)\dot{\tilde{\beta}}(t- \\
&\tau-\delta_0,1) + q((\rho-\tilde{\rho})\epsilon+\tilde{\rho})\dot{\beta}(t-\bar{\tau},1) + (\tilde{\rho}-\rho)q\epsilon\dot{\beta}(t \\
&\quad - \bar{\tau}-\delta_1,1) - (1-\epsilon)\tilde{\rho} q\dot{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) - \tilde{\rho}\epsilon q \\
&\dot{\beta}(t-\bar{\tau}-\delta_0-\delta_1,1) + k_I q\lambda l_1(1)(\beta(t-\bar{\tau}-\delta_0,1) \\
&\quad - \beta(t-\bar{\tau},1)) + k_I q(\beta(t-\bar{\tau}-\delta_0-\delta_1,1) \\
&\quad - \beta(t-\bar{\tau}-\delta_1)) + \mathcal{O}(\dot{X}_t) + \mathcal{O}(\beta(\cdot,1)_t). \qquad (75)
\end{aligned}$$

Finally, using equation (70) we have

$$\begin{aligned}
&\dot{V}(t) = \tilde{\rho}\epsilon q(\dot{\beta}(t-\bar{\tau},1) - \dot{\beta}(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon)q\dot{\tilde{\beta}}(t \\
&\quad - \tau,1) - \tilde{\rho} q\dot{\beta}(t-\bar{\tau},1) + \int_0^\tau \tilde{F}(\nu)\dot{\tilde{\beta}}(t-\nu,1)d\nu \\
&\quad + \int_0^{\bar{\tau}} \tilde{N}(\xi)d\xi\dot{\beta}(t-\xi,1) + k_I q(\beta(t-\bar{\tau}-\delta_1,1) \\
&\quad + \lambda l_1(1)\beta(t-\bar{\tau},1)) + \mathcal{O}(\beta(\cdot,1)_t) + \mathcal{O}(\dot{X}_t). \qquad (76)
\end{aligned}$$

Consider system (74)-(76), the objective is now to prove that the first component of the solution X_t, i.e. $\beta(t,1)$, exponentially converges to zero. Note that only the convergence of $\beta(t,1)$ to zero is required and that $\tilde{\beta}(t,1)$ does not necessarily converge to zero (due to the presence of the integral term). Thus, we choose to consider the new state $Y(t)$ defined by

$$Y(t) = \Big(\beta(t,1)\ \dot{\beta}(t,1)\ \dot{\tilde{\beta}}(t,1), \dot{V}(t)\Big). \qquad (77)$$

The stability proof is achieved considering the characteristic function associated to the system (74)-(76).

4.4 Complex stability analysis

We start writing the Laplace transform of the equations satisfied by the state Y. We can then easily obtain the characteristic equation associated to this system. The stability is granted provided this characteristic equation does not have any roots in the open complex right plane $\mathbb{C}^+$. The first objective is to express in a simple way the Laplace transform of the system (74)-(76). Let us introduce the following holomorphic matrices

$$\begin{aligned}
&F_0(s) = (\rho q e^{-\bar{\tau}s} - \tilde{\rho} q(1-\epsilon)e^{-(\bar{\tau}+\delta_0)s} - \tilde{\rho} q\epsilon e^{-(\bar{\tau}+\delta_0+\delta_1)s})E^4_{2,2} \\
&\quad + (\tilde{\rho} q(1-\epsilon)e^{-(\tau+\delta_0)s})E^4_{2,3} + (q((\rho-\tilde{\rho})\epsilon+\tilde{\rho})e^{-\bar{\tau}s} + (\tilde{\rho}-\rho) \\
&\quad q\epsilon e^{-(\bar{\tau}+\delta_1)s} - (1-\epsilon)\tilde{\rho} q e^{-(\bar{\tau}+\delta_0)s} - \tilde{\rho} q\epsilon e^{-(\bar{\tau}+\delta_0+\delta_1)s})E^4_{3,2} \\
&\quad + ((\rho-\tilde{\rho})q(1-\epsilon)e^{-\tau s} + \tilde{\rho} q(1-\epsilon)e^{-(\tau+\delta_0)s})E^4_{3,3} + E^4_{1,2} \\
&\quad + (\tilde{\rho}\epsilon q e^{-\bar{\tau}s} - \tilde{\rho}\epsilon q e^{-(\bar{\tau}+\delta_1)s} - \tilde{\rho} q e^{-\bar{\tau}s})E^4_{4,2} + (\tilde{\rho}(1-\epsilon)q e^{-\tau s})E^4_{4,3}, \\
&C_0(s) = (k_I q e^{-(\bar{\tau}+\delta_1+\delta_0)s} + k_I q\lambda l_1(1)e^{-(\bar{\tau}+\delta_1)s})E^{4,4}_{2,1} + (k_I q \\
&\quad \lambda l_1(1)e^{-\bar{\tau}s}(e^{-\delta_0 s}-1) + k_I q e^{-(\bar{\tau}+\delta_1)s}(e^{-\delta_0 s}-1))E^{4,4}_{3,1} + \\
&\quad (k_I q e^{-\bar{\tau}s}(1+\lambda l_1(1)e^{-\bar{\tau}s}))E^{4,4}_{4,1}, \\
&E_0(s) = \Big(\int_0^\tau \tilde{F}(\nu)e^{-(\nu+\delta_0)s}d\nu\Big)E^4_{2,3} + \Big(\int_0^\tau \tilde{F}(\nu)e^{-\nu s}d\nu\Big)E^4_{4,3} \\
&\quad + \Big(\int_0^{\bar{\tau}} s\tilde{N}(\nu)\tilde{F}(\nu)e^{-\nu s}d\nu\Big)E^4_{4,2}, \\
&I_0(s) = sE^4_{1,1} + E^4_{2,2} + E^4_{3,3} + E^4_{4,4}.
\end{aligned}$$

where the matrices $E^k_{i,j}$ have been defined while giving Condition 10. With these notations, the Laplace transform of (74)-(76) for the state Y defined by (77) is given by

$$I_0(s)\hat{Y}(s) = (F_0(s) + C_0(s) + E_0(s))\hat{Y}(s) + \hat{\mathcal{O}}(s), \qquad (78)$$

where we have (abusively) denoted $\hat{\mathcal{O}}(s)$ the transfer function associated to the operator $\hat{\mathcal{O}}(\hat{Y}_t)(s)$. The characteristic equation associated to (74)-(76) can be expressed as

$$\det(I_0(s) - F_0(s) - C_0(s) - E_0(s) - \hat{\mathcal{O}}(s)) = 0, \qquad (79)$$

In what follows, we denote

$$P(s) = \Big|\det\Big(I_0(s) - F_0(s) - C_0(s) - E_0(s) - \hat{\mathcal{O}}(s)\Big)\Big|.$$

We have the following theorem

Theorem 22 *There exist $\delta_m > 0$ and $\kappa_m > 0$ such that if $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$ then, $P(s)$ does not have any zero in $\mathbb{C}^+$.*

The proof of this Theorem is done in Appendix D. Theorem 22 implies the function $\beta(t,1)$ converges to zero. Using the transport equations (48)-(51) and the transformation (47), we can conclude to the convergence of the state (u,v) to its zero-equilibrium. This proves Theorem 11

5 Input-to-State Stability: proof of Theorem 12

We have proved that in the absence of disturbances and noise, there exist $\delta_m > 0$ and $\kappa_m > 0$ such that if $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$ the state (u,v) exponentially converges to zero (and thus the output regulation is ensured). We consider in this section the influence of the disturbances and of the noise on the regulation. The objective is to prove Theorem 12. The methodology we use is the same as the one developed in [24]. It requires doing computations which are extremely similar to the one done in Section 4. Using the backstepping transformations (47) and (60)-(61) we prove (using similar computations as the ones done in Section 4), that the extended states $\beta(t,1)$, $\tilde{\beta}(t,1)$ and $V(t)$ still satisfy (74)-(76) in which are added some additional terms that vanish if the disturbances are constant (with respect to t). For sake of brevity, we choose not to give these equations but to directly jump to the following lemma

Lemma 23 *Let us denote* $\zeta(t,x) = (d_1(t,x)\ d_2(t,x)\ d_3(t)\ d_4(t)\ \dot{d}_1(t,x)\ \dot{d}_2(t,x)\ \dot{d}_3(t)\ \dot{d}_4(t)\ n(t))^T$. *Let us denote* $\mathcal{P}$ *the operator associated to* (74)-(76) *(i.e* (74)-(76) *can be rewritten* $X_t = \mathcal{P}(X_t)$*). Then, there exists a linear operator* K *such that*

$$\dot{X}_t = \mathcal{P}(\dot{X}_t) + K(\dot{\zeta}(t,x), n(t)).$$

The variation-of-constant formula for this system reads (see [21] page 173)

$$X\left((\alpha_0,\beta_0),K\right)(t) = X\left((\alpha_0,\beta_0),0\right)(t) + \int_0^t X_0(t-s)K(\dot{\zeta}(s),n(s))ds\,, \quad (80)$$

where $X\left((\alpha_0,\beta_0),0\right)(t)$ denotes the solution of the homogeneous NDE system $X_t = \mathcal{P}(X_t)$ in term of the fundamental solution X_0 (see [21] for a definition of the fundamental solution). As $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$, the first component of X converges to zero. As we can rewrite for all $x \in [0,1]$ $\alpha(t,x)$ and $\beta(t,x)$ as functions of $\beta(t,1)$ (equations (48)-(49) and using the backstepping transformation (47)), we immediately get the ISS of the system. To conclude the proof, we now have to show that equation (38) holds when $\dot{d}_1(t,x) = \dot{d}_2(t,x) = \dot{d}_3(t) = \dot{d}_4(t) = n(t) = 0$. In this case the operator K satisfies $K \equiv 0$. Thus, $\beta(t,1)$ exponentially converges to zero. This implies the convergence to zero of $\alpha(t,x)$ and $\beta(t,x)$ for every $x \in [0,1]$. Finally, using the transformation (47) we obtain

$$\lim_{t\to\infty} |u(t,1)| = \lim_{t\to\infty} \left| \alpha(t,1) + \int_0^1 \bar{L}^{\alpha\alpha}(1,\xi)\alpha(t,\xi)d\xi + \int_0^1 \bar{L}^{\alpha\beta}(1,\xi)\beta(t,\xi)d\xi \right| = 0. \quad (81)$$

This concludes the proof of Theorem 12. Using the same computations, we can write the following Corollary.

Corollary 24 *Assume that*

$$\sup_{\theta_k \in [0,2\pi]^4} Sp(\sum_{k=1}^4 A_k \exp(i\theta_k)) > 1, \quad (82)$$

For any $\delta_0 > 0$, $\delta_1 > 0$ *and* $\kappa > 0$ *the output of the closed loop system* (1)-(4) *along with the control law* (16) *and the observer* (10)-(13) *diverges.*

More precisely, if (82) holds, then, the function $\det(Id - F_0(s))$ has an infinite number of zeros in the right half plane (see [21] for instance). Thus, using [4, Lemma 3] and Theorem 18 yields the expected result.

6 Simulation results on a toy problem and robustness trade-offs

In this section, we numerically illustrate the results of this paper by explicitly computing the admissible $\tilde{\rho}$, ϵ and k_I that guarantee robustness in the case of a simple example. Let us consider the following set of parameters

$$\lambda = \mu = q = 1,\ \sigma^{-+} = -1,\ \sigma^{+-} = 0,\ \rho = 0.6.$$

Solving the PDEs given in [33], we obtain $L^{\alpha\alpha}(x,\xi) = L^{\alpha\beta}(x,\xi) = 0$ and consequently, for any $x \in [0,1]$, $l_1(x) = l_2(x) = 0$. With this set of parameters, Assumption 3 is obviously satisfied. It has been proved in [4] that the maximal amount of reflection that can be canceled is given by $|\tilde{\rho}_{\max}| = \frac{1-|\rho q|}{|q|} = 0.4$. In what follows we consider $\tilde{\rho} \in [0, \tilde{\rho}_{\max}[$. Condition 4 implies the following condition for $0 > k_I$:

$$k_I > -\frac{\sqrt{1-(0.6-\tilde{\rho})^2}}{2} \arctan\left(\frac{\sqrt{1-(0.6-\tilde{\rho})^2}}{(0.6-\tilde{\rho})}\right) \quad (83)$$

Figure 1 pictures the domain associated to (36), condition that the parameters $\tilde{\rho}$ and ϵ have to satisfy (we have only pictured the positive values) while Figure 2 pictures the domain associated to condition (83), condition that the coefficient k_I has to satisfy. These conditions are required to guarantee the existence of robustness margins. The $(\tilde{\rho}, \epsilon)$ domain is obtained using an iterative algorithm that computes condition (36). This algorithm uses some convexity properties of the stability domain. It is also based on the fact that it is easier to check that (36) is false rather than the converse. This domain is compared with the stability domain we would have obtained considering only a delay in the actuation δ_0 and neglecting the influence of the terms δ_1, δ_λ, and δ_μ (blue line). This illustrates the property stated in [8] that uncertainties on the velocities have a non-negligible impact on delay-robustness and emphasizes the necessity to study these two problems simultaneously while tuning the parameters ρ and ϵ. In particular, the blue line somehow corresponds to the w-stability condition. The $(k_I, \tilde{\rho})$ domain is obtained by computing the inequalities given in (83).

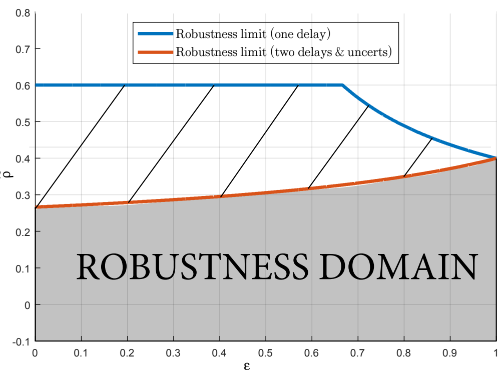

Fig. 1. Representation of the robustness domain in the plane $(\epsilon, \tilde{\rho})$ for $q = 1$ and $\rho = 0.6$

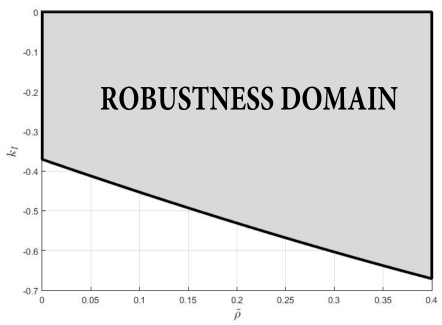

Fig. 2. Representation of the robustness domain in the plane $(\tilde{\rho}, k_I)$ for $q = 1$ and $\rho = 0.6$.

To analyze the effect of the tuning parameters k_I and ϵ, we picture in Figure 3-9 the temporal response of the output for different situations:

- In Figures 3-4, we picture the evolution of the output in the absence of disturbance or noise and without any integral compensation for different values of ϵ and $\tilde{\rho}$.
- In Figures 5-6, we picture the evolution of the output in the absence of noise and with a disturbance $d_3(t) = 1$ for different values of ϵ and $\tilde{\rho}$. The integral gain k_I is set to zero.
- In Figures 7-8, we picture the evolution of the output in the absence of disturbance with a high frequency noise for different values of ϵ and $\tilde{\rho}$. The integral gain k_I is set to zero.
- In Figure 9, we fix the tuning parameters $\tilde{\rho} = 0.3$ and $\epsilon = 1$. We consider the constant disturbances $d_1 = d_2 = n = 0$, $d_3 = d_4 = 1$. We picture the evolution of the output for different admissible values of k_I.

From these figures, we can make the following remarks

(1) The coefficient k_I has a direct impact on disturbance rejection but also on the raise-time and on the settle time. Classically, increasing k_I improves the disturbance rejection but generates oscillations.
(2) Choosing a high absolute value for k_I implies to

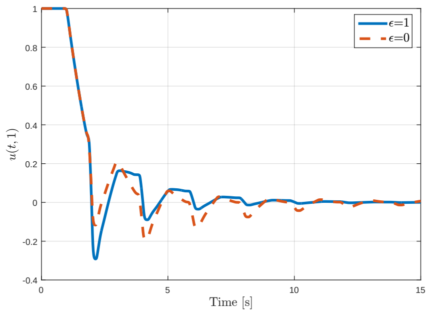

Fig. 3. Evolution of the output $u(t, 1)$ in the absence of noise and disturbance for different values of ϵ with $k_I = 0$ and $\tilde{\rho} = 0.2$.

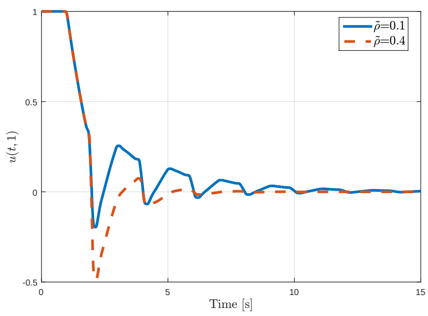

Fig. 4. Evolution of the output $u(t, 1)$ in the absence of noise and disturbance for different values of $\tilde{\rho}$ with $k_I = 0$ and $\epsilon = 1$.

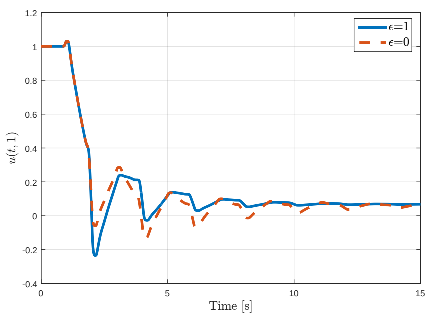

Fig. 5. Evolution of the output $u(t, 1)$ in the absence of noise with a disturbance $d_3 = 0.1$ for different values of ϵ with $k_I = 0$ and $\tilde{\rho} = 0.2$.

have $\tilde{\rho}$ large enough (right part of Figure 1). Thus, as $\tilde{\rho}$ enables a trade-off between performance and robustness, choosing an arbitrary value for k_I may have some negative impact on robustness. Consequently, there is a trade-off between disturbance rejection and (delay-)robustness.
(3) Choosing a small value for ϵ seems to improve the noise rejection (even if the convergence is slower). However, a reduction of ϵ may cause a loss of phase margin which must be amended by also reducing the integral gain k_I to avoid a potential unacceptably high controller induced resonance. There is consequently a complex trade-off between performance and robustness, noise sensitivity and disturbance rejection.

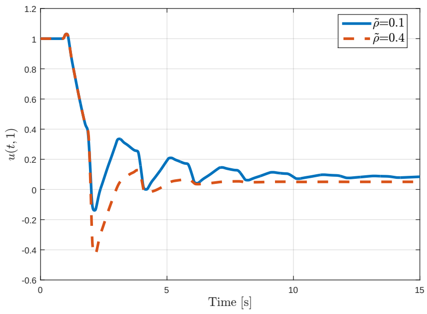

Fig. 6. Evolution of the output $u(t, 1)$ in the absence of noise with a disturbance $d_3 = 0.1$ for different values of $\tilde{\rho}$ with $k_I = 0$ and $\epsilon = 1$.

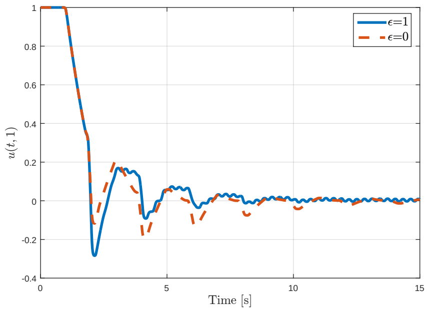

Fig. 7. Evolution of the output $u(t, 1)$ in the absence of disturbance along with a high frequency noise for different values of ϵ with $k_I = 0$ and $\tilde{\rho} = 0.2$.

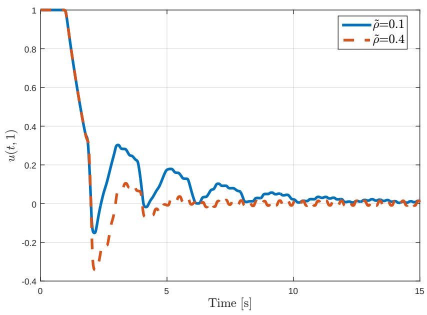

Fig. 8. Evolution of the output $u(t, 1)$ in the absence of disturbance along with a high frequency noise for different values of $\tilde{\rho}$ with $k_I = 0$ and $\epsilon = 1$.

The different effects on the tuning parameters are summarized in Table 1. These remarks illustrate the fact that the degrees of freedom introduced in this paper enable various trade-offs and have to be specifically tuned depending on the problem considered. A deeper analysis can only be done for a case by case basis. More precisely, deriving the transfer function of the controller and of the observer, using classical controller analysis techniques (including the analysis of the rise time, of the response time, computing Nyquist charts, or the Gang of Six [3] for instance), it is possible to derive some tuning heuristics giving a trade-off between noise sensitivity versus disturbance rejection performance or between delay-robustness (especially for high frequencies) and nominal performance.

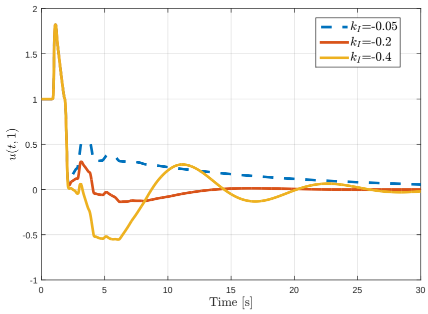

Fig. 9. Evolution of the output $u(t, 1)$ for different values of k_I with $\tilde{\rho} = 0.3$ and $\epsilon = 1$ in presence of a disturbance $d_3 = 0.1$.

Coef.	Cond.	Pros	Cons
k_I	4	$\nearrow$ Dist. reject. $\nearrow$ Conv. rate	$\nearrow$ Oscillations $\searrow$ delay-robustness $\nearrow$ Raise/Settle-time
$\tilde{\rho}$	4,10	$\nearrow$ Conv. rate	$\searrow$ delay-robustness
ϵ	10	$\nearrow$ Conv. rate	$\searrow$ noise rejection

Table 1
Consequences of choosing large values for the different tuning parameters $(\tilde{\rho}, k_I, \epsilon)$ under their respective admissibility domains (non exhaustive list).

7 Concluding Remarks

The control law derived in [24] introduces three degrees of freedom. In this paper we have given explicit conditions on these degrees of freedom to guarantee the robust stabilization of a system of two coupled hyperbolic PDEs. The proposed condition is stronger (and more restrictive in its requirements) than the w-stability condition given in [14] as the concept of w-stability does not encompass uncertainties on the transport velocities. We have highlighted how these degrees of freedom can be tuned to enable various trade-offs (convergence rate-robustness, noise sensibility-disturbance rejection). However, the proposed approach is **qualitative** as only a robust stability criterion has been given. The next step towards a real implementation of the backstepping controllers consists in developing **quantitative** tools to tune these degrees of freedom. In particular, their impact on the size of the robustness margins should be the purpose of further investigations. Considering higher-dimensional problems, we think that combining the approach proposed in this paper with the results developed in [7] would lead to promising results.

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A Proof of Lemma 19

We recall that since the kernels $R^{\cdot\cdot}$ are the inverse kernels of the kernels $P^{\cdot\cdot}$, we have [23]

$$\begin{aligned} R^{\alpha\alpha}(x,1) &= P^{uu}(x,1) + \int_x^1 R^{\alpha\alpha}(x,\xi)P^{uu}(\xi,1)d\xi \\ &\quad + \int_x^1 R^{\alpha\beta}(x,\xi)P^{vu}(\xi,1)d\xi, \qquad (A.1) \\ R^{\alpha\beta}(x,1) &= P^{uv}(x,1) + \int_x^1 R^{\alpha\alpha}(x,\xi)P^{uv}(\xi,1)d\xi \\ &\quad + \int_x^1 R^{\alpha\beta}(x,\xi)P^{vv}(\xi,1)d\xi. \qquad (A.2) \end{aligned}$$

Differentiating (60) with respect to space and time and integrating by part and using the equations satisfied by the kernels $R^{\cdot,\cdot}$, we obtain equation (62) where the function f is defined by

$$f(x) = -R^{\alpha\alpha}(x,1) - \rho\epsilon\mu R^{\alpha\beta}(x,1).$$

A similar proof can be done to derive equation (63).

B Proof of Lemma 20

We start giving the expression of $u(t,x)$, $\partial_x u(t,x)$, $v(t,x)$ and $\partial_x v(t,x)$ in terms of $\beta(t,1)$. Using equations (48)-(51) and the Volterra transformation (47), we have that for all $t \le \bar{\tau}$ and all $x \in [0,1]$,

$$u(t,x) = \bar{q}\beta(t - \frac{x}{\bar{\lambda}} - \frac{1}{\bar{\mu}}, 1) + \int_0^1 (\bar{q}\bar{L}^{\alpha\alpha}(x,\xi) \beta(t - \frac{\xi}{\bar{\lambda}} - \frac{1}{\bar{\mu}}, 1) + \bar{L}^{\alpha\beta}(x,\xi)\beta(t - \frac{1-\xi}{\bar{\mu}}, 1)d\xi).$$

Using the notations of Section 3, for every $x \in [0,1]$, there exist $\mathcal{I}_u(x) \in \mathfrak{I}$ and $\mathcal{I}_{u_x}(x) \in \mathfrak{I}$ such that

$$u(t,x) = \bar{q}\beta(t - \frac{1}{\bar{\mu}} - \frac{x}{\bar{\lambda}}, 1) + \mathcal{I}_u(x)(\beta(\cdot,1)_t), \tag{B.1}$$

$$\partial_x u(t,x) = -\frac{\bar{q}}{\bar{\lambda}}\partial_x\beta(t - \frac{1}{\bar{\mu}} - \frac{x}{\bar{\lambda}}, 1) + \mathcal{I}_{u_x}(x)(\beta(\cdot,1)_t) \tag{B.2}$$

Similarly we obtain the existence of $\mathcal{I}_v(x) \in \mathfrak{I}$ and $\mathcal{I}_{v_x}(x) \in \mathfrak{I}$ such that

$$v(t,x) = \beta(t - \frac{1-x}{\bar{\mu}}, 1) + \mathcal{I}_v(x)(\beta(\cdot,1)_t), \tag{B.3}$$

$$\partial_x v(t,x) = \frac{1}{\bar{\mu}}\partial_x\beta(t - \frac{1-x}{\bar{\mu}}, 1) + \mathcal{I}_{v_x}(x)(\beta(\cdot,1)_t). \tag{B.4}$$

The objective is to use the method of characteristics and the formalism introduced in Section 3 to express $\tilde{\beta}(t,1)$ as the solution of a neutral equation. Regarding the terms in (62)-(65) that are functions of $u(t,x)$ or $v(t,x)$, one can use equations (B.1)-(B.3) to express them as functions of X_t. We have for all $t > \bar{\tau}$ and all $x \in [0,1]$, $\beta(t,x) = \beta(t - \frac{1-x}{\bar{\mu}}, 1)$. This implies

$$\partial_x\beta(t,x) = \frac{1}{\bar{\mu}}\partial_t\beta(t - \frac{1-x}{\bar{\mu}}, 1). \tag{B.5}$$

Combining this with (B.2), we obtain

$$-\delta_\lambda\partial_x u(t,x) = -\delta_\lambda(-\frac{1}{\bar{\lambda}}\bar{q}\partial_x\beta(t - \frac{1}{\bar{\mu}} - \frac{x}{\bar{\lambda}}, 1) + \mathcal{I}_{u_x} (x)(\beta(\cdot,1)_t)) = \delta_\lambda\frac{1}{\bar{\lambda}}\bar{q}\frac{1}{\bar{\mu}}\partial_t\beta(t - \frac{1}{\bar{\mu}} - \frac{x}{\bar{\lambda}}, 1) + \mathcal{O}(X_t).$$

Thus, $-\delta_\lambda \int_0^{\frac{1}{\lambda}} \partial_x u(t-s, x-\bar{\lambda}s)ds = \mathcal{O}(X_t)$. Using (B.4) and (B.5), we obtain

$$\int_0^{\frac{1}{\lambda}}\int_{x-\bar{\lambda}s}^1 R^{\alpha\beta}(x-\bar{\lambda}s,\xi)\delta_\mu\partial_x v(t-s,\xi)d\xi ds = \int_0^{\frac{1}{\lambda}}\int_{x-\bar{\lambda}s}^1 R^{\alpha\beta}(x-\bar{\lambda}s,\xi)\delta_\mu\frac{1}{\bar{\mu}^2}\partial_t\beta(t-s-\frac{1-\xi}{\bar{\mu}},1)d\xi ds + \mathcal{O}(X_t).$$

Integrating by part, we get

$$\int_0^{\frac{1}{\lambda}}\int_{x-\bar{\lambda}s}^1 R^{\alpha\beta}(x-\bar{\lambda}s,\xi)\delta_\mu\partial_x v(t-s,\xi)d\xi ds = \int_0^{\frac{1}{\lambda}}\int_{x-\bar{\lambda}s}^1 \partial_\xi R^{\alpha\beta}(x-\bar{\lambda}s,\xi)\delta_\mu\frac{1}{\bar{\mu}}\beta(t-s-\frac{1-\xi}{\bar{\mu}},1)d\xi ds + \mathcal{O}(X_t)$$
$$+ \int_0^{\frac{1}{\lambda}}\frac{\delta_\mu}{\bar{\mu}}(R^{\alpha\beta}(x-\bar{\lambda}s,1)\beta(t-s,1) - R^{\alpha\beta}(x-\bar{\lambda}s,x-\bar{\lambda}s) \beta(t-s-\frac{1-x+\bar{\lambda}s}{\bar{\mu}}))ds + \mathcal{O}(X_t) = \mathcal{O}(X_t).$$

Similar computations can be done to obtain $\int_0^{\frac{1}{\lambda}}\int_{x-\bar{\lambda}s}^1 R^{\alpha\alpha} (x-\bar{\lambda}s,\xi)\delta_\lambda\partial_x u(t-s,\xi)d\xi ds = \mathcal{O}(X_t)$. This yields

$$\int_0^{\frac{1}{\lambda}} -\delta_\lambda\partial_x u(t-s,x-\bar{\lambda}s) + \int_{x-\bar{\lambda}s}^1 R^{\alpha\beta}(x-\bar{\lambda}s,\xi) \delta_\mu\partial_x v(t-s,\xi) - R^{\alpha\alpha}(x-\bar{\lambda}s,\xi)\delta_\lambda\partial_x u(t-s,\xi)d\xi ds = \mathcal{O}(X_t) \tag{B.6}$$

Similarly, we obtain $\int_0^{\frac{1}{\bar{\mu}}} \delta_\lambda\partial_x v(t-s,x+\bar{\mu}s) + \int_{x+\bar{\mu}s}^1 R^{\beta\beta}(x+\bar{\mu}s,\xi)\delta_\mu\partial_x v(t-s,\xi) - R^{\beta\alpha}(x+\bar{\mu}s,\xi)\delta_\lambda\partial_x u(t-s,\xi)d\xi ds = \mathcal{O}(X_t)$. Using the method of characteristics on equations (B.1)-(B.4), we obtain for any $t \ge \tau + \delta_0 + \delta_1$

$$\tilde{\beta}(t,1) = \rho q(1-\epsilon)\tilde{\beta}(t-\tau,1) + \rho q\epsilon(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + V(t-\delta_0) - V(t) + \mathcal{O}(X_t). \tag{B.7}$$

C Proof of Lemma 21

We start by expressing the terms $\tilde{u}(t,x)$ and $\tilde{v}(t,x)$ as functions of the state X_t. Using the notations of Section 3, the relations (B.1)-(B.4), for every $x \in [0,1]$, there exist $\mathcal{I}_{\tilde{u}}(x) \in \mathfrak{I}$ and $\mathcal{I}_{\tilde{v}}(x) \in \mathfrak{I}$ such that

$$\tilde{u}(t,x) = q\tilde{\beta}(t - \frac{1}{\mu} - \frac{x}{\lambda}, 1) + \mathcal{I}_{\tilde{u}}(x)(X_t) + \mathcal{O}(X_t) \tag{C.1}$$

$$\tilde{v}(t,x) = \tilde{\beta}(t - \frac{1-x}{\mu}, 1) + \mathcal{I}_{\tilde{v}}(x)(X_t) + \mathcal{O}(X_t). \tag{C.2}$$

Let us denote $V_0(t) = V_{BS}(t) + k_I V_I(t)$. Using (B.1)-(B.3) with (C.1)-(C.2), we obtain

$$V_0(t) = \tilde{\rho}\epsilon q(\beta(t-\bar{\tau},1) - \beta(t-\bar{\tau}-\delta_1,1)) + \tilde{\rho}(1-\epsilon) q\tilde{\beta}(t-\tau,1) - \tilde{\rho}q\beta(t-\bar{\tau},1) - k_I\int_0^1(\bar{l}_1(\xi)\alpha(t,\xi) + \bar{l}_2(\xi) \beta(t,\xi))d\xi + \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu,1)d\nu + \int_0^{\bar{\tau}}\tilde{N}(\xi) \beta(t-\xi,1)d\xi + \mathcal{O}(X_t). \tag{C.3}$$

Using (69), we obtain the expected expression for the control law $V(t)$. Differentiating (69) and integrating by part yields

$$\begin{aligned}\dot{\gamma}(t) &= \dot{\eta} - \int_0^1 \bar{l}_1(t,\xi)\partial_t\alpha(t,\xi) + l_2(t,\xi)\partial_t\beta(t,\xi)d\xi \\ &= \alpha(t-\delta_1,1) + \int_0^1 \bar{L}^{\alpha\alpha}(x,\xi)\alpha(t-\delta_1,\xi) \\ &\quad + \bar{L}^{\alpha\beta}(x,\xi)\alpha(t-\delta_1,\xi)d\xi - \int_0^1 \bar{\lambda}\bar{l}_1'(\xi)\alpha(t,\xi) \\ &\quad - \bar{\mu}\bar{l}_2'(\xi)\beta(t,\xi)d\xi + \bar{\lambda}\bar{l}_1(1)\alpha(t,1) \\ &\quad - \bar{\lambda}\bar{l}_1(0)\alpha(t,0) - \bar{\mu}\bar{l}_2(1)\beta(t,1) + \bar{\mu}\bar{l}_2(0)\beta(t,0).\end{aligned}$$

Using the definition of $\bar{l}_1$ and $\bar{l}_2$ given in equations (54)-(55) and the boundary conditions (50)-(51), we obtain $\dot{\gamma}(t) = \alpha(t-\delta_1,1) + \bar{\lambda}\bar{l}_1(1)\alpha(t,1) + \mathcal{O}(\beta(\cdot,1)_t)$. Finally, $\alpha(t-\delta_1,1) + \bar{\lambda}\bar{l}_1(1)\alpha(t,1) = \bar{q}(\beta(t-\bar{\tau}-\delta_1,1) + \bar{\lambda}\bar{l}_1(1))\alpha(t-\bar{\tau},1)$. Using the continuity of the function $\bar{l}_1$ when the uncertainties go to zero, we obtain the expected expression for $\dot{\gamma}$.

D Proof of Theorem 22

We need to prove that all the solutions of the characteristic equation (79) are located in the complex left-half plane if the delays and uncertainties are small enough. The analysis is different depending on the magnitude of $|s|$. For large values of $|s|$, the strategy is the following. As the principal term $F_0(s)$ may be the main limitation for stability (see Theorem 18 for details), we start by proving that the function $|\det(I_0(s) - sF_0(s))|$ is positively bounded on the complex right-half plane as long as the modulus of s is large enough. Then, we focus on the influence of the integral components of the matrix E_0, which do not belong to $\mathfrak{W}$. We prove that they do not have any incidence in terms of stability if the uncertainties and delays are small enough. Finally, we consider the influence of the term C_0, that corresponds to the integral action. We show that the influence of this term is negligible. For small values of $|s|$, we prove that due to the choice of k_I in Condition 4, if the delays and uncertainties are small enough then the characteristic function $P(s)$ cannot vanish.

D.1 Analysis of the function $\det(I_0(s) - F_0(s))$

We first consider the influence of the principal term F_0 on stability when $|s|$ is large enough. More precisely we consider the subsystem

$$(I_0(s) - F_0(s))\begin{pmatrix} y_1(s) & y_2(s) & y_3(s) & y_4(s)\end{pmatrix}^T = 0. \quad \text{(D.1)}$$

Due to the structure of the matrix F_0 (the last column is equal to zero), there is a cascade from the first three lines to the last one. Consequently the last line does not play any role in terms of stability. The characteristic equation associated to (D.1) is given by $s[(1-(F_0)_{22}(s))(1-(F_0)_{33}(s)) - (F_0)_{23}(s)(F_0)_{32}(s)] = 0$. As we consider that $|s|$ is large enough, we can simplify this equation dividing it by s. The resulting equation corresponds to the characteristic equation associated to the system

$$\begin{pmatrix} z_1(s) \\ z_2(s)\end{pmatrix} = \begin{pmatrix}(F_0)_{22}(s) & (F_0)_{23}(s) \\ (F_0)_{32}(s) & (F_0)_{33}(s)\end{pmatrix}\begin{pmatrix} z_1(s) \\ z_2(s)\end{pmatrix}. \quad \text{(D.2)}$$

The corresponding temporal system is a neutral system. However, since the different delays involved in this neutral system are not rationally independent (we have the delays $\bar{\tau}$, $\bar{\tau}-\delta_1$, $\bar{\tau}-\delta_0$, and $\bar{\tau}-\delta_0-\delta_1$), we cannot directly use [21, Theorem 6.1] to conclude to the exponential stability of the temporal system associated to (D.2). Thus, we consider the extended state.

$$Z_p(t) = \begin{pmatrix} z_1(t) & z_1(t-\delta_0) & z_2(t) & z_2(t-\delta_0)\end{pmatrix}^T.$$

With these notations, equation (D.2) rewrites

$$\hat{Z}_p(s) = (A_1\mathrm{e}^{-\bar{\tau}s} + A_2\mathrm{e}^{-\tau s} + A_3\mathrm{e}^{-\delta_0 s} + A_4\mathrm{e}^{-(\bar{\tau}+\delta_1)s})\hat{Z}_p(s),$$

where the A_i are defined by (32)-(35). Since the delays involved in this equation are now rationally independent, combining Condition 10 and [21, Theorem 6.1], we can conclude that there exist $\gamma_0 > 0$ such that the roots of the characteristic equation associated to (D.2) are on the open complex half plane $\{s \in \mathbb{C} \mid \Re(s) < -\gamma_0\}$. Moreover, adjusting the proof of [20, Lemma 2.1], there exists $M_0 > 0$ such that the function $|\det(I_0(s)-F_0(s))|$ is lower-bounded by a constant $\omega_0 > 0$ on the complex set $\Omega_0 = \{s \in \mathbb{C} \mid \Re(s) \geq 0 \text{ and } |s| > M_0\}$.

D.2 Stability analysis of the equation: $I_0(s)\hat{Y}(s) = (F_0(s) + E_0(s))\hat{Y}(s) + \hat{\mathcal{O}}(s)$

We now prove that the integral terms of the matrix E_0 and the operator $\hat{\mathcal{O}}((\hat{Y})_t)(s)$ do not affect the stability properties of the previous system if the delays and uncertainties are chosen small enough. As the three first lines of E_0 do not have any component on the last column, there is still a cascade from the three first lines to the last one. Consequently, we only need to focus on the term $\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu$. Using successive iterations on this term we prove that, provided the delays and uncertainties are small, its norm is small enough. More precisely, using equation (73), direct computations yield $\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu = \rho q(1-\epsilon)\cdot \int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\tau-\delta_0,1)d\nu + \mathcal{O}(\hat{Y}_t)$. Let us consider an integer N_0 that still has to be defined and consider a time $t > (N_0+2)\tau$. Iterating N_0 times the previous computations we obtain $\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-\delta_0,1)d\nu = \rho q(1-\epsilon)^{N_0}\cdot\int_0^\tau \tilde{F}(\nu)\tilde{\beta}(t-\nu-N_0\tau-\delta_0,1)d\nu + \mathcal{O}(\hat{Y}_t)$. Consequently, the system $I_0(s)\hat{Y}(s) = (F_0(s)+E_0(s))\hat{Y}(s) + \hat{\mathcal{O}}((\hat{Y})_t)(s)$ can be rewritten

$$I_0(s)\hat{Y}(s) = (F_0(s) + E_1(s))\hat{Y}(s) + \hat{\mathcal{O}}_2((\hat{Y})_t)(s), \quad \text{(D.3)}$$

where $E_1(s) = ((\rho q(1-\epsilon))^{N_0} \int_0^\tau \tilde{F}(\nu) e^{-(\nu+\tau+\delta_0)s} d\nu) E^4_{4,2} + (\int_0^\tau s\tilde{N}(\nu)\tilde{F}(\nu)e^{-\nu s} d\nu) E^4_{2,3} + \int_0^\tau \tilde{F}(\nu)e^{-\nu s} d\nu E^4_{3,3}$ and where we have used the notation $\hat{\mathcal{O}}_2$ to highlight the fact that this $\mathfrak{W}$-term is not the same as before. Since $|(\rho q(1-\epsilon))| < 1$, for N_0 large enough, the term $(\rho q(1-\epsilon))^{N_0}$ can be chosen as small as desired. Combining the proof of [5, Theorem 2] and the proof of Theorem 17, there exist $\delta_{m_1} > 0$ and $\kappa_{m_1} > 0$ such that if $\delta_0 < \delta_{m_1}$, $\delta_1 < \delta_{m_1}$ and $\kappa < \kappa_{m_1}$ then the function $\det(I_0 - F_0(s) - E_0(s) - \hat{\mathcal{O}}(s))$ is positively bounded by a constant $\omega_1 > 0$ on Ω_0.

D.3 Stability analysis of equation (78)

We can now state the stability properties of equation (78). We first prove that for large values of $|s|$, the function $P(s)$ cannot be equal to 0. If $s \in \Omega_0$, the function $|\det(s(I_0 - F_0(s) - E_0(s) - \mathcal{O}(s)))|$ is lower-bounded by $M_0\omega_1$. In the same time, the function $\frac{1}{s}C_0(s)$ converges to zero for $|s|$ large enough. Thus, using the continuity of the determinant, there exists $M_1 > 0$, such that $\forall s \in \Omega_1 = \{s \in \mathbb{C} \mid \Re(s) \geq 0, \text{ and } |s| \geq M_1\}$,

$$|\det\Big((I_0 - F_0(s) - E_0(s) - \hat{\mathcal{O}}(s) - C_0(s)\Big)| > 0.$$

We now have to prove that $P(s)$ is not equal to zero on $\mathbb{C}\backslash\Omega_1 \cup \mathbb{C}^+$. Let us consider $s \in \mathbb{C}^+$ such that $|s| \leq M_1$. Let us define $L(s)$ and $H(s)$ as

$$L(s) = \begin{pmatrix} 0 & k_I q(1+\lambda l_1(1)))e^{-\tau s} & 0 & C_3(s) \\ 1 & (\rho - \tilde{\rho})q e^{-\tau s} & 0 & (F_0)_{42}(s) \\ 0 & \tilde{\rho} q(1-\epsilon)e^{-\tau s} & \rho q(1-\epsilon)e^{-\tau s} & (F_0)_{43}(s) \\ 0 & 0 & 0 & 0 \end{pmatrix}^T$$

$$H(s) = F_0(s) - L(s) + E_0(s) + C_0(s) + \hat{\mathcal{O}}(\cdot)(s),$$

the function $P(s)$ can be expressed as $P(s) = |\det(I_0(s) - L(s) - H(s))|$. As we have seen above, the function $F_0(s)$, as the principal part of the system, imposes the root location for large values of $|s|$, i.e. when $s \in \Omega_1$. However, if s does not belong to Ω_1, then the function $L(s)$ is predominant for the root location. Considering the function $I_0(s) - L(s)$, as $(1 - \rho q(1-\epsilon)) \neq 0$ the associated characteristic equation is given by

$$\big(s - sq(\rho - \tilde{\rho})e^{-\tau s} - k_I q(1 + \lambda l_1(1)))e^{-\tau s}\big) = 0.$$

This corresponds to the characteristic function associated to the system $\dot{z}(t) = (\rho - \tilde{\rho})q\dot{z}(t-\tau) + k_I q(1 + \lambda l_1(1))z(t-\tau)$. Thus, using Condition 4, the function $\det(I_0(s) - L(s))$ does not have any zero in the open Right Half Plane. Adjusting the proof of [5, Theorem 2], we can conclude to the existence of $\delta_m > 0$ and $\kappa_m > 0$ such that if $\delta_0 < \delta_m$, $\delta_1 < \delta_m$ and $\kappa < \kappa_m$ then, $\det(I_0 - L(s) - sH(s)) = \det(I_0 - F_0(s) - C_0(s) - E_0(s) - \hat{\mathcal{O}}(s))$ does not have any zero in $\mathbb{C}^+$.