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An extended conjugate duality for generalized semi-infinite programming problems via a convex decomposition

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ABSTRACT

We present an extended conjugate duality for a generalized semi-infinite programming problem (\mathcal{P}). The extended duality is defined in the context of the absence of convexity of problem (\mathcal{P}), by means of a decomposition into a family of convex subproblems and a conjugate dualization of the subproblems. Under appropriate assumptions, we establish strong extended duality and provide necessary and sufficient optimality conditions for problem (\mathcal{P}). These extended conjugate duality and optimality conditions are new in the literature of generalized semi-infinite programming.

KEYWORDS

Generalized semi-infinite programming, reverse convex problems, DC problems, convex analysis, conjugate duality.

AMS subject classification. 90C34, 90C26, 46N10, 90C46.

1. Introduction

We are concerned with the following generalized semi-infinite programming problem

$$(\mathcal{P}) : \quad \min F(x)$$
$$\text{subject to } \begin{cases} x \in X \\ f(x, y) \geq 0, \forall y \in Y(x) \end{cases}$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ are convex functions, X is a full-dimensional convex compact subset of \mathbb{R}^n , and $Y(x)$ is an index subset of \mathbb{R}^m depending on x defined by

$$Y(x) = \{y \in \mathbb{R}^m : g(x, y) \leq 0\}$$

where $g = (g_1, \dots, g_p)^T : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a convex function. Note that when the multifunction $Y(\cdot)$ is constant, (\mathcal{P}) becomes a semi-infinite programming problem.

As it is well known, generalized semi-infinite programming (GSIP) problems have several concrete and theoretical applications; we cite for example engineering, transportation problems, economics, optimal control and approximation theory (see [24]). Due to its importance, in the last three decades, generalized semi-infinite programming has attracted many researchers in different areas. In particular, several works on optimality conditions have been devoted to this class of problems. For papers dealing with this topic, we cite for example [2], [7], [10]-[14], [20]-[23], [30] and [31].

The problem (\mathcal{P}) with the same convex data has been considered, e.g., in [2] and [13]. The case where the function f is assumed to be convex has been also considered in [14], [25] and [31]. Let us summarize some works on optimality conditions for the class of generalized semi-infinite programming problems. In [2], for problem (\mathcal{P}) necessary and sufficient global optimality conditions are given by using tools from convex analysis combined with a notion of stability of optimization problems. In particular, by means of an optimality condition, the initial problem is reduced to a min-max problem. In [13] and [14], Kanzi provided for problem (\mathcal{P}) necessary optimality conditions of Fritz-John type. The study in these papers was based on the respective use of the subdifferential of marginal functions and some constraint qualifications. In [30], the authors first established optimality conditions for a class of semi-infinite programming problems by using penalty functions. Then they derived optimality conditions for a class of generalized semi-infinite programming problems via semi-infinite programming ones. In [31], Ye and Wu provided first order optimality conditions for a class of generalized semi-infinite programming problems where all functions are assumed to be continuously differentiable. These optimality conditions are obtained under various extended well-known constraint qualifications.

Our aim in this paper is to define an extended conjugate duality and provide optimality conditions for the global optimization problem (\mathcal{P}) . The duality that we will use in our study is the so-called Fenchel-Lagrange duality. It was introduced by Boţ and Wanka in [29] for convex programming problems. But our problem (\mathcal{P}) is not convex, and consequently this duality cannot be applied directly. However, according to our data, and via some transformations using reverse convex and DC problems (that is problems with objective and/or constraint functions are difference of two convex functions) we decompose it into a family of convex minimization subproblems $(\mathcal{P}_{x^*})_{x^* \in \mathbb{R}^n \setminus \{0\}}$. These subproblems have the same objective function F as (\mathcal{P}) , and their constraints are expressed in terms of the conjugate of the function f . As a first step in our procedure of dualization of (\mathcal{P}) , we give the Fenchel-Lagrange duality and provide necessary and sufficient global optimality conditions for every convex subproblem (\mathcal{P}_{x^*}) , $x^* \in \mathbb{R}^n \setminus \{0\}$. Then by means of the duality given for the family of subproblems, we define an extended duality and provide necessary and sufficient global optimality conditions for (\mathcal{P}) . This optimality conditions are expressed in terms of subdifferentials and normal cones in the sense of convex analysis. We note that these extended duality and optimality conditions are new in the literature of generalized semi-infinite programming. An illustrative example is given at the end of the paper.

The paper is organized as follows. In Section 2, we recall some definitions and results related to convex analysis that we will need for our investigation. In Section 3, we give the decomposition of problem (\mathcal{P}) into the family of convex minimization subproblems. In Section 4, after studying the subproblems, we define the extended Fenchel-Lagrange duality and provide necessary and sufficient global optimality con-

ditions for (\mathcal{P}) .

2. Preliminaries

In what follows, the set \mathbb{R}^n is equipped with its usual topology and the following conventions in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, will be adopted:

$$(+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = +\infty$$

$$\begin{cases} 0 \times (+\infty) = +\infty, & \begin{cases} r \times (-\infty) = -\infty, & r \times (+\infty) = +\infty, \text{ for } r \in \mathbb{R}_+^*, \\ r \times (-\infty) = 0, & r \times (+\infty) = -\infty, \text{ for } r \in \mathbb{R}_-^*. \end{cases} \end{cases}$$

For the definitions and details concerning such conventions in $\overline{\mathbb{R}}$ we refer to [18]. Let A be a nonempty subset of \mathbb{R}^n . We shall denote by ψ_A and σ_A the indicator and the support functions of the set A respectively, defined on \mathbb{R}^n by

$$\psi_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad \sigma_A(x) = \sup_{s \in A} \langle s, x \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in \mathbb{R}^n , i.e., for $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T \in \mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. Let $\text{aff}(A)$ denote the smallest affine manifold containing A . Then the relative interior of A denoted by $\text{ri}A$ is the interior of A relative to $\text{aff}(A)$. Note that if A is convex, then $\text{ri}A$ is nonempty ([19]).

Let $\hat{f} : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a function. The conjugate function of \hat{f} relative to the set A is denoted by \hat{f}_A^* and defined on \mathbb{R}^n by

$$\hat{f}_A^*(x^*) = \sup_{x \in A} \{ \langle x^*, x \rangle - \hat{f}(x) \}.$$

When $A = \mathbb{R}^n$, we obtain the classical Legendre-Fenchel conjugate function of \hat{f} denoted by \hat{f}^* . We denote by $\text{dom}(\hat{f}) = \{x \in \mathbb{R}^n : \hat{f}(x) < +\infty\}$ the effective domain of \hat{f} . The function \hat{f} is called *proper* if $\hat{f}(x) > -\infty$, for all $x \in \mathbb{R}^n$ and $\text{dom}(\hat{f}) \neq \emptyset$.

Let now $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function and $\bar{x} \in \text{dom}(\hat{f})$. The subdifferential of \hat{f} at \bar{x} is the set defined by

$$\partial \hat{f}(\bar{x}) = \{x^* \in \mathbb{R}^n : \langle x^*, x - \bar{x} \rangle \leq \hat{f}(x) - \hat{f}(\bar{x}), \forall x \in \mathbb{R}^n\}.$$

The elements of $\partial \hat{f}(\bar{x})$ are called subgradients of \hat{f} at \bar{x} .

Remark 1. We have the following properties ([19]):

- i) $x^* \in \partial \hat{f}(\bar{x}) \iff \hat{f}^*(x^*) + \hat{f}(\bar{x}) = \langle x^*, \bar{x} \rangle$,
- ii) $\langle x^*, x \rangle \leq \hat{f}^*(x^*) + \hat{f}(x)$, for all $x, x^* \in \mathbb{R}^n$, called the Fenchel inequality.

Let C be a nonempty convex subset of \mathbb{R}^n and \bar{x} be an element of C . The normal cone of C at \bar{x} denoted by $\mathcal{N}_C(\bar{x})$ is defined by

$$\mathcal{N}_C(\bar{x}) = \{x^* \in \mathbb{R}^n : \langle x^*, x - \bar{x} \rangle \leq 0, \forall x \in C\}.$$

It is not difficult to verify that $\partial\psi_C(\bar{x}) = \mathcal{N}_C(\bar{x})$, and when $\text{int}C \neq \emptyset$ and $\bar{x} \in \text{int}C$, we have $\mathcal{N}_C(\bar{x}) = \{0_{\mathbb{R}^n}\}$, where $\text{int}C$ denotes the topological interior of C .

We recall the following result on the addition rule of subdifferential calculus that will be used later.

Theorem 2.1. ([19]) *Let \hat{f} and $\hat{g} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be two proper convex functions. Assume that $\text{dom}(\hat{f}) \cap \text{int}(\text{dom}(\hat{g})) \neq \emptyset$. Then for all $x \in \mathbb{R}^n$, we have*

$$\partial(\hat{f} + \hat{g})(x) = \partial\hat{f}(x) + \partial\hat{g}(x).$$

3. The convex decomposition of problem (\mathcal{P})

In this section, via some transformations using reverse convex and DC problems, we give a decomposition of (\mathcal{P}) into a family of convex minimization subproblems. These subproblems have the same objective function as (\mathcal{P}) and their feasible sets are expressed in terms of the conjugate of the function f .

Set

$$v(x) = \inf_{y \in Y(x)} f(x, y).$$

Then problem (\mathcal{P}) can be rewritten as follows

$$(\mathcal{P}) : \quad \min_{\substack{x \in X \\ v(x) \geq 0}} F(x).$$

Throughout the paper we will make the following assumptions:

$$(\mathcal{H}_1) \quad \inf_{x \in X} F(x) < \inf(\mathcal{P}),$$

(\mathcal{H}_2) For every $x \in X$, $Y(x)$ is a nonempty compact subset of \mathbb{R}^m .

When assumption (\mathcal{H}_1) is satisfied, the reverse convex constraint $v(x) \geq 0$ is called essential, and (\mathcal{P}) is termed a reverse convex problem. Note that such an assumption is natural. In fact, in its absence we have $\inf_{x \in X} F(x) = \inf(\mathcal{P})$. So, any solution \hat{x} of problem

$$(\hat{\mathcal{P}}) : \quad \min_{x \in X} F(x)$$

satisfying the constraint $v(\hat{x}) \geq 0$ is a solution of problem (\mathcal{P}) .

Let us summarize the following properties that will be useful for our investigation.

Remark 2. 1) Since F and f are convex functions, then they are continuous on the interior of their effective domains which are equal respectively to \mathbb{R}^n and $\mathbb{R}^n \times \mathbb{R}^m$ (see [19]).

2) Assume that assumption (\mathcal{H}_2) is satisfied.

i) From the convexity of X , we have $\text{ri}X$ is nonempty, and since $\text{aff}(X) = \mathbb{R}^n$, then $\text{ri}X = \text{int}X$.

- ii) Since for every $x \in X$, the function $f(x, \cdot)$ is lower semi-continuous, the problem $\min_{y \in Y(x)} f(x, y)$ admits at least one solution. So, for every $x \in X$, $v(x)$ is a finite real number. Hence $X \subset \text{dom}v(\cdot)$. Then $\text{aff}(\text{dom}v(\cdot)) = \mathbb{R}^n$, and $\text{int}X \subset \text{int}(\text{dom}v(\cdot))$.
- iii) We have $v(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper convex function. So, it is continuous on $\text{int}(\text{dom}v(\cdot))$ (see [19]) and hence it is continuous on $\text{int}X$.

Proposition 3.1. ([1], [26]) *Let assumptions (\mathcal{H}_1) and (\mathcal{H}_2) be fulfilled and let $\hat{x} \in X$ be a solution of problem (\mathcal{P}) . Then $v(\hat{x}) = 0$.*

Proof. Assume the contrary, that is $v(\hat{x}) > 0$. Assumption (\mathcal{H}_1) implies that there exists $\bar{x} \in X$ such that

$$F(\bar{x}) < \inf(\mathcal{P}) \quad \text{and} \quad v(\bar{x}) < 0.$$

Since $v(\bar{x}) < 0$ and $v(\hat{x}) > 0$, then by the continuity of the marginal function $v(\cdot)$ on $\text{int}X$, there exists $\tilde{x} \in]\bar{x}, \hat{x}[$ such that $v(\tilde{x}) = 0$. Let $\alpha \in]0, 1[$ such that

$$\tilde{x} = \alpha\bar{x} + (1 - \alpha)\hat{x}$$

which belongs to X . Hence, \tilde{x} is a feasible point of problem (\mathcal{P}) . Furthermore, the convexity of F yields

$$\begin{aligned} F(\tilde{x}) &\leq \alpha F(\bar{x}) + (1 - \alpha)F(\hat{x}) \\ &< \inf(\mathcal{P}) \end{aligned}$$

which contradicts the optimality of \hat{x} . □

Remark 3. 1) According to Proposition 3.1 the formulation of (\mathcal{P}) is reduced to the following

$$(\mathcal{P}) : \quad \min_{\substack{x \in X \\ v(x)=0}} F(x).$$

- 2) According to the proof, the result of Proposition 3.1 remains true if we replace the marginal function $v(\cdot)$ by a continuous function $\tilde{v}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$. More precisely, let the problem

$$(\tilde{\mathcal{P}}) : \quad \min_{\substack{x \in \tilde{X} \\ \tilde{v}(x) \geq 0}} \tilde{F}(x)$$

where \tilde{X} is a nonempty convex compact subset of \mathbb{R}^n (not necessarily full-dimensional) and $\tilde{F}, \tilde{v}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ are respectively convex and continuous functions. Then under the following assumption

$$\inf_{x \in \tilde{X}} \tilde{F}(x) < \min_{\substack{x \in \tilde{X} \\ \tilde{v}(x) \geq 0}} \tilde{F}(x).$$

the constraint $\tilde{v}(x) \geq 0$ is active at the solution set of problem $(\tilde{\mathcal{P}})$.

Let

$$\mathcal{A} = \{x \in X : v(x) \geq 0\}$$

denote the feasible set of problem (\mathcal{P}) and define the following functions on \mathbb{R}^n

$$\hat{g}_1(x) = \psi_X(x), \quad \hat{g}_2(x) = v(x) \quad \text{and} \quad h(x) = 0.$$

Then the constraint set \mathcal{A} can be rewritten as

$$\mathcal{A} = \{x \in \mathbb{R}^n : \hat{g}_1(x) - h(x) \leq 0, h(x) - \hat{g}_2(x) \leq 0\}$$

and accordingly, (\mathcal{P}) can be viewed as a DC problem. Let $GrY(\cdot)$ be the graph of the multifunction $Y(\cdot)$, i.e.,

$$GrY(\cdot) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in Y(x)\}.$$

The following result expresses the set \mathcal{A} by means of conjugate functions.

Proposition 3.2. *We have*

$$\mathcal{A} = \bigcup_{\substack{(x^*, t^*) \in \mathbb{R}^n \times \mathbb{R}^m \\ h^*(x^*) - \hat{g}_1^*(x^*) \leq 0 \\ \hat{g}_2^*(t^*) - h^*(t^*) \leq 0}} \{x \in \mathbb{R}^n : \hat{g}_1(x) + h^*(x^*) - \langle x^*, x \rangle \leq 0, h(x) + \hat{g}_2^*(t^*) - \langle t^*, x \rangle \leq 0\}.$$

Proof. The result is a direct application of Lemma 2.1 in [16].

Let us calculate the conjugates on \mathbb{R}^n of the constraint functions \hat{g}_i , $i = 1, 2$, and h .

Proposition 3.3. *Let $x^* \in \mathbb{R}^n$. Then we have*

- i) $\hat{g}_1^*(x^*) = \sigma_X(x^*)$,
- ii) $\hat{g}_2^*(x^*) = f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m})$,
- iii) $h^*(x^*) = \psi_{\{0_{\mathbb{R}^n}\}}(x^*)$.

Proof. We have

$$i) \hat{g}_1^*(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - \psi_X(x)\} = \sup_{x \in X} \langle x^*, x \rangle = \sigma_X(x^*),$$

ii)

$$\begin{aligned} \hat{g}_2^*(x^*) &= \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - v(x)\} = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - \inf_{y \in Y(x)} f(x, y)\} \\ &= \sup_{x \in \mathbb{R}^n, y \in Y(x)} \{\langle x^*, x \rangle - f(x, y)\} = \sup_{(x, y) \in GrY(\cdot)} \{\langle x^*, x \rangle - f(x, y)\} \\ &= \sup_{(x, y) \in GrY(\cdot)} \left\{ \left\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^m} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle - f(x, y) \right\} = f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}), \end{aligned}$$

$$iii) h^*(x^*) = \sup_{x \in \mathbb{R}^n} \langle x^*, x \rangle = \begin{cases} +\infty, & \text{if } x^* \neq 0, \\ 0, & \text{if } x^* = 0. \end{cases}$$

$$\text{So, } h^*(x^*) = \psi_{\{0_{\mathbb{R}^n}\}}(x^*).$$

□

For $x^* \in \mathbb{R}^n$, set

$$\mathcal{A}_{x^*} = \{x \in X : f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, x \rangle \leq 0\}$$

which is a convex set. By a simple calculation and using the expression of the set \mathcal{A} given in Proposition 3.2, we obtain the following new expression of \mathcal{A} using the sets \mathcal{A}_{x^*} , $x^* \in \mathbb{R}^n$.

Proposition 3.4. *We have $\mathcal{A} = \bigcup_{x^* \in \mathbb{R}^n \setminus \{0\}} \mathcal{A}_{x^*}$.*

Proof. According to Propositions 3.2 and 3.3 the set \mathcal{A} is defined by the following two systems

$$\begin{cases} \psi_X(x) + \psi_{\{0_{\mathbb{R}^n}\}}(x^*) - \langle x^*, x \rangle \leq 0, & (1) \\ \psi_{\{0_{\mathbb{R}^n}\}}(x^*) - \sigma_X(x^*) \leq 0, & (2) \end{cases}$$

and

$$\begin{cases} f_{GrY(\cdot)}^*(t^*, 0_{\mathbb{R}^m}) - \langle t^*, x \rangle \leq 0, & (3) \\ f_{GrY(\cdot)}^*(t^*, 0_{\mathbb{R}^m}) - \psi_{\{0_{\mathbb{R}^n}\}}(t^*) \leq 0. & (4) \end{cases}$$

Then the inequalities (1) and (2) yield $x^* = 0$ and $x \in X$. On the other hand, according to our hypothesis (\mathcal{H}_1) the inequality (4) yields that $t^* \neq 0$. In fact, assume that $t^* = 0$. Then inequality (4) yields

$$f_{GrY(\cdot)}^*(t^*, 0_{\mathbb{R}^m}) = - \inf_{x \in \mathbb{R}^n} v(x) \leq 0.$$

So $v(x) \geq 0$, for all $x \in \mathbb{R}^n$. This contradicts the hypothesis (\mathcal{H}_1) which says that the constraint $v(x) \geq 0$ is essential. That is (see the proof of Proposition 3.1) there exists $\bar{x} \in X$ such that

$$F(\bar{x}) < \inf(\mathcal{P}) \quad \text{and} \quad v(\bar{x}) < 0.$$

Therefore, $\mathcal{A} = \bigcup_{t^* \in \mathbb{R}^n \setminus \{0\}} \mathcal{A}_{t^*}$. □

For $x^* \in \mathbb{R}^n \setminus \{0\}$, we consider the following convex minimization subproblem

$$(\mathcal{P}_{x^*}) : \min_{x \in \mathcal{A}_{x^*}} F(x).$$

Remark 4. Let \bar{x} be a solution of problem (\mathcal{P}) . Then from Proposition 3.4, there exists $x^* \in \mathbb{R}^n \setminus \{0\}$ such that \bar{x} solves problem (\mathcal{P}_{x^*}) , and $\inf(\mathcal{P}) = \inf(\mathcal{P}_{x^*})$. Therefore, we get a decomposition of problem (\mathcal{P}) into the family of convex subproblems (\mathcal{P}_{x^*}) , $x^* \in \mathbb{R}^n \setminus \{0\}$, which we term a convex decomposition.

4. Extended Fenchel-Lagrange duality and optimality conditions for problem (\mathcal{P})

In this section, we define an extended conjugate duality and provide necessary and sufficient global optimality conditions for the generalized semi-infinite programming problem (\mathcal{P}) . This extended duality is defined by means of the so-called Fenchel-Lagrange duality. This latter was first introduced by Boç and Wanka in [29] for convex programming problems, and afterwards extended to some generalized convex programming problems (see [5] and [6]). In fact, since our study is based on convexity and problem (\mathcal{P}) is not convex, this duality cannot be applied directly. So, using the above convex decomposition of (\mathcal{P}) , we first give the Fenchel-Lagrange dual to each convex minimization subproblem (\mathcal{P}_{x^*}) , $x^* \in \mathbb{R}^n \setminus \{0\}$, and then define the extended Fenchel-Lagrange duality for our initial global optimization problem (\mathcal{P}) .

4.1. Fenchel-Lagrange duality and optimality conditions for the subproblems

In this subsection, first we give the Fenchel-Lagrange duality for the subproblems (\mathcal{P}_{x^*}) , $x^* \in \mathbb{R}^n \setminus \{0\}$. Before going further, let us recall this duality. Consider the following minimization problem

$$(\tilde{\mathcal{P}}) : \quad \min_{\substack{x \in \tilde{X} \\ \tilde{g}(x) \leq 0}} \tilde{f}(x)$$

where \tilde{X} is a nonempty convex subset of \mathbb{R}^n , $\tilde{f} : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ and $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_m)^T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are convex functions, with $\text{dom}(\tilde{f}) = \tilde{X}$, and $m \geq 1$. The Fenchel-Lagrange dual $(\tilde{\mathcal{D}})$ of problem $(\tilde{\mathcal{P}})$ is defined as follows ([29])

$$(\tilde{\mathcal{D}}) : \quad \sup_{\substack{p \in \mathbb{R}^n \\ q \in \mathbb{R}_+^m}} \left\{ -\tilde{f}^*(p) - (q^T \tilde{g})_{\tilde{X}}^*(-p) \right\}$$

where for $q = (q_1, \dots, q_m)^T$, the function $q^T \tilde{g}(\cdot)$ is defined on \mathbb{R}^n by $q^T \tilde{g}(x) = \sum_{i=1}^m q_i \tilde{g}_i(x)$.

We say that we have strong Fenchel-Lagrange duality between problems $(\tilde{\mathcal{P}})$ and $(\tilde{\mathcal{D}})$, if $\inf(\tilde{\mathcal{P}}) = \sup(\tilde{\mathcal{D}})$ and the dual problem $(\tilde{\mathcal{D}})$ admits a solution. We note that the Fenchel-Lagrange duality is a combination of the well-known Lagrange and Fenchel dualities.

Let us return to our problem (\mathcal{P}) . Define on \mathbb{R}^n the function

$$g_{x^*}(\cdot) = f_{GY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, \cdot \rangle$$

and write the problem (\mathcal{P}_{x^*}) under the formulation

$$(\mathcal{P}_{x^*}) : \quad \min_{\substack{x \in X \\ g_{x^*}(x) \leq 0}} F(x).$$

Let (\mathcal{D}_{x^*}) denote the Fenchel-Lagrange dual to (\mathcal{P}_{x^*}) . Then according to the above definition of duality, (\mathcal{D}_{x^*}) has the following form

$$(\mathcal{D}_{x^*}) : \quad \sup_{\substack{p \in \mathbb{R}^n \\ q \in \mathbb{R}_+}} \{-F^*(p) - (qg_{x^*})_X^*(-p)\}.$$

By using the explicit expression of $(qg_{x^*})_X^*$, we obtain

$$(\mathcal{D}_{x^*}) : \quad \sup_{\substack{p \in \mathbb{R}^n \\ q \in \mathbb{R}_+}} \inf_{x \in X} \left\{ -F^*(p) + \langle p - q x^*, x \rangle + q f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) \right\}.$$

Then we have the following results on weak and strong dualities for the primal-dual pair (\mathcal{P}_{x^*}) - (\mathcal{D}_{x^*}) .

Theorem 4.1. (Weak Fenchel-Lagrange duality) *Let $x^* \in \mathbb{R}^n \setminus \{0\}$ and assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) are fulfilled. Then weak duality always holds between problems (\mathcal{P}_{x^*}) and (\mathcal{D}_{x^*}) , i.e.,*

$$\sup(\mathcal{D}_{x^*}) \leq \inf(\mathcal{P}_{x^*}).$$

Proof. See for example [29].

In order to establish our main results, we will use the following classical Slater constraint qualification ([4]):

(CQ) For every $x^* \in \mathbb{R}^n \setminus \{0\}$, there exists $x_{x^*} \in X$ such that:

$$f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, x_{x^*} \rangle < 0.$$

Theorem 4.2. (Strong Fenchel-Lagrange duality) *Let $x^* \in \mathbb{R}^n \setminus \{0\}$ and assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) and the constraint qualification (CQ) are fulfilled. Then strong duality holds between problems (\mathcal{P}_{x^*}) and (\mathcal{D}_{x^*}) , i.e.,*

$$\inf(\mathcal{P}_{x^*}) = \sup(\mathcal{D}_{x^*})$$

and the dual (\mathcal{D}_{x^*}) has a solution.

Proof. See Theorem 3.3 in [8].

4.2. Optimality conditions for the subproblems

In this subsection, using the previous result on strong duality, we provide necessary and sufficient global optimality conditions for the primal-dual pair (\mathcal{P}_{x^*}) - (\mathcal{D}_{x^*}) , $x^* \in \mathbb{R}^n \setminus \{0\}$.

Theorem 4.3 (Necessary optimality conditions). *Let $x^* \in \mathbb{R}^n \setminus \{0\}$ and \bar{x} be a solution of problem (\mathcal{P}_{x^*}) . Assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) and the constraint qualification condition (CQ) are fulfilled. Then there exists $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+^*$ a solution of the dual problem (\mathcal{D}_{x^*}) such that the following optimality conditions are satisfied*

$$i) \quad \bar{p} \in \partial F(\bar{x}),$$

- ii) there exists $\bar{y} \in Y(\bar{x})$ such that $\begin{pmatrix} x^* \\ 0_{\mathbb{R}^m} \end{pmatrix} \in \partial f(\bar{x}, \bar{y}) + \mathcal{N}_{GrY(\cdot)}(\bar{x}, \bar{y})$,
- iii) $\bar{q}x^* - \bar{p} \in \mathcal{N}_X(\bar{x})$.

Proof. Since the constraint qualification condition (CQ) is fulfilled, then by adopting for the dual problem (\mathcal{D}_{x^*}) the formulation

$$(\mathcal{D}_{x^*}) : \quad \sup_{\substack{p \in \mathbb{R}^n \\ q \in \mathbb{R}_+}} \{-F^*(p) - (qg_{x^*})_X^*(-p)\}$$

we deduce from Theorem 3.2 in [4] that there exists $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+$ such that the following optimality conditions are satisfied

- a) $F^*(\bar{p}) + F(\bar{x}) = \langle \bar{p}, \bar{x} \rangle$,
- b) $\bar{q}g_{x^*}(\bar{x}) = \bar{q}(f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, \bar{x} \rangle) = 0$,
- c) $(\bar{q}g_{x^*})_X^*(-\bar{p}) = \langle -\bar{p}, \bar{x} \rangle$.

We have

$$\begin{aligned} (\bar{q}g_{x^*})_X^*(-\bar{p}) &= \sup_{x \in X} \left\{ \langle -\bar{p}, x \rangle - \bar{q}(f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, x \rangle) \right\} \\ &= \sup_{x \in X} \{ \langle \bar{q}x^* - \bar{p}, x \rangle \} - \bar{q}f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) \\ &= \sigma_X(\bar{q}x^* - \bar{p}) - \bar{q}f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}). \end{aligned}$$

Then condition c) can be rewritten as

$$\sigma_X(\bar{q}x^* - \bar{p}) - \bar{q}f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) = \langle -\bar{p}, \bar{x} \rangle. \quad (5)$$

Let us show that $\bar{q} > 0$. Assume that $\bar{q} = 0$. Then

$$\sigma_X(-\bar{p}) = \langle -\bar{p}, \bar{x} \rangle.$$

Since $\bar{x} \in X$ and $\sigma_X = (\psi_X)^*$, we have

$$(\psi_X)^*(-\bar{p}) + \psi_X(\bar{x}) = \langle -\bar{p}, \bar{x} \rangle.$$

That is

$$-\bar{p} \in \partial\psi_X(\bar{x}) = \mathcal{N}_X(\bar{x}). \quad (6)$$

Moreover, equation a) is equivalent to the following

$$\bar{p} \in \partial F(\bar{x}). \quad (7)$$

Then, adding (6) and (7), we obtain

$$0 \in \partial F(\bar{x}) + \mathcal{N}_X(\bar{x}).$$

That is \bar{x} is a solution of the following problem

$$\min_{x \in X} F(x).$$

So that

$$\inf(\mathcal{P}_{x^*}) = \inf_{x \in X} F(x).$$

Using the fact that $\inf(\mathcal{P}) \leq \inf(\mathcal{P}_{x^*})$, we obtain

$$\inf(\mathcal{P}) \leq \inf_{x \in X} F(x)$$

which contradicts the hypothesis (\mathcal{H}_1). So, $\bar{q} > 0$. Therefore, the complementary slackness condition b) yields

$$f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, \bar{x} \rangle = 0.$$

Let $\bar{y} \in Y(\bar{x})$ such that $v(\bar{x}) = f(\bar{x}, \bar{y})$ [such a point exists according to 2)-ii) of Remark 2]. By using Proposition 3.1, we have $v(\bar{x}) = 0$. Hence

$$f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) + f(\bar{x}, \bar{y}) = \left\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^m} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle.$$

By the subdifferential calculus, the above equation can be written as

$$\begin{aligned} \begin{pmatrix} x^* \\ 0_{\mathbb{R}^m} \end{pmatrix} &\in \partial(f + \psi_{GrY(\cdot)})(\bar{x}, \bar{y}) \\ &= \partial f(\bar{x}, \bar{y}) + \mathcal{N}_{GrY(\cdot)}(\bar{x}, \bar{y}) \end{aligned}$$

where the equality follows from Theorem 2.1 ($\text{dom}(f) = \mathbb{R}^n \times \mathbb{R}^m$). That is the property ii) is satisfied.

In order to show the property iii), let us write the equation (5) as follows

$$(\psi_X)^*(\bar{q}x^* - \bar{p}) - \underbrace{\langle \bar{q}x^*, \bar{x} \rangle + \bar{q} f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \bar{q} f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m})}_{=0} = \langle -\bar{p}, \bar{x} \rangle$$

where the equality $-\langle \bar{q}x^*, \bar{x} \rangle + \bar{q} f_{GrY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) = 0$ follows from property b). Then

$$(\psi_X)^*(\bar{q}x^* - \bar{p}) = \langle \bar{q}x^* - \bar{p}, \bar{x} \rangle$$

which is equivalent to

$$(\psi_X)^*(\bar{q}x^* - \bar{p}) + \psi_X(\bar{x}) = \langle \bar{q}x^* - \bar{p}, \bar{x} \rangle.$$

That is

$$\bar{q}x^* - \bar{p} \in \partial\psi_X(\bar{x}) = \mathcal{N}_X(\bar{x}).$$

Hence, property *iii*) is satisfied. \square

Corollary 4.4. *Let $x^* \in \mathbb{R}^n \setminus \{0\}$ and \bar{x} be a solution of problem (\mathcal{P}_{x^*}) . Let assumptions of Theorem 4.3 be satisfied. Then there exists $\bar{q} \in \mathbb{R}_+^*$ such that \bar{x} solves the problem*

$$\min_{x \in X} \{F(x) - \langle \bar{q}x^*, x \rangle\}.$$

Proof. From Theorem 4.3, there exists $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+^*$, such that

$$\bar{p} \in \partial F(\bar{x}) \quad \text{and} \quad \bar{q}x^* - \bar{p} \in \mathcal{N}_X(\bar{x}).$$

So

$$\bar{q}x^* \in \partial F(\bar{x}) + \mathcal{N}_X(\bar{x}).$$

Then

$$0 \in \partial F(\bar{x}) + \mathcal{N}_X(\bar{x}) - \bar{q}x^*.$$

That is \bar{x} solves the problem

$$\min_{x \in X} \{F(x) - \langle \bar{q}x^*, x \rangle\}. \quad \square$$

For a given $x^* \in \mathbb{R}^n \setminus \{0\}$, the following result gives sufficient optimality conditions for the subproblem (\mathcal{P}_{x^*}) .

Theorem 4.5 (Sufficient optimality conditions). *Let $x^* \in \mathbb{R}^n \setminus \{0\}$. Assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) are fulfilled. Let $\bar{x} \in X$ and $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+^*$ satisfy the conditions *i*) – *iii*) in Theorem 4.3. Then \bar{x} and (\bar{p}, \bar{q}) solve respectively the problems (\mathcal{P}_{x^*}) and (\mathcal{D}_{x^*}) . Furthermore, strong Fenchel-Lagrange duality holds between them.*

Proof. In the same way as in the proof of Theorem 4.3, we show that

$$f_{GY(\cdot)}^*(x^*, 0_{\mathbb{R}^m}) - \langle x^*, \bar{x} \rangle = 0.$$

So, $\bar{x} \in \mathcal{A}_{x^*}$. That is \bar{x} is a feasible point of problem (\mathcal{P}_{x^*}) . On the other hand, conditions *i*) in Theorem 4.3 is equivalent to

$$F^*(\bar{p}) + F(\bar{x}) = \langle \bar{p}, \bar{x} \rangle. \quad (8)$$

Moreover, from the last part of the proof of Theorem 4.3, the condition *iii*) in this theorem can be rewritten as

$$(\bar{q}g_{x^*})_X^*(-\bar{p}) = \langle -\bar{p}, \bar{x} \rangle. \quad (9)$$

Then adding the equations (8) and (9), we obtain

$$F^*(\bar{p}) + F(\bar{x}) + (\bar{q}g_{x^*})_X^*(-\bar{p}) = 0.$$

Therefore

$$\inf(\mathcal{P}_{x^*}) \leq F(\bar{x}) = -F^*(\bar{p}) - (\bar{q}g_{x^*})_X^*(-\bar{p}) \leq \sup(\mathcal{D}_{x^*}).$$

By weak duality, we deduce that

$$\begin{aligned} \inf(\mathcal{P}_{x^*}) &= F(\bar{x}) \\ &= \sup(\mathcal{D}_{x^*}) \\ &= -F^*(\bar{p}) - (\bar{q}g_{x^*})_X^*(-\bar{p}). \end{aligned}$$

It follows that \bar{x} and (\bar{p}, \bar{q}) solve (\mathcal{P}_{x^*}) and (\mathcal{D}_{x^*}) respectively. Moreover, (\mathcal{P}_{x^*}) and (\mathcal{D}_{x^*}) are in strong Fenchel-Lagrange duality. \square

4.3. *Extended Fenchel-Lagrange duality and optimality conditions for problem (\mathcal{P})*

As we have mentioned in the introduction, since problem (\mathcal{P}) is not convex, we cannot apply directly the Fenchel-Lagrange duality. However, thanks to the convex decomposition of (\mathcal{P}) into the family of subproblems $(\mathcal{P}_{x^*})_{x^* \in \mathbb{R}^n \setminus \{0\}}$ we can define an extended Fenchel-Lagrange duality for (\mathcal{P}) in the following sense.

Definition 4.6. We say that the generalized semi-infinite programming problem (\mathcal{P}) is in

- i) weak extended Fenchel-Lagrange duality if there exists $x^* \in \mathbb{R}^n \setminus \{0\}$ such that $\inf(\mathcal{P}) \geq \sup(\mathcal{D}_{x^*})$,
- ii) strong extended Fenchel-Lagrange duality if there exists $x^* \in \mathbb{R}^n \setminus \{0\}$ such that $\inf(\mathcal{P}) = \sup(\mathcal{D}_{x^*})$, and problem (\mathcal{D}_{x^*}) admits a solution,

where we recall that (\mathcal{D}_{x^*}) denotes the dual of the subproblem (\mathcal{P}_{x^*}) .

Theorem 4.7 (Weak extended Fenchel-Lagrange duality). *Assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) are fulfilled. Then the generalized semi-infinite programming problem (\mathcal{P}) is always in weak extended Fenchel-Lagrange duality.*

Proof. Let $\bar{x} \in \mathcal{A}$ be a solution of problem (\mathcal{P}) . Since

$$\mathcal{A} = \bigcup_{x^* \in \mathbb{R}^n \setminus \{0\}} \mathcal{A}_{x^*}$$

then there exists $x^* \in \mathbb{R}^n \setminus \{0\}$, such that $\bar{x} \in \mathcal{A}_{x^*}$ and \bar{x} solves the problem

$$(\mathcal{P}_{x^*}) : \quad \min_{x \in \mathcal{A}_{x^*}} F(x).$$

On the other hand, Theorem 4.1 implies that

$$\inf(\mathcal{P}_{x^*}) \geq \sup(\mathcal{D}_{x^*}).$$

Finally, using the fact that $\inf(\mathcal{P}) = \inf(\mathcal{P}_{x^*})$, we obtain the result. \square

Theorem 4.8 (Strong extended Fenchel-Lagrange duality). *Assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) and the constraint qualification condition (CQ) are fulfilled. Then the problem (\mathcal{P}) is in strong extended Fenchel-Lagrange duality.*

Proof. The proof is similar to the one of Theorem 4.7 by using Theorem 4.2 instead of Theorem 4.1. \square

4.4. Optimality conditions for problem (\mathcal{P})

In this subsection, by means of the duality given for the family of subproblems, we provide necessary and sufficient optimality conditions for problem (\mathcal{P}) .

Theorem 4.9 (Necessary optimality conditions). *Assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) and the constraint qualification condition (CQ) are fulfilled. Let \bar{x} be a solution of the generalized semi-infinite programming problem (\mathcal{P}) . Then there exist $x^* \in \mathbb{R}^n \setminus \{0\}$ and $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+^*$ a solution of the dual problem (\mathcal{D}_{x^*}) such that*

- i) $\bar{p} \in \partial F(\bar{x})$,
- ii) there exists $\bar{y} \in Y(\bar{x})$ verifying $\begin{pmatrix} x^* \\ 0_{\mathbb{R}^m} \end{pmatrix} \in \partial f(\bar{x}, \bar{y}) + \mathcal{N}_{GrY(\cdot)}(\bar{x}, \bar{y})$,
- iii) $\bar{q}x^* - \bar{p} \in \mathcal{N}_X(\bar{x})$.

Proof. Using the fact that $\mathcal{A} = \bigcup_{x^* \in \mathbb{R}^n \setminus \{0\}} \mathcal{A}_{x^*}$, we deduce that there exists $x^* \in \mathbb{R}^n \setminus \{0\}$ such that $\bar{x} \in \mathcal{A}_{x^*}$ and \bar{x} solves problem (\mathcal{P}_{x^*}) . Then the result follows from Theorem 4.3. \square

Corollary 4.10. *Assume that assumptions of Theorem 4.9 are fulfilled. Let \bar{x} be a solution of problem (\mathcal{P}) . Then there exist $x^* \in \mathbb{R}^n \setminus \{0\}$ and $\bar{q} \in \mathbb{R}_+^*$ such that \bar{x} solves the problem*

$$\min_{x \in X} \{F(x) - \langle \bar{q}x^*, x \rangle\}.$$

Proof. From Theorem 4.9, there exist $x^* \in \mathbb{R}^n \setminus \{0\}$ and $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+^*$ a solution of the dual problem (\mathcal{D}_{x^*}) of (\mathcal{P}_{x^*}) such that

$$\bar{p} \in \partial F(\bar{x}) \quad \text{and} \quad \bar{q}x^* - \bar{p} \in \mathcal{N}_X(\bar{x}).$$

Then the rest of the proof is identical to the one of Corollary 4.4. \square

For $x \in \mathbb{R}^n$, define on \mathbb{R}^m the function $f_x(\cdot)$ by

$$f_x(y) = f(x, y).$$

The following theorem provides sufficient optimality conditions for problem (\mathcal{P}) .

Theorem 4.11 (Sufficient optimality conditions). *Let $\bar{x} \in X$. Assume that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) are fulfilled. If moreover, there exist $\bar{y} \in \mathbb{R}^m$, $x^* \in \mathbb{R}^n \setminus \{0\}$, and $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+^*$, such that the following conditions are satisfied*

- i) $0_{\mathbb{R}^m} \in \partial f_{\bar{x}}(\bar{y}) + \mathcal{N}_{Y(\bar{x})}(\bar{y})$, $g(\bar{x}, \bar{y}) \leq 0$, and $f(\bar{x}, \bar{y}) = 0$,
- ii) $\bar{p} \in \partial F(\bar{x})$,
- iii) $\bar{q}x^* - \bar{p} \in \mathcal{N}_X(\bar{x})$,

iv) (\bar{x}, \bar{y}) solves the problem

$$(\mathcal{Q}_{x^*}) : \max_{\substack{(x,y) \in X \times \mathbb{R}^m \\ g(x,y) \leq 0}} \{f(x,y) - \langle x^*, x \rangle\}$$

then \bar{x} is a solution of the generalized semi-infinite programming problem (\mathcal{P}) .

Proof. Feasibility: According to Proposition 3.1, problem (\mathcal{P}) can be rewritten as

$$(\mathcal{P}) : \min_{\substack{x \in X \\ v(x)=0}} F(x).$$

From condition i), we have

$$0_{\mathbb{R}^m} \in \partial f_{\bar{x}}(\bar{y}) + \mathcal{N}_{Y(\bar{x})}(\bar{y}) \quad \text{and} \quad g(\bar{x}, \bar{y}) \leq 0.$$

That is \bar{y} solves the problem

$$\min_{y \in Y(\bar{x})} f_{\bar{x}}(y).$$

It follows that $v(\bar{x}) = f(\bar{x}, \bar{y}) = 0$. Therefore, \bar{x} is a feasible point of (\mathcal{P}) .

Optimality: In order to prove the optimality of \bar{x} for problem (\mathcal{P}) , let $x \in X$ be such that $v(x) = 0$ [see 1) of Remark 3]. From the property $\bar{p} \in \partial F(\bar{x})$, we have

$$F(x') \geq F(\bar{x}) + \langle \bar{p}, x' - \bar{x} \rangle, \quad \forall x' \in \mathbb{R}^n.$$

In particular

$$F(x) \geq F(\bar{x}) + \langle \bar{p}, x - \bar{x} \rangle. \tag{10}$$

Property iii) yields

$$\langle x^*, x - \bar{x} \rangle \leq \left\langle \frac{\bar{p}}{\bar{q}}, x - \bar{x} \right\rangle.$$

Furthermore, since (\bar{x}, \bar{y}) solves the problem (\mathcal{Q}_{x^*}) , it follows that

$$f(\bar{x}, \bar{y}) - \langle x^*, \bar{x} \rangle \geq f(x, y) - \langle x^*, x \rangle, \quad \forall y \in Y(x).$$

Let $y \in Y(x)$. We have

$$\begin{aligned} f(\bar{x}, \bar{y}) - v(x) &\geq f(\bar{x}, \bar{y}) - f(x, y) \\ &\geq \langle x^*, \bar{x} - x \rangle \\ &\geq \left\langle \frac{\bar{p}}{\bar{q}}, \bar{x} - x \right\rangle \end{aligned} \tag{11}$$

where the first inequality follows from that $v(x) \leq f(x, y)$. On the other hand, we

have $v(x) = 0$ and $f(\bar{x}, \bar{y}) = 0$. Then, from (11) we deduce that

$$\begin{aligned} 0 &\leq f(x, y) \\ &\leq \langle x^*, x - \bar{x} \rangle \\ &\leq \left\langle \frac{\bar{p}}{q}, x - \bar{x} \right\rangle. \end{aligned}$$

Hence

$$\langle \bar{p}, x - \bar{x} \rangle \geq 0.$$

Finally, using (10) we obtain

$$F(x) \geq F(\bar{x}).$$

Therefore, \bar{x} solves problem (\mathcal{P}) . □

To illustrate our study, let us consider the following simple example in which we apply Theorem 4.11.

Example 4.12. Let $X = [-\frac{1}{2}, \frac{1}{2}]$, $F : \mathbb{R} \rightarrow \mathbb{R}$, and $f, g_i : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $i = 1, \dots, 4$, be the functions defined by

$$F(x) = x^2 + |x| + 1, \quad f(x, y) = x^2 + 2x + y_1 + y_2,$$

$$\begin{cases} g_1(x, y) = y_1 + \frac{3}{4}, & g_2(x, y) = x^2 - 1 - y_1, \\ g_3(x, y) = y_2 + \frac{1}{2}, & g_4(x, y) = x - 1 - y_2. \end{cases}$$

Then the problem that we are concerned with is

$$(\mathcal{P}) : \quad \min(x^2 + |x| + 1)$$

$$\text{subject to } \begin{cases} x \in X \\ x^2 + 2x + y_1 + y_2 \geq 0 \\ \text{for all } (y_1, y_2) \in \mathbb{R}^2 \text{ such that} \\ y_1 + \frac{3}{4} \leq 0, x^2 - 1 - y_1 \leq 0 \\ y_2 + \frac{1}{2} \leq 0, x - 1 - y_2 \leq 0. \end{cases}$$

For $x \in X$, we have $Y(x) = [x^2 - 1, -\frac{3}{4}] \times [x - 1, -\frac{1}{2}]$, which is a compact subset of \mathbb{R}^2 . That is assumption (\mathcal{H}_2) is satisfied. Moreover, X is a convex compact set and F, g_i , $i \in \{1, \dots, 4\}$, and f are convex functions. Let us apply our duality approach to solve problem (\mathcal{P}) .

Let $\bar{x} \in X$. We will impose on \bar{x} the sufficient optimality conditions $i) - iv)$ in Theorem 4.11 (if such a point exists).

Condition i): Let $\bar{y} = (\bar{y}_1, \bar{y}_2)^T \in \mathbb{R}^2$ be a solution of problem $\min_{y \in Y(\bar{x})} f(\bar{x}, y)$, verifying $f(\bar{x}, \bar{y}) = 0$ (if it exists). By simple calculation, we find $\bar{y} = (\bar{y}_1, \bar{y}_2)^T = (\bar{x}^2 - 1, \bar{x} - 1)^T$, and then $f(\bar{x}, \bar{y}) = 2\bar{x}^2 + 3\bar{x} - 2$. The equation $f(\bar{x}, \bar{y}) = 0$ gives two values $\bar{x} = -2$, and $\bar{x} = \frac{1}{2}$. But only $\bar{x} = \frac{1}{2}$ is feasible for (\mathcal{P}) . It follows that $\bar{y}_1 = -\frac{3}{4}$, and $\bar{y}_2 = -\frac{1}{2}$.

Now, let us verify if there exist $\bar{p} \in \mathbb{R}$, $\bar{q} > 0$ and $x^* \in \mathbb{R} \setminus \{0\}$, such that the conditions *ii*) – *iv*) are satisfied.

$$\text{Condition ii): We have } \partial F(x) = \begin{cases} \{2x - 1\}, & \text{if } x \in [-\frac{1}{2}, 0[, \\ [-1, 1], & \text{if } x = 0, \\ \{2x + 1\}, & \text{if } x \in]0, \frac{1}{2}]. \end{cases}$$

For $\bar{x} = \frac{1}{2}$, the condition $\bar{p} \in \partial F(\bar{x})$ yields $\bar{p} = 2$.

Condition iii): We must have $\bar{q}x^* - \bar{p} \in \mathcal{N}_X(\frac{1}{2})$. We have $\mathcal{N}_X(\frac{1}{2}) = \mathbb{R}_+$. So, $\bar{q}x^* \geq \bar{p} = 2$.

Condition iv): The element x^* must be chosen such that (\bar{x}, \bar{y}) solves the problem

$$(\mathcal{Q}_{x^*}) : \quad \max_{\substack{x \in [-\frac{1}{2}, \frac{1}{2}] \\ y_1 \in [x^2 - 1, \frac{-3}{4}] \\ y_2 \in [x - 1, \frac{-1}{2}]}} \{x^2 + 2x + y_1 + y_2 - x^*x\}.$$

We have

$$\begin{aligned} \sup(\mathcal{Q}_{x^*}) &= \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} \{x^2 + 2x - x^*x + \sup_{\substack{y_1 \in [x^2 - 1, \frac{-3}{4}] \\ y_2 \in [x - 1, \frac{-1}{2}]}} (y_1 + y_2)\} \\ &= \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} \{x^2 + (2 - x^*)x - \frac{5}{4}\} \end{aligned}$$

which corresponds to a maximization problem of a convex function over a convex compact set. Then the supremum is attained at one of the extremal points of the set $[-\frac{1}{2}, \frac{1}{2}]$, which are $\frac{1}{2}$ and $-\frac{1}{2}$. The values of the function: $x \rightarrow x^2 + (2 - x^*)x - \frac{5}{4}$ at these points are respectively $-\frac{1}{2}x^*$ and $\frac{1}{2}x^* - 2$. Then according to condition i) above, we must choose x^* such that $-\frac{1}{2}x^* > \frac{1}{2}x^* - 2$. That is $x^* < 2$. Since we must have $\bar{q} > 0$ and $\bar{q}x^* \geq 2$, we take for example $x^* = 1$ and $\bar{q} = 3$. Therefore, according to Theorem 4.11, $\bar{x} = \frac{1}{2}$ solves the problem (\mathcal{P}) with $\min(\mathcal{P}) = \frac{7}{4}$ (we verify that actually assumption (\mathcal{H}_1) is satisfied).

5. Conclusion

In this paper, for a generalized semi-infinite programming problem (\mathcal{P}) we have defined an extended Fenchel-Lagrange duality and provided necessary and sufficient conditions for global optimality. This extended duality is defined in the context that the Fenchel-Lagrange duality cannot be applied directly to the nonconvex minimization problem (\mathcal{P}) . So, in a first step, we have decomposed (\mathcal{P}) into a family of convex minimization subproblems. Then for each subproblem we have given its Fenchel-Lagrange dual and provided necessary and sufficient optimality conditions. Thanks to the convex decomposition and the duality given for the family of subproblems, we have defined the extended conjugate duality and provided necessary and sufficient optimality conditions for problem (\mathcal{P}) . The optimality conditions are expressed in terms of subdifferentials and normal cones in the sense of convex analysis. We note that these extended duality and optimality conditions are new in the literature of generalized semi-infinite programming. It will be interesting to construct some algorithms from the obtained optimality conditions to solve (\mathcal{P}) , and also to test our approach on concrete examples. This will be the subject of a forthcoming work.

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