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A NONSMOOTH APPROACH FOR THE MODELLING OF A MECHANICAL ROTARY DRILLING SYSTEM WITH FRICTION

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Dedicated to 70th birthday of Professor Meir Shillor.

ABSTRACT. In this paper, we show how the approach of nonsmooth dynamical systems can be used to develop a suitable method for the modelling of a rotary oil drilling system with friction. We study different kinds of frictions and analyse the mathematical properties of the involved dynamical systems. We show that using a general Stribeck model for the frictional contact, we can formulate the rotary drilling system as a well-posed evolution variational inequality. Several numerical simulations are also given to illustrate both the model and the theoretical results.

1. Introduction.

The first systematic study of friction is due to Leonardo da Vinci (1452-1519) (see e.g. [12]). The famous italian scientific discovered indeed that the friction force is proportional to load, opposes the motion and is independent of the contact area. These fundamental results have been rediscovered by Guillaume Amontons (1663-1705) and developed by Charles-Augustin de Coulomb (1736-1806) ([4], [11]). The Coulomb friction force F is a function of the load and direction of the sliding velocity v . Arthur Morin (1795-1880) found that the friction at zero sliding speed (static friction) is larger than the Coulomb friction (dynamic friction) [26]. Osborne Reynolds (1842-1912) introduced the concept of viscous friction in relation to lubricated contact [34] and Richard Stribeck (1861-1950) observed that friction force decreases with the increase of the sliding speed from the static friction to the Coulomb friction [35]. All these fundamental discoveries on the friction phenomena have since been the subject of much research by the engineering community (see e.g. [3], [6], [7], [17], [21], [27], [28], [32]).

Most models of friction are nonsmooth in the sense that the function involved in the model $v \mapsto F(v)$ is not continuous at $v = 0$. The pioneering works of Jean-Jacques Moreau (1923-2014) and Panagiotis D. Panagiotopoulos (1950-1998) initiated the development of a mathematical framework applicable to the study of nonsmooth mechanical problems in using advanced tools from modern convex analysis and set-valued analysis (see e.g. [1], [2],

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[14], [15], [20], [23], [24], [25], [29], [30]). The approach of Moreau and Panagiotopoulos can in particular be used to write a precise and rigorous mathematical model describing the friction force and the stick-slip phenomena. This approach using set-valued functions left aside the complicated transition processes between "stick" and "slip" but lead however to rigorous mathematical models like differential inclusions, variational inequalities and hemivariational inequalities. There is an abundant literature for the mathematical modelling and analysis of friction problems in the context of contact mechanics. For a comprehensive reference on this subject, we refer to the book by F. Shillor, M. Sofonea and J. Telega [33].

In this paper, we show how the approach of nonsmooth dynamical systems can be used to develop a suitable method for the modelling of a rotary oil drilling system with friction. The drilling technique is used in the oil industry. The mechanical system involves a rotating bit to produce a bore-hole in the rock, a drill-string to rotate the bit and a rotary drive at surface to rotate the drill-string. In this paper, we consider the model of M. Kidouche and R. Riane [19] where the interaction between the bit and the rock is considered as an combination of cutting of the rock and frictional contact. We show that in using a general Stribeck model for the frictional contact, we can formulate the rotary drilling system as a well-posed evolution variational inequality. We analyse the mathematical properties of the variational inequality and give some numerical simulations to illustrate the theory.

2. Mathematical preliminaries. Let $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ be a real Hilbert space. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product on H and $\| \cdot \|$ is the associated norm: $(\forall x \in H) : \|x\| = \sqrt{\langle x, x \rangle}$. A set-valued map $\mathcal{F} : H \rightrightarrows H$ is a multifunction that associates to any $x \in H$ a subset $\mathcal{F}(x) \subset H$. Given a set-valued map $\mathcal{F} : H \rightrightarrows H$. The domain $D(\mathcal{F})$ of \mathcal{F} is defined by: $D(\mathcal{F}) = \{x \in H : \mathcal{F}(x) \neq \emptyset\}$. The range $R(\mathcal{F})$ of \mathcal{F} is given by: $R(\mathcal{F}) = \bigcup_{x \in H} \mathcal{F}(x)$. The graph $G(\mathcal{F})$ of \mathcal{F} is defined by: $G(\mathcal{F}) = \{(x, y) \in H \times H : y \in \mathcal{F}(x)\}$. The inverse \mathcal{F}^{-1} of \mathcal{F} is given by the relation: $y \in \mathcal{F}(x) \iff x \in \mathcal{F}^{-1}(y)$. One says that \mathcal{F} is monotone if and only if $\langle x^* - y^*, x - y \rangle \geq 0, \forall (x, x^*) \in G(\mathcal{F}), \forall (y, y^*) \in G(\mathcal{F})$. One says that \mathcal{F} is maximal monotone if and only if it is monotone and its graph is maximal in the sense of inclusion, i.e., $G(\mathcal{F})$ is not properly contained in the graph of any other monotone operator.

One extensively-used maximal monotone operator is the subdifferential of a proper convex and lower semicontinuous function. Let $\Phi : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Phi(x)$ be a proper convex and lower semicontinuous function. The effective domain of Φ is defined by: $\text{dom}(\Phi) = \{x \in H : \Phi(x) < +\infty\}$. We say that $w \in H$ is a subgradient of Φ at $x \in H$ if $\Phi(v) - \Phi(x) \geq \langle w, v - x \rangle, \forall v \in H$. The set of subgradients of Φ at x , denoted by $\partial\Phi(x)$, is called the subdifferential of Φ at x :

$$\partial\Phi(x) = \{w \in H : \Phi(v) - \Phi(x) \geq \langle w, v - x \rangle, \forall v \in H\}.$$

The set-valued function: $\partial\Phi : H \rightrightarrows H; x \mapsto \partial\Phi(x)$ is a maximal monotone operator. The domain of $\partial\Phi$ is defined by $D(\partial\Phi) = \{x \in H : \partial\Phi(x) \neq \emptyset\}$. It is clear that $D(\partial\Phi) \subset \text{dom}(\Phi)$.

Remark 1. If $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}; x \mapsto \Phi(x)$ is a convex and Gateaux differentiable, then

$$(\forall x \in \mathbb{R}^n) : \partial\Phi(x) = \{\nabla\Phi(x)\}.$$

Remark 2. Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto \Phi(x)$ be a convex function. Then

$$(\forall x \in \mathbb{R}) : \partial\Phi(x) = [\Phi'_-(x), \Phi'_+(x)]$$

where $\Phi'_-(x)$ and $\Phi'_+(x)$ denote the left and the right derivative of Φ at x , respectively.

Example 2.1. For example, if $\Phi : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto \Phi(x) = |x|$, then

$$\partial\Phi(x) = \begin{cases} -1 & \text{if } x < 0, \\ [-1, +1] & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

If $\Phi : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto \Phi(x) = \max\{0, x^2 - 4\}$, then

$$\partial\Phi(x) = \begin{cases} 2x & \text{if } x < -2, \\ [-4, 0] & \text{if } x = -2, \\ 0 & \text{if } x \in]-2, +2[, \\ [0, +4] & \text{if } x = +2, \\ 2x & \text{if } x > +2. \end{cases}$$

Let us here recall the basic rules of subdifferential calculus. Let $\Phi : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Phi(x)$ be a proper convex and lower semicontinuous function and $\lambda > 0$. Then for every $x \in D(\partial\Phi) : \partial(\lambda\Phi)(x) = \lambda\partial\Phi(x)$. Let $\Phi_1 : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Phi_1(x)$ and $\Phi_2 : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Phi_2(x)$ be two proper convex and lower semicontinuous functions. Then for every $x \in D(\partial\Phi_1) \cap D(\partial\Phi_2)$, we have : $\partial\Phi_1(x) + \partial\Phi_2(x) \subset \partial(\Phi_1 + \Phi_2)(x)$. Let $\Phi_1 : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Phi_1(x)$ be a proper convex and lower semicontinuous function and $\Phi_2 : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Phi_2(x)$ a Gateaux differentiable function. Then for every $x \in D(\partial\Phi_1)$, we have $\partial\Phi_1(x) + \partial\Phi_2(x) = \partial\Phi_1(x) + \nabla\Phi_2(x)$.

Let us now recall an existence and uniqueness result, due to Kato [10, 18], for a general nonlinear Cauchy problem involving a maximal monotone operator in a Hilbert space.

Theorem 2.1. *Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be a real Hilbert space and let $\mathcal{F} : D(\mathcal{F}) \subset H \rightrightarrows H$ be a maximal monotone operator. Let $t_0 \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $x_0 \in D(\mathcal{F})$ be given and suppose that $f : [t_0, +\infty[\rightarrow H$ satisfies*

$$f \in C^0([t_0, +\infty[; H), \frac{df}{dt} \in L^1_{\text{loc}}([t_0, +\infty[; H).$$

Then there exists a unique $x \in C^0([t_0, +\infty[; H)$ satisfying

$$\frac{dx}{dt} \in L^\infty_{\text{loc}}([t_0, +\infty[; H);$$

x is right-differentiable on $[t_0, +\infty[;$

$$x(t) \in D(\mathcal{F}), t \geq t_0;$$

$$x(t_0) = x_0;$$

$$\alpha x(t) + f(t) \in \frac{dx}{dt}(t) + \mathcal{F}(x(t)), \text{ a.e. } t \geq t_0.$$

Let us here recall that an operator $A : H \rightarrow H$ is *hemicontinuous* if the functional $t \mapsto \langle A(u + tv), w \rangle$ is continuous on $[0, 1]$ for all $u, v, w \in H$.

Corollary 1. *Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be a real Hilbert space and let $\Psi : H \rightarrow \mathbb{R} \cup \{+\infty\}; x \mapsto \Psi(x)$ be a proper, convex and lower semicontinuous function. Let $A : H \rightarrow H$ be a hemicontinuous operator such that for some $\alpha_1 \geq 0$, $A + \alpha_1 I$ is monotone. Let $F : H \rightarrow H$ be an operator such that*

$$(\forall x, y \in H) : \|F(x) - F(y)\| \leq \alpha_2 \|x - y\|,$$

for some $\alpha_2 > 0$. Let $t_0 \in \mathbb{R}$ and $x_0 \in D(\partial\Psi)$ be given and suppose that $f : [t_0, +\infty) \rightarrow H$ satisfies

$$f \in C^0([t_0, +\infty); H), \quad \frac{df}{dt} \in L^1_{\text{loc}}([t_0, +\infty[; H).$$

Then there exists a unique trajectory $x \in C^0([t_0, +\infty[; H)$ such that

$$\frac{dx}{dt} \in L^\infty_{\text{loc}}([t_0, +\infty[; H); \quad (1)$$

$$x \text{ is right-differentiable on } [t_0, +\infty[; \quad (2)$$

$$x(t) \in D(\partial\Psi), \quad t \geq t_0; \quad (3)$$

$$x(t_0) = x_0; \quad (4)$$

$$\begin{aligned} & \left\langle \frac{dx}{dt}(t) + A(x(t)) + F(x(t)) - f(t), y - x(t) \right\rangle \\ & + \Psi(y) - \Psi(x(t)) \geq 0, \quad \forall y \in H, \text{ a.e. } t \geq t_0. \end{aligned} \quad (5)$$

Proof. Let us notice that the evolution variational inequality (5) can be rewritten as the following differential inclusion:

$$\frac{dx}{dt}(t) + A(x(t)) + F(x(t)) - f(t) \in -\partial\Psi(x(t)).$$

We set $\mathcal{F}_1 = A + \alpha_1 I$ and $\mathcal{F}_2 = F + \alpha_2 I$. The operator \mathcal{F}_1 is single-valued, hemicontinuous and monotone (by assumptions of Corollary 1). Since F is α_2 -Lipschitz continuous, then $\mathcal{F}_2 = F + \alpha_2 I$ is monotone and hemicontinuous. Therefore, $\mathcal{F}_1 + \mathcal{F}_2$ is single-valued, monotone and hemicontinuous. Setting $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \partial\Psi$, by a classical result (see for instance [31], [36]), it is clear that the operator \mathcal{F} is maximal monotone. Applying Theorem 2.1 for $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \partial\Psi$ and $\alpha = \alpha_1 + \alpha_2$, we get the existence and uniqueness of a trajectory $x \in C^0([t_0, +\infty[; H)$ satisfying (1)-(5). \blacksquare

3. A general friction model. We consider the set-valued relation between friction force F and sliding velocity v :

$$F \in \mathcal{F}(v),$$

with

$$\mathcal{F}(v) = \begin{cases} \varphi_-(v) & \text{if } v < 0, \\ [-F_S, F_S] & \text{if } v = 0, \\ \varphi_+(v) & \text{if } v > 0, \end{cases} \quad (6)$$

where $F_S > 0$, $\varphi_+ : [0, +\infty[\rightarrow \mathbb{R}; x \mapsto \varphi_+(x)$ is a function that is assumed to satisfy the following conditions:

(H1) φ_+ is differentiable on $[0, +\infty[$,

(H2) $\varphi_+(0) = F_S$,

(H3) $(\exists K > 0) (\forall x \in [0, +\infty[) : |\varphi'_+(x)| \leq K$,

and $\varphi_- :] - \infty, 0] \rightarrow \mathbb{R}; x \mapsto \varphi_-(x)$ is defined by:

$$(\forall x \in] - \infty, 0]) : \varphi_-(x) = -\varphi_+(-x).$$

In this model, the value of the friction force F is not specified for zero sliding velocity ($v = 0$), it can take any value in the interval $[-F_S, +F_S]$ where $F_S > 0$ is the maximum static friction force. The functions φ_- and φ_+ are used to define the dynamic friction force, i.e. the friction force during the slip phase.

It follows from these assumptions that φ_- is differentiable on $] - \infty, 0]$. Indeed, we have: $(\forall x \in] - \infty, 0]) : \varphi'_-(x) = \varphi'_+(-x)$. In particular: $\varphi'_-(0^-) = \varphi'_+(0^+)$. We have also: $(\forall x \in] - \infty, 0]) : |\varphi'_-(x)| \leq K$ and $\varphi_-(0) = -\varphi_+(0) = -F_S$. Let us now set

$$\varphi(x) = \begin{cases} \varphi_-(x) + F_S & \text{if } x < 0, \\ \varphi_+(x) - F_S & \text{if } x \geq 0. \end{cases} \quad (7)$$

The function φ is differentiable on $] - \infty, 0[\cup] 0, +\infty[$. It is also continue at 0 since: $\varphi(0^-) = F_S + \varphi_-(0^-) = F_S - F_S = 0$ and $\varphi(0^+) = -F_S + \varphi_+(0^+) = -F_S + F_S = 0$. The function φ is thus continuous on \mathbb{R} and differentiable on $\mathbb{R} \setminus \{0\}$. Moreover $\lim_{x \rightarrow 0, x \neq 0} \varphi'(x) = L < +\infty$. This last result holds indeed with $L = \varphi'_-(0^-) = \varphi'_+(0^+)$ since $\lim_{x \rightarrow 0^-} \varphi'(x) = \lim_{x \rightarrow 0^-} \varphi'_-(x) = \varphi'_-(0^-) = \varphi'_+(0^+) = \lim_{x \rightarrow 0^+} \varphi'_+(x) = \lim_{x \rightarrow 0^+} \varphi'(x)$. It results that $\varphi'(0) = L$ and the function φ is differentiable on \mathbb{R} . We have $(\forall x \in \mathbb{R}) : |\varphi'(x)| \leq K$ and the function φ is thus Lipschitz continuous on \mathbb{R} , i.e.

$$(\forall x, y \in \mathbb{R}) : |\varphi(x) - \varphi(y)| \leq K|x - y|. \quad (8)$$

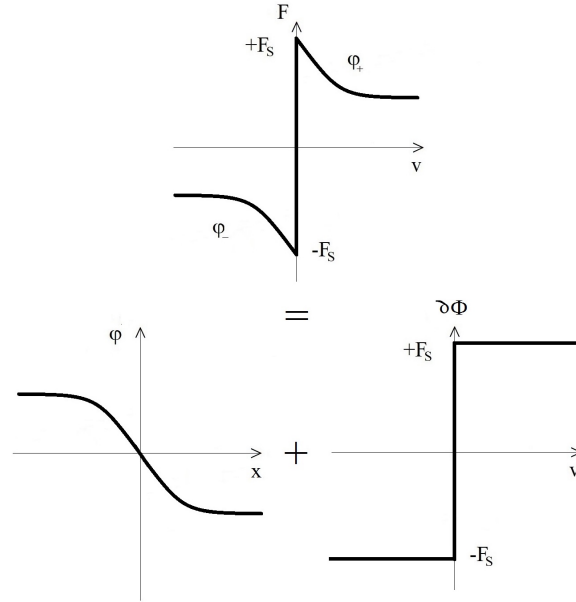


FIGURE 1. General friction model as in (6) and the function φ as in (7).

We have

$$\begin{aligned} \mathcal{F}(v) &= \begin{cases} \varphi_-(v) & \text{if } v < 0, \\ [-F_S, F_S] & \text{if } v = 0, \\ \varphi_+(v) & \text{if } v > 0, \end{cases} = \begin{cases} \varphi(v) - F_S & \text{if } v < 0, \\ [-F_S, F_S] & \text{if } v = 0, \\ \varphi(v) + F_S & \text{if } v > 0, \end{cases} \\ &= \varphi(v) + \begin{cases} -F_S & \text{if } v < 0, \\ [-F_S, F_S] & \text{if } v = 0, \\ +F_S & \text{if } v > 0, \end{cases} = \varphi(v) + \partial\Phi_S(v) \end{aligned}$$

where

$$\Phi_S(v) = F_S|v|.$$

The general set-valued friction model can thus be written as the sum of a Lipschitz continuous function and the convex subdifferential of a proper, convex and lower semicontinuous function (which is a maximal monotone set-valued map in \mathbb{R}^2).

Example 3.1 (Set-valued Coulomb Friction Model). The Coulomb model is a very simple mathematical formulation of the frictional phenomena. It is widely used by engineers to study systems with dry friction. Coulomb friction model is also called Amontons-Coulomb friction model so as to refer to the work by Guillaume Amontons and Charles-Augustin de Coulomb (see e.g. [5], [21]). The Coulomb model expresses that friction force F opposes motion and that its magnitude is independent of the sliding velocity v . The model is

$$F(v) = \begin{cases} -F_C & \text{if } v < 0, \\ +F_C & \text{if } v > 0, \end{cases}$$

where F_C is the Coulomb friction force proportionnal to the normal load $F_N = mg$ in the contact, i.e. $F_C = \mu F_N$ with $\mu > 0$. The coefficient μ is called the Coulomb friction coefficient also called the dynamic friction coefficient. The constant $g = 9.81$ (m/s²) is the acceleration of gravity. In this model, the value of the friction force is not specified for zero sliding velocity ($v = 0$), it can take any value in the interval $[-F_C, +F_C]$, i.e. $v = 0 \implies F \in [-F_C, +F_C]$. We may thus write $F \in \mathcal{F}(v)$, where \mathcal{F} is defined as in (6) with $F_S = F_C$, $(\forall x \leq 0) : \varphi_-(x) = -F_C$ and $(\forall x \geq 0) : \varphi_+(x) = +F_C$.

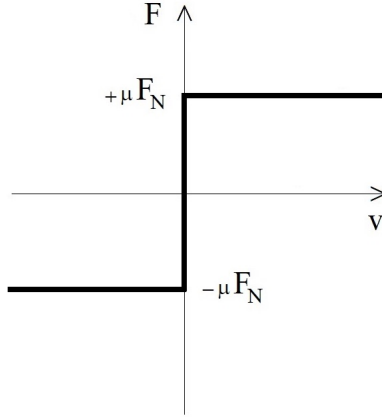


FIGURE 2. Coulomb friction model.

The graph of \mathcal{F} is depicted in Fig. 2. We may reduce the Coulomb model to the mathematical formula: $F \in \partial\Phi_C(v)$, where $\Phi_C(v) = F_C|v|$. In using this model, one left aside the complicated transition processes between "slip" and "stick".

Example 3.2 (The linearized exponential model of Bo and Pavelescu [9]). Friction acts like a spring when a small force is applied. This phenomena is called "stiction". A model of stiction consists to express that the transition from stick to slip has to occur via the maximum static friction force $F_S = \mu_S F_N$ that may be higher than the maximum dynamic friction $F_C = \mu F_N$. Here $\mu_S > 0$ denotes the friction coefficient in the slip phase

and F_S is called the stiction force. Most sliding contacts are lubricated and Stribeck [35] observed that the friction force does not drop suddenly when velocity increases but follows a continuous curve as depicted in Figure 3.

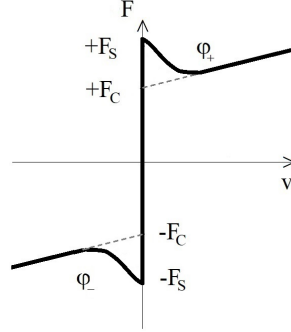


FIGURE 3. Stribeck friction model

The friction decreases with increased sliding speed until a mixed or full film situation is reached. Then the friction can either be constant, increase, or decrease somewhat with increased sliding speed due to viscous and thermal effects. The velocity at which the friction force is minimal is called the Stribeck velocity. A modern set-valued formulation of the Stribeck friction is given by the linearized exponential model of Bo and Pavelescu [9]: $F \in \mathcal{F}(v)$, with \mathcal{F} defined as in (6) and where the functions φ_- and φ_+ are given by the formula:

$$(\forall x \leq 0) : \varphi_-(x) = k_v x - F_C - (F_S - F_C)e^{-\left|\frac{x}{v_s}\right|^\sigma} \quad (9)$$

and

$$(\forall x \geq 0) : \varphi_+(x) = k_v x + F_C + (F_S - F_C)e^{-\left|\frac{x}{v_s}\right|^\sigma} \quad (10)$$

where $\sigma > 0$ is an empirical exponent and $v_s > 0$ is an empirical coefficient called the sliding speed coefficient. Note that the model of Bo and Pavelescu has been originally stated with $k_v = 0$. The viscous friction term has been added by Armstrong-Hélouvy in [6]. Different values for σ have been used in the engineering literature [6]. Armstrong-Hélouvy employs $\sigma = 2$. Čerkala and Jadlovska use $\sigma = 1$ in the study of a two-wheel robot dynamic with differential chassis. The parameter $0.00001 \leq v_s \leq 0.1$ (m/s) depends upon the contact geometry and loading.

Let us first consider the case $\sigma = 1$. The function φ_+ is differentiable on $[0, +\infty[$ and we have:

$$(\forall x \in [0, +\infty[) : \varphi'_+(x) = k_v - \frac{1}{v_s}(F_S - F_C)e^{-\frac{x}{v_s}}$$

Assumption (H1) is thus satisfied. We have also $\varphi_+(0) = F_S$ and assumption (H2) is satisfied. We note finally that

$$\begin{aligned} (\forall x \in [0, +\infty[) : |\varphi'_+(x)| &\leq |k_v| + \left|\frac{1}{v_s}(F_S - F_C)e^{-\frac{x}{v_s}}\right| \\ &= k_v + \frac{1}{v_s}(F_S - F_C)e^{-\frac{x}{v_s}} \leq k_v + \frac{F_S - F_C}{v_s}, \end{aligned}$$

since $e^{-\frac{x}{v_s}} \leq 1$ for $x \geq 0$, assumption (H3) is thus satisfied.

Let us now suppose that $\sigma > 1$. The function φ_+ is differentiable on $[0, +\infty[$ and we have

$$(\forall x \in [0, +\infty[) : \varphi'_+(x) = k_v - \frac{\sigma}{v_s^\sigma} (F_S - F_C) x^{\sigma-1} e^{-\left(\frac{x}{v_s}\right)^\sigma}.$$

Assumption (H1) is thus satisfied. We have also $\varphi_+(0) = F_S$ and assumption (H2) is satisfied. We have

$$\begin{aligned} (\forall x \in [0, +\infty[) : |\varphi'_+(x)| &\leq |k_v| + \left| \frac{\sigma}{v_s^\sigma} (F_S - F_C) x^{\sigma-1} e^{-\left(\frac{x}{v_s}\right)^\sigma} \right| \\ &= k_v + \frac{\sigma}{v_s^\sigma} (F_S - F_C) x^{\sigma-1} e^{-\left(\frac{x}{v_s}\right)^\sigma}. \end{aligned}$$

We have

$$\lim_{x \rightarrow +\infty} x^{\sigma-1} e^{-\left(\frac{x}{v_s}\right)^\sigma} = 0.$$

It results that there exists $H > 0$ such that $(\forall x \geq H) : x^{\sigma-1} e^{-\left(\frac{x}{v_s}\right)^\sigma} \leq 1$. The function φ'_+ is continuous on $[0, H]$ and there exists thus a constant $M > 0$ such that $(\forall x \in [0, H]) : |\varphi'_+(x)| \leq M$. Thus

$$(\forall x \in [0, +\infty[) : |\varphi'_+(x)| \leq \max\{M, k_v + \frac{\sigma}{v_s^\sigma} (F_S - F_C)\}.$$

Assumption (H3) is thus satisfied.

Example 3.3 (The model of Hess and Soom [16]). The function φ_- and φ_+ are given by the formula:

$$(\forall x \leq 0) : \varphi_-(x) = F_v x - F_C - \frac{(F_S - F_C)}{1 + \left(\frac{x}{v_s}\right)^2}$$

and

$$(\forall x \geq 0) : \varphi_+(x) = F_v x + F_C + \frac{(F_S - F_C)}{1 + \left(\frac{x}{v_s}\right)^2},$$

where $F_v \geq 0$ is a viscous friction coefficient and $v_s > 0$ is a characteristic velocity of the Stribeck curve. The function φ_+ is differentiable on $[0, +\infty[$. We have

$$(\forall x \in [0, +\infty[) : \varphi'_+(x) = F_v - \frac{2(F_S - F_C)x}{v_s^2 \left(1 + \left(\frac{x}{v_s}\right)^2\right)^2}.$$

Assumption (H1) is thus satisfied. We have also $\varphi_+(0) = F_S$ and assumption (H2) is satisfied. We note finally that

$$(\forall x \in [0, +\infty[) : |\varphi'_+(x)| \leq |F_v| + \left| \frac{2(F_S - F_C)x}{v_s^2 \left(1 + \left(\frac{x}{v_s}\right)^2\right)^2} \right| = F_v + \frac{2(F_S - F_C)x}{v_s^2 \left(1 + \left(\frac{x}{v_s}\right)^2\right)^2}.$$

We have

$$\lim_{x \rightarrow +\infty} \frac{x}{\left(1 + \left(\frac{x}{v_s}\right)^2\right)^2} = 0.$$

It results that there exists $H > 0$ such that $(\forall x \geq H) : \frac{x}{\left(1 + \left(\frac{x}{v_s}\right)^2\right)^2} \leq 1$. The function φ'_+ is continuous on $[0, H]$ and there exists thus a constant $M > 0$ such that $(\forall x \in [0, H]) : |\varphi'_+(x)| \leq M$. Thus

$$(\forall x \in [0, +\infty[) : |\varphi'_+(x)| \leq \max\{M, F_v + \frac{2(F_S - F_C)}{v_s^2}\}.$$

Assumption (H3) is thus satisfied.

Example 3.4 (Stiction model). A basic model of stiction is given by (see Figure 4): $F \in \mathcal{F}(v)$, with

$$\mathcal{F}(v) = \begin{cases} -F_C & \text{if } v < 0, \\ [-F_S, F_S] & \text{if } v = 0, \\ +F_C & \text{if } v > 0. \end{cases}$$

We note that in the case $F_S > F_C$, the set-valued function \mathcal{F} (in Figure 4) does not

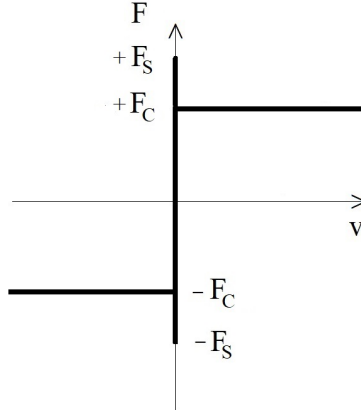


FIGURE 4. Stiction model

possess good mathematical properties. It can in particular not be formulated as the sum of a Lipschitz continuous function and the convex subdifferential of proper, convex and lower semicontinuous function. Moreover, a transition from stick to slip which is not mechanically consistent is possible ([8], [20]). It is however convenient to use the model of Example 3.2 with $k_v = 0$, $\sigma = 1$ and a small value for v_s to get a suitable model.

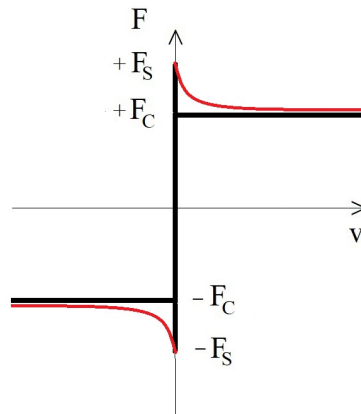


FIGURE 5. Consistent stiction model as in (6) with $\varphi_+(x) = F_C + (F_S - F_C)e^{-\frac{x}{v_s}}$.

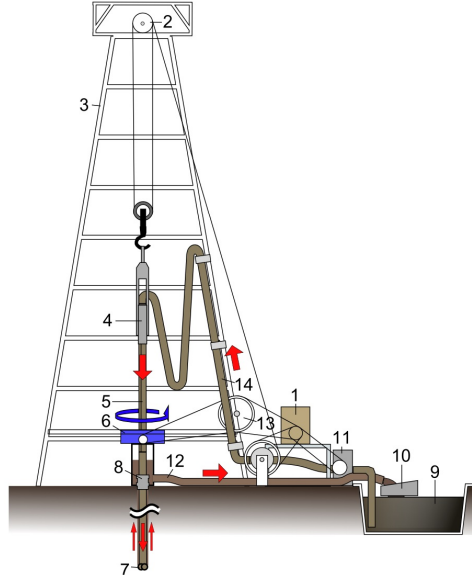


FIGURE 6. Oil drilling rig illustration - 1. Mud tank, 2. Shale shakers, 3. Suction line (mud pump), 4. Mud pump, 5. Motor or power source, 6. Vibrating hose, 7. Draw-works (winch), 8. Standpipe 9. Kelly hose, 10. Goose-neck, 11. Traveling block, 12. Drill line, 13. Crown block 14. Derrick - **Author: Tosaka - Attribution 3.0 Unported (CC BY 3.0) - <https://creativecommons.org/licenses/by/3.0/deed.en> (https://commons.wikimedia.org/wiki/File:Oil_Rig_NT.PNG).**

4. Mathematical analysis of a rotary drilling system. Let us consider the model of a rotary drilling system (see Figure 7) consisting of a motor, drill pipe represented by a torsional spring and drill collar represented by a rigid body (see [19]). Let us denote by φ_1 (resp. φ_2) the angular displacement of the rotor (resp. rigid body), $\omega_1 = \dot{\varphi}_1$ (resp. $\omega_2 = \dot{\varphi}_2$) the angular velocity of the motor (resp. rigid body), $J_1 > 0$ (resp. $J_2 > 0$) the moment of inertia of the motor (resp. rigid body), $d_1 > 0$ (resp. $d_2 > 0$) the viscous damping coefficient of the motor (resp. load) and $k > 0$ the torsional stiffness coefficient of the shaft. The applied torque is denoted by T_1 . The torque inherent in the drill-collar is denoted by T_2 . It is a combination of the cutting of the rock process and the frictional contact. We write

$$T_2 = T_{\text{CUT}} + T_F$$

where T_{CUT} is the cutting torque resulting from the cutting process and T_F is the frictional torque resulting from the frictional contact. The cutting torque is given by

$$T_{\text{CUT}} = \frac{1}{2} \delta R_B^2 E$$

where $E > 0$ is the amount of energy required to cut a unit volume of rock, $R_B > 0$ is the drill bit radius and $\delta > 0$ is the depth of cut. For the frictional contact torque, we use a Stribeck law as described by Bo and Pavelescu [9]:

$$T_F \in \mathcal{F}(\omega_2),$$

where

$$\mathcal{F}(\omega_2) = \begin{cases} \varphi_-(\omega_2) & \text{if } \omega_2 < 0, \\ [-T_S, T_S] & \text{if } \omega_2 = 0, \\ \varphi_+(\omega_2) & \text{if } \omega_2 > 0, \end{cases}$$

with

$$(\forall x \leq 0) : \varphi_-(x) = -T_C - (T_S - T_C)e^{-|\frac{x}{\omega_s}|^\sigma} \quad (11)$$

and

$$(\forall x \geq 0) : \varphi_+(x) = +T_C + (T_S - T_C)e^{-|\frac{x}{\omega_s}|^\sigma} \quad (12)$$

where $T_S > 0$ is the static friction torque, $T_C \in [0, T_S]$ is the dynamic friction torque and $\sigma \geq 1, \omega_s > 0$ are empirical coefficients. We have

$$T_C = \frac{1}{2}\mu_C R_B W, \quad T_S = \frac{1}{2}\mu_S R_B W \quad (13)$$

where $\mu_C > 0$ is the dynamic friction coefficient, $\mu_S > 0$ is the static dynamic friction and $W > 0$ is the Weight-On-Bit. We set

$$\Xi(x) = \begin{cases} \varphi_-(x) + T_S & \text{if } x < 0, \\ \varphi_+(x) - T_S & \text{if } x \geq 0. \end{cases} \quad (14)$$

Then

$$\mathcal{F}(\omega_2) = \Xi(\omega_2) + \partial\Pi_F(\omega_2), \quad (15)$$

where for all $\omega_2 \in \mathbb{R}$:

$$\Pi_F(\omega_2) = T_S|\omega_2| \text{ and } \partial\Pi_F(\omega_2) = \begin{cases} -T_S & \text{if } \omega_2 < 0, \\ [-T_S, +T_S] & \text{if } \omega_2 = 0, \\ +T_S & \text{if } \omega_2 > 0. \end{cases}$$

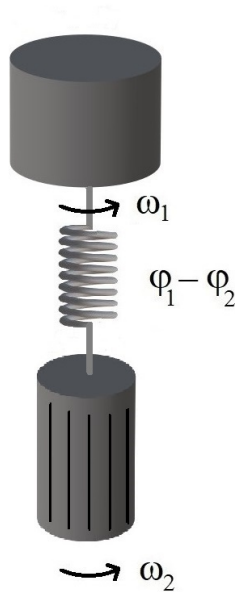


FIGURE 7. Rotary drilling system.

4.1. Mechanical system. Newton's second law, when applied to rotational motion expresses that the torque equals the product of the moment of inertia and the angular acceleration. The rotor (resp. the rigid body) produces through the shaft a stiffness resistance to the rotary movement that is given by $-k(\varphi_1 - \varphi_2)$ (resp. $-k(\varphi_2 - \varphi_1)$). The rotor (resp. the rigid body) produces also a viscous friction to the rotary motion that is given by $-d_1\dot{\omega}_1$ (res. $-d_2\dot{\omega}_2$). The equations of motion for our problem are thus:

$$J_1\dot{\omega}_1 = -k(\varphi_1 - \varphi_2) - d_1\dot{\omega}_1 + T_1$$

and

$$J_2\dot{\omega}_2 = k(\varphi_1 - \varphi_2) - d_2\dot{\omega}_2 - T_2.$$

Or equivalently,

$$J_1\ddot{\varphi}_1 + k\varphi_1 - k\varphi_2 + d_1\dot{\varphi}_1 - T_1 = 0$$

and

$$J_2\ddot{\varphi}_2 - k\varphi_1 + k\varphi_2 + d_2\dot{\varphi}_2 = -T_2$$

with

$$T_2 \in T_{\text{CUT}} + \Xi(\dot{\varphi}_2) + \partial\Pi_F(\dot{\varphi}_2).$$

Consequently,

$$\begin{aligned} \begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix} + \begin{bmatrix} \frac{d_1}{J_1} & 0 \\ 0 & \frac{d_2}{J_2} \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{J_1} & -\frac{k}{J_1} \\ -\frac{k}{J_2} & \frac{k}{J_2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} + \\ + \begin{bmatrix} -\frac{T_1}{J_1} \\ \frac{T_{\text{CUT}}}{J_2} + \frac{1}{J_2}\Xi(\dot{\varphi}_2) \end{bmatrix} \in - \begin{bmatrix} 0 \\ \frac{1}{J_2}\partial\Pi_F(\dot{\varphi}_2) \end{bmatrix}. \end{aligned} \quad (16)$$

Let us now set

$$X_1 = \varphi_1, X_2 = \varphi_2, X_3 = \dot{\varphi}_1, X_4 = \dot{\varphi}_2.$$

The inclusion in (16) is therefore equivalent to the following first order dynamic

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{k}{J_1} & -\frac{k}{J_1} & \frac{d_1}{J_1} & 0 \\ -\frac{k}{J_2} & \frac{k}{J_2} & 0 & \frac{d_2}{J_2} \end{bmatrix}}_A \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}}_X + \\ + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -\frac{T_1}{J_1} \\ \frac{T_{\text{CUT}}}{J_2} + \frac{1}{J_2}\Xi(X_4) \end{bmatrix}}_{F(X)} \in - \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2}\partial\Pi_F(X_4) \end{bmatrix}}_{\partial\Psi(X)}, \end{aligned} \quad (17)$$

where

$$(\forall X = (X_1, X_2, X_3, X_4) \in \mathbb{R}^4) : \Psi(X) = \frac{1}{J_2}\Pi_F(X_4) = \frac{T_S}{J_2}|X_4|. \quad (18)$$

We note that the function Ψ is convex and continuous on \mathbb{R}^4 . Consequently, the model can be formulated as the following first-order differential inclusion:

$$\dot{X}(t) + AX(t) + F(X(t)) \in -\partial\Psi(X(t)), \quad t \in [t_0, +\infty[\quad (19)$$

with the initial condition $X(t_0) = X_0 \in \mathbb{R}^4$. The matrix A is given in (17). The function $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $X = (X_1, X_2, X_3, X_4) \mapsto F(X)$ is defined by

$$(\forall X = (X_1, X_2, X_3, X_4) \in \mathbb{R}^4) : F(X) = (0, 0, -\frac{T_1}{J_1}, \frac{T_{\text{CUT}}}{J_2} + \frac{1}{J_2}\Xi(X_4)), \quad (20)$$

and the function $\Psi : \mathbb{R}^4 \rightarrow \mathbb{R}$, $X = (X_1, X_2, X_3, X_4) \mapsto \Psi(X)$ is given by (18). We recall that the function Ξ is defined in (14).

4.2. Electromechanical system. The torque T_1 is the torque T_M delivered by the motor to the system multiplied by the gearbox ratio $N : T_1 = NT_M$. Let us now denote by $L > 0$, $R > 0$, $i > 0$ and $V > 0$ the motor inductance, motor resistance, motor current and motor input voltage, respectively. We have

$$V = L \frac{di}{dt} + Ri + V_{cem},$$

where V_{cem} is the counter-electromotive force. The counter-electromotive force, and the motor torque, are linearly related to the motor speed and the motor current by the following relations:

$$T_M = K_M i \quad \text{and} \quad V_{cem} = NK_M \omega_1,$$

where $K_M > 0$ is the motor constant. Let us set $K = NK_M$. We have $T_1 = Ki$ and $V_{cem} = K\omega_1 = K\dot{\varphi}_1$. Let us now set $X_5 = i$. We have $T_1 = KX_5$ and

$$\dot{X}_5 + \frac{K}{L}X_3 + \frac{R}{L}X_5 - \frac{V}{L} = 0.$$

For a comprehensive reference about the electromechanical systems and their simulations, we refer for example to [22]. We obtain the system:

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \\ \dot{X}_5 \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \frac{k}{J_1} & -\frac{k}{J_1} & \frac{d_1}{J_1} & 0 & -\frac{K}{J_1} \\ -\frac{k}{J_2} & \frac{k}{J_2} & 0 & \frac{d_2}{J_2} & 0 \\ 0 & 0 & \frac{K}{L} & 0 & \frac{R}{L} \end{bmatrix}}_{A^*} \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}}_X - \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{T_{cut}}{J_2} \\ \frac{V(t)}{L} \end{bmatrix}}_{g(t)} + \\ + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2}\Xi(X_4) \\ 0 \end{bmatrix}}_{G(X)} \in - \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2}\partial\Pi_F(X_4) \\ 0 \end{bmatrix}}_{\partial\Upsilon(X)}, \end{aligned} \quad (21)$$

where

$$(\forall X = (X_1, X_2, X_3, X_4, X_5) \in \mathbb{R}^5) : \Upsilon(X) = \frac{1}{J_2}\Pi_F(X_4) = \frac{T_S}{J_2}|X_4|. \quad (22)$$

The model can thus be formulated as the following first-order differential inclusion:

$$\dot{X}(t) + A^*X(t) + G(X(t)) - g(t) \in -\partial\Upsilon(X(t)), \quad t \geq t_0, \quad (23)$$

with the initial condition $X(t_0) = X_0 \in \mathbb{R}^5$. The matrix A^* and the function $t \mapsto g(t)$ are given in (21). The function $G : \mathbb{R}^5 \rightarrow \mathbb{R}^5$, $X = (X_1, X_2, X_3, X_4, X_5) \mapsto G(X)$ is defined by

$$(\forall X = (X_1, X_2, X_3, X_4, X_5) \in \mathbb{R}^5) : G(X) = (0, 0, 0, \frac{1}{J_2}\Xi(X_4), 0) \quad (24)$$

and the function $\Upsilon : \mathbb{R}^5 \rightarrow \mathbb{R}$, $X = (X_1, X_2, X_3, X_4, X_5) \mapsto \Upsilon(X)$ is given by (22).

4.3. Mathematical analysis. In this section, we will show that problems (19) and (23) have a unique solution for every initial data. The following theorem is in this sense. Let us first consider the mechanical problem in (19).

Theorem 4.1. *Let $t_0 \in \mathbb{R}$. For every initial condition $X_0 \in \mathbb{R}^4$, there exists a unique trajectory $X \in C^0([t_0, +\infty[; \mathbb{R}^4)$ such that*

$$\frac{dX}{dt} \in L_{\text{loc}}^\infty([t_0, +\infty[; \mathbb{R}^4);$$

X is right-differentiable on $[t_0, +\infty[$;

$$X(t_0) = X_0;$$

$$\dot{X}(t) + AX(t) + F(X(t)) \in -\partial\Psi(X(t)), \text{ a.e. } t \in [t_0, +\infty[,$$

with A , Ψ and F defined respectively in (17), (18) and (20).

Proof. Let us check that all assumptions of Corollary 1 are satisfied. Since A is Lipschitz continuous, then $A + \alpha_1 I$ is monotone with $\alpha_1 = \|A\|$. Let us check that the function F defined in (20) is Lipschitz continuous. Since the functions φ_- and φ_+ defined respectively in (11) and (12) are of the form of φ_- and φ_+ defined respectively in (9) and (10) (with $k_V = 0$), we deduce from (8) that the function Ξ defined in (14) is Lipschitz continuous with constant $\frac{K}{J_2} > 0$. The conclusion follows from Corollary 1. ■

Let us now consider the electromechanical problem in (23).

Theorem 4.2. *Let $t_0 \in \mathbb{R}$ be given. Suppose that the function $V : [t_0, +\infty[\rightarrow \mathbb{R}$ (motor input voltage in (21)) satisfies*

$$V \in C^0([t_0, +\infty); \mathbb{R}), \quad \frac{dV}{dt} \in L_{\text{loc}}^1([t_0, +\infty[; \mathbb{R}).$$

Then for every initial condition $X_0 \in \mathbb{R}^5$, there exists a unique trajectory $X \in C^0([t_0, +\infty[; \mathbb{R}^5)$ such that

$$\frac{dX}{dt} \in L_{\text{loc}}^\infty([t_0, +\infty[; \mathbb{R}^5);$$

X is right-differentiable on $[t_0, +\infty[$;

$$X(t_0) = X_0;$$

$$\dot{X}(t) + A^*X(t) + G(X(t)) - g(t) \in -\partial\Upsilon(X(t)), \text{ a.e. } t \in [t_0, +\infty[$$

Proof. It is clear that all the data A^* , g , Υ and G defined in (21), (22), (24) satisfy all assumptions of Corollary 1. ■

5. Numerical simulations. In this section, we perform some numerical simulations for the models (19) and (23) using Matlab. The set of numerical values was taken from [19] and is listed in the following tables.

Remark 3. To solve numerically the differential inclusions (19) and (23), we used a convergent multistep Euler method with a minimum norm selection strategy (we refer to the survey by Dontchev and Lempio [13] for more details about the convergence results). We could use here also an implicit Euler method since the proximal operator associated to the convex functions given in (18) and (22) could be computed exactly (in a closed form). For example if $\Psi(x) = \gamma|x|$, $\gamma > 0$ and $x \in \mathbb{R}$, then the classical soft thresholding operator

J_1	999.35 ($kg.m^2$)
J_2	127.27 ($kg.m^2$)
d_1	51.38 ($N.m.s/rad$)
d_2	39.79 ($N.m.s$)
k	481.29 ($N.m/rad$)
R	0.01 (Ω)
L	0.005 (H)
K_M	6 ($N.m/A$)
N	7.20
$K = NK_M$	43.20 ($N.m/A$)
E	130 (MJ/m^3)
δ	0.64×10^{-3} (m/rad)
R_B	0.10 m
μ_C	0.4
μ_S	0.6

TABLE 1. Parameters.

σ	1
ω_s	10^{-3} (rad/s)

TABLE 2. Empirical coefficients.

$$V(t) = \begin{cases} 125 & \text{if } t \in [0, 25[, \\ 5t & \text{if } t \in [25, 30[, \\ 150 & \text{if } t \in [30, +\infty[. \end{cases}$$

TABLE 3. Motor voltage. Augmentation of DC motor voltage from 125 (V) to 150 (V) at $t = 30$ (s) (see Figure 8).

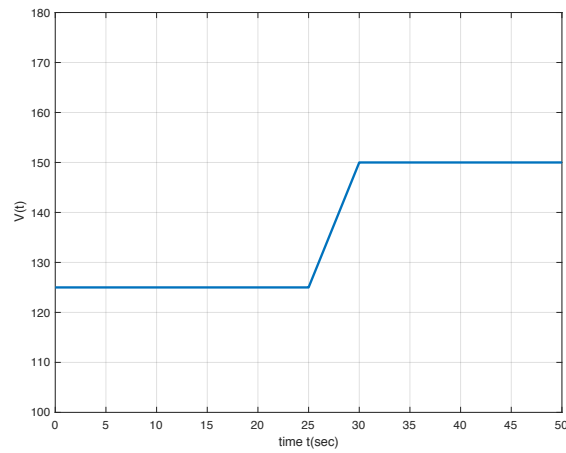


FIGURE 8. Graph of the function $V(t)$ in Table 3.

W	15000 (<i>kg</i>)
$T_C = \frac{1}{2}\mu_C R_B W$	300 (<i>kg.m</i>)
$T_S = \frac{1}{2}\mu_S R_B W$	450 (<i>kg.m</i>)
$T_{CUT} = \frac{1}{2}\delta R_B^2 E$	$4.16 \cdot 10^{-4}$ (<i>MJ/rad</i>)

TABLE 4. Weight-On-Bit and corresponding friction torques.

$X_1(0) = \varphi_1(0)$	-10 (<i>rad</i>)	$X_1(0) = \varphi_1(0)$	0 (<i>rad</i>)
$X_2(0) = \varphi_2(0)$	0 (<i>rad</i>)	$X_2(0) = \varphi_2(0)$	0 (<i>rad</i>)
$X_3(0) = \dot{\varphi}_1(0)$	-10 (<i>rad/s</i>)	$X_3(0) = \dot{\varphi}_1(0)$	0 (<i>rad/s</i>)
$X_4(0) = \dot{\varphi}_2(0)$	20 (<i>rad/s</i>)	$X_4(0) = \dot{\varphi}_2(0)$	0 (<i>rad/s</i>)
		$X_5(0) = i(0)$	0 (<i>A</i>)

TABLE 5. Initial conditions for problems (19) and (23).

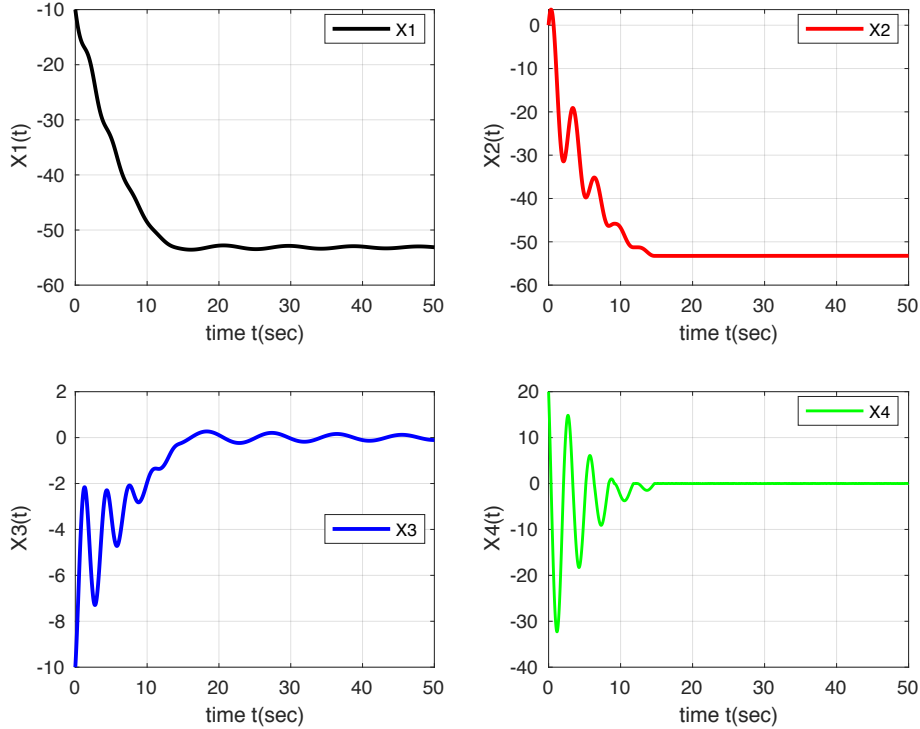


FIGURE 9. Numerical solution of the evolution variational inequality (19) with the initial conditions given in Table 5.

$\text{prox}_{\lambda\Psi} = (I + \lambda\partial\Psi)^{-1}$, which is used in the FISTA method for sparse convex optimization, is given by

$$\text{prox}_{\lambda\Psi}(x) = \text{sign}(x)(|x| - \lambda\gamma)_+ = \begin{cases} x - \lambda\gamma & \text{if } x \geq \lambda\gamma; \\ 0 & \text{if } -\lambda\gamma \leq x \leq \lambda\gamma; \\ x + \lambda\gamma & \text{if } x \leq -\lambda\gamma. \end{cases}$$

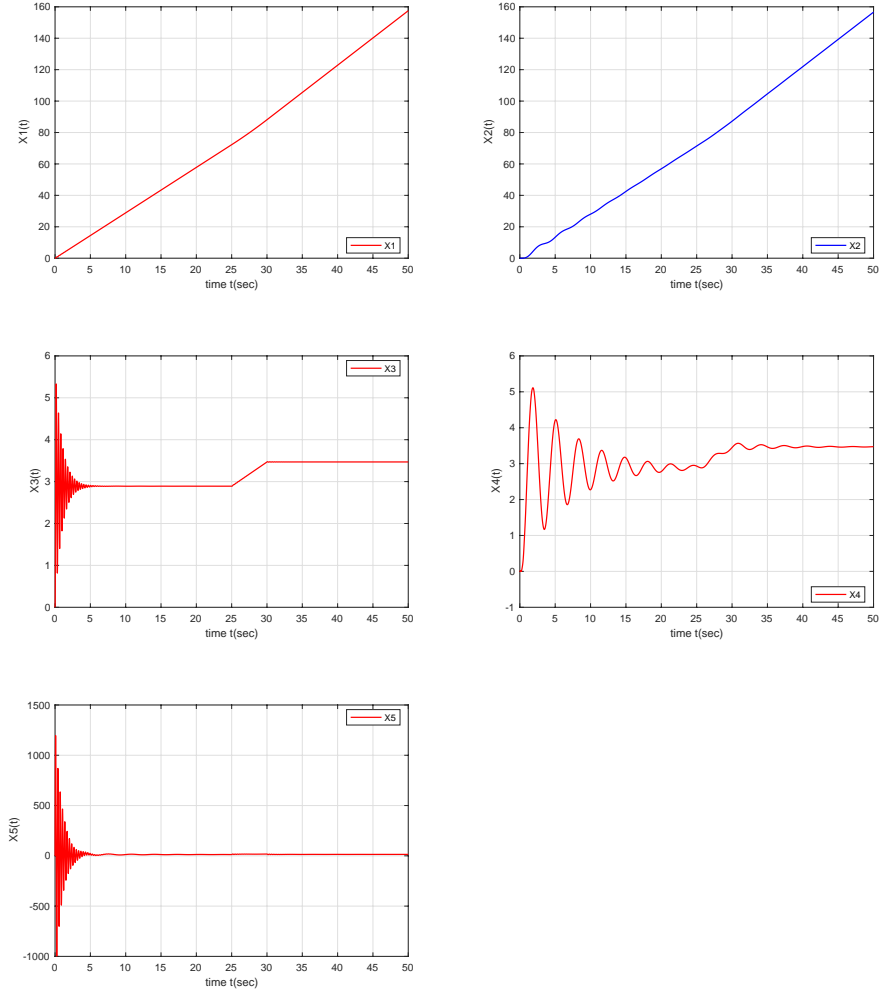


FIGURE 10. Numerical solution of the evolution variational inequality (23) with the initial conditions given in Table 5.

For the multidimensional case given by $\Psi(x) = \gamma \|x\|_1 = \gamma \sum_{i=1}^n |x_i|$, $x \in \mathbb{R}^n$, the proximal mapping of Ψ can be computed componentwise by applying the one-dimensional soft thresholding operator to each component.

Comment on the numerical simulations. The stick-slip oscillations phenomena is a real drawback in many mechanical systems and particularly in the rotary drilling system. It is considered as a real obstacle which decreases the drilling efficiency and may increase the cost of the drilling operation. According to [19], in order to reduce the stick-slip vibrations on the ground in practice, the driller operators try to control some parameters involved in

the model such as: the motor voltage function defined in Table 3, the weight on the bit W defined in (13), the speed at the surface and the viscosity of the drilling fluid.

For the evolution variational inequality (19), we observe in Figure 9 that the oscillation are damped and that the angular velocity of the rigid body $\omega_2 = \dot{\varphi}_2 = X_4$ goes to zero as the time t increases. For of the electromechanical system (23) with the initial conditions given in Table 5, with the augmentation of the DC motor voltage from 125 Volts to 150 Volts at $t = 30s$, we observe also that the oscillations are also damped (see Figure 10). It would be interesting to validate this model by decreasing the weight on the bit W from 15 tonnes to $W = 8$ tonnes at some fixed t (as suggested in [19]). This means that the functions T_S and T_C would depend on time t in the models (19) and (23).

6. Concluding Remarks. Motivated by the modelling of a mechanical rotary oil drilling system, we gave a general panorama of different types of friction that can be fit in a general set-valued framework. This includes the classical set-valued Coulomb friction model, the linearized exponential model of Bo and Pavelescu [9] (the Stribeck friction model), the model of Hess and Soom [16] and the stiction model. Using tools from convex and set-valued analysis, the rotary drilling system is formulated as a nonsmooth second-order dynamic in finite dimensional spaces. The mechanical and the electromechanical models can be rewritten as two evolution variational inequalities where existence and uniqueness results are given. Many open questions need further investigations such as for example the Lyapunov stability and the invariance properties of the associated stationary solutions of the dynamics (19) and (23). Due to the fact that the stiffness matrix given in (16) is singular and to the fact that the function Ξ in (14) is not monotone, the approaches developed in [1] and [2] can not be directly applicable. It would be interesting to investigate the stability analysis as well as all the numerical strategies to reduce the stick-slip oscillations of the nonsmooth dynamics (19) and (23). This is out of the scope of the current manuscript and will be the subject of a new research project.

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