



**HAL**  
open science

# Shape optimization of a weighted two-phase Dirichlet eigenvalue

Idriss Mazari, Grégoire Nadin, Yannick Privat

► **To cite this version:**

| Idriss Mazari, Grégoire Nadin, Yannick Privat. Shape optimization of a weighted two-phase Dirichlet eigenvalue. 2020. hal-02432387v1

**HAL Id: hal-02432387**

**<https://hal.science/hal-02432387v1>**

Preprint submitted on 8 Jan 2020 (v1), last revised 18 Oct 2021 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Shape optimization of a weighted two-phase Dirichlet eigenvalue\*

Idriss Mazari<sup>†</sup> Grégoire Nadin<sup>‡</sup> Yannick Privat<sup>§</sup>

January 9, 2020

## Abstract

Let  $m$  be a bounded function and  $\alpha$  a nonnegative parameter. This article is concerned with the first eigenvalue  $\lambda_\alpha(m)$  of the drifted Laplacian type operator  $\mathcal{L}_m$  given by  $\mathcal{L}_m(u) = -\operatorname{div}((1 + \alpha m)\nabla u) - mu$  on a smooth bounded domain, with Dirichlet boundary conditions. Assuming uniform pointwise and integral bounds on  $m$ , we investigate the issue of minimizing  $\lambda_\alpha(m)$  with respect to  $m$ . Such a problem is related to the so-called “two phase extremal eigenvalue problem” and arises naturally, for instance in population dynamics where it is related to the survival ability of a species in a domain. We prove that unless the domain is a ball, this problem has no “regular” solution. We then provide a careful analysis in the case of a ball by: (1) characterizing the solution among all radially symmetric resources distributions, with the help of a new method involving a homogenized version of the problem; (2) proving in a more general setting, a stability result for the centered distribution of resources with the help of a monotonicity principle for second order shape derivatives which significantly simplifies the analysis.

**Keywords:** shape derivatives, drifted Laplacian, bang-bang functions, spectral optimization, homogenization, reaction-diffusion equations.

**AMS classification:** 35K57, 35P99, 49J20, 49J50, 49Q10

## Contents

<b>1</b>	<b>Introduction and main results</b>	<b>2</b>
1.1	Mathematical Set Up	3
1.2	Main results	4
1.3	A biological application of the problem	6
1.4	Notations and notational conventions, technical properties of the eigenfunctions	7
<b>2</b>	<b>Preliminaries</b>	<b>8</b>
2.1	Switching function	8
2.2	Proof of Proposition 1	9

---

\*The authors were partially supported by the Project “Analysis and simulation of optimal shapes - application to lifesciences” of the Paris City Hall. I. Mazari and Y. Privat were partially supported by the ANR Project “SHAPE Optimization - SHAPO”.

<sup>†</sup>Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France ([idriss.mazari@sorbonne-universite.fr](mailto:idriss.mazari@sorbonne-universite.fr)).

<sup>‡</sup>CNRS, Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France ([gregoire.nadin@sorbonne-universite.fr](mailto:gregoire.nadin@sorbonne-universite.fr))

<sup>§</sup>Université de Strasbourg, CNRS UMR 7501, INRIA, Institut de Recherche Mathématique Avancée (IRMA), 7 rue René Descartes, 67084 Strasbourg, France ([yannick.privat@unistra.fr](mailto:yannick.privat@unistra.fr)).

<b>3</b>	<b>Proof of Theorem 1</b>	<b>10</b>
3.1	Background material on homogenization and bibliographical comments . . . . .	10
3.2	Proof of Theorem 1 . . . . .	12
<b>4</b>	<b>Proof of Theorem 2</b>	<b>14</b>
4.1	Steps of the proof for the stationarity . . . . .	15
4.2	Step 1: convergence of quasi-minimizers and of sequences of eigenfunctions . . . . .	15
4.3	Step 2: reduction to particular resource distributions close to $m_0^*$ . . . . .	18
4.4	Step 3: conclusion, by the mean value theorem . . . . .	19
<b>5</b>	<b>Sketch of the proof of Corollary 1</b>	<b>21</b>
<b>6</b>	<b>Proof of Theorem 3</b>	<b>22</b>
6.1	Preliminaries . . . . .	22
6.2	Computation of the first and second order shape derivatives . . . . .	23
6.3	Analysis of the quadratic form $\mathcal{F}_\alpha$ . . . . .	27
6.4	Conclusion . . . . .	32
<b>A</b>	<b>Proof of Lemma 1</b>	<b>32</b>
	<b>Bibliography</b>	<b>35</b>

## 1 Introduction and main results

In recent decades, much attention has been paid to extremal problems involving eigenvalues, and in particular to shape optimization problems in which the unknown is the domain where the eigenvalue problem is solved (see e.g. [32, 33] for a survey). The study of these last problems is motivated by stability issues of vibrating bodies, wave propagation in composite environments, or also on conductor thermal insulation.

In this article, we are interested in studying a particular extremal eigenvalues problem, involving a drift term. The influence of drift terms on optimal design problems is not so well understood. Such problems naturally arise for instance when looking for optimal shape design for two-phase composite materials. In that case, a possible formulation reads: given  $\Omega$ , a bounded connected open subset of  $\mathbb{R}^n$  and a set of admissible non-negative densities  $\mathcal{M}$  in  $\Omega$ , solve the optimal design problem

$$\inf_{m \in \mathcal{M}} \hat{\lambda}_\alpha(m) \quad (\hat{P}_\alpha)$$

where  $\hat{\lambda}_\alpha(m)$  denotes the first eigenvalue of the elliptic operator

$$\hat{\mathcal{L}}_\alpha^m : W_0^{1,2}(\Omega) \ni u \mapsto -\nabla \cdot ((1 + \alpha m)\nabla u).$$

Restricting the set of admissible densities to *bang-bang* ones (in other words to functions taking only two different values) is known to be relevant for the research of structures optimizing the compliance. We refer to Section 3 for detailed bibliographical comments.

Mathematically, the main issues regarding Problem  $(\hat{P}_\alpha)$  concern the existence of optimal densities in  $\mathcal{M}$ , possibly the existence of optimal *bang-bang* densities (i.e characteristic functions). In this case, it is interesting to try to describe minimizers in a qualitative way.

In what follows, we will consider a refined version of Problem  $(\hat{P}_\alpha)$ , where the operator  $\hat{\mathcal{L}}_\alpha^m$  is replaced by

$$\mathcal{L}_m^\alpha : W_0^{1,2}(\Omega) \ni u \mapsto -\nabla \cdot ((1 + \alpha m)\nabla u) - mu. \quad (1)$$

Besides its intrinsic mathematical interest, the issue of minimizing the first eigenvalue of  $\mathcal{L}_m^\alpha$  with respect to densities  $m$  is motivated by a model of population dynamics (see Section 1.3).

Before providing a precise mathematical frame of the questions we raise in what follows, let us roughly describe the main results and contributions of this article:

- by adapting the methods developed by Murat and Tartar, [48], and Cox and Lipton, [23], we show that the first eigenvalue of  $\mathcal{L}_m^\alpha$  has no regular minimizer in  $\mathcal{M}$  unless  $\Omega$  is a ball;
- if  $\Omega$  is a ball, denoting by  $m^*$  a minimizer of  $\mathcal{L}_m^0$  over  $\mathcal{M}$  (known to be *bang-bang* and radially symmetric), we show the following stationarity result:  $m^*$  still minimizes  $\mathcal{L}_m^\alpha$  over radially symmetric distributions of  $\mathcal{M}$  whenever  $\alpha$  is small enough and in small dimension ( $n = 1, 2, 3$ ). Such a result appears unexpectedly difficult to prove. Our approach is based on the use of a well chosen path of quasi-minimizers and on a new type of local argument.
- if  $\Omega$  is a ball, we investigate the local optimality of ball centered distributions among all distributions and prove a quantitative estimate on the second order shape derivative by using a new approach relying on a kind of comparison principle for second order shape derivatives.

Precise statements of these results are given in Section 1.2.

## 1.1 Mathematical Set Up

Throughout this article,  $m_0, \kappa$  are fixed positive parameters. Since in our work we want to extend the results of [39], let us define the set of admissible functions

$$\mathcal{M}_{m_0, \kappa}(\Omega) = \left\{ m \in L^\infty(\Omega), 0 \leq m \leq \kappa, \int_{\Omega} m = m_0 \right\},$$

where  $\int_{\Omega} m$  denotes the average value of  $m$  (see Section 1.4) and assume that  $m_0 < \kappa$  so that  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is non-empty. Given  $\alpha \geq 0$  and  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , the operator  $\mathcal{L}_m^\alpha$  is symmetric and compact. According to the spectral theorem, it is diagonalizable in  $L^2(\Omega)$ . In what follows, let  $\lambda_\alpha(m)$  be the first positive eigenvalue for this problem. According to the Krein-Rutman theorem,  $\lambda_\alpha(m)$  is simple and its associated  $L^2(\Omega)$ -normalized eigenfunction  $u_{\alpha, m}$  has a constant sign, say  $u_{\alpha, m} \geq 0$ . Let  $R_{\alpha, m}$  be the associated Rayleigh quotient given by

$$R_{\alpha, m} : W_0^{1,2}(\Omega) \ni u \mapsto \frac{\frac{1}{2} \int_{\Omega} (1 + \alpha m) |\nabla u|^2 - \int_{\Omega} m u^2}{\int_{\Omega} u^2}. \quad (2)$$

We recall that  $\lambda_\alpha(m)$  can also be defined through the variational formulation

$$\lambda_\alpha(m) := \inf_{u \in W_0^{1,2}(\Omega), u \neq 0} R_{\alpha, m}(u) = R_{\alpha, m}(u_{\alpha, m}). \quad (3)$$

and that  $u_{\alpha, m}$  solves

$$\begin{cases} -\nabla \cdot \left( (1 + \alpha m) \nabla u_{\alpha, m} \right) - m u_{\alpha, m} = \lambda_\alpha(m) u_{\alpha, m} & \text{in } \Omega, \\ u_{\alpha, m} = 0 & \text{on } \Omega. \end{cases} \quad (4)$$

in a weak  $W_0^{1,2}(\Omega)$  sense. In this article, we address the optimization problem

$$\boxed{\inf_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \lambda_\alpha(m)}. \quad (P_\alpha)$$

This problem is a modified version of the standard two-phase problem. It is notable that it is relevant in the framework of population dynamics, when looking for optimal resources configurations in a heterogeneous environment for species survival, see Section 1.3.

## 1.2 Main results

Before providing the main results of this article, we state a first fundamental property of the investigated model, reducing in some sense the research of general minimizers to the one of *bang-bang* densities. It is notable that, although the set of *bang-bang* densities is known to be dense in the set of all densities for the weak-star topology, such a result is not obvious since it rests upon continuity properties of  $\lambda_\alpha$  for this topology. We overcome this difficulty by exploiting a convexity-like property of  $\lambda_\alpha$ .

**Proposition 1** (weak *bang-bang* property). *Let  $\Omega$  be a bounded connected subset of  $\mathbb{R}^n$  with a Lipschitz boundary and let  $\alpha > 0$  be given. For every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , there exists a bang-bang function  $\tilde{m} \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that*

$$\lambda_\alpha(m) \geq \lambda_\alpha(\tilde{m}).$$

*Moreover, if  $m$  is not bang-bang, then we can choose  $\tilde{m}$  so that the previous inequality is strict.*

In other words, given any resources distribution  $m$ , it is always possible to construct a *bang-bang* function  $\tilde{m}$  that improves the criterion.

**Non-existence for general domains.** In a series of paper, [14, 15, 16], Casado-Diaz proved that the problem of minimizing the first eigenvalue of the operator  $u \mapsto -\nabla \cdot (1 + \alpha m) \nabla u$  with respect to  $m$  does not have a solution when  $\partial\Omega$  is connected. His proof relies on a study of the regularity for this minimization problem, on homogenization and on a Serrin type argument. The following result is in the same vein, with two differences: it is weaker than his in the sense that it needs to assume higher regularity of the optimal set, but stronger in the sense that we do not make any strong assumption on  $\partial\Omega$ . For further details regarding this literature, we refer to Section 3.1.

**Theorem 1.** *Let  $\Omega$  be a bounded connected subset of  $\mathbb{R}^n$  with a Lipschitz boundary, let  $\alpha > 0$  and  $n \geq 2$ . If the optimization problem  $(P_\alpha)$  has a solution  $\hat{m} \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , then this solution writes  $\hat{m} = \kappa \chi_{\hat{E}}$ , where  $\hat{E}$  is a measurable subset of  $\Omega$ . Moreover, if  $\partial\hat{E}$  is a  $\mathcal{C}^2$  hypersurface and if  $\Omega$  is connected, then  $\Omega$  is a ball.*

The proof of this Theorem relies on methods developed by Murat and Tartar, [48], Cox and Lipton, [23], and on a Theorem of Serrin [53].

**Analysis of optimal configurations in a ball.** According to Theorem 1, existence of regular solutions fail when  $\Omega$  is not a ball. This suggest to investigate the case  $\Omega = \mathbb{B}(0, R)$ , which is the main goal of what follows.

Let us stress that proving the existence of a minimizer in this setting and characterizing it is a hard task. Indeed, to underline the difficulty, notice in particular that none of the usual rearrangement techniques (the Schwarz rearrangement or the Alvino-Trombetti one, see Section 3.1), that enable in general to reduce the research of solutions to radially symmetric densities, and thus to get compactness properties, can be applied here.

### The case of radially symmetric distributions

Here, we assume that  $\Omega$  denotes the ball  $\mathbb{B}(0, R)$  with  $R > 0$ . Let

$$m_0^* = \kappa \mathbb{1}_{\mathbb{B}(0, r_0^*)} = \kappa \mathbb{1}_{E_0^*}$$

be the centered distribution known to be the unique minimizer of  $\lambda_0$  in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  (see e.g. [39]).

In what follows, we restrict ourselves to the case of radially symmetric resources distributions.

**Theorem 2.** *Let  $\mathcal{M}_{rad}$  be the subset of radially symmetric distributions of  $\mathcal{M}_{m_0, \kappa}(\Omega)$ . The optimization problem*

$$\inf_{m \in \mathcal{M}_{rad}} \lambda_\alpha(m)$$

*has a solution. Furthermore, when  $n = 1, 2, 3$ , there exists  $\alpha^* > 0$  such that, for any  $\alpha < \alpha^*$ , there holds*

$$\min_{m \in \mathcal{M}_{rad}} \lambda_\alpha(m) = \lambda_\alpha(m_0^*). \quad (5)$$

The proof of the existence part of the theorem relies on rearrangement techniques that were first introduced by Alvino and Trombetti in [3] and then refined in [21]. The stationarity result, i.e the fact that  $m^*$  is a minimizer among radially symmetric distributions, was proved in the one-dimensional case in [17]. To extend this result to higher dimensions, we developed an approach involving a homogenized version of the problem under consideration. The small dimensions hypothesis is due to a technical reason, which arises when dealing with elliptic regularity for this equation.

Restricting ourselves to radially symmetric distributions might appear surprising since one could expect this result to be true without restriction, in  $\mathcal{M}_{m_0, \kappa}(\Omega)$ . For instance, a similar result has been shown in the framework of two-phase eigenvalues [21], as a consequence of the Alvino-Trombetti rearrangement. Unfortunately, regarding Problem  $(P_\alpha)$ , no standard rearrangement technique leads to the expected conclusion, because of the specific form of the involved Rayleigh quotient. A first attempt in the investigation of the ball case is then to consider the case of radially symmetric distributions. It is notable that, even in this case, the proof appears unexpectedly difficult.

Finally, we note that, as a consequence of the methods developed to prove Theorem 2, when a small amount of resources is available, the centered distribution  $m_0^*$  is optimal.

**Corollary 1.** *Under the same assumptions as in Theorem 2, there exists  $\underline{m} > 0$ ,  $\underline{\alpha} > 0$  such that, if  $m_0 \leq \underline{m}$  and  $\alpha < \underline{\alpha}$ , then the unique solution of  $(P_\alpha)$  is  $m_0^* = \kappa \mathbb{1}_{E_0^*}$ .*

### Local minimality of the centered distribution among all resources distributions

In what follows, we tackle the issue of the local minimality of  $m_0^*$  in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  with the help of a shape derivative approach. We obtain partial results in dimension  $n = 2$ .

Let  $\Omega$  be a bounded connected domain with a Lipschitz boundary, and consider a *bang-bang* function  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  writing  $m = \kappa \mathbb{1}_E$ , for a measurable subset  $E$  of  $\Omega$  such that  $\kappa|E| = m_0|\Omega|$ . Let us introduce  $\lambda_\alpha(E) := \lambda_\alpha(\mathbb{1}_E)$ , with a slight abuse of notation. Let us assume that  $E$  has a  $\mathcal{C}^2$  boundary. Let  $V : \Omega \rightarrow \mathbb{R}^n$  be a  $W^{3, \infty}$  vector field with compact support, and define for every  $t$  small enough,  $E_t := (\text{Id} + tV)E$ . For  $t$  small enough,  $\phi_t := \text{Id} + tV$  is a smooth diffeomorphism from  $E$  to  $E_t$ , and  $E_t$  is an open connected set with a  $\mathcal{C}^2$  boundary. If  $\mathcal{F} : E \mapsto \mathcal{F}(E)$  denotes a shape functional, the first (resp. second) order shape derivative of  $\mathcal{F}$  at  $E$  in the direction  $V$  is

$$\mathcal{F}'(E)[V] := \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(E_t) \quad \left( \text{resp.} \quad \left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{F}(E_t) \right)$$

whenever these quantities exist.

For further details regarding the notion of shape derivative, we refer to [34, Chapter 5].

Since one wants to ensure that  $|E_t| = V_0$ , we impose the condition  $\int_E \nabla \cdot V = 0$  on the vector field  $V$ . We call *admissible at  $E$*  such vector fields and introduce

$$\mathcal{X}(E) := \left\{ V \in W^\infty(\mathbb{R}^n; \mathbb{R}^n), \int_E \nabla \cdot V = 0, \|V\|_{W^{3, \infty}} \leq 1 \right\}. \quad (6)$$

A shape  $E \subset \Omega$  with a  $\mathcal{C}^2$  boundary such that  $\kappa|E| = m_0|\Omega|$  is said to be critical if

$$\forall V \in \mathcal{X}(E), \quad \lambda'_\alpha(E)[V] = 0. \quad (7)$$

or, similarly, if there exist a Lagrange multiplier  $\Lambda_\alpha$  such that  $(\lambda_\alpha - \Lambda_\alpha \text{Vol})'(E)[V] = 0$  for all  $V \in \mathcal{X}(E)$ , where  $\text{Vol} : \Omega \mapsto |\Omega|$  denotes the volume functional. Furthermore, if  $E$  is a local minimizer for Problem  $(P_\alpha)$ , then one has

$$\forall V \in \mathcal{X}(E), \quad (\lambda_\alpha - \Lambda_\alpha \text{Vol})''(E)[V, V] \geq 0. \quad (8)$$

**Theorem 3.** *Let us assume that  $n = 2$ . The ball  $E = \mathbb{B}(0, r_0^*) = \mathbb{B}^*$  satisfies the shape optimality conditions (7)-(8). Furthermore, if  $\Lambda_\alpha$  is the Lagrange multiplier associated with the volume constraint, there exists two constants  $\bar{\alpha} > 0$  and  $C > 0$  such that, for any  $\alpha \in [0, \bar{\alpha})$  and any vector field  $V \in \mathcal{X}(\mathbb{B}^*)$  normal to  $\partial\mathbb{B}^* = \mathbb{S}^*$  there holds*

$$(\lambda_\alpha - \Lambda_\alpha \text{Vol})''(\mathbb{B}^*)[V, V] \geq C \|V \cdot \nu\|_{L^2(\mathbb{S}^*)}^2.$$

**Remark 1.** The proof requires explicit computation of the shape derivative of the eigenfunction. We note that in [24] such computations are carried out for the two-phase problem and that in [36] such an approach is undertaken to investigate the stability of certain configurations for a weighted Neumann eigenvalue problem.

The main contribution of this result is to shed light on a monotonicity principle that enables one to lead a careful asymptotic analysis of the second order shape derivative of the functional as  $\alpha \rightarrow 0$ . It is important to note that, although this allows us to deeply analyze the second order optimality conditions, it is expected that the optimal coercivity norm in the right-hand side above is expected to be  $H^{\frac{1}{2}}$  whenever  $\alpha > 0$ , which we do not recover with our method. When  $\alpha = 0$ , we know that the optimal coercivity norm is  $L^2$  (see [44]).

The rest of this article is dedicated to proofs of the results we have just outlined.

### 1.3 A biological application of the problem

Equation (4) arises naturally when dealing with simple population dynamics in heterogeneous spaces.

Let  $\varepsilon \geq 0$  be a parameter of the model. We consider a population density whose flux is given by

$$\mathcal{J}_\varepsilon = -\nabla u + \varepsilon u \nabla m.$$

Since  $\nabla m$  might not make sense if  $m$  is assumed to be only measurable, we temporarily omit this difficulty by assuming it smooth enough so that the expression above makes sense. The term  $u \nabla m$  stands for a bias in the population movement, modeling a tendency of the population to disperse along the gradient of resources and hence move to favorable regions. The parameter  $\varepsilon$  quantifies the influence of the resources distribution on the movement of the species. The complete associated reaction diffusion equation, called “logistic diffusive equation”, reads

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \varepsilon u \nabla m) + m u - u^2 \quad \text{in } \Omega,$$

completed with suitable boundary conditions. In what follows, we will focus on Dirichlet boundary conditions meaning that the boundary of  $\Omega$  is lethal for the population living inside. Plugging the change of variable  $v = e^{-\varepsilon m} u$  in this equation leads to

$$\frac{\partial v}{\partial t} = \Delta v + \varepsilon \langle \nabla m, \nabla u \rangle + m v - e^{\varepsilon m} v \quad \text{in } \Omega.$$

It is known (see e.g. [5, 4, 49]) that the asymptotic behavior of this equation is driven by the principal eigenvalue of the operator  $\tilde{\mathcal{L}} : u \mapsto -\Delta u - \varepsilon \langle \nabla m, \nabla u \rangle - mu$ . The associated principal eigenfunction  $\psi$  satisfies

$$-\nabla \cdot (e^{\varepsilon m} \nabla \psi) - m e^{\varepsilon m} \psi = \tilde{\lambda}_\varepsilon \psi e^{\varepsilon m} \quad \text{in } \Omega.$$

Following the approach developed in [39], optimal configurations of resources correspond to the ones ensuring the fastest convergence to the steady-states of the PDE above, which comes to minimizing  $\tilde{\lambda}_\varepsilon(m)$  with respect to  $m$ .

By using Proposition 1, which enables us to only deal with *bang-bang* densities  $m$ , one shows easily that minimizing  $\tilde{\lambda}_\varepsilon(m)$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is equivalent to minimizing  $\lambda_\varepsilon(m)$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , in other words to Problem  $(P_\alpha)$  with  $\alpha = \varepsilon$ . Theorem 1 can thus be interpreted as follows in this framework: assuming that the population density moves along the gradient of the resources, it is not possible to lay the resources in an optimal way. Note that the conclusion is completely different in the case  $\alpha = 0$  (see [39]) or in the one-dimensional case (i.e.  $\Omega = (0; 1)$ ) with  $\alpha > 0$  (see [17]), where minimizers exist. In the last case, optimal configurations for three kinds boundary conditions (Dirichlet, Neumann, Robin) have been obtained, by using a new rearrangement technique. Finally, let us mention the related result [30, Theorem 2.1], dealing with Faber-Krahn type inequalities for general operators of the form

$$\mathcal{K} : u \mapsto -\nabla \cdot (A \nabla u) - \langle V, \nabla u \rangle - mu$$

where  $A$  is a positive symmetric matrix. Let us denote the first eigenvalue of  $\mathcal{K}$  by  $E(A, V, m)$ . It is shown, by using new rearrangements, that there exist radially symmetric elements  $A^*, V^*, m^*$  such that

$$0 < \inf A \leq A^* \leq \|A\|_\infty, \quad \|A^{-1}\|_{L^1} = \|(A^*)^{-1}\|_{L^1}, \quad \|V^*\|_{L^\infty} \leq \|V\|_{L^\infty}$$

and  $E(A, V, m) \geq E(A^*, V^*, m^*)$ . We note that applying this result directly to our problem would not allow us to conclude. Indeed, we would get that for every  $\Omega$  of volume  $V_1$  and every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , if  $\Omega^*$  is the ball of volume  $V_1$ , there exists two radially symmetric functions  $m_1$  and  $m_2$  satisfying  $m_1, m_2 \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $\lambda_\alpha(m) \geq \mu_\alpha(m_1, m_2)$ , where  $\mu_\alpha(m_1, m_2)$  is the first eigenvalue of the operator  $-\nabla \cdot ((1 + \alpha m_1) \nabla) - m_2$ . We note that this result could also be obtained by using the symmetrization techniques of [3].

Finally, let us mention that an optimal control problem involving a similar model but a different cost functional, related to optimal harvesting of a marine resource, has been investigated in the series of articles [10, 11, 19].

## 1.4 Notations and notational conventions, technical properties of the eigenfunctions

Let us sum-up the notations used throughout this article.

- $\mathbb{R}_+$  is the set of non-negative real numbers.  $\mathbb{R}_+^*$  is the set of positive real numbers.
- $n$  is a fixed positive integer and  $\Omega$  is a bounded connected domain in  $\mathbb{R}^n$ .
- if  $E$  denotes a subset of  $\Omega$ , the notation  $\chi_E$  stands for the characteristic function of  $E$ , equal to 1 in  $E$  and 0 elsewhere.
- the notation  $\|\cdot\|$  used without subscript refers to the standard Euclidean norm in  $\mathbb{R}^n$ . When referring to the norm of a Banach space  $\mathcal{X}$ , we write it  $\|\cdot\|_{\mathcal{X}}$ .
- The average of every  $f \in L^1(\Omega)$  is denoted by  $f_\Omega$   $f := \frac{1}{|\Omega|} \int_\Omega f$ .
- $\nu$  stands for the outward unit normal vector on  $\partial\Omega$ .



## 2 Preliminaries

### 2.1 Switching function

In view of deriving optimality conditions for Problem  $(P_\alpha)$ , we introduce the tangent cone to  $\mathcal{M}_{m_0, \kappa}(\Omega)$  at any point of this set.

**Definition 1.** ([34, chapter 7]) For every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , the tangent cone to the set  $\mathcal{M}_{m_0, \kappa}(\Omega)$  at  $m$ , also called the admissible cone to the set  $\mathcal{M}_{m_0, \kappa}(\Omega)$  at  $m$ , denoted by  $\mathcal{T}_m$  is the set of functions  $h \in L^\infty(\Omega)$  such that, for any sequence of positive real numbers  $\varepsilon_n$  decreasing to 0, there exists a sequence of functions  $h_n \in L^\infty(\Omega)$  converging to  $h$  as  $n \rightarrow +\infty$ , and  $m + \varepsilon_n h_n \in \mathcal{M}_{m_0, \kappa}(\Omega)$  for every  $n \in \mathbb{N}$ .

Notice that, as a consequence of this definition, any  $h \in \mathcal{T}_m$  satisfies  $\int_\Omega h = 0$ .

**Lemma 1.** Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $h \in \mathcal{T}_m$ . The mapping  $\mathcal{M}_{m_0, \kappa}(\Omega) \ni m \mapsto u_{\alpha, m}$  is twice differentiable at  $m$  in direction  $h$  in a strong  $L^2(\Omega)$  sense and in a weak  $W_0^{1,2}(\Omega)$  sense, and the mapping  $\mathcal{M}_{m_0, \kappa}(\Omega) \ni m \mapsto \lambda_\alpha$  is twice differentiable in a strong  $L^2(\Omega)$  sense.

The proof of this lemma is technical and is postponed to Appendix A.

For  $t$  small enough, let us introduce the mapping  $g_h : t \mapsto \lambda_\alpha([m + th])$ . Hence,  $g_h$  is twice differentiable. The first and second order derivatives of  $\lambda_\alpha$  at  $m$  in direction  $h$ , denoted by  $\dot{\lambda}_\alpha(m)[h]$  and  $\ddot{\lambda}_\alpha(m)[h]$ , are defined by

$$\dot{\lambda}_\alpha(m)[h] := g'_h(0) \quad \text{and} \quad \ddot{\lambda}_\alpha(m)[h] := g''_h(0).$$

**Lemma 2.** Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $h \in \mathcal{T}_m$ . The mapping  $m \mapsto \lambda_\alpha(m)$  is differentiable at  $m$  in direction  $h$  in  $L^2$  and its differential reads

$$\dot{\lambda}_\alpha(m)[h] = \int_\Omega h \psi_{\alpha, m}, \quad \text{with} \quad \psi_{\alpha, m} := \alpha |\nabla u_{\alpha, m}|^2 - u_{\alpha, m}^2. \quad (9)$$

The function  $\psi_{\alpha, m}$  is called switching function.

*Proof.* According to Lemma 1, we can differentiate the variational formulation associated to (4) and get that the differential  $\dot{u}_{\alpha, m}[h]$  of  $m \mapsto u_{\alpha, m}$  at  $m$  in direction  $h$  satisfies

$$\begin{cases} -\nabla \cdot (\sigma_\alpha \nabla \dot{u}_{\alpha, m}[h]) - \alpha \nabla \cdot (h \nabla u_{\alpha, m}) = \dot{\lambda}_\alpha(m)[h] u_{\alpha, m} + \lambda_\alpha(m) \dot{u}_{\alpha, m}[h] \\ \quad + m \dot{u}_{\alpha, m}[h] + h u_{\alpha, m} & \text{in } \Omega, \\ \dot{u}_{\alpha, m}[h] = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h] = 0. \end{cases} \quad (10)$$

Multiplying this equation by  $u_{\alpha, m}$ , integrating by parts and using that  $u_{\alpha, m}$  is normalized in  $L^2(\Omega)$  leads to

$$\begin{aligned} \dot{\lambda}_\alpha(m)[h] &= \underbrace{\int_\Omega \sigma_\alpha \langle \nabla \dot{u}_{\alpha, m}[h], \nabla u_{\alpha, m} \rangle - \lambda_\alpha(m) \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h] - \int_\Omega m u_{\alpha, m} \dot{u}_{\alpha, m}[h]}_{=0 \text{ according to (4)}} \\ &\quad + \int_\Omega \alpha h |\nabla u_{\alpha, m}|^2 - \int_\Omega h u_{\alpha, m}^2. \end{aligned}$$

□

## 2.2 Proof of Proposition 1

The proof relies on concavity properties of the functional  $\lambda_\alpha$ . More precisely, let  $m_1, m_2 \in \mathcal{M}_{m_0, \kappa}(\Omega)$ . We will show that the map  $f : [0, 1] \ni t \mapsto \lambda_\alpha((1-t)m_1 + tm_2)$  is strictly concave, i.e that  $f'' < 0$  on  $[0, 1]$ .

Note that the characterization of the concavity in terms of second order derivatives makes sense, according to Lemma 1, since  $\lambda_\alpha$  is twice differentiable. Before showing this concavity property, let us first explain why it implies the conclusion of Proposition 1 (the weak *bang-bang* property). Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  assumed to be not *bang-bang*. The set  $\mathcal{I} = \{0 < m < \kappa\}$  is then of positive Lebesgue measure and  $m$  is therefore not extremal in  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , according to [34, Prop. 7.2.14]. We then infer the existence of  $t \in (0, 1)$  as well as two distinct elements  $m_1$  and  $m_2$  of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $m = (1-t)m_1 + tm_2$ . Because of the strict concavity of  $\lambda_\alpha$ , the solution of the optimization problem  $\min\{\lambda_\alpha((1-t)m_1 + tm_2)\}$  is either  $m_1$  or  $m_2$ , and moreover,  $m$  cannot solve this problem. Assume that  $m_1$  solves this problem without loss of generality. One thus has  $\lambda_\alpha(m_1) < \lambda_\alpha(m)$ . Since the subset of *bang-bang* functions of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is dense in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  for the weak-star topology of  $L^\infty(\Omega)$ , there exists a sequence of *bang-bang* functions  $(m^k)_{k \in \mathbb{N}}$  of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  converging weakly-star to  $m_1$  in  $L^\infty(\Omega)$ . Furthermore,  $\lambda_\alpha$  is upper semicontinuous for the weak-star topology of  $L^\infty(\Omega)$ , since it reads as the infimum of continuous linear functionals for this topology. Let  $\varepsilon > 0$ . We infer the existence of  $k_\varepsilon \in \mathbb{N}$  such that  $\lambda_\alpha(m^{k_\varepsilon}) \leq \lambda_\alpha(m_1) + \varepsilon$ . By choosing  $\varepsilon$  small enough, we infer that  $\lambda_\alpha(m^{k_\varepsilon}) < \lambda_\alpha(m)$ , whence the result.

It now remains to prove that  $f$  is strictly concave. Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , and set  $m_1 = m$ ,  $h = m_2 - m_1$ , we observe that  $f''(t) = \ddot{\lambda}_\alpha((1-t)m_1 + tm_2)[h]$  for all  $t \in [0, 1]$ . The differential  $\dot{u}_{\alpha, m}[h]$  of  $m \mapsto u_{\alpha, m}$  at  $m$  in direction  $h$ , denoted  $\dot{u}_{\alpha, m}[h]$ , satisfies (10) and the second order Gâteaux-derivatives  $\ddot{u}_{\alpha, m}[h]$  and  $\ddot{\lambda}_\alpha(m)[h]$  solve

$$\begin{cases} -\nabla \cdot (\sigma_\alpha \nabla \dot{u}_{\alpha, m}[h]) - 2\alpha \nabla \cdot (h \nabla \dot{u}_{\alpha, m}[h]) = & \ddot{\lambda}_\alpha(m)[h] u_{\alpha, m} + 2\dot{\lambda}_\alpha(m)[h] \dot{u}_{\alpha, m}[h] \\ & + \lambda_\alpha(m) \ddot{u}_{\alpha, m}[h] + m \ddot{u}_{\alpha, m}[h] + 2h \dot{u}_{\alpha, m}[h] & \text{in } \Omega, \\ \ddot{u}_{\alpha, m}[h] = 0 & & \text{on } \partial\Omega. \end{cases} \quad (11)$$

Multiplying this Equation by  $u_{\alpha, m}$ , using that  $u_{\alpha, m}$  is normalized in  $L^2(\Omega)$  and integrating by parts yields

$$\begin{aligned} \ddot{\lambda}_\alpha(m)[h] &= \underbrace{\int_\Omega \sigma_\alpha \langle \nabla \ddot{u}_{\alpha, m}[h], \nabla u_{\alpha, m} \rangle - \lambda_\alpha(m) \int_\Omega u_{\alpha, m} \ddot{u}_{\alpha, m}[h] - \int_\Omega m u_{\alpha, m} \ddot{u}_{\alpha, m}[h]}_{=0 \text{ according to (4)}} \\ &+ 2\alpha \int_\Omega h \langle \nabla \dot{u}_{\alpha, m}[h], \nabla u_{\alpha, m} \rangle - 2 \int_\Omega h u_{\alpha, m} \dot{u}_{\alpha, m}[h] \\ &= 2 \left( - \int_\Omega \sigma_\alpha |\nabla \dot{u}_{\alpha, m}[h]|^2 + \int_\Omega m \dot{u}_{\alpha, m}[h]^2 + \lambda_\alpha(m) \int_\Omega \dot{u}_{\alpha, m}[h]^2 \right) + \underbrace{2 \dot{\lambda}_\alpha(m)[h] \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h]}_{=0 \text{ since } \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h] = 0} \\ &= 2 \int_\Omega \dot{u}_{\alpha, m}[h]^2 (-R_{\alpha, m}[\dot{u}_{\alpha, m}[h]] + \lambda_\alpha(m)) < 0, \end{aligned}$$

where the last inequality comes from the observation that, whenever  $h \neq 0$ , one has  $\dot{u}_{\alpha, m}[h] \neq 0$  and  $\dot{u}_{\alpha, m}[h]$  is in the orthogonal space to the first eigenfunction  $u_{\alpha, m}$  in  $L^2(\Omega)$ . Since the first eigenvalue is simple, the Rayleigh quotient of  $\dot{u}_{\alpha, m}[h]$  is greater than  $\lambda_\alpha(m)$ .

### 3 Proof of Theorem 1

This proof is based on a homogenization argument, inspired from the notions and techniques introduced in [48]. In the next section, we gather the preliminary tools and material involved in what follows.

#### 3.1 Background material on homogenization and bibliographical comments

Let us recall several usual definitions and results in homogenization theory we will need hereafter.

**Definition 2** (*H-convergence*). Let  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)^{\mathbb{N}}$  and for every  $k \in \mathbb{N}$ , define respectively  $\sigma_k$  and  $u_k(f)$  by  $\sigma_k = 1 + \alpha m_k$  and as the unique solution of

$$\begin{cases} -\nabla \cdot (\sigma_k \nabla u_k(f)) = f & \text{in } \Omega \\ u_k(f) = 0 & \text{on } \Omega \end{cases}$$

where  $f \in L^2(\Omega)$  is given. We say that the sequence  $(\sigma_k)_{k \in \mathbb{N}}$  *H-converges* to  $A : \Omega \rightarrow M_n(\mathbb{R})$  if, for every  $f \in L^2(\Omega)$ , the sequence  $(u_k(f))_{k \in \mathbb{N}}$  converges weakly to  $u_\infty$  in  $W_0^{1,2}(\Omega)$  and the sequence  $(\sigma_k \nabla u_k)_{k \in \mathbb{N}}$  converges weakly to  $A \nabla u_\infty$  in  $L^2(\Omega)$ , where  $u_\infty$  solves

$$\begin{cases} -\nabla \cdot (A \nabla u_\infty) = f & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \Omega \end{cases}$$

In that case, we will write  $\sigma_k \xrightarrow[k \rightarrow \infty]{H} A$ .

**Definition 3** (*arithmetic and geometric means*). Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $\sigma = 1 + \alpha m$ . We define the arithmetic mean of  $\sigma$  by  $\Lambda_+(m) = \sigma$ , and its harmonic mean by  $\Lambda_-(m) = \frac{1 + \alpha \kappa}{1 + \alpha(\kappa - m)}$ . One has  $\Lambda_-(m) \leq \Lambda_+(m)$ , according to the arithmetic-harmonic inequality, with equality if and only if  $m$  is a bang-bang function.

**Proposition.** [48, Proposition 10] Let  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)^{\mathbb{N}}$  and  $(\sigma_k)_{k \in \mathbb{N}}$  given by  $\sigma_k = 1 + \alpha m_k$ . Up to a subsequence, there exists  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)^{\mathbb{N}}$  converges to  $m$  for the weak-star topology of  $L^\infty$ .

Assume moreover that the sequence  $(\sigma_k)_{k \in \mathbb{N}}$  *H-converges* to a matrix  $A$ . Then,  $A$  is a symmetric matrix, its spectrum  $\Sigma(A)$  is real, and

$$\Lambda_-(m) \leq \min \Sigma(A) \leq \max \Sigma(A) \leq \Lambda_+(m). \quad (J_1)$$

$$\sum_{j=1}^n \frac{1}{\lambda_j - 1} \leq \frac{1}{\Lambda_-(m) - 1} + \frac{n-1}{\Lambda_+(m) - 1}, \quad (J_2)$$

$$\sum_{j=1}^n \frac{1}{1 + \alpha \kappa - \lambda_j} \leq \frac{1}{1 + \alpha \kappa - \Lambda_-(m)} + \frac{n-1}{1 + \alpha \kappa - \Lambda_+(m)}. \quad (J_3)$$

For a given  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , we introduce

$$M_m^\alpha = \{A : \Omega \rightarrow S_n(\mathbb{R}), A \text{ satisfies } (J_1)-(J_2)-(J_3)\}.$$

For a matrix-valued application  $A \in M_m^\alpha$  for some  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , it is possible to define the principal eigenvalue of  $A$  via Rayleigh quotients as

$$\zeta_\alpha(m, A) := \inf_{u \in W_0^{1,2}(\Omega), \int_\Omega u^2 = 1} \int_\Omega \langle A \nabla u, \nabla u \rangle - \int_\Omega m u^2. \quad (12)$$

Note that the dependence of  $\zeta_\alpha$  on the parameter  $\alpha$  is implicitly contained in the condition  $A \in M_m^\alpha$ . We henceforth focus on the following relaxed version of the optimization problem:

$$\inf_{m \in \mathcal{M}_{m_0, \kappa}(\Omega), A \in M_m^\alpha} \zeta_\alpha(m, A). \quad (13)$$

for which we have the following result.

**Theorem.** [48, Proposition 10]

1. For every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $A \in M_m^\alpha$ , there exists a sequence  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $(m_k)_{k \in \mathbb{N}}$  converges to  $m$  for the weak-star topology of  $L^\infty$ , and the sequence  $(\sigma_k)_{k \in \mathbb{N}}$  defined by  $\sigma_k = 1 + \alpha m_k$   $H$ -converges to  $A$ , as  $k \rightarrow +\infty$ .
2. The mapping  $(m, A) \mapsto \lambda_\alpha(m, A)$  is continuous with respect to the  $H$ -convergence (see in particular [51]).
3. The variational problem (13) has a solution  $(\bar{m}, \bar{A})$ ; by definition,  $\bar{A} \in M_{\bar{m}}^\alpha$ . Furthermore, if  $\bar{u}$  is the associated eigenfunction, then  $\bar{A} \nabla \bar{u} = \Lambda_-(\bar{m}) \nabla \bar{u}$ .

This theorem allows us to solve Problem (13).

**Corollary 2.** [48] If Problem  $(P_\alpha)$  has a solution  $\bar{m}$ , then the couple  $(\bar{m}, 1 + \alpha \bar{m})$  solves Problem (13).

*Proof of Corollary 2.* Assume that the solution of (13) is  $(\bar{m}, \bar{A})$  and that  $\bar{A} \neq 1 + \alpha \bar{m}$ . Then there exists a sequence  $(m_k)_{k \in \mathbb{N}}$  converging weak-star in  $L^\infty$  to  $\bar{m}$  and such that the sequence  $(1 + \alpha m_k)_{k \in \mathbb{N}}$   $H$ -converges to  $\bar{A}$ . This means that

$$\lambda_\alpha(\bar{m}) = \zeta_\alpha(\bar{m}, 1 + \alpha \bar{m}) > \zeta_\alpha(\bar{m}, \bar{A}) = \lim_{k \rightarrow \infty} \lambda_\alpha(m_k)$$

which immediately yields a contradiction.  $\square$

Let us end this section with several bibliographical comments on such problems.

**Bibliographical comments on the two-phase conductors problem.** Problem  $(\hat{P}_\alpha)$  has drawn a lot of attention in the last decades, since the seminal works by Murat and Tartar, [47, 48]. Roughly speaking, this optimal design problem is, in general, ill-posed and one needs to introduce a relaxed formulation to get existence. We refer to [1, 23, 48, 51].

Let us provide the main lines strategy to investigate existence issues for Problem  $(\hat{P}_\alpha)$ , according to [47, 48]. If the solution  $(\bar{m}, 1 + \alpha \bar{m})$  to the relaxed problem (13) is a solution to the original problem  $(\hat{P}_\alpha)$ , then there exists a measurable subset  $\bar{E}$  of  $\Omega$  such that  $\bar{m} = \kappa \mathbb{1}_{\bar{E}}$ . If furthermore  $\bar{E}$  is assumed to be smooth enough, then, denoting by  $\bar{u}$  the principal eigenfunction associated with  $(\bar{m}, \lambda_\alpha(m)) = (\bar{m}, \zeta_\alpha(\bar{m}, 1 + \alpha \bar{m}))$ , we get that  $\bar{u}$  and  $(1 + \alpha \bar{m}) \frac{\partial \bar{u}}{\partial \nu}$  must be constant on  $\partial \bar{E}$ . The function  $1 + \alpha \bar{m}$  being discontinuous across  $\partial E$ , the optimality condition above has to be understood in the following sense: the function  $(1 + \alpha \bar{m}) \frac{\partial \bar{u}}{\partial \nu}$ , a priori discontinuous, is in fact continuous across  $\partial E$  and even constant on it. Note that these arguments have been generalized in [23]. These optimality conditions, combined with Serrin's Theorem [53], suggest that Problem  $(\hat{P}_\alpha)$  could have a solution if, and only if  $\Omega$  is a ball. The best results known to date are the following ones.

**Theorem.** (i) Let  $\Omega$  be an open set such that  $\partial \Omega$  is  $\mathcal{C}^2$  and connected. Problem  $(\hat{P}_\alpha)$  has a solution if and only if  $\Omega$  is a ball [16].

(ii) If  $\Omega$  is a ball, then Problem  $(\hat{P}_\alpha)$  has a solution which is moreover radially symmetric [21].

Regarding the second part of the theorem, the authors used a particular rearrangement coming to replace  $1 + \alpha m$  by its harmonic mean on each level-set of the eigenfunction. Such a rearrangement has been first introduced by Alvino and Trombetti [3]. This drives the author to reduce the class of admissible functions to radially symmetric ones, which allow them to conclude thanks to a compactness argument [2]. These arguments are mimicked to derive the existence part of Theorem 2.

Finally, let us mention [20, 40], where the optimality of annular configurations in the ball is investigated. A complete picture of the situation is then depicted in the case where  $\alpha$  is small, which is often referred to as the "low contrast regime". We also mention [24], where a shape derivative approach is undertaken to characterize minimizers when  $\Omega$  is a ball.

### 3.2 Proof of Theorem 1

Let us assume the existence of a solution to Problem  $(P_\alpha)$ , denoted  $\bar{m}$ . According to Proposition 1, there exists a measurable subset  $\bar{E}$  of  $\Omega$  such that  $\bar{m} = \kappa \chi_{\bar{E}}$ . Let us introduce  $\bar{\sigma} := 1 + \alpha \bar{m}$  and  $\bar{u}$ , the  $L^2$ -normalized eigenfunction associated to  $\bar{m}$ .

Let us now assume that  $\partial E$  is  $\mathcal{C}^2$ .

**Step 1: derivation of optimality conditions.** What follows is an adaptation of [23]. For this reason, we only recall the main lines. Let us write the optimality condition for the problem

$$\min_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \min_{A \in M_m^\alpha} \zeta_\alpha(m, A) = \lambda_\alpha(\bar{m}),$$

where  $\zeta_\alpha$  is given by (12). Let  $h$  be an admissible perturbation at  $\bar{m}$ . In [48] it is proved that for every  $\varepsilon > 0$  small enough, there exists a matrix-valued application  $A_\varepsilon \in M_{\bar{m} + \varepsilon h}$  such that

$$A_\varepsilon \nabla \bar{u} = \Lambda_-(\bar{m} + \varepsilon h) \nabla \bar{u} \quad \text{in } \Omega,$$

where  $\Lambda_-$  has been introduced in Definition 3. Fix  $\varepsilon$  as above. Since  $(\bar{m}, 1 + \alpha \bar{m})$  is a solution of the Problem (13), one has

$$\begin{aligned} \int_{\Omega} \langle A_\varepsilon \nabla \bar{u}, \nabla \bar{u} \rangle - \int_{\Omega} (m + \varepsilon h) \bar{u}^2 &\geq \zeta_\alpha(m + \varepsilon h, A_\varepsilon) \\ &\geq \zeta_\alpha(\bar{m}, 1 + \alpha \bar{m}) = \int_{\Omega} \bar{\sigma} |\nabla \bar{u}|^2 - \int_{\Omega} \bar{m} \bar{u}^2. \end{aligned}$$

where one used the Rayleigh quotient definition of  $\zeta_\alpha$  as well as the minimality of  $(\bar{m}, 1 + \alpha \bar{m})$ . Dividing the last inequality by  $\varepsilon$  and passing to the limit yields

$$\int_{\Omega} h \left. \frac{d\Lambda_-(m)}{dm} \right|_{m=\bar{m}} |\nabla \bar{u}|^2 - h \bar{u}^2 \geq 0.$$

Using that  $d\Lambda_-/dm = \alpha \Lambda_-(m)^2 / (1 + \alpha \kappa)$ , and that  $\bar{m}$  is a *bang-bang* function (so that  $\Lambda_-(\bar{m}) = \bar{\sigma}$ ), we infer that the first order optimality conditions read: there exists  $\mu \in \mathbb{R}$  such that

$$\bar{E} = \{\Psi_\alpha \leq \mu\} \quad \text{where} \quad \Psi_\alpha := \frac{\alpha}{1 + \alpha \kappa} \bar{\sigma}^2 |\nabla \bar{u}|^2 - \bar{u}^2. \quad (14)$$

Since the flux  $\bar{\sigma} \frac{\partial \bar{u}}{\partial \nu}$  is continuous across  $\partial \bar{E}$ , one has necessarily  $\Psi_\alpha = \mu$  on  $\partial \bar{E}$ .

Now, let us follow the approach used in [48] and [16] to simplify the writing of the optimality conditions. Notice first that  $\bar{u}$  and  $\bar{\sigma}^2 \left| \frac{\partial \bar{u}}{\partial \nu_{\bar{E}}} \right|^2$  are continuous across  $\partial \bar{E}$ . Let  $\nabla_{\tau} \bar{u}$  denote the tangential gradient of  $\bar{u}$  on  $\partial \bar{E}$ . For the sake of clarity, the quantities computed on  $\partial \bar{E}$  seen as the boundary of  $\bar{E}$  will be denoted with the subscript *int*, whereas the ones computed on  $\partial \bar{E}$  seen as part of the boundary of  $\bar{E}^c$  will be denoted with the subscript *ext*. According to the optimality conditions (14), one has

$$\frac{\alpha}{1+\alpha\kappa} \bar{\sigma}^2 |\nabla_{\tau} \bar{u}|^2 + \frac{\alpha}{1+\alpha\kappa} \bar{\sigma}^2 \left( \frac{\partial \bar{u}}{\partial \nu} \right)^2 - \bar{u}^2 \Big|_{int} \leq \frac{\alpha}{1+\alpha\kappa} \bar{\sigma}^2 |\nabla_{\tau} \bar{u}|^2 + \frac{\alpha}{1+\alpha\kappa} \bar{\sigma}^2 \left( \frac{\partial \bar{u}}{\partial \nu} \right)^2 - \bar{u}^2 \Big|_{ext}$$

on  $\partial \bar{E}$ . By continuity of the flux  $\bar{\sigma} \frac{\partial \bar{u}}{\partial \nu}$ , we infer that  $\alpha \bar{\sigma}^2 |\nabla_{\tau} \bar{u}|^2 \Big|_{int} \leq \alpha \bar{\sigma}^2 |\nabla_{\tau} \bar{u}|^2 \Big|_{ext}$  which comes to  $(1+\alpha\kappa) |\nabla_{\tau} \bar{u}|^2 \Big|_{int} \leq |\nabla_{\tau} \bar{u}|^2 \Big|_{ext}$ . Since  $|\nabla_{\tau} \bar{u}|^2 \Big|_{int} = |\nabla_{\tau} \bar{u}|^2 \Big|_{ext}$ , we have  $\nabla_{\tau} \bar{u}|_{\Sigma} = 0$ . Therefore,  $\bar{u}$  is constant on  $\partial \bar{E}$  and since  $\Psi_{\alpha}$  is constant on  $\partial \bar{E}$ , it follows that  $\left| \frac{\partial \bar{u}}{\partial \nu} \right|_{int}^2$  is constant as well on  $\partial \bar{E}$ .

To sum-up, the first order necessary conditions drive to the following condition:

$$\text{The functions } \bar{u} \text{ and } |\nabla u| \text{ are constant on } \partial \bar{E}. \quad (15)$$

**Step 2: proof that  $\Omega$  is necessarily a ball.** To prove that  $\Omega$  is a ball, we will use Serrin's Theorem, that we recall hereafter.

**Theorem.** [53, Theorem 2] *Let  $\mathcal{E}$  be a connected domain with a  $\mathcal{C}^2$  boundary,  $h$  a  $\mathcal{C}^1(\mathbb{R}; \mathbb{R})$  function and let  $f \in \mathcal{C}^2(\bar{\mathcal{E}})$  be a function satisfying*

$$-\Delta f = h(f), \quad f > 0 \text{ in } \mathcal{E}, \quad f = 0 \text{ on } \partial \mathcal{E}, \quad \frac{\partial f}{\partial \nu} \text{ is constant on } \partial \mathcal{E}.$$

*Then  $\mathcal{E}$  is a ball and  $f$  is radially symmetric.*

According to (15), let us introduce  $\bar{\mu} = \bar{u}|_{\partial \bar{E}}$ . One has  $\bar{\mu} > 0$  by using the maximum principle. Let us set  $f = \bar{u} - \bar{\mu}$ ,  $h(z) = (\lambda_{\alpha}(\bar{m}) + \kappa)z$  and call  $\mathcal{E}$  a given connected component of  $\bar{E}$ . By assumption,  $\bar{E}$  is a  $\mathcal{C}^2$  set, and, according to (15), the function  $\partial(\bar{u} - \bar{\mu})/\partial \nu$  is constant on  $\partial \bar{E}$ .

The next result allows us to verify the last assumption of Serrin's theorem.

**Lemma 3.** *There holds  $\bar{u} > \bar{\mu}$  in  $\bar{E}$ .*

For the sake of clarity, the proof of this lemma is postponed to the end of this section.

Let us now pick a connected component of  $\bar{E}$  and write it  $\bar{E}_1$ . Applying Serrin's Theorem yields that  $\bar{E}_1$  is a ball centered at some point  $x_0$  and that  $\bar{u}$  is radially symmetric in  $\bar{E}_1$ . Let us introduce  $\mathcal{O} = \cup_{r_1 > 0, \bar{u}}$  radially symmetric in  $\mathbb{B}(x_0, r_1)$  so that  $\mathcal{O}$  is the maximal set in which  $\bar{u}$  is radially symmetric. Let us now show that one has necessarily  $\mathcal{O} = \Omega$ . Since  $\bar{u}$  is radially symmetric in  $\bar{E}_1$ ,  $\mathcal{O}$  is non-empty and there exists  $\tilde{\mu} \geq \bar{\mu}$  such that  $\Psi_{\alpha} = \tilde{\mu}$  on  $\partial \mathcal{O}$ . We argue by contradiction, assuming that  $\mathcal{O} \neq \Omega$ . It follows that the set  $U_{\delta} = \{\partial \mathcal{O} + \mathbb{B}(0; \delta)\}$  is contained in  $\Omega$  for  $\delta > 0$  small enough. Let us fix such a  $\delta$ . To get a contradiction we will show that  $\bar{m}$  is in fact radially symmetric in  $U_{\delta}$ .

If  $\tilde{\mu} = \bar{\mu}$ , then  $\partial \mathcal{O} \subset \partial \bar{E}$  and there exists  $\delta > 0$  such that  $U_{\delta} \setminus \bar{E} \subset \{\bar{m} = 0\}$  and  $U_{\delta} \cap \bar{E} \subset \{\bar{m} = \kappa\}$ . In any case,  $\bar{m}$  is radially symmetric in  $U_{\delta}$  which contradicts the maximality of  $\mathcal{O}$ . If  $\tilde{\mu} < \bar{\mu}$  then, by continuity of  $\Psi_{\alpha}$  it follows that, for  $\delta > 0$  small enough,  $U_{\delta} \subset \{\bar{m} = \kappa\}$ , so that  $\bar{m}$  is radially symmetric in  $U_{\delta}$  and we conclude as before. The conclusion follows. Hence,  $\mathcal{O} = \Omega$ , and  $\Omega$  is a ball.

*Proof of Lemma 3.* Let us set  $v = \bar{u} - \bar{\mu}$ , hence  $v$  solves

$$\begin{cases} -\Delta v = (\lambda_\alpha(\bar{m}) + \kappa)v + (\lambda_\alpha(\bar{m}) + \kappa)\bar{\mu} & \text{in } \bar{E}, \\ v = 0 & \text{on } \partial\bar{E}. \end{cases} \quad (16)$$

and we are led to show that  $v > 0$  in  $\bar{E}$ . Let  $\lambda^D(\Omega)$  be the first Dirichlet eigenvalue<sup>1</sup> of the Laplace operator in  $E$ . By using the Rayleigh quotient (3) we have

$$\lambda_\alpha(\bar{m}) = \min_{u \in W_0^{1,2}(\Omega), u \neq 0} R_{\alpha, \bar{m}}(u) > \min_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\frac{1}{2} \int_\Omega |\nabla u|^2 - \kappa \int_\Omega u^2}{\int_\Omega u^2} = \lambda^D(\Omega) - \kappa,$$

so that  $\lambda_\alpha(\bar{m}) + \kappa > \lambda^D(\Omega) > 0$ . Now, since  $v = 0$  on  $\partial\bar{E}$  and that  $\bar{E}$  is a  $\mathcal{C}^2$  open subset of  $\Omega$ , the extension  $\tilde{v}$  of  $v$  by zero outside  $\bar{E}$  belongs to  $W_0^{1,2}(\Omega)$ . Since  $(\lambda_\alpha(\bar{m}) + \kappa)$  and  $\bar{\mu}$  are non-negative, we get

$$-\Delta \tilde{v} \geq (\lambda_\alpha(\bar{m}) + \kappa)\tilde{v} \text{ in } \bar{E} \quad \text{and} \quad -\Delta \tilde{v} = 0 = (\lambda_\alpha(\bar{m}) + \bar{m})\tilde{v} \text{ in } (\bar{E})^c.$$

We thus have

$$\begin{cases} -\Delta \tilde{v} \geq (\lambda_\alpha(\bar{m}) + \kappa)\tilde{v} & \text{in } \Omega \\ \tilde{v} = 0 & \text{on } \partial\bar{E} \\ \tilde{v} \in W_0^{1,2}(\Omega). \end{cases} \quad (18)$$

Splitting  $\tilde{v}$  into its positive and negative parts as  $\tilde{v} = \tilde{v}_+ - \tilde{v}_-$  and multiply (18) by  $\tilde{v}_-$  we get after an integration by parts

$$-\int_\Omega |\nabla \tilde{v}_-|^2 \geq -(\lambda_\alpha(\bar{m}) + \kappa) \int_\Omega \tilde{v}_-^2.$$

Using that  $\lambda_\alpha(\bar{m}) + \kappa > \lambda^D(\Omega) > 0$ , we get

$$\int_\Omega |\nabla \tilde{v}_-|^2 \leq (\lambda_\alpha(\bar{m}) + \kappa) \int_\Omega \tilde{v}_-^2 < \lambda^D(\Omega) \int_\Omega \tilde{v}_-^2,$$

which, combined with the Rayleigh quotient formulation of  $\lambda^D(\Omega)$  yields  $\tilde{v}_- = 0$ . Hence  $v$  is nonnegative in  $\bar{E}$ . Using moreover that  $(\lambda_\alpha(\bar{m}) + \kappa) \geq 0$  and  $\bar{\mu} \geq 0$  yields that  $-\Delta v \geq 0$  in  $\bar{E}$ . Notice that  $v$  does not vanish identically in  $\bar{E}$ . Indeed,  $u$  would otherwise be constant in  $\bar{E}$  which cannot arise because of (4). According to the strong maximum principle, we infer that  $v > 0$  in  $\bar{E}$ .  $\square$

**Remark 2.** Following the arguments by Casado-Diaz in [16], it would be possible to weaken the regularity assumption on  $E$  provided that we assume the stronger hypothesis that  $\partial\Omega$  is simply connected. Indeed, in that case, assuming that  $E$  is only of class  $\mathcal{C}^1$  leads to the same conclusion.

## 4 Proof of Theorem 2

Throughout this section,  $\Omega$  will denote the ball  $\mathbb{B}(0, R)$ , which will also be denoted  $\mathbb{B}$  for the sake of simplicity. Let  $r_0^* \in (0, R)$  be chosen in such a way that  $m_0^* = \kappa \mathbb{1}_{\mathbb{B}(0, r_0^*)}$  belongs to  $\mathcal{M}_{m_0, \kappa}(\Omega)$ . Let us introduce the notation  $E_0^* = \mathbb{B}(0, r_0^*)$ .

The existence part of the Theorem follows from a straightforward adaptation of [21]. In what follows, we focus on the second part of this theorem, that is, the stationarity of minimizers provided  $\alpha$  is small enough.

<sup>1</sup>In other words

$$\lambda^D(\Omega) = \inf_{u \in W_0^{1,2}(\Omega), \int_\Omega u^2 = 1} \int_\Omega |\nabla u|^2 > 0. \quad (17)$$

## 4.1 Steps of the proof for the stationarity

We argue by contradiction, assuming that, for any  $\alpha > 0$ , there exists a radially symmetric distribution  $\tilde{m}_\alpha$  such that  $\lambda_\alpha(\tilde{m}_\alpha) < \lambda_\alpha(m_0^*)$ . Consider the resulting sequence  $\{\tilde{m}_\alpha\}_{\alpha>0}$ .

- **Step 1:** we prove that  $\{\tilde{m}_\alpha\}_{\alpha\rightarrow 0}$  converges strongly to  $m_0^*$  in  $L^1$ , as  $\alpha \rightarrow 0$ . Regarding the associated eigenfunction, we prove that  $\{u_{\alpha, m_\alpha}\}_{\alpha>0}$  converges strongly to  $u_{0, m_0^*}$  in  $\mathcal{C}^0$  and that  $\alpha \nabla u_{\alpha, m_\alpha}$  converges to 0 in  $L^\infty(\mathbb{B})$ , as  $\alpha \rightarrow 0$ .
- **Step 2:** by adapting [40, Theorem 3.7], we prove that we can reduce ourselves to considering *bang-bang* radially symmetric distributions of resources  $\tilde{m}_\alpha = \kappa \mathbb{1}_{\tilde{E}}$  such that the Hausdorff distance  $d_H(\tilde{E}, E_0^*)$  is arbitrarily small.
- **Step 3:** this is the main innovation of the proof. Introduce  $h_\alpha = \tilde{m}_\alpha - m_0^*$ , and consider the path  $\{m_t\}_{t\in[0,1]}$  from  $m_\alpha$  to  $m_0^*$  defined by  $m_t = m_0^* + th_\alpha$ . We then consider the mapping

$$f_\alpha : t \mapsto \zeta_\alpha(m_t, \Lambda_-(m_t))$$

where  $\zeta_\alpha$  and  $\Lambda_-(m_t)$  are respectively given in Def. 3 and Eq. (12). Notice that, since  $m_0^*$  and  $\tilde{m}_\alpha$  are *bang-bang*,  $f_\alpha(0) = \lambda_\alpha(m_0^*)$  and  $f_\alpha(1) = \lambda_\alpha(m_\alpha)$  according to Def. 3. Let  $u_t$  be a  $L^2$  normalized eigenfunction associated with  $(m_t, \Lambda_-(m_t))$ , in other words a solution to the equation

$$\begin{cases} -\nabla \cdot (\Lambda_-(m_t) \nabla u_t) = \zeta_\alpha(m_t, \Lambda_-(m_t)) u_t + m_t u_t & \text{in } \mathbb{B} \\ u_t = 0 & \text{on } \partial \mathbb{B} \\ \int_{\mathbb{B}} u_t^2 = 1. \end{cases} \quad (19)$$

According to the proof of the optimality conditions (14), one has

$$f'_\alpha(t) = \int_{\mathbb{B}} h_\alpha \left( \frac{\alpha}{1 + \alpha \kappa} \Lambda_-(m_t)^2 |\nabla u_t|^2 - u_t^2 \right).$$

Applying the mean value Theorem yields the existence of  $t_1 \in (0, 1)$  such that  $\lambda_\alpha(\tilde{m}_\alpha) - \lambda_\alpha(m_0^*) = f'(t_1)$ . This enables us to show that, for  $t \in [0, 1]$  and  $\alpha$  small enough, one has

$$f'_\alpha(t) \geq C \int_{\mathbb{B}} |h_\alpha| \text{dist}(\cdot, \mathbb{S}(0, r_0^*))$$

for some  $C > 0$ , giving in turn  $\lambda_\alpha(m_\alpha) - \lambda_\alpha(m_0^*) \geq C \int_{\mathbb{B}} |h_\alpha| \text{dist}(\cdot, \mathbb{S}(0, r_0^*))$ . (we note that the same quantity is obtained in [40]. Nevertheless, we obtain it in a more straightforward manner which bypasses the exact decomposition of eigenfunctions and eigenvalues used there.)

Let us now provide the details of each step.

## 4.2 Step 1: convergence of quasi-minimizers and of sequences of eigenfunctions

We first investigate the convergence of quasi-minimizers.

**Lemma 4.** *Let  $\{m_\alpha\}_{\alpha>0}$  be a sequence in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  such that,*

$$\forall \alpha > 0, \quad \lambda_\alpha(m_\alpha) \leq \lambda_\alpha(m_0^*). \quad (20)$$

*Then,  $\{m_\alpha\}_{\alpha>0}$  converges strongly to  $m_0^*$  in  $L^1(\Omega)$ .*



*Proof of Lemma 4.* The sequence  $(\lambda_\alpha(m_\alpha))_{\alpha>0}$  is bounded from above. Indeed, choosing any test function  $\psi \in W_0^{1,2}(\Omega)$  such that  $\int_\Omega \psi^2 = 1$ , it follows from (3) that  $\lambda_\alpha(m_\alpha) \leq (1 + \alpha\kappa)\|\nabla\psi\|_2^2 + \kappa\|\psi\|_2^2$ . Similarly, using once again (3), we get that if  $\xi_\alpha$  is the first eigenvalue associated to the operator  $-(1 + \alpha\kappa)\Delta - \kappa$ , then  $\lambda_\alpha(m_\alpha) \geq \xi_\alpha$ . Since  $(\xi_\alpha)_{\alpha>0}$  converges to the first eigenvalue of  $-\Delta - \kappa$  as  $\alpha \rightarrow 0$ ,  $(\xi_\alpha)_{\alpha>0}$  is bounded from below whenever  $\alpha$  is small enough. Combining these facts yields that the sequence  $(\lambda_\alpha(m_\alpha))_{\alpha>0}$  is bounded by some positive constant  $M$  and converges, up to a subfamily, to  $\tilde{\lambda}$ . For any  $\alpha > 0$ , let us denote by  $u_\alpha$  the associated  $L^2$ -normalized eigenfunction associated to  $\lambda_\alpha(m_\alpha)$ . From the weak formulation of equation (4) and the normalization condition  $\int_\Omega u_\alpha^2 = 1$ , we infer that

$$\|\nabla u_\alpha\|_2^2 = \int_\Omega |\nabla u_\alpha|^2 \leq \int_\Omega (1 + \alpha m) |\nabla u_\alpha|^2 = \int_\Omega m_\alpha u_\alpha^2 + \lambda_\alpha(m_\alpha) \int_\Omega u_\alpha^2 \leq (M + \kappa).$$

According to the Poincaré inequality and the Rellich-Kondrachov Theorem, the sequence  $(u_\alpha)_{\alpha>0}$  is uniformly bounded in  $W_0^{1,2}(\Omega)$  and converges, up to subfamily, to  $\tilde{u} \in W_0^{1,2}(\Omega)$  weakly in  $W_0^{1,2}(\Omega)$  and strongly in  $L^2(\Omega)$ , and moreover  $\tilde{u}$  is also normalized in  $L^2(\Omega)$ .

Furthermore, since  $L^2$  convergence implies pointwise convergence (up to a subfamily),  $\tilde{u}$  is necessarily nonnegative in  $\Omega$ . Let  $\tilde{m}$  be a closure point of  $(m_\alpha)_{\alpha>0}$  for the weak-star topology of  $L^\infty$ . Passing to the weak limit in the weak formulation of the equation solved by  $u_\alpha$ , namely Eq. (4), one gets

$$-\Delta \tilde{u} - \tilde{m} \tilde{u} = \tilde{\lambda} \tilde{u} \quad \text{in } \Omega.$$

Since  $\tilde{u} \geq 0$  and  $\int_{\mathbb{B}(0,R)} \tilde{u}^2 = 1$ , it follows that  $\tilde{u}$  is the principal eigenfunction of  $-\Delta - \tilde{m}$ , so that  $\tilde{\lambda} = \lambda_0(m^*)$ .

Mimicking this reasoning enables us to show in a similar way that, up to a subfamily,  $(\lambda_\alpha(m_\alpha^*))_{\alpha>0}$  converges to  $\lambda_0(m_0^*)$  and  $(u_{\alpha,m_0^*})_{\alpha>0}$  converges to  $u_{0,m_0^*}$  as  $\alpha \rightarrow 0$ . Passing to the limit in the inequality (20) and since  $m_0^*$  is the only minimizer of  $\lambda_0$  in  $\mathcal{M}_{m_0,\kappa}(\Omega)$  according to the Faber-Krahn inequality, we infer that necessarily,  $\tilde{m} = m_0^*$ . Moreover,  $m_0^*$  being an extreme point of  $\mathcal{M}_{m_0,\kappa}(\Omega)$ , the subfamily  $(m_\alpha)_{\alpha>0}$  converges to  $\tilde{m} = m^*$  (see [34, Proposition 2.2.1]), strongly in  $L^1(\Omega)$ .  $\square$

A straightforward adaptation of Lemma 4's proof yields that both sets  $\{\lambda_\alpha(m)\}_{m \in \mathcal{M}_{m_0,\kappa}(\Omega)}$  and  $\{\|u_{\alpha,m}\|_{W^{1,2}(\Omega)}\}_{m \in \mathcal{M}_{m_0,\kappa}(\Omega)}$  are uniformly bounded whenever  $\alpha \leq 1$ . Let us hence introduce  $M_0 > 0$  such that

$$\forall \alpha \in [0, 1], \quad \max\{|\lambda_\alpha(m)|, \|u_{\alpha,m}\|_{W_0^{1,2}(\Omega)}\} \leq M_0. \quad (21)$$

The next result is the only ingredient of the proof of Theorem 2 where the low dimension assumption on  $n$  is needed.

**Lemma 5.** *Let us assume that  $n = 1, 2, 3$ . There exists  $M_1 > 0$  such that, for every radially symmetric distribution  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$  and every  $\alpha \in [0, 1]$ , there holds*

$$\|u_{\alpha,m}\|_{W^{1,\infty}(\mathbb{B})} \leq M_1.$$

Furthermore, define  $\tilde{\sigma}_{\alpha,m}$ ,  $\tilde{m}$  and  $\varphi_{\alpha,m} : (0, R) \rightarrow \mathbb{R}$  by

$$\forall x \in \mathbb{B}, \quad u_{\alpha,m}(x) = \varphi_{\alpha,m}(|x|), \quad \sigma_{\alpha,m}(x) = \tilde{\sigma}_{\alpha,m}(|x|), \quad m(x) = \tilde{m}(|x|),$$

then  $\tilde{\sigma}_{\alpha,m}(\varphi_{\alpha,m})'$  belongs to  $W^{1,\infty}(0, R)$ .

*Proof of Lemma 5.* This proof is inspired by [40, Proof of Theorem 3.3]. It is standard that for every  $\alpha \in [0, 1]$  and every radially symmetric distribution  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$ , the eigenfunction  $u_{\alpha,m}$

is itself radially symmetric. By rewriting the equation (4) on  $u_{\alpha,m}$  in polar coordinates, one sees that  $\varphi_{\alpha,m}$  solves

$$\begin{cases} -\frac{d}{dr} (r^{n-1} \tilde{\sigma}_{\alpha,m} \frac{d}{dr} \varphi_{\alpha,m}) = (\lambda_{\alpha}(m) \varphi_{\alpha,m} + \tilde{m} \varphi_{\alpha,m}) r^{n-1} & \text{in } (0, R) \\ \varphi_{\alpha,m}(R) = 0. \end{cases} \quad (22)$$

By applying the Hardy Inequality<sup>2</sup> on  $f = \varphi_{\alpha,m}$ , we get

$$\begin{aligned} \int_0^R \varphi_{\alpha,m}^2(r) dr &\leq 4 \int_0^R x^2 \varphi_{\alpha,m}'(x)^2 dx \\ &\leq 4R^2 \int_0^R \left(\frac{x}{R}\right)^{n-1} \varphi_{\alpha,m}'(x)^2 dx = 4R^{4-2n} \|\nabla u_{\alpha,m}\|_{L^2(\mathbb{B})}^2 \leq M, \end{aligned}$$

since  $n \in \{1, 2, 3\}$ . Hence, there exists  $C > 0$  such that

$$\|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \leq C. \quad (23)$$

We will successively prove that  $\varphi_{\alpha,m}$  is uniformly bounded in  $W_0^{1,2}(0, R)$ , then in  $L^\infty(0, R)$  to infer that  $\varphi_{\alpha,m}'$  is bounded in  $L^\infty(0, R)$ . This proves in particular that  $\sigma_{\alpha,m} \varphi_{\alpha,m}' \in L^\infty(0, R)$ . We will then conclude that  $\sigma_{\alpha,m} \varphi_{\alpha,m}' \in W^{1,\infty}(0, R)$  by using that it is a continuous function whose derivative is uniformly bounded in  $L^\infty$  by the equation on  $\varphi_{\alpha,m}$ .

According to (21),  $\|r^{\frac{n-1}{2}} \varphi_{\alpha,m}'\|_{L^2(0,R)} = \|\nabla u_{\alpha,m}\|_{L^2(\mathbb{B})}$  is bounded and therefore,  $r^{n-1} \varphi_{\alpha,m}'(r)$  converges to 0 as  $r \rightarrow 0$ . Hence, integrating Eq. (22) between 0 and  $r > 0$  yields

$$\tilde{\sigma}_{\alpha,m}(r) \varphi_{\alpha,m}'(r) = -\frac{1}{r^{n-1}} \int_0^r t^{n-1} (\lambda_{\alpha}(m) \varphi_{\alpha,m}(t) + \tilde{m}(t) \varphi_{\alpha,m}(t)) dt.$$

By using the Cauchy-Schwarz inequality and (23), we get the existence of  $\tilde{M} > 0$  such that

$$\begin{aligned} \|\varphi_{\alpha,m}'\|_{L^2(0,R)}^2 &\leq \int_0^1 (\tilde{\sigma}_{\alpha,m} \varphi_{\alpha,m}')^2(r) dr \\ &= \int_0^1 \frac{1}{r^{2(n-1)}} \left( \int_0^r t^{n-1} (\lambda_{\alpha}(m) \varphi_{\alpha,m}(t) + \tilde{m}(t) \varphi_{\alpha,m}(t)) dt \right)^2 dr \\ &\leq \int_0^1 \frac{1}{r^{2(n-1)}} (\lambda_{\alpha}(m) + \kappa)^2 \|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \|t^{n-1}\|_{L^2(0;r)}^2 dr \\ &\leq \frac{\tilde{M}}{4n-2} \|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \leq \tilde{M} \|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \leq \tilde{M}C, \end{aligned}$$

Hence,  $\varphi_{\alpha,m}$  is uniformly bounded in  $W_0^{1,2}(0, R)$ .

It follows from standard Sobolev embedding's theorems that there exists a constant  $M_2 > 0$ , such that  $\|\varphi_{\alpha,m}\|_{L^\infty(0,R)} \leq M_2$ .

Finally, plugging this estimate in the equality

$$\tilde{\sigma}_{\alpha,m}(r) \varphi_{\alpha,m}'(r) = -\frac{1}{r^{n-1}} \int_0^r t^{n-1} (\lambda_{\alpha}(m) \varphi_{\alpha,m}(t) + \tilde{m}(t) \varphi_{\alpha,m}(t)) dt$$

and since  $t^{n-1} \leq r^{n-1}$  on  $(0, r)$ , we get that  $\varphi_{\alpha,m}'$  is uniformly bounded in  $L^\infty(0, R)$ . □

---

<sup>2</sup>This inequality reads (see e.g. [52, Lemma 1.3] or [31]): for any non-negative  $f$ ,

$$\int_0^\infty f(x)^2 dx \leq 4 \int_0^\infty x^2 f'(x)^2 dx.$$

The next lemma is a direct corollary of Lemma 4, Lemma 5 and the Arzela-Ascoli Theorem.

**Lemma 6.** *Let  $(m_\alpha)_{\alpha>0}$  be a sequence of radially symmetric functions of  $\mathcal{M}_{m_0,\kappa}(\Omega)$  such that, for every  $\alpha \in [0, 1]$ ,  $\lambda_\alpha(m_\alpha) \leq \lambda_\alpha(m_0^*)$ . Then, up to a subfamily,  $u_{\alpha,m^*}$  converges to  $u_{0,m_0^*}$  for the strong topology of  $\mathcal{C}^0(\bar{\Omega})$  as  $\alpha \rightarrow 0$ .*

### 4.3 Step 2: reduction to particular resource distributions close to $m_0^*$

Let us consider a sequence of radially symmetric distributions  $(m_\alpha)_{\alpha>0}$  such that, for every  $\alpha \in [0, 1]$ ,  $\lambda_\alpha(m_\alpha) \leq \lambda_\alpha(m_0^*)$ . According to Proposition 1, we can assume that each  $m_\alpha$  is a *bang-bang*, in other words that  $m_\alpha = \kappa \chi_{E_\alpha}$  where  $E_\alpha$  is a measurable subset of  $\mathbb{B}(0, R)$ . For every  $\alpha \in [0, 1]$ , one introduces  $d_\alpha = d_H(E_\alpha, E_0^*)$ , the Hausdorff distance of  $E_\alpha$  to  $E_0^*$ .

**Lemma 7.** *For every  $\varepsilon > 0$  small enough, there exists  $\bar{\alpha} > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ , there exists a measurable subset  $\tilde{E}_\alpha$  of  $\Omega$  such that*

$$\lambda_\alpha(\kappa \chi_{E_\alpha}) \geq \lambda_\alpha(\kappa \chi_{\tilde{E}_\alpha}), \quad |E_\alpha| = |\tilde{E}_\alpha| \quad \text{and} \quad d_H(\tilde{E}_\alpha, E_0^*) \leq \varepsilon.$$

*Proof of Lemma 7.* Let  $\alpha \in [0, 1]$ . Observe first that  $\lambda_\alpha(m) = \int_{\mathbb{B}} |\nabla u_{\alpha,m}|^2 + \alpha \int_{\mathbb{B}} m |\nabla u_{\alpha,m}|^2 - \int_{\mathbb{B}} m u_{\alpha,m}^2 = \int_{\mathbb{B}} |\nabla u_{\alpha,m}|^2 + \int_{\mathbb{B}} m \psi_{\alpha,m}$ , where  $\psi_{\alpha,m}$  has been introduced in Lemma 2. We will first construct  $\tilde{m}_\alpha$  in such a way that

$$\lambda_\alpha(m_\alpha) \geq \int_{\mathbb{B}} |\nabla u_{\alpha,m_\alpha}|^2 + \int_{\Omega} \psi_{\alpha,m_\alpha} \tilde{m}_\alpha \geq \lambda_\alpha(\tilde{m}_\alpha),$$

and, to this aim, we will define  $\tilde{m}_\alpha$  as a suitable level set of  $\psi_{\alpha,m_\alpha}$ . Thus, we will evaluate the Hausdorff distance of these level sets to  $E_0^*$ . The main difficulty here rests upon the lack of regularity of the switching function  $\psi_{\alpha,m_\alpha}$ , which is even not continuous.

According to Lemmas 5 and 6,  $\psi_{\alpha,m_\alpha}$  converges to  $-\varphi_{0,m_0^*}^2$  for the strong topology of  $L^\infty(\mathbb{B})$ . Recall that  $m_0^* = \kappa \chi_{\mathbb{B}(0,r_0^*)}$  and let  $V_0$  be defined by  $V_0 = |\mathbb{B}(0, r_0^*)|$ . Let us define  $\mu_\alpha^*$  by dichotomy, as the only real number such that

$$|\underline{\omega}_\alpha| \leq V_0 \leq |\bar{\omega}_\alpha|,$$

where  $\underline{\omega}_\alpha = \{\psi_{\alpha,m_\alpha} < \mu_\alpha^*\}$  and  $\bar{\omega}_\alpha = \{\psi_{\alpha,m_\alpha} \leq \mu_\alpha^*\}$ .

Since  $|\{\psi_{0,m_0^*} < -\varphi_{0,m_0^*}^2(r_0^*)\}| = V_0$ , we deduce that  $(\mu_\alpha^*)$  converges to  $-\varphi_{0,m_0^*}^2(r_0^*)$  as  $\alpha \rightarrow 0$ . Since  $\varphi_{0,m_0^*}$  is decreasing, we infer that for any  $\varepsilon > 0$  small enough, there exists  $\bar{\alpha} > 0$  such that: for every  $\alpha \in [0, \bar{\alpha}]$ ,  $\mathbb{B}(0; r_0^* - \varepsilon) \subset \underline{\omega}_\alpha \subset \bar{\omega}_\alpha \subset \mathbb{B}(0; r_0^* + \varepsilon)$ . Therefore, there exists a radially symmetric set  $B_\varepsilon^\alpha$  such that

$$\underline{\omega}_\alpha \subset B_\varepsilon^\alpha \subset \bar{\omega}_\alpha, \quad |B_\varepsilon^\alpha| = V_0, \quad d_H(B_\varepsilon^\alpha, E_0^*) \leq \varepsilon.$$

Since  $E_\alpha$  and  $B_\varepsilon^\alpha$  have the same measure, one has  $|(E_\alpha)^c \cap B_\varepsilon^\alpha| = |E_\alpha \cap (B_\varepsilon^\alpha)^c|$ , we introduce  $\tilde{m}_\alpha = \kappa \chi_{B_\varepsilon^\alpha}$  so that  $\tilde{m}_\alpha$  belongs to  $\mathcal{M}_{m_0,\kappa}(\Omega)$ .

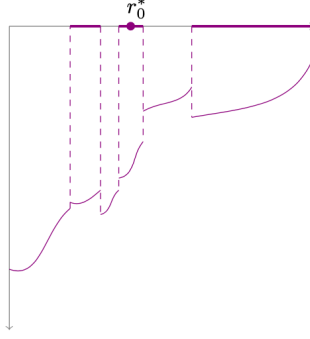


Figure 1: Possible graph of the discontinuous function  $\psi_{\alpha, m_\alpha}$ . The bold intervals on the  $x$  axis correspond to  $\{m_\alpha = 0\}$ .

By construction, one has

$$\begin{aligned}
\lambda_\alpha(m_\alpha) &= \int_{\mathbb{B}} (1 + \alpha m_\alpha) |\nabla u_{\alpha, m_\alpha}|^2 - \int_{\mathbb{B}} m_\alpha u_{\alpha, m_\alpha}^2 = \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \int_{\mathbb{B}} \psi_{\alpha, m_\alpha} m_\alpha \\
&= \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \int_{E_\alpha} \psi_{\alpha, m_\alpha} = \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \int_{E_\alpha \cap (B_\varepsilon^\alpha)^c} \psi_{\alpha, m_\alpha} + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} \\
&\geq \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \mu_\alpha^* |E_\alpha \cap (B_\varepsilon^\alpha)^c| + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} \\
&= \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \mu_\alpha^* |(E_\alpha)^c \cap B_\varepsilon^\alpha| + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} \\
&\geq \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \int_{(E_\alpha)^c \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} = \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \int_{\mathbb{B}} \tilde{m}_\alpha \psi_{\alpha, m_\alpha} \\
&= \int_{\mathbb{B}} \sigma_{\alpha, \tilde{m}_\alpha} |\nabla u_{\alpha, m_\alpha}|^2 - \int_{\mathbb{B}} \tilde{m}_\alpha u_{\alpha, m_\alpha}^2 \geq \lambda_\alpha(\tilde{m}_\alpha),
\end{aligned}$$

the last inequality coming from the variational formulation (3). The expected conclusion thus follows.  $\square$

From now on we will replace  $m_\alpha$  by  $\kappa \chi_{\tilde{E}_\alpha}$  and still denote this function by  $m_\alpha$  with a slight abuse of notation.

#### 4.4 Step 3: conclusion, by the mean value theorem

Recall that, according to Section 4.1, for every  $\alpha \in [0, 1]$ , the mapping  $f_\alpha$  is defined by  $f_\alpha(t) := \zeta_\alpha(m_t, \Lambda_-(m_t))$  for all  $t \in [0, 1]$ . We claim that  $f_\alpha$  belongs to  $\mathcal{C}^1$ . This follows from similar arguments to those of the  $L^2$  differentiability of  $m \mapsto \lambda_\alpha(m)$  in Appendix A. Following the proof of (14), it is also straightforward that for every  $t \in [0, 1]$ , one has

$$f'_\alpha(t) = \int_{\mathbb{B}} \left( \frac{\alpha}{1 + \alpha \kappa} \Lambda_-(m_t)^2 |\nabla u_t|^2 - u_t^2 \right) h_\alpha. \quad (24)$$

Finally, since  $m_0^*$  and  $m_\alpha$  are *bang-bang*, it follows from Definition 3 that  $f_\alpha(0) = \lambda_\alpha(m^*)$  and  $f_\alpha(1) = \lambda_\alpha(m_\alpha)$ .

Since  $m_\alpha$  is assumed to be radially symmetric, so is  $m_t$  for every  $t \in [0, 1]$  thanks to a standard reasoning, and, therefore, so is  $u_t$ . With a slight abuse of notation, we identify  $m_t$ ,  $u_t$  and  $\Lambda_-(m_t)$  with their radially symmetric part  $\tilde{m}_t$ ,  $\tilde{u}_t$ ,  $\tilde{\Lambda}_-(m_t)$  defined on  $[0, R]$  by

$$u_t(x) = \tilde{u}_t(|x|), \quad m_t(x) = \tilde{m}_t(|x|), \quad \Lambda_-(m_t)(x) = \tilde{\Lambda}_-(\tilde{m}_t)(|x|).$$

Then the function  $u_t$  (defined on  $[0, R]$ ) solves the equation

$$\begin{cases} -\frac{d}{dr} \left( r^{n-1} \Lambda_-(m_t) \frac{du_t}{dr} \right) = (\zeta_\alpha(m_t, \Lambda_-(m_t)) u_t + m_t u_t) r^{n-1} & r \in [0; R] \\ u_t(R) = 0 \\ \int_0^R r^{n-1} u_t(r)^2 dr = \frac{1}{c_n}, \end{cases} \quad (25)$$

where  $c_n = |\mathbb{S}(0, 1)|$ . As a consequence, an immediate adaptation of the proof of Lemma 5 yields:

**Lemma 8.** *There exists  $M > 0$  such that*

$$\max \left\{ \|u_t\|_{W^{1,\infty}}, \left\| \Lambda_-(m_t) u_t' \right\|_{W^{1,\infty}} \right\} \leq M.$$

Furthermore,  $\Lambda_-(m_t) u_t'$  converges to  $u'_{0,m^*}$  in  $L^\infty(0, R)$  and uniformly with respect to  $t \in [0, 1]$ , as  $\alpha \rightarrow 0$ .

According to the mean value Theorem, there exists  $t_1 = t_1(\alpha) \in [0, 1]$  such that

$$\lambda_\alpha(m_\alpha) - \lambda_\alpha(m^*) = f_\alpha(1) - f_\alpha(0) = f'_\alpha(t_1)$$

and by using Eq. (24), one has

$$f'_\alpha(t_1) = \int_{\mathbb{B}} \left( \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_{t_1})^2 |\nabla u_{t_1}|^2 - u_{t_1}^2 \right) h_\alpha,$$

where  $h_\alpha = m_\alpha - m_0^*$ . Let us introduce  $I_\alpha^\pm$  as the two subsets of  $[0, R]$  given by  $I_\alpha^\pm = \{h_\alpha = \pm 1\}$ . Let  $\varepsilon > 0$ . According to Lemma 7, we have, for  $\alpha$  small enough,  $I_\alpha^+ \subset [r_0^*, r_0^* + \varepsilon]$  and  $I_\alpha^- \subset [r_0^* - \varepsilon, r_0^*]$ . Finally, let us introduce

$$\mathfrak{F}_1 := \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_{t_1})^2 |\nabla u_{t_1}|^2 - u_{t_1}^2.$$

According to Lemma 8,  $\mathfrak{F}_1$  belongs to  $W^{1,\infty}$  and  $\mathfrak{F}_1 + u_{\alpha, m_0^*}^2$  converges to 0 as  $\alpha \rightarrow 0$ , for the strong topology of  $W^{1,\infty}(0, R)$ . Moreover, there exists  $M > 0$  independent of  $\alpha$  such that for  $\varepsilon > 0$  small enough,

$$-M \leq 2u_{\alpha, m_0^*} \frac{du_{\alpha, m_0^*}}{dr} \leq -M \quad \text{in } [r_0^* - \varepsilon, r_0^* + \varepsilon]$$

and it follows that

$$\frac{M}{2} \leq \frac{d\mathfrak{F}_1}{dr} \leq 2M \quad \text{in } [r_0^* - \varepsilon, r_0^* + \varepsilon]$$

for  $\alpha$  small enough. Hence, since  $\mathfrak{F}_1$  is Lipschitz continuous and thus absolutely continuous, one has for every  $y \in [0, \varepsilon]$ ,

$$\begin{aligned} \mathfrak{F}_1(r_0^* + y) &= \mathfrak{F}_1(r_0^*) + \int_{r_0^*}^{r_0^* + y} \mathfrak{F}_1'(s) ds \geq \mathfrak{F}_1(r_0^*) + \frac{M}{2} y \\ \text{and } \mathfrak{F}_1(r_0^* - y) &= \mathfrak{F}_1(r_0^*) + \int_{r_0^* - y}^{r_0^*} (-\mathfrak{F}_1'(s)) ds \leq \mathfrak{F}_1(r_0^*) - \frac{M}{2} y. \end{aligned}$$

Since  $h_\alpha \leq 0$  in  $[r_0^* - \varepsilon, r_0^*]$  and  $h_\alpha \geq 0$  in  $[r_0^*, r_0^* + \varepsilon]$ , we have

$$\begin{aligned} h_\alpha(r_0^* + y)\mathfrak{F}_1(r_0^* + y) &\geq h_\alpha(r_0^* + y)\mathfrak{F}_1(r_0^*) + \frac{|h_\alpha|(r_0^* + y)M}{2}y \\ \text{and } h_\alpha(r_0^* - y)\mathfrak{F}_1(r_0^* - y) &\geq h_\alpha(r_0^* - y)\mathfrak{F}_1(r_0^*) + \frac{|h_\alpha|(r_0^* - y)M}{2}y. \end{aligned}$$

for every  $y \in [0, \varepsilon]$ . Hence, using that  $\int_{\mathbb{B}} h_\alpha = 0$ , we infer that

$$\begin{aligned} f'_\alpha(t_1) &= \int_{\mathbb{B}} \left( \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_{t_1})^2 |\nabla u_{t_1}|^2 - u_{t_1}^2 \right) h_\alpha = c_n \int_0^R h_\alpha(s)\mathfrak{F}_1(s)s^{n-1} ds \\ &= c_n \left( \int_{r_0^* - \varepsilon}^{r_0^*} h_\alpha\mathfrak{F}_1(s)s^{n-1} ds + \int_{r_0^*}^{r_0^* + \varepsilon} h_\alpha\mathfrak{F}_1(s)s^{n-1} ds \right) \\ &\geq c_n \left( \int_{r_0^* - \varepsilon}^{r_0^*} h_\alpha(s)\mathfrak{F}_1(r_0^*)s^{n-1} ds + \int_{r_0^*}^{r_0^* + \varepsilon} h_\alpha(s)\mathfrak{F}_1(r_0^*)s^{n-1} ds \right) \\ &\quad + \frac{c_n M}{2} \left( \int_{r_0^* - \varepsilon}^{r_0^*} |h_\alpha|(s)|r_0^* - s|s^{n-1} ds + \int_{r_0^*}^{r_0^* + \varepsilon} |h_\alpha|(s)|r_0^* - s|s^{n-1} ds \right) \\ &= \frac{c_n M}{2} \int_{\mathbb{B}} |h_\alpha| \text{dist}(\cdot, \mathbb{S}(0, r_0^*)), \end{aligned}$$

which concludes Step 3. Theorem 2 is thus proved.

**Remark 3.** Regarding the proof of Theorem 2, it would have been more natural to consider the path  $t \mapsto (\lambda_\alpha(m_t), m_t)$  rather than  $t \mapsto (\zeta_\alpha(m_t, \Lambda_-(m_t)), m_t)$ . However, we would have been led to consider  $\mathfrak{G}_t = \alpha\kappa|\nabla u_{\alpha, m_t}|^2 - u_{\alpha, m_t}^2$  instead of  $\mathfrak{F}_t$ . Unfortunately, this would have been more intricate because of the regularity of  $\mathfrak{G}_t$ , which is discontinuous and thus, no longer a  $W^{1, \infty}$  function, so that a Lemma analogous to Lemma 8 would not be true. Adapting step by step the arguments of [40] would nevertheless be possible although much more technical.

## 5 Sketch of the proof of Corollary 1

We do not give all details since the proof is then very similar to the ones written previously. We only underline the slight differences in every step.

To prove this result, we consider the following relaxation of our problem, which is reminiscent of the problems considered in [30]. Let us consider, for any pair  $(m_1, m_2) \in \mathcal{M}_{m_0, \kappa}(\Omega)^2$ , the first eigenvalue of the operator  $\mathcal{N} : u \mapsto -\nabla \cdot ((1 + \alpha m_1)\nabla u) - m_2 u$ , and write it  $\eta_\alpha(m_1, m_2)$ . Let  $m^* := \kappa \mathbb{1}_{\mathbb{B}(0, R)}$ . By using the results of [30] or alternatively, applying the rearrangement of Alvino and Trombetti, [3] as it has been done in [21], one proves the existence of a radially symmetric function  $\tilde{m}_1$  such that

$$\eta_\alpha(m_1, m_2) \geq \eta_\alpha(\tilde{m}_1, m^*),$$

so that we are done if we can prove that, for any  $m \in \mathcal{M}(\Omega)$  there holds

$$\eta_\alpha(m, m^*) \geq \eta_\alpha(m^*, m^*). \quad (26)$$

We claim that (26) holds for any  $m \in \mathcal{M}_{m_0, \kappa}$ , provided that  $m_0$  and  $\alpha$  be small enough. Let us describe the main steps of the proof:

- **Step 1:** mimicking the compactness argument used in [21], one shows that there exists a solution  $m_\alpha$  to the problem

$$\inf_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \eta_\alpha(m, m^*),$$

which is radially symmetric and *bang-bang*. We write it  $m_\alpha = \kappa \mathbb{1}_{E_\alpha}$ .

- **Step 2:** let  $\mu_0$  and  $r_0^*$  be the unique real numbers such that

$$|\{|\nabla u_{0, m^*}|^2 \leq \mu_0\}| = V_0 = |\mathbb{B}(0, r_0^*)|.$$

Introducing  $E_0 = \{|\nabla u_{0, m^*}|^2 \leq \mu_0\}$ , we prove that  $m_\alpha$  converges in  $L^1(\Omega)$  to  $\kappa \mathbb{1}_{E_0}$  as  $\alpha \rightarrow 0$ .

- **Step 3:** we establish that if  $m_0$  is small enough, then  $E_0 = \mathbb{B}(0, r_0^*)$ . This is done by proving that  $u_{0, m^*}$  converges in  $\mathcal{C}^1$  to the first Dirichlet eigenfunction of the ball as  $r_0^* \rightarrow 0$  and by determining the level-sets of this first eigenfunction, as done in [20, Section 2.2].
- **Step 4:** once this limit identified, we mimic the steps of the proof of Theorem 2 (reduction to a small Hausdorff distance and mean value Theorem for a well-chosen auxiliary function) to conclude that one necessarily has  $m_\alpha = m^*$  for  $\alpha$  small enough.

## 6 Proof of Theorem 3

Throughout this section, we will denote by  $\mathbb{B}^*$  the ball  $\mathbb{B}(0, r_0^*)$ , where  $r_0^*$  is chosen so that  $m_0^* = \kappa \chi_{\mathbb{B}^*}$  belongs to  $\mathcal{M}_{m_0, \kappa}(\Omega)$ .

When it makes sense, we will write  $f|_{int}(y) = \lim_{x \in \mathbb{B}^*, x \rightarrow y} f(x)$ ,  $f|_{ext}(y) := \lim_{x \in (\mathbb{B}^*)^c, x \rightarrow y} f(y)$ , so that  $[f] = f|_{ext} - f|_{int}$  denotes the jump of  $f$  at the boundary  $\mathbb{S}(0, r_0^*)$ .

### 6.1 Preliminaries

For  $\varepsilon > 0$ , let us introduce  $\mathbb{B}_\varepsilon^* := (\text{Id} + \varepsilon V)\mathbb{B}^*$  and define  $u_\varepsilon$  as the  $L^2$ -normalized first eigenfunction associated with  $m_\varepsilon = \kappa \chi_{\mathbb{B}_\varepsilon^*}$ .

It is well known (see e.g. [32, 34]) that  $u_\varepsilon$  expands as

$$u_\varepsilon = u_{0, \alpha} + \varepsilon u_{1, \alpha} + \varepsilon^2 \frac{u_{2, \alpha}}{2} + \underset{\varepsilon \rightarrow 0}{o}(\varepsilon^2) \quad \text{in } H^1(\mathbb{B}^*) \text{ and in } H^1(\Omega \setminus \mathbb{B}^*), \quad (27)$$

where, in particular,  $u_{0, \alpha} = u_{\alpha, m_0^*}$ , whereas  $\lambda_\alpha(\mathbb{B}_\varepsilon^*)$  expands as

$$\begin{aligned} \lambda_\alpha(\mathbb{B}_\varepsilon^*) &= \lambda_{0, \alpha} + \varepsilon \lambda'_\alpha(\mathbb{B}^*)[V] + \varepsilon^2 \lambda''_\alpha(\mathbb{B}^*)[V] + \underset{\varepsilon \rightarrow 0}{o}(\varepsilon^2) \\ &= \lambda_{0, \alpha} + \varepsilon \lambda_{1, \alpha} + \varepsilon^2 \lambda_{2, \alpha} + \underset{\varepsilon \rightarrow 0}{o}(\varepsilon^2). \end{aligned} \quad (28)$$

By mimicking the proof of Lemma 5, one shows the following symmetry result.

**Lemma 9.**  $u_{\alpha, m_0^*}$  is a radially symmetric function. Let  $\varphi_{\alpha, m_0^*}$  and  $\tilde{m}$  be such that  $u_{\alpha, m_0^*} = \varphi_{\alpha, m_0^*}(|\cdot|)$  and  $m_0^* = \tilde{m}(|\cdot|)$ , then  $\varphi_{\alpha, m_0^*}$  satisfies the ODE

$$\begin{cases} -\frac{d}{dr} \left( r^{n-1} \tilde{\sigma}_{\alpha, m_0^*} \varphi'_{\alpha, m_0^*} \right) = (\lambda_\alpha(m_0^*) + \tilde{m}) \varphi_{\alpha, m_0^*} r^{n-1} & \text{in } (0, R) \\ \varphi_{\alpha, m_0^*}(R) = 0 \end{cases} \quad (29)$$

and one has the following jump conditions

$$[\varphi_{\alpha, m_0^*}](r_0^*) = [\sigma_{\alpha, m_0^*} \varphi'_{\alpha, m_0^*}](r_0^*) = 0, \quad [\sigma_\alpha \varphi''_{\alpha, m_0^*}](r_0^*) = \kappa \varphi_{\alpha, m_0^*}(r_0^*). \quad (30)$$

Furthermore,  $\varphi_{\alpha, m_0^*}$  converges to  $\varphi_{0, m_0^*}$  for the strong topology of  $\mathcal{C}^1$  as  $\alpha \rightarrow 0$ .

## 6.2 Computation of the first and second order shape derivatives

Hadamard's structure theorem enables us to work with only normal vector fields  $V$  to compute the second order derivative. Since we are working in dimension 2, this means that one can deal with vector fields  $V$  given in polar coordinates by

$$V(r_0^*, \theta) = g(\theta)(\cos \theta, \sin \theta).$$

The proof of the shape differentiability at the first and second order of  $\lambda_\alpha$ , based on the method of [46], is exactly similar to [24, Proof of Theorem 2.2]. For this reason, we admit it.

**Computation and analysis of the first order shape derivative.** Let us prove that  $\mathbb{B}^*$  is a critical shape in the sense of (7).

**Lemma 10.** *The first order shape derivative of  $\lambda_\alpha$  at  $\mathbb{B}^*$  in direction  $V$  reads*

$$\lambda_{1,\alpha} = \lambda'_\alpha(\mathbb{B}^*)[V] = \int_{\mathbb{S}(0,r_0^*)} V \cdot \nu. \quad (31)$$

For all  $V \in \mathcal{X}(\mathbb{B}^*)$  (defined by (6)), one has  $\lambda_{1,\alpha} = 0$  meaning that  $\mathbb{B}^*$  satisfies (7).

*Proof of Lemma 10.* First, elementary computations show that  $u_{1,\alpha}$  solves

$$\begin{cases} -\nabla \cdot (\sigma_\alpha \nabla u_{1,\alpha}) = \lambda_{1,\alpha} u_{0,\alpha} + \lambda_{0,\alpha} u_{1,\alpha} + m^* u_{1,\alpha} & \text{in } \mathbb{B}(0, R), \\ \left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial \nu} \right] (r_0^* \cos \theta, r_0^* \sin \theta) = -\kappa g(\theta) u_{0,\alpha}, \\ [u_{1,\alpha}] (r_0^* \cos \theta, r_0^* \sin \theta) = -g(\theta) \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] (r_0^* \cos \theta, r_0^* \sin \theta), \end{cases} \quad (32)$$

where the jumps denote the jumps of the functions at  $\mathbb{S}(0, r_0^*)$ . The derivation of the main equation of (32) is an adaptation of the computations in [24]. To derive the jump on  $u_{1,\alpha}$ , we follow [24] and differentiate the continuity equation  $[u_\varepsilon]_{\partial \mathbb{B}_\varepsilon^*} = 0$ . Formally plugging (27) in this equation yields

$$u_{1,\alpha}|_{int}(r^*, \theta) + g(\theta) \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} = u_{1,\alpha}|_{ext}(r^*, \theta) + g(\theta) \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{ext},$$

and hence

$$[u_{1,\alpha}] = u_{1,\alpha}|_{ext} - u_{1,\alpha}|_{int} = -g(\theta) \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right].$$

Note that the same goes for the normal derivative: we differentiate the continuity equation

$$\left[ (1 + \alpha m_\varepsilon) \frac{\partial u_{\varepsilon,\alpha}}{\partial \nu} \right]_{\partial \mathbb{B}_\varepsilon^*} = 0,$$

yielding

$$\left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] = -g(\theta) \left[ \sigma_\alpha \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right].$$

According to the equation  $-\sigma_\alpha \Delta u_{0,\alpha} = \lambda_\alpha(m^*) u_{0,\alpha} + m^* u_{0,\alpha}$  in  $\mathbb{B}^*$ , this rewrites

$$\left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] = -\kappa g(\theta) u_{0,\alpha}. \quad (33)$$



Now, using  $u_{0,\alpha}$  as a test function in (32), we get

$$\begin{aligned}
\lambda_{1,\alpha} &= - \int_{\mathbb{B}(0,R)} \sigma_\alpha u_{0,\alpha} \Delta u_{1,\alpha} - \int_{\mathbb{B}(0,R)} m_0^* u_{1,\alpha} u_{0,\alpha} = - \int_{\mathbb{B}(0,R)} \sigma_\alpha u_{0,\alpha} \Delta u_{1,\alpha} + \int_{\mathbb{B}(0,R)} \sigma_\alpha u_{1,\alpha} \Delta u_{0,\alpha} \\
&= \int_{\mathbb{S}(0,r_0^*)} u_{0,\alpha} \left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial \nu} \right] - \int_{\mathbb{S}(0,r_0^*)} \left[ \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} u_{1,\alpha} \right] \\
&= -r_0^* \int_0^{2\pi} \kappa g(\theta) u_{0,\alpha}(r_0^*)^2 d\theta + r_0^* \int_0^{2\pi} g(\theta) \left( \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \right) \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] d\theta \\
&= r_0^* \int_0^{2\pi} g(\theta) \left( -\kappa u_{0,\alpha}(r_0^*)^2 + \left[ \sigma_\alpha \left( \frac{\partial u_{0,\alpha}}{\partial r} \right)^2 \right] \right) d\theta.
\end{aligned}$$

by using that  $\int_{\mathbb{B}(0,R)} u_\varepsilon^2 = 1$ , so that  $\int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{1,\alpha} = 0$  by differentiation.

Since  $u_{0,\alpha}$  is radially symmetric according to Lemma 9, we introduce the two real numbers

$$\eta_\alpha = -\kappa u_{0,\alpha}(r_0^*)^2 + \left[ \sigma_\alpha \left( \frac{\partial u_{0,\alpha}}{\partial r} \right)^2 \right] \quad \text{and} \quad \lambda_{1,\alpha} = r_0^* \eta_\alpha \int_0^{2\pi} g(\theta) d\theta. \quad (34)$$

It is easy to see that  $V$  belongs to  $\mathcal{X}(\mathbb{B}^*)$  if, and only if  $\int_0^{2\pi} g = 0$  so that we finally have  $\lambda_{1,\alpha} = 0$ .  $\square$

**Computation of the Lagrange multiplier.** The existence of a Lagrange multiplier  $\Lambda_\alpha \in \mathbb{R}$  related to the volume constraint is standard, and one has

$$\forall V \in \mathcal{X}(\mathbb{B}^*), \quad (\lambda_\alpha - \Lambda_\alpha \text{Vol})'(\mathbb{B}^*)[V] = 0,$$

Since

$$\text{Vol}'(\mathbb{B}^*)[V] = \int_{\mathbb{S}(0,r_0^*)} V \cdot \nu = r_0^* \int_0^{2\pi} g(\theta) d\theta.$$

(see e.g. [34, chapitre 5]) and since

$$\lambda'_\alpha(\mathbb{B}^*)[V] = r_0^* \eta_\alpha \int_0^{2\pi} g(\theta) d\theta,$$

where  $\eta_\alpha$  is defined by (34), the Lagrange multiplier reads

$$\Lambda_\alpha = \eta_\alpha = -\kappa u_{0,\alpha}(r_0^*)^2 + \left[ \sigma_\alpha \left( \frac{\partial u_{0,\alpha}}{\partial r} \right)^2 \right].$$

**Computation of the second order derivative and second order optimality conditions.**

**Lemma 11.** *For every  $V \in \mathcal{X}(\mathbb{B}^*)$ , one has for the coefficient  $\lambda_{2,\alpha} = \lambda''_\alpha(\mathbb{B}^*)[V, V]$  introduced in (28) the expression*

$$\begin{aligned}
\lambda_{2,\alpha} &= 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{int} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\
&\quad + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2.
\end{aligned}$$

*Proof of Lemma 11.* In the computations below, we do not need to make the equation satisfied by  $u_{2,\alpha}$  explicit, but we nevertheless will need several times the knowledge of  $[u_{2,\alpha}]$  at  $\mathbb{S}(0, r_0^*)$ . In the same fashion that we obtained the jump conditions on  $u_{1,\alpha}$  Let us differentiate two times the continuity equation  $[u_\varepsilon]_{\partial\mathbb{B}_\varepsilon^*} = 0$ . We obtain

$$[u_{2,\alpha}]_{\partial\mathbb{B}_\varepsilon^*} = -2g(\theta) \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] - g(\theta)^2 \left[ \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right]. \quad (35)$$

Now, according to Hadamard second variation formula (see [34, Chapitre 5, page 227] for a proof), if  $\Omega$  is a  $\mathcal{C}^2$  domain and  $f$  is two times differentiable at 0 and taking values in  $W^{2,2}(\Omega)$ , then one has

$$\frac{d^2}{dt^2} \Big|_{t=0} \int_{(\text{Id}+tV)\Omega} f(t) = \int_{\Omega} f''(0) + 2 \int_{\partial\Omega} f'(0) V \cdot \nu + \int_{\partial\Omega} \left( Hf(0) + \frac{\partial f(0)}{\partial \nu} \right) (V \cdot \nu)^2, \quad (36)$$

where  $H$  denotes the mean curvature. We apply it to  $f(\varepsilon) = \sigma_{\alpha,\varepsilon} |\nabla u_\varepsilon|^2 - m_\varepsilon u_\varepsilon^2$  on  $\mathbb{B}(0, R)$ , since  $\lambda_\alpha(m_\varepsilon) = \int_{\mathbb{B}(0,R)} f_\varepsilon$ . Let us distinguish between the two subdomains  $\mathbb{B}_\varepsilon^*$  and  $(\mathbb{B}_\varepsilon^*)^c$ . We introduce

$$D_1 = \frac{d^2}{d\varepsilon^2} \Big|_{\varepsilon=0} \int_{\mathbb{B}_\varepsilon^*} (\sigma_{\alpha,\varepsilon} |\nabla u_\varepsilon|^2 - \kappa u_\varepsilon^2) \quad \text{and} \quad D_2 = \frac{d^2}{d\varepsilon^2} \Big|_{\varepsilon=0} \int_{(\mathbb{B}_\varepsilon^*)^c} (\sigma_{\alpha,\varepsilon} |\nabla u_\varepsilon|^2)$$

so that  $\lambda_\alpha''(\mathbb{B}^*)[V, V] = D_1 + D_2$ .

One has

$$\begin{aligned} D_1 &= \int_{\mathbb{B}^*} 2(1 + \alpha\kappa) \nabla u_{2,\alpha} \cdot \nabla u_{1,\alpha} + 2 \int_{\mathbb{B}_\varepsilon^*} (1 + \alpha\kappa) |\nabla u_{1,\alpha}|^2 \\ &\quad - 2\kappa \int_{\mathbb{B}^*} u_{2,\alpha} u_{0,\alpha} - 2\kappa \int_{\mathbb{B}^*} u_{1,\alpha} u_{0,\alpha} \\ &\quad + 4 \int_{\mathbb{S}(0, r_0^*)} (1 + \alpha\kappa) (\nabla u_{1,\alpha}|_{int} \cdot \nabla u_{0,\alpha}|_{int}) V \cdot \nu - 4\kappa \int_{\mathbb{S}(0, r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\ &\quad + \int_{\mathbb{S}(0, r_0^*)} \left( \frac{1}{r_0^*} (1 + \alpha\kappa) |\nabla u_{0,\alpha}|_{int}^2 - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 + 2(1 + \alpha\kappa) \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \Big|_{int} \right. \\ &\quad \left. - 2\kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} \right) (V \cdot \nu)^2, \end{aligned}$$

and taking into account that the mean curvature has a sign on  $(\mathbb{B}_\varepsilon^*)^c$ , one has

$$\begin{aligned} D_2 &= \int_{(\mathbb{B}^*)^c} 2\nabla u_{2,\alpha} \cdot \nabla u_{1,\alpha} + 2 \int_{(\mathbb{B}^*)^c} |\nabla u_{1,\alpha}|^2 \\ &\quad - 4 \int_{\mathbb{S}(0, r_0^*)} (\nabla u_{1,\alpha}|_{ext} \cdot \nabla u_{0,\alpha}|_{ext}) V \cdot \nu \\ &\quad + \int_{\mathbb{S}(0, r_0^*)} \left( -\frac{1}{r_0^*} |\nabla u_{0,\alpha}|_{ext}^2 - 2 \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{ext} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \Big|_{ext} \right) (V \cdot \nu)^2. \end{aligned}$$

Summing these two quantities, we get

$$\begin{aligned} \lambda_\alpha''(\mathbb{B}^*)[V, V] &= 2 \int_{\mathbb{B}(0,R)} \sigma_\alpha \nabla u_{0,\alpha} \cdot \nabla u_{2,\alpha} - 2 \int_{\mathbb{B}(0,R)} m_0^* u_{0,\alpha} u_{2,\alpha} \\ &\quad + 2 \int_{\mathbb{B}(0,R)} \sigma_\alpha |\nabla u_{1,\alpha}|^2 - 2 \int_{\mathbb{B}(0,R)} m_0^* u_{1,\alpha}^2 \end{aligned}$$

$$\begin{aligned}
& -4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu - 4\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\
& + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 \\
& - 2 \int_{\mathbb{S}(0,r_0^*)} \left[ \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2 - 2\kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2.
\end{aligned}$$

To simplify this expression, let us use Eq. (30). Introducing

$$D_3 = \int_{\mathbb{B}(0,R)} \sigma_\alpha \nabla u_{0,\alpha} \cdot \nabla u_{2,\alpha} - \int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha} - \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha},$$

one has

$$D_3 = \int_{\mathbb{S}(0,r_0^*)} [u_{2,\alpha}] \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r}$$

and hence, by using Equation (35), one has

$$\begin{aligned}
D_3 &= -2 \int_{\mathbb{S}(0,r_0^*)} [u_{2,\alpha}] \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \\
&= 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu + 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2.
\end{aligned}$$

Similarly, let

$$D_4 = \int_{\mathbb{B}(0,R)} \sigma_\alpha |\nabla u_{1,\alpha}|^2 - \int_{\mathbb{B}(0,R)} m_0^* u_{1,\alpha}^2.$$

By using Eq. (32) and the fact that  $\lambda_{1,\alpha} = 0$ , one has

$$\begin{aligned}
D_4 &= \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 - \int_{\mathbb{S}(0,r_0^*)} \left[ u_{1,\alpha} \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] \\
&= \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 - \int_{\mathbb{S}(0,r_0^*)} [u_{1,\alpha}] \left( \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right) \Big|_{ext} - \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} \left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] \\
&= \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 + \int_{\mathbb{S}(0,r_0^*)} \left( \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] + \kappa u_{1,\alpha}|_{int} u_{0,\alpha} \right) V \cdot \nu.
\end{aligned}$$

Finally, by differentiating the normalization condition  $\int_{\mathbb{B}(0,R)} u_\varepsilon^2 = 1$ , we get

$$\int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha} + \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 = 0. \quad (37)$$

Combining the equalities above, one gets

$$\begin{aligned}
\lambda_{2,\alpha} &= 2\lambda_\alpha(\mathbb{B}^*) \left( \int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha} + \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 \right) + 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu \\
&+ 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2 + 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu \\
&+ 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu - 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu
\end{aligned}$$

$$\begin{aligned}
& -4\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu - \int_{\mathbb{S}(0,r_0^*)} \left( \frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 \\
& - 2 \int_{\mathbb{S}(0,r_0^*)} \left[ \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2 - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \\
= & 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\
& - \int_{\mathbb{S}(0,r_0^*)} \left( \frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2
\end{aligned}$$

We have then obtained the desired expression.  $\square$

**Strong stability** The second derivative of the volume is known to be

$$\text{Vol}''(\mathbb{B}^*)[V, V] = \int_{\mathbb{S}(0,r_0^*)} H(V \cdot \nu)^2. \quad (38)$$

Hence, introducing  $D_5 = (\lambda_\alpha - \eta_\alpha \text{Vol})''(\mathbb{B}^*)[V, V]$  and taking into account Lemma 11, (34) and (38), we have

$$\begin{aligned}
D_5 &= 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \\
& - 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{ext} u_{0,\alpha} V \cdot \nu + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 \\
& + \kappa \int_{\mathbb{S}(0,r_0^*)} \frac{1}{r_0^*} u_{0,\alpha}^2 (V \cdot \nu)^2 - \int_{\mathbb{S}(0,r_0^*)} \frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] (V \cdot \nu)^2 \\
= & 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \\
& - 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu - \int_{\mathbb{S}(0,r_0^*)} \frac{2}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] (V \cdot \nu)^2
\end{aligned}$$

We are then led to determine the signature of the quadratic form

$$\begin{aligned}
\mathcal{F}_\alpha[V, V] &= \frac{1}{2} (\lambda_\alpha(\mathbb{B}^*) - \eta_\alpha \text{Vol})''[V, V] \\
&= \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - \kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\
& + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{2}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] \right) (V \cdot \nu)^2 - \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2.
\end{aligned} \quad (39)$$

### 6.3 Analysis of the quadratic form $\mathcal{F}_\alpha$

**Separation of variables and first simplification.** Each perturbation  $g \in L^2(0, 2\pi)$  such that  $\int g = 0$  expands as

$$g = \sum_{k=1}^{\infty} (\gamma_k \cos(k \cdot) + \beta_k \sin(k \cdot)), \quad \text{with } \gamma_0 = 0.$$

For every  $k \in \mathbb{N}^*$ , let us introduce  $g_k := \cos(k \cdot)$  and  $\tilde{g}_k := \sin(k \cdot)$ . For any  $k \in \mathbb{N}^*$ , let  $u_{1,\alpha}^{(k)}$  be the solution of Eq. (32) associated with the perturbation  $g_k$ . It is readily checked that there exists a function  $\psi_{k,\alpha} : [0; R] \rightarrow \mathbb{R}$  such that

$$\forall (r, \theta) \in [0; R] \times [0; 2\pi], \quad u_{1,\alpha}^{(k)}(r, \theta) = g_k(\theta)\psi_{k,\alpha}(r).$$

Furthermore,  $\psi_{k,\alpha}$  solves the ODE

$$\begin{cases} -\sigma_\alpha \psi_{k,\alpha}'' - \frac{\sigma_\alpha}{r} \psi_{k,\alpha}'(r) = \left( \lambda_{0,\alpha} - \frac{k^2}{r^2} \right) \psi_{k,\alpha} + m^* \psi_{k,\alpha} & \text{in } (-R; R), \\ \left[ \sigma_\alpha \psi_{k,\alpha}' \right] (r_0^*) = -\kappa u_{0,\alpha}(r_0^*) \\ \left[ \psi_{k,\alpha} \right] (r_0^*) = - \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] (r_0^*), \\ \psi_{k,\alpha}(-R) = \psi_{k,\alpha}(R) = 0. \end{cases} \quad (40)$$

Regarding  $\tilde{g}_k$ , if we define  $\tilde{u}_{1,\alpha}^{(k)}$  in a similar fashion, it is readily checked that

$$\forall (r, \theta) \in [0; R] \times [0; 2\pi], \quad \tilde{u}_{1,\alpha}^{(k)}(r, \theta) = \tilde{g}_k(\theta)\psi_{k,\alpha}(r).$$

Therefore, any admissible perturbation  $g$  writes

$$g = \sum_{k=1}^{\infty} \{ \gamma_k g_k + \beta_k \tilde{g}_k \} \quad \text{with } \gamma_0 = 0$$

and the solution  $u_{1,\alpha}$  associated with  $g$  writes

$$u_{1,\alpha} = \sum_{k=1}^{\infty} \{ \gamma_k u_{1,\alpha}^{(k)} + \beta_k \tilde{u}_{1,\alpha}^{(k)} \}.$$

Using (39) and the orthogonality properties of the family  $\{g_k\}_{k \in \mathbb{N}^*} \cup \{\tilde{g}_k\}_{k \in \mathbb{N}}$ , one gets

$$\begin{aligned} \mathcal{F}_\alpha[V, V] &= \frac{r_0^*}{2} \sum_{k=1}^{\infty} \left( \sigma_\alpha \psi_{k,\alpha}'(r_0^*) \Big|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] - \kappa \psi_{k,\alpha} \Big|_{int} u_{0,\alpha}(r_0^*) \right) (\gamma_k^2 + \beta_k^2) \\ &\quad - \frac{r_0^*}{2} \sum_{k=0}^{\infty} \kappa u_{0,\alpha}(r_0^*) \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) (\gamma_k^2 + \beta_k^2) - \sum_{k=1}^{\infty} 2 [\sigma_\alpha |\nabla u_{0,\alpha}|^2] (\gamma_k^2 + \beta_k^2) \\ &= \frac{r_0^* \kappa u_{0,\alpha}(r_0^*)}{2} \sum_{k=1}^{\infty} \left( - \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int} - \psi_{k,\alpha} \Big|_{int} \right) (\gamma_k^2 + \beta_k^2) \\ &\quad + \sum_{k=1}^{\infty} \left( -2 [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \sigma_\alpha \psi_{k,\alpha}'(r_0^*) \Big|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] \right) (\gamma_k^2 + \beta_k^2) \end{aligned} \quad (41)$$

Define, for any  $k \in \mathbb{N}$ ,

$$\omega_{k,\alpha} := - \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int} - \psi_{k,\alpha} \Big|_{int} \quad \text{and} \quad \zeta_{k,\alpha} := -2 [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \sigma_\alpha \psi_{k,\alpha}'(r_0^*) \Big|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right].$$

Thus,

$$\mathcal{F}_\alpha[V, V] = \sum_{k=1}^{\infty} (\omega_{k,\alpha} + \zeta_{k,\alpha}) (\gamma_k^2 + \beta_k^2).$$

The end of the proof is devoted to proving the local shape minimality of the centered ball, which relies on an asymptotic analysis of the sequences  $\{\omega_{k,\alpha}\}_{k \in \mathbb{N}}$  and  $\{\zeta_{k,\alpha}\}_{k \in \mathbb{N}}$  as  $\alpha$  converges to 0.

**Proposition 2.** *There exists  $C > 0$  and  $\bar{\alpha} > 0$ , there exists  $M \in \mathbb{R}$  such that for any  $\alpha \leq \bar{\alpha}$  and any  $k \in \mathbb{N}$ , one has*

$$\omega_{k,\alpha} \geq C > 0, \quad \text{and} \quad \zeta_{k,\alpha} \geq -M\alpha. \quad (42)$$

The last claim of Theorem 3 is then an easy consequence of this proposition. The rest of the proof is devoted to the proof of Proposition 2, which follows from the combination of the following series of lemmas.

**Lemma 12.** *There exists  $\bar{\alpha} > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ ,  $\psi_{1,\alpha}$  is nonnegative on  $(0, R)$ .*

*Proof of Lemma 12.* For the sake of notational simplicity, we temporarily drop the dependence on  $\alpha$  and denote  $\psi_{1,\alpha}$  by  $\psi_\alpha$ . The function  $\psi_\alpha$  solves the ODE

$$\begin{cases} -\sigma_\alpha \psi_\alpha'' - \frac{\sigma_\alpha}{r} \psi_\alpha'(r) = \left(\lambda_{0,\alpha} - \frac{1}{r^2}\right) \psi_\alpha + m^* \psi_\alpha & \text{in } (0; R), \\ [\sigma_\alpha \psi_\alpha'](r_0^*) = -\kappa u_{0,\alpha}(r_0^*) \\ [\psi_\alpha](r_0^*) = -\left[\frac{\partial u_{0,\alpha}}{\partial r}\right](r_0^*), \\ \psi_\alpha(R) = 0. \end{cases}$$

Let us introduce  $p_\alpha = \psi_\alpha / u_{0,\alpha}$ . One checks easily that  $p_\alpha$  solves the ODE

$$-\sigma_\alpha p_\alpha'' - \frac{\sigma_\alpha}{r} p_\alpha' = \frac{1}{r^2} p_\alpha + 2p_\alpha' \frac{u_{0,\alpha}'}{u_{0,\alpha}} \quad \text{in } (0, R).$$

Furthermore,  $p_\alpha$  satisfies the jump conditions

$$[p_\alpha](r_0^*) = -\frac{[\partial_r u_{0,\alpha}](r_0^*)}{u_{0,\alpha}(r_0^*)} = \frac{-\alpha \kappa \partial_r u_{0,\alpha}|_{int}}{u_{0,\alpha}(r_0^*)} > 0 \quad \text{and} \quad [\sigma_\alpha p_\alpha'](r_0^*) = -\kappa + \frac{\sigma_\alpha \partial_r u_{0,\alpha}}{u_{0,\alpha}(r_0^*)^2} \left[\frac{\partial u_{0,\alpha}}{\partial r}\right].$$

To show that  $\psi_\alpha$  is nonnegative, we argue by contradiction and consider first the case where a negative minimum is reached at an interior point  $r_- \neq r_0^*$ . Then,  $p_\alpha$  is  $\mathcal{C}^2$  in a neighborhood of  $r_-$  and we have

$$0 \geq -p_\alpha''(r_-) = -\frac{p_\alpha(r_-)}{(r_-)^2} > 0,$$

whence the contradiction.

To exclude the case  $r_- = R$ , let us notice that, according to L'Hospital's rule, one has  $p_\alpha(R) = \psi_\alpha'(R) / u_{0,\alpha}'(R)$ . According to the Hopf lemma applied to  $u_{0,\alpha}$ , this quotient is well-defined. If  $p_\alpha(R) < 0$  then it follows that  $\psi_\alpha'(R) > 0$ . However, one has  $p_\alpha'(r) \sim \psi_\alpha'(r) / (2u_{0,\alpha}(r)) > 0$  as  $r \rightarrow R$ , which contradicts the fact that a minimum is reached at  $R$ .

Let us finally exclude the case where  $r_- = r_0^*$ . Mimicking the elliptic regularity arguments used in the proofs of Lemmas 5 and 6, we get that  $p_\alpha$  converges to  $p_0$  as  $\alpha \rightarrow 0$  for the strong topologies of  $\mathcal{C}^0([0, r_0^*])$  and  $\mathcal{C}^0([r_0^*, R])$ .

To conclude, it suffices hence to prove that  $p_0$  is positive in a neighborhood of  $r_0^*$ . We once again argue by contradiction and assume that  $p_0$  reaches a negative minimum at  $r_- \in [0, R]$ . Notice that  $r_- \neq r_0^*$  since  $[p_0](r_0^*) = 0$  and  $[p_0'](r_0^*) = -\kappa < 0$ .

If  $r_- \in (0, R)$ , since  $r_- \neq r_0^*$ , we claim that  $p_0$  is  $\mathcal{C}^2$  in a neighborhood of  $r_-$  and, if  $p_0(r_-) < 0$ , the contradiction follows from

$$0 \geq -p_0''(r_-) = -\frac{p_0(r_-)}{(r_-)^2} > 0.$$

For the same reason, a negative minimum cannot be reached at  $r = 0$ .

If  $r_- = R$ , we observe that  $p_0(R) = \psi'_0(R)/u'_{0,0}(R)$ . According to the Hopf lemma applied to  $u_{0,0}$ , this quantity is well-defined. If  $p_0(R) < 0$ , then it follows that  $\psi'_0(R) > 0$ . However,  $p'_0(r) \sim \psi'_0(r)/(2u(r)) > 0$  as  $r \rightarrow R$ , which contradicts the fact that  $R$  is a minimizer.

Therefore  $p_0$  is positive in a neighborhood of  $r_0^*$  and we infer that  $p_\alpha$  is non-negative, so that, in turn,  $\psi_\alpha \geq 0$  in  $[0, R]$ .  $\square$

**Lemma 13.** *Let  $\bar{\alpha}$  be defined as in Lemma 12. Then, for every  $\alpha \in [0, \bar{\alpha}]$  and every  $k \in \mathbb{N}$ ,*

$$\psi_{k,\alpha} \leq \psi_{1,\alpha}. \quad (43)$$

As a consequence, for any  $\alpha \leq \bar{\alpha}$  and any  $k \in \mathbb{N}$ , there holds  $\omega_{k,\alpha} \geq \omega_{1,\alpha}$ .

*Proof of Lemma 13.* Since  $\omega_{k,\alpha} - \omega_{1,\alpha} = -\psi_{k,\alpha}|_{int}(r_0^*) + \psi_{1,\alpha}|_{int}(r_0^*)$ , the fact that  $\omega_{k,\alpha} \geq \omega_{1,\alpha}$  will follow from (43), on which we now focus on. Let us set  $\Psi_k = \psi_{1,\alpha} - \psi_{k,\alpha}$ . From the jump conditions on  $\psi_{1,\alpha}$  and  $\psi_{k,\alpha}$ , one has  $[\Psi_k](r_0) = [\sigma_\alpha \Psi'_k](r_0) = 0$ . The function  $\Psi_k$  satisfies

$$\begin{aligned} -\sigma_\alpha \Psi''_k - \sigma_\alpha \frac{\Psi'_k}{r} &= -\left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \psi_{k,\alpha} - m_0^* \psi_{k,\alpha} + \left(\lambda_{0,\alpha} - \frac{1}{r^2}\right) \psi_{1,\alpha} + m_0^* \psi_{1,\alpha} \\ &> \left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \psi_{k,\alpha} - m_0^* \psi_{k,\alpha} + \left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \psi_{1,\alpha} + m_0^* \psi_{1,\alpha} \\ &> \left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \Psi_k + m_0^* \Psi_k. \end{aligned}$$

since  $\psi_{1,\alpha} \geq 0$ , according to Lemma 12. Since  $\Psi_k$  satisfies Dirichlet boundary conditions,  $\Psi_k \geq 0$  in  $(0, R)$ .  $\square$

**Lemma 14.** *There exists  $C > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ , where  $\bar{\alpha}$  is introduced on Lemma (12), one has  $\omega_{1,\alpha} \geq C$ .*

*Proof of Lemma 14.* Let us introduce  $\Psi = -\partial u_{0,\alpha}/\partial r - \psi_{k,\alpha}$ . According to (29), we have  $[\Psi](r_0^*) = [\Psi'](r_0^*) = 0$ . Furthermore,  $\Psi(R) = -\frac{\partial u_{0,\alpha}}{\partial r}(R) > 0$  according to the Hopf Lemma and  $\Psi(0) = 0$ . Finally, since  $\Psi$  solves the ODE, one has

$$-\frac{1}{r}(\sigma_\alpha \Psi')' = \left(\lambda_{0,\alpha} - \frac{1}{r^2}\right) \Psi + m^* \Psi \quad \text{in } (0, R),$$

it follows that  $\Psi$  is positive in  $(0, R]$ . Furthermore,  $\Psi$  converges to  $\Psi_0$  for the strong topology of  $\mathcal{C}^0([0, R])$  and  $\Psi_0$  solves the ODE

$$\begin{cases} -\frac{1}{r}(\Psi'_0)' = \left(\lambda_{0,0} - \frac{1}{r^2}\right) \Psi_0 + m_0^* \Psi_0 & \text{in } (0, R) \\ \Psi_0(R) = -\frac{\partial u_{0,0}}{\partial r}(R) > 0. \end{cases}$$

Hence there exists  $C > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ , one has  $\Psi(r_0^*) \geq C > 0$ .  $\square$

It remains to prove the second inequality of (42). As a consequence of the convergence result stated in Lemma 9, one has

$$[\sigma_\alpha |\nabla u_{0,\alpha}|^2] = O(\alpha), \quad \left[\frac{\partial u_{0,\alpha}}{\partial r}\right] = \alpha \kappa \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} < 0. \quad (44)$$

It follows that we only need to prove that there exists a constant  $M > 0$  such that, for any  $\alpha \in [0, \bar{\alpha}]$ , and any  $k \in \mathbb{N}^*$ ,

$$M \geq \sigma_\alpha \psi'_{k,\alpha}|_{ext}(r_0^*) \quad (45)$$

so that

$$\zeta_{k,\alpha} = O(\alpha) + \sigma_\alpha \psi'_{k,\alpha}|_{ext}(r_0^*) \alpha \kappa \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int} \geq O(\alpha) - M \alpha \kappa \left| \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int}$$

To show the estimate (45), let us distinguish between small and large values of  $k$ . To this aim, we introduce  $N \in \mathbb{N}$  as the smallest integer such that

$$\lambda_{0,\alpha} + m_0^* - \frac{k^2}{r^2} < 0 \text{ in } (0, R) \quad (46)$$

for every  $k \geq N$  and  $\alpha \in [0, \bar{\alpha}]$ . The existence of such an integer follows immediately from the convergence of  $(\lambda_{0,\alpha})_{\alpha>0}$  to  $\lambda_0(m_0^*)$  as  $\alpha \rightarrow 0$ .

First, we will prove that, for every  $k \geq N$ ,

$$\psi'_{k,\alpha}(r_0^*)|_{ext} < 0 \quad (47)$$

and that there exists  $M > 0$  such that, for every  $k \leq N$ ,

$$|\psi'_{k,\alpha}(r_0^*)|_{ext}| \leq M \quad (48)$$

which will lead to (45) and thus yield the desired conclusion.

To show (47), let us argue by contradiction, assuming that  $\psi'_{k,\alpha}(r_0^*)|_{ext} > 0$ . Since the jump  $[\sigma_\alpha \psi'_{k,\alpha}] = -\kappa u_{0,\alpha}(r_0^*)$  is negative, it follows that

$$(1 + \alpha \kappa) \psi'_{k,\alpha}(r_0^*)|_{int} = \psi'_{k,\alpha}(r_0^*)|_{ext} - [\sigma_\alpha \psi'_{k,\alpha}] > 0.$$

By mimicking the reasonings in the proof of Lemma 12,  $\psi_{k,\alpha}$  cannot reach a negative minimum on  $(0, r_0^*)$  since (46) holds true. Therefore, since  $\psi_{k,\alpha}(0) = 0$  and  $\psi'_{k,\alpha}(r_0^*)|_{int} > 0$ , one has necessarily  $\psi_{k,\alpha}(r_0^*)|_{int} > 0$ , which in turn gives  $\psi_{k,\alpha}(r_0^*)|_{ext} > 0$  since  $[\psi_{k,\alpha}] = -\alpha \kappa \frac{\partial u_{0,\alpha}}{\partial r} > 0$ .

Furthermore,  $\psi_{k,\alpha}(R) = 0$ . Since  $\psi_{k,\alpha}(r_0^*)|_{ext} > 0$  and  $\psi'_{k,\alpha}(r_0^*)|_{ext} > 0$ , it follows that  $\psi_{k,\alpha}$  reaches a positive maximum at some interior point  $r_1$ , satisfying hence

$$0 \leq -\psi''_{k,\alpha}(r_-) = \left( \lambda_{0,\alpha} + m_0^* - \frac{k^2}{r^2} \right) \psi_{k,\alpha}(r_-) < 0,$$

leading to a contradiction.

Let us now deal with small values of  $k$ , by assuming  $k \leq N$ . We will prove that (48) holds true. To this aim, we will compute  $\psi_{k,\alpha}$ . Let  $J_k$  (resp.  $Y_k$ ) be the  $k$ -th Bessel function of the first (resp. the second) kind. One has

$$\psi_{k,\alpha}(r) = \begin{cases} A_{k,\alpha} J_k\left(\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha \kappa}} \frac{r}{R}\right) & \text{if } r \leq r_0^*, \\ B_{k,\alpha} J_k\left(\sqrt{\lambda_{0,\alpha}} \frac{r}{R}\right) + C_{k,\alpha} Y_k\left(\sqrt{\lambda_{0,\alpha}} \frac{r}{R}\right) & \text{if } r_0^* \leq r \leq R, \end{cases}$$

where  $X_{k,\alpha} = (B_{k,\alpha}, C_{k,\alpha}, A_{k,\alpha})$  solves the linear system

$$\mathcal{A}_{k,\alpha} X_{k,\alpha} = b_\alpha$$

where

$$b_\alpha = \begin{pmatrix} 0 \\ -\kappa u_{0,\alpha}(r_0^*) \\ -[\partial_R u_{0,\alpha}] \end{pmatrix}$$



and

$$\mathcal{A}_{k,\alpha} = \begin{pmatrix} J_k(\sqrt{\lambda_0}) & Y_k(\sqrt{\lambda_0}) & 0 \\ \sqrt{\lambda_{0,\alpha}} J'_k(\sqrt{\lambda_{0,\alpha}} + \kappa \frac{r_0^*}{R}) & \sqrt{\lambda_{0,\alpha}} Y'_k(\sqrt{\lambda_{0,\alpha}} + \kappa \frac{r_0^*}{R}) & -\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha \kappa}} J'_k(\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha \kappa}} \frac{r_0^*}{R}) \\ J_k(\sqrt{\lambda_{0,\alpha}} \frac{r_0^*}{R}) & Y_k(\sqrt{\lambda_{0,\alpha}} \frac{r_0^*}{R}) & -J_k(\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha \kappa}} \frac{r_0^*}{R}) \end{pmatrix}.$$

It is easy to check that

$$\|\mathcal{A}_{k,\alpha} - \mathcal{A}_{k,0}\| \leq M\alpha$$

where  $M$  only depends<sup>3</sup> on  $N$ . Hence it is enough to prove that  $|\psi'_{k,0}(r_0^*)| \leq M$  for some  $M > 0$  depending only on  $N$ , which is straightforward since the set of indices is finite. The expected conclusion follows.

## 6.4 Conclusion

From Eq. (42) and Lemma 14, there exists  $C > 0$  and  $M > 0$  such that  $\omega_{k,\alpha} \geq C > 0$  and  $\zeta_{k,\alpha} \geq -M\alpha$  for every  $\alpha \in [0, \bar{\alpha}]$  and  $k \in \mathbb{N}$ , from which we infer that

$$\mathcal{F}_\alpha[V, V] \geq (C - M\alpha) \sum_{k=1}^{\infty} (\gamma_k^2 + \beta_k^2) \geq \frac{C}{2} \|V \cdot \nu\|_{L^2}^2.$$

according to Eq. (41).

## A Proof of Lemma 1

We prove hereafter that the mapping  $m \mapsto (u_{\alpha,m}, \lambda_\alpha(m))$  is twice differentiable (and even  $\mathcal{C}^\infty$ ) in the  $L^2$  sense, the proof of the differentiability in the weak  $W^{1,2}(\Omega)$  sense being similar. Let  $m^* \in \mathcal{M}_{m_0, \kappa}(\Omega)$ ,  $\sigma_\alpha := 1 + \alpha m^*$ , and  $(u_0, \lambda_0)$  be the eigenpair associated with  $m^*$ . Let  $h \in \mathcal{T}_{m^*}$  (see Def. 1). Let  $m_h^* := m^* + h$  and  $\sigma_{m^*+h} := 1 + \alpha(m^* + h)$ . Let  $(u_h, \lambda_h)$  be the eigenpair associated with  $m_h^*$ . Let us introduce the mapping  $G$  defined by

$$G : \begin{cases} \mathcal{T}_{m^*} \times W_0^{1,2}(\Omega) \times \mathbb{R} \rightarrow W^{-1,2}(\Omega) \times \mathbb{R}, \\ (h, v, \lambda) \mapsto (-\nabla \cdot (\sigma_{m^*+h} \nabla v)) - \lambda v - m_h^* v, \int_\Omega v^2 - 1). \end{cases}$$

From the definition of the eigenvalue, one has  $G(0, u_0, \lambda_0) = 0$ . Moreover,  $G$  is  $\mathcal{C}^\infty$  in  $\mathcal{T}_{m^*} \cap B \times W_0^{1,2}(\Omega) \times \mathbb{R}$ , where  $B$  is an open ball centered at 0. The differential of  $G$  at  $(0, u_0, \lambda_0)$  reads

$$D_{v,\lambda} G(0, u_0, \lambda_0)[w, \mu] = \left( -\nabla \cdot (\sigma_\alpha \nabla w) - \mu u_0 - \lambda_0 w - m^* w, \int_\Omega 2u_0 w \right).$$

Let us show that this differential is invertible. We will show that, if  $(z, k) \in W^{-1,2}(\Omega) \times \mathbb{R}$ , then there exists a unique pair  $(w, \mu)$  such that  $D_{v,\lambda} G(0, u_0, \lambda_0)[w, \mu] = (z, k)$ . According to the Fredholm alternative, one has necessarily  $\mu = -\langle z, u_0 \rangle$  and for this choice of  $\mu$ , there exists a solution  $w_1$  to the equation

$$-\nabla \cdot (\sigma_\alpha \nabla w) - \mu u_0 - \lambda_0 w - m^* w = z \quad \text{in } \Omega.$$

<sup>3</sup>Indeed,  $\{J_k, Y_k\}_{k \leq N}$  are uniformly bounded in  $\mathcal{C}^2([r_0^*/R - \varepsilon, R])$  for every  $\varepsilon > 0$  small enough. Since we consider a finite number of indices  $k$ , there exists  $\delta > 0$  (depending only on  $N$ ) such that

$$\forall k \in \{0, \dots, N\}, \quad \det(\mathcal{A}_{k,\alpha}) \geq \delta > 0.$$

Then, since  $\|X_\alpha - X_0\| \leq M\alpha$ , it follows from the Cramer formula that there exists  $M$  (depending only on  $N$ ) such that  $\|X_{k,\alpha} - X_{k,0}\|_{L^\infty} \leq M\alpha$ .

Moreover, since  $\lambda_0$  is simple, any other solution is of the form  $w = w_1 + tu_0$  with  $t \in \mathbb{R}$ . From the equation  $2 \int_{\Omega} u_0 w = k$ , we get  $t = k/2 - \int_{\Omega} w_1 u_0$ . Hence, the pair  $(w, \mu)$  is uniquely determined. According to the implicit function theorem, the mapping  $h \mapsto (u_h, \lambda_h)$  is  $\mathcal{C}^\infty$  in a neighbourhood of  $\vec{0}$ .

## References

- [1] G. Allaire. *Shape Optimization by the Homogenization Method*. Springer New York, 2002.
- [2] A. Alvino, P.-L. Lions, and G. Trombetti. Comparison results for elliptic and parabolic equations via symmetrization: a new approach. *Differential Integral Equations*, 4(1):25–50, 1991.
- [3] A. Alvino and G. Trombetti. A lower bound for the first eigenvalue of an elliptic operator. *Journal of Mathematical Analysis and Applications*, 94(2):328 – 337, 1983.
- [4] F. Belgacem. *Elliptic Boundary Value Problems with Indefinite Weights, Variational Formulations of the Principal Eigenvalue, and Applications*. Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis, 1997.
- [5] F. Belgacem and C. Cosner. The effect of dispersal along environmental gradients on the dynamics of populations in heterogeneous environment. *Canadian Applied Mathematics Quarterly*, 3:379–397, 01 1995.
- [6] H. Berestycki, F. Hamel, and L. Roques. Analysis of the periodically fragmented environment model : I – species persistence. *Journal of Mathematical Biology*, 51(1):75–113, 2005.
- [7] H. Berestycki and T. Lachand-Robert. Some properties of monotone rearrangement with applications to elliptic equations in cylinders. *Math. Nachr.*, 266:3–19, 2004.
- [8] P. Bernhard and A. Rapaport. On a theorem of danskin with an application to a theorem of von neumann-sion. *Nonlinear Analysis: Theory, Methods & Applications*, 24(8):1163–1181, apr 1995.
- [9] B. Brandolini, F. Chiacchio, A. Henrot, and C. Trombetti. Existence of minimizers for eigenvalues of the dirichlet-laplacian with a drift. *Journal of Differential Equations*, 259(2):708 – 727, 2015.
- [10] A. Bressan, G. M. Coclite, and W. Shen. A multidimensional optimal-harvesting problem with measure-valued solutions. *SIAM J. Control Optim.*, 51(2):1186–1202, 2013.
- [11] A. Bressan and W. Shen. Measure-valued solutions for a differential game related to fish harvesting. *SIAM J. Control Optim.*, 47(6):3118–3137, 2008.
- [12] R. S. Cantrell and C. Cosner. Diffusive logistic equations with indefinite weights: Population models in disrupted environments II. *SIAM Journal on Mathematical Analysis*, 22(4):1043–1064, jul 1991.
- [13] R. S. Cantrell and C. Cosner. Diffusive logistic equations with indefinite weights: Population models in disrupted environments II. *SIAM Journal on Mathematical Analysis*, 22(4):1043–1064, jul 1991.
- [14] J. Casado-Díaz. Smoothness properties for the optimal mixture of two isotropic materials: The compliance and eigenvalue problems. *SIAM J. Control and Optimization*, 53:2319–2349, 2015.
- [15] J. Casado-Díaz. Some smoothness results for the optimal design of a two-composite material which minimizes the energy. *Calculus of Variations and Partial Differential Equations*, 53(3):649–673, 2015.
- [16] J. Casado-Díaz. A characterization result for the existence of a two-phase material minimizing the first eigenvalue. *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis*, 34, 10 2016.
- [17] F. Caubet, T. Deheuvels, and Y. Privat. Optimal location of resources for biased movement of species: the 1D case. *SIAM Journal on Applied Mathematics*, 77(6):1876–1903, 2017.
- [18] A. Cianchi, L. Esposito, N. Fusco, and C. Trombetti. A quantitative polya?szego principle. *J. Reine Angew. Math.*
- [19] G. M. Coclite and M. Garavello. A time-dependent optimal harvesting problem with measure-valued solutions. *SIAM J. Control Optim.*, 55(2):913–935, 2017.

- [20] C. Conca, A. Laurain, and R. Mahadevan. Minimization of the ground state for two phase conductors in low contrast regime. *SIAM Journal of Applied Mathematics*, 72:1238–1259, 2012.
- [21] C. Conca, R. Mahadevan, and L. Sanz. An extremal eigenvalue problem for a two-phase conductor in a ball. *Applied Mathematics and Optimization*, 60(2):173–184, Oct 2009.
- [22] C. Cosner and Y. Lou. Does movement toward better environments always benefit a population? *Journal of Mathematical Analysis and Applications*, 277(2):489–503, jan 2003.
- [23] S. Cox and R. Lipton. Extremal eigenvalue problems for two-phase conductors. *Archive for Rational Mechanics and Analysis*, 136:101–117, 12 1996.
- [24] M. Dambrine and D. Kateb. On the shape sensitivity of the first dirichlet eigenvalue for two-phase problems. *Applied Mathematics & Optimization*, 63(1):45–74, jul 2010.
- [25] M. Dambrine and J. Lamboley. Stability in shape optimization with second variation. *Journal of Differential Equations*, Apr. 2019.
- [26] J. L. Ericksen, D. Kinderlehrer, R. Kohn, and J.-L. Lions, editors. *Homogenization and Effective Moduli of Materials and Media*. Springer New York, 1986.
- [27] A. Ferone and R. Volpicelli. Minimal rearrangements of sobolev functions: a new proof. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 20(2):333–339, mar 2003.
- [28] R. A. Fisher. The wave of advances of advantageous genes. *Annals of Eugenics*, 7(4):355–369, 1937.
- [29] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin Heidelberg, 1983.
- [30] F. Hamel, N. Nadirashvili, and E. Russ. Rearrangement inequalities and applications to isoperimetric problems for eigenvalues. *Annals of Mathematics*, 174(2):647–755, sep 2011.
- [31] G. H. Hardy. Note on a theorem of hilbert. *Mathematische Zeitschrift*, 6(3):314–317, Sep 1920.
- [32] A. Henrot. *Extremum Problems for Eigenvalues of Elliptic Operators*. Birkhäuser Basel, 2006.
- [33] A. Henrot. *Shape Optimization and Spectral Theory*. De Gruyter Open, 2017.
- [34] A. Henrot and M. Pierre. *Shape Variation and Optimization*. European Mathematical Society Publishing House, feb 2018.
- [35] P. Hess and T. Kato. On some linear and nonlinear eigenvalue problems with an indefinite weight function. *Communications in Partial Differential Equations*, 5(10):999–1030, 1980.
- [36] C.-Y. Kao, Y. Lou, and E. Yanagida. Principal eigenvalue for an elliptic problem with indefinite weight on cylindrical domains. *Mathematical biosciences and engineering : MBE*, 5 2:315–35, 2008.
- [37] B. Kawohl. *Rearrangements and Convexity of Level Sets in PDE*. Springer Berlin Heidelberg, 1985.
- [38] A. Kolmogoroff, I. Petrovsky, and N. Piscounoff. études de l'équation avec croissance de la quantité de matière et son application à un problème biologique. *Moscow University Bulletin Of Mathematics*, 1:1–25, 01 1937.
- [39] J. Lamboley, A. Laurain, G. Nadin, and Y. Privat. Properties of optimizers of the principal eigenvalue with indefinite weight and Robin conditions. *Calculus of Variations and Partial Differential Equations*, 55(6), Dec. 2016.
- [40] A. Laurain. Global minimizer of the ground state for two phase conductors in low contrast regime. *ESAIM: Control, Optimisation and Calculus of Variations*, 20(2):362–388, 2014.
- [41] Y. Lou. *Some Challenging Mathematical Problems in Evolution of Dispersal and Population Dynamics*, pages 171–205. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.
- [42] Y. Lou and E. Yanagida. Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics. *Japan Journal of Industrial and Applied Mathematics*, 23(3):275, Oct 2006.
- [43] Y. Lou and E. Yanagida. Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics. *Japan J. Indust. Appl. Math.*, 23(3):275–292, 10 2006.

- [44] I. Mazari. Quantitative inequality for the first eigenvalue of a schrödinger operator in the ball.
- [45] N. G. Meyers. An  $l^p$ -estimate for the gradient of solutions of second order elliptic divergence equations. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 3, 17(3):189–206, 1963.
- [46] F. Mignot, J. Puel, and F. Murat. Variation d’un point de retournement par rapport au domaine. *Communications in Partial Differential Equations*, 4(11):1263–1297, 1979.
- [47] F. Murat. Contre-exemples pour divers problèmes où le contrôle intervient dans les coefficients. *Annali di Matematica Pura ed Applicata*, 112(1):49–68, Dec 1977.
- [48] F. Murat and L. Tartar. *Calculus of Variations and Homogenization*, pages 139–173. Birkhäuser Boston, Boston, MA, 1997.
- [49] J. D. Murray. *Mathematical Biology*. Springer Berlin Heidelberg, 1993.
- [50] G. Nadin. The effect of the schwarz rearrangement on the periodic principal eigenvalue of a nonsymmetric operator. *SIAM Journal on Mathematical Analysis*, 41(6):2388–2406, jan 2010.
- [51] O. Oleinik, A. Shamaev, and G. Yosifian. *Mathematical Problems in Elasticity and Homogenization, Volume 26 (Studies in Mathematics and its Applications)*. North Holland, 1992.
- [52] B. Opic and A. Kufner. *Hardy-type Inequalities*. Pitman research notes in mathematics series. Longman Scientific & Technical, 1990.
- [53] J. Serrin. A symmetry problem in potential theory. *Archive for Rational Mechanics and Analysis*, 43(4):304–318, Jan 1971.
- [54] J. G. Skellam. Random dispersal in theoretical populations. *Biometrika*, 38(1-2):196–218, 06 1951.