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Frank Loray, Frédéric Touzet, Sergei M Voronin

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TWO DIMENSIONAL NEIGHBORHOODS OF ELLIPTIC CURVES: ANALYTIC CLASSIFICATION.

FRANK LORAY, FRÉDÉRIC TOUZET, AND SERGEI M. VORONIN

ABSTRACT. We investigate the analytic classification of two dimensional neighborhoods of an elliptic curve C with torsion normal bundle. We provide the complete analytic classification for those neighborhoods in the simplest formal class.

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1. INTRODUCTION AND RESULTS

Let C be a smooth elliptic curve: $C = \mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbb{Z} + \tau\mathbb{Z}$, with $\Im(\tau) > 0$. Given an embedding $\iota : C \hookrightarrow U$ of C into a smooth complex surface U , we would like to understand the germ $(U, \iota(C))$ of neighborhood of $\iota(C)$ in U . Precisely, we will say that two embeddings $\iota, \iota' : C \hookrightarrow U, U'$ are (formally/analytically) equivalent if there is a (formal/analytic) isomorphism $\Psi : (U, \iota(C)) \rightarrow (U', \iota'(C))$ between germs of neighborhoods making commutative the following diagram

$$(1.1) \quad \begin{array}{ccc} C & \xrightarrow{\iota} & U \\ \mathrm{id} \downarrow & & \downarrow \Psi \\ C & \xrightarrow{\iota'} & U' \end{array}$$

By abuse of notation, we will still denote by C the image $\iota(C)$ of its embedding in U , and we will simply denote by (U, C) the germ of neighborhood.

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1.1. Some historical background. The problem of analytic classification of neighborhoods of compact complex curves in complex surfaces goes back at least to the celebrated work of Grauert [6]. There, he considered the normal bundle N_C of the curve in U . The neighborhood of the zero section in the total space of N_C , that we denote $(N_C, 0)$, can be viewed¹ as the linear part of (U, C) . A coarse invariant is given by the degree $\deg N_C$ which is also the self-intersection of the curve $C \cdot C$. In this paper, Grauert proved that the germ of neighborhood is “linearizable”, i.e. analytically equivalent to the germ of neighborhood $(N_C, 0)$, provided that $\deg(N_C)$ is negative enough, namely $\deg(N_C) < 4 - 4g$ for a curve of genus $g > 0$, and $\deg(N_C) < 0$ for a rational curve $g = 0$. It was also clear from his work that even the formal classification was much more complicated when $\deg(N_C) > 0$. At the same period, Kodaira investigated the deformation of compact submanifolds of complex manifolds in [10]. His result, in the particular case of curves in surfaces, says that the curve can be deformed provided that $\deg(N_C)$ is positive enough, namely $\deg(N_C) > 2g - 2$ for a curve of genus $g > 0$, and $\deg(N_C) \geq 0$ for a rational curve $g = 0$. Using these deformations, it is possible to provide a complete set of invariants for analytic classification for $g = 0$: (U, C) is linearizable when $\deg(N_C) \leq 0$ (Grauert for < 0 and Savelev [21] for $= 0$), and there is a functional moduli² when $\deg(N_C) > 0$ following Mishustin [15] (see also [4]). Also, when $g > 0$ and $\deg(N_C) > 2g - 2$, the analytic classification has been carried out by Ilyashenko [9] and Mishustin [16]. In all these results, it is important to notice that formally equivalent neighborhoods are also analytically equivalent: the two classifications coincide for such neighborhoods.

The case of an elliptic³ curve $g = 1$ with $\deg(N_C) = 0$, which is still open today, has been investigated by Arnold [1] in another celebrated work. In this case, the normal bundle N_C belongs to the Jacobian curve $\text{Jac}(C) \simeq C = \mathbb{C}/\Gamma_\tau$ and can be torsion⁴ or not. Torsion points correspond to the image of $\mathbb{Q} + \tau\mathbb{Q} \subset C$ in the curve. Arnold investigated the non torsion case and proved in that case

- if N_C is non torsion, then (U, C) is formally linearizable;
- if N_C is generic⁵ enough in $\text{Jac}(C)$, then (U, C) is analytically linearizable;
- for non generic (and still non torsion) N_C , there is a huge⁶ moduli space for the analytic classification.

However, we are still far, nowadays, to expect a complete description of the analytic classification in that non torsion case. It is the first case where the divergence of formal and analytic classification occur. Also, it is interesting to note that the study of neighborhoods of elliptic curves in the case $\deg(N_C) = 0$ has strong reminiscence with the classification of germs of diffeomorphisms up to conjugacy. It will be more explicit later when describing the torsion case.

The goal of this paper is to investigate the analytic classification when the normal bundle is torsion, and show that we can expect to provide a complete description of the moduli space in that case. More precisely, the formal classification of such neighborhoods has

¹Strictly speaking, the linear part is more complicated in general, as it needs not fiber over the curve, as it is the case for a conic in \mathbb{P}^2 .

²The moduli space is comparable with the ring of convergent power series $\mathbb{C}\{X, Y\}$.

³Elliptic means $g = 1$ and that we have moreover fixed a (zero) point on the curve, to avoid considering automorphisms of the curve in our study.

⁴Torsion means that some iterate for the group law \otimes is the trivial bundle \mathcal{O}_C .

⁵i.e. belongs to some subset of total Lebesgue measure defined by a certain diophantine condition

⁶Thanks to the works of Yoccoz [32] and Perez-Marco [20], we can embed at least $\mathbb{C}\{X\}$ in the moduli space with a huge degree of freedom.

been achieved in [11]; we provide the analytic classification inside the simplest formal class, and we explain how we think it generalizes for all other formal classes.

1.2. Formal classification. An important formal invariant has been introduced by Ueda [27] in the case $\deg(N_C) = 0$ and $g > 0$. There, among other results, he investigates the obstruction for the curve to be the fiber of a fibration (as it would be in the linear case $(N_C, 0)$ when N_C is torsion). The Ueda type $\kappa \in \mathbb{Z}_{>0}$ is the largest integer for which the aforementioned fibration⁷ of N_C can be extended to the κ^{th} infinitesimal neighborhood of C (see [3, section 2] for a short exposition). When $\kappa = \infty$, then we have a formal fibration, that can be proved to be analytic; the classification in that case goes back to the works of Kodaira, in particular in the elliptic case $g = 1$.

Inspired by Ueda's approach, it has been proved by Claudon, Pereira and the two first authors of this paper (see [3]) that a formal neighborhoods (U, C) with $\deg(N_C) = 0$ carry many regular (formal) foliations such that C is a compact leaf. This construction has been improved in [11, 25] showing that one can choose two of these foliations in a canonical way and use them to produce a complete set of formal invariants. In the elliptic case $g = 1$, there are $\frac{\kappa}{\nu} + 1$ independant formal invariants for fixed Ueda type κ where ν is the torsion order $N_C^{\otimes \nu} = \mathcal{O}_C$ (see [11]); for $g > 1$ and $N_C = \mathcal{O}_C$ (the trivial bundle), Thom founds infinitely many independant formal invariants in [25].

In this paper, we only consider the case $g = 1$ where $N_C = \mathcal{O}_C$ is the trivial bundle, to which we can reduce by covering whenever N_C was torsion. Let us recall the formal classification in that case. For each Ueda type $\kappa \in \mathbb{Z}_{>0}$, let $P \in \mathbb{C}[X]$ be any polynomial of degree $< \kappa$ and $\nu \in \mathbb{C}$ a scalar. To these data, we associate a germ of neighborhood $(U_{\kappa, \nu, P}, C)$ as follows. Writing C as a quotient of \mathbb{C}^* by a contraction:

$$C = \mathbb{C}_z^* / \langle z \mapsto qz \rangle \quad \text{with } z = e^{2i\pi x} \text{ and } q = e^{2i\pi\tau}, \quad (|q| < 1)$$

we similarly define $(U_{\kappa, \nu, P}, C)$ as the quotient of the germ of neighborhood

$$(\mathbb{C}_z^* \times \mathbb{C}_y, \{y = 0\})$$

by the germ of diffeomorphism

$$F_{\kappa, \nu, P} = \exp(v_0 + v_\infty), \quad \text{where } \begin{cases} v_0 &= \frac{y^{\kappa+1}}{1+\nu y^\kappa} \partial_y + 2i\pi\tau \frac{yP(y)}{1+\nu y^\kappa} z \partial_z \\ v_\infty &= 2i\pi\tau z \partial_z \end{cases}$$

The two vector fields v_0 and v_∞ span a commutative Lie algebra, and therefore an infinitesimal \mathbb{C}^2 -action on the quotient neighborhood. By duality, we have a 2-dimensional vector space of closed meromorphic 1-forms spanned by

$$\omega_0 = \frac{dy}{y^{\kappa+1}} + \nu \frac{dy}{y} \quad \text{and} \quad \omega_\infty = \frac{1}{2i\pi\tau} \frac{dz}{z} - \frac{P(y)}{y^\kappa} dy.$$

In particular, we get a pencil of foliations \mathcal{F}_t , $t \in \mathbb{P}^1$, by considering⁸ $\omega_t = 0$ where

$$\omega_t = \omega_0 - t\omega_\infty.$$

When $P = 0$, \mathcal{F}_∞ defines a fibration transversal to the curve C and the neighborhood is the suspension⁹ of a representation $\varrho : \pi_1(C) \rightarrow \text{Diff}(\mathbb{C}, 0)$ taking values into the one-parameter group generated by $v_0 = \frac{y^{\kappa+1}}{1+\nu y^\kappa} \partial_y$. For $t \in \mathbb{C}$ finite, \mathcal{F}_t is always (smooth)

⁷More generally, in the non torsion case, we may try to extend the foliation defined by the unitary connection on N_C .

⁸or equivalently the phase portrait of the vector fields $v_t = tv_0 + v_\infty$

⁹in the sense of foliations

tangent to C , i.e. C is a compact leaf; when $P \neq 0$, the same holds for \mathcal{F}_∞ . For $m + \tau n \in \Gamma$ in the lattice, $\mathcal{F}_{\frac{\tau n}{m + \tau n}}$ is the unique foliation of the pencil whose holonomy along the corresponding loop $m + \tau n$ in $\pi_1(C) \sim \Gamma$ is trivial. As proved in [11, Theorem 1.3], the neighborhoods $(U_{\kappa, \nu, P}, C)$ span all formal classes of neighborhoods with trivial normal bundle $N_C = \mathcal{O}_C$ and finite Ueda type κ ; moreover, any two such neighborhoods are formally equivalent $(U_{\kappa, \nu, P}, C) \sim^{\text{for}} (U_{\kappa', \nu', P'}, C)$ if, and only if there is a κ^{th} -root of unity ζ such that:

$$\kappa = \kappa', \quad \nu = \nu' \quad \text{and} \quad P'(y) = \zeta P(\zeta y), \quad \zeta^\kappa = 1.$$

As explained in [11, Theorem 1.5], the moduli space of those neighborhoods with two convergent foliations in a given formal class up to analytic conjugacy is infinite dimensional¹⁰, comparable with $\mathbb{C}\{X\}$. *A contrario*, if a third foliation is convergent, then the neighborhood is analytically equivalent to its formal model $(U_{\kappa, \nu, P}, C)$. However, an example of a neighborhood without convergent foliation is given by Mishustin in [17], and it is expected to be a generic property. In this paper, we describe the analytic classification of neighborhoods with Ueda type $\kappa = 1$; we expect that a similar result holds more generally for torsion normal bundle $N_C^{\otimes m} = \mathcal{O}_C$, and finite Ueda type $\kappa < \infty$. As we shall see, the moduli space is comparable with $\mathbb{C}\{X, Y\}$.

1.3. The fundamental isomorphism. In order to explain our classification result, it is convenient to recall the following classical construction. For the simplest formal type $(\kappa, \nu, P) = (1, 0, 0)$, the neighborhood $(U_{1,0,0}, C)$ actually embeds into a ruled surface $S_0 \rightarrow C$, namely one of the two indecomposable ruled surfaces over C after Atiyah [2]. Indeed, setting $y = -1/\xi$, the ruled surface is defined as the quotient

$$S_0 = \tilde{U}_0 / \langle F_0 \rangle \quad \text{where} \quad \tilde{U}_0 = \mathbb{C}_z^* \times \overline{\mathbb{C}}_\xi \quad \text{and} \quad F_0(z, \xi) = (qz, \xi + 1)$$

and the infinity section $\xi = \infty$ defines the embedding of the curve $C \subset S_0$. The complement of the curve $S_0 \setminus C$ is known to be isomorphic to the moduli space of flat line bundles¹¹ over the elliptic curve, and has the structure of an affine bundle. The Riemann-Hilbert correspondance provides an analytic isomorphism with the space of characters $\text{Hom}(\pi_1(C), \text{GL}_1(\mathbb{C}))$, which is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Explicitely, the isomorphism is induced on the quotient S_0 by the following map

$$\Pi : S_0 \setminus C \xrightarrow{\sim} \mathbb{C}_X^* \times \mathbb{C}_Y^*; \quad (z, \xi) \mapsto (e^{2i\pi\xi}, z^{-1}e^{2i\pi\tau\xi}).$$

In this sense, we can view S_0 and $\mathbb{P}_X^1 \times \mathbb{P}_Y^1 \subset \mathbb{C}_X^* \times \mathbb{C}_Y^*$ as two non algebraically equivalent compactifications of the same analytic variety. In fact, the algebraic structures of the two open sets are different as $\mathbb{C}_X^* \times \mathbb{C}_Y^*$ is affine, while $S_0 \setminus C$ is not: there is no non constant regular function on it. This construction, due to Serre, provides an example of a Stein quasiprojective variety which is not affine (see [7, page 232]). Denote by $D \subset \mathbb{P}_X^1 \times \mathbb{P}_Y^1$ the compactifying divisor, union of four projective lines:

$$D = L_1 \cup L_2 \cup L_3 \cup L_4 \quad \text{with}$$

$$L_1 : \{Y = 0\}, \quad L_2 : \{X = \infty\}, \quad L_3 : \{Y = \infty\} \quad \text{and} \quad L_4 : \{X = 0\}$$

Logarithmic one-forms with poles supported on D correspond to the space of closed one-forms considered above via the isomorphism:

$$\begin{cases} \omega_0 = d\xi \\ \omega_\infty = \frac{1}{2i\pi\tau} \frac{dz}{z} \end{cases} \quad \text{and} \quad \begin{cases} \frac{1}{2i\pi} \frac{dX}{Y} = \omega_0 \\ \frac{1}{2i\pi} \frac{dY}{X} = \tau(\omega_0 - \omega_\infty) \end{cases}$$

¹⁰isomorphic to Écalle-Voronin moduli spaces

¹¹i.e. line bundles together with a holomorphic connection

Therefore, at the level of foliations, we have the following correspondance

$$(1.2) \quad \mathcal{F}_t : \{\omega_0 - t\omega_\infty = 0\} \longleftrightarrow (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y} = 0.$$

In particular, for $m + \tau n \in \Gamma \sim \pi_1(C)$ in the lattice, the unique foliation with trivial holonomy along $m + \tau n$ corresponds to the one with rational first integral $X^m Y^n$:

$$\mathcal{F}_{\frac{\tau n}{m+\tau n}} \longleftrightarrow m \frac{dX}{X} + n \frac{dY}{Y} = 0$$

and the ruling corresponds to a foliation with transcendental leaves:

$$\mathcal{F}_\infty \longleftrightarrow \tau \frac{dX}{X} - \frac{dY}{Y} = 0.$$

Let us now study the isomorphism $\Pi : S_0 \setminus C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$ near the compactifying divisors. Denote by V_i a tubular neighborhood of L_i in $\mathbb{P}_X^1 \times \mathbb{P}_Y^1$, of the form $L_i \times \text{disc}$ say, $V = V_1 \cup V_2 \cup V_3 \cup V_4$ the corresponding neighborhood of D . Denote by $V_i^* = V_i \setminus (V_i \cap D)$ the complement of the divisor and by $U_i = \Pi^{-1}(V_i^*)$ the preimage: we have a decomposition neighborhood $U \setminus C = U_1 \cup U_2 \cup U_3 \cup U_4$. One can show that U_i 's look like sectorial domains of opening π in the variable y saturated by variable z (see section 3.1). Our main result is that this sectorial decomposition together with isomorphisms $\Pi_i : U_i \rightarrow V_i^*$ persists for general neighborhoods (U, C) in the formal class $(U_{1,0,0}, C)$; we conjecture a similar result holds true for all formal types, whenever N_C is torsion.

1.4. Analytic classification: main result. A general neighborhood (U, C) formally conjugated to $(U_{1,0,0}, C)$ can be described as quotient (see Proposition 2.3)

$$U = \tilde{U} / \langle F \rangle \quad \text{where } \tilde{U} \subset \mathbb{C}_z^* \times \mathbb{C}_y$$

is a neighborhood of the zero section $\tilde{C} = \{y = 0\}$, and

$$F(z, y) = (qz + O(y^2), y + y^2 + y^3 + O(y^4)).$$

There is a formal isomorphism

$$\hat{\Psi} = \left(z + \sum_{m \geq 2} a_m(z) y^m, y + \sum_{n \geq 4} b_n(z) y^n \right)$$

such that $\hat{\Psi} \circ F = F_{1,0,0} \circ \hat{\Psi}$; we have $a_m, b_n \in \mathcal{O}(\mathbb{C}_z^*)$ and no convergence assumption in y -variable. We can also consider $\hat{\Psi}$ as a formal diffeomorphism $(U, C) \rightarrow (U_{1,0,0}, C)$. The main ingredient of our classification result, proved in section 9, is the

Lemma A. Sectorial normalization. *Denote $\varpi = \arg \tau$. For each interval*

$$(1.3) \quad I_1 =]\varpi, \varpi + \pi[, \quad I_2 =]-\pi, 0[, \quad I_3 = I_1 + \pi, \quad I_4 = I_2 + \pi$$

there is a transversely sectorial domain¹² $U_i \subset U$ of opening I_i and a diffeomorphism

$$\Psi_i : U_i \rightarrow U_{1,0,0}$$

¹²Given an interval $I = [\theta_1, \theta_2] \subset \mathbb{R}$, an open subset $U_0 \subset U$ is said transversely sectorial of opening I if the lift $\tilde{U}_0 \subset \tilde{U} \subset \mathbb{C}_z^* \times \mathbb{C}_y$ contains, for arbitrary large open set $\mathcal{C} \Subset \mathbb{C}^*$ and arbitrary small $\epsilon > 0$, a sector $\mathcal{C} \times \mathcal{S}(I_\epsilon, r)$ where

$$\mathcal{S}(I_\epsilon, r) = \{y \in \mathbb{C} ; \arg(y) \in I_\epsilon, 0 < |y| < r\}, \quad I_\epsilon =]\theta_1 + \epsilon, \theta_2 - \epsilon[$$

for some $r > 0$.

(onto its image) having $\hat{\Psi}$ as asymptotic expansion¹³ along C , satisfying

$$\Psi_i \circ F = F_{1,0,0} \circ \Psi_i.$$

After composition with the fundamental isomorphism $\Pi : U_{1,0,0} \rightarrow \mathbb{C}_X^* \times \mathbb{C}_Y^*$, we get

Corollary B. *The composition $\Pi_i = \Pi \circ \Psi_i$ provides an isomorphism germ*

$$\Pi_i : (U_i, C) \rightarrow (V_i^*, L_i) \quad \text{such that} \quad \Pi_i = \varphi_{i,i+1} \circ \Pi_{i+1} \quad \text{on} \quad U_i \cap U_{i+1}$$

for some diffeomorphism germs $\varphi_{i,i+1} \in \text{Diff}(V_{i,i+1}^*, p_{i,i+1})$.

After patching copies of germs $(V_i, L_i) \simeq (\mathbb{P}_X^1 \times \mathbb{P}_Y^1, L_i)$ by the $\varphi_{i,i+1} : (V_{i+1}, p_{i,i+1}) \rightarrow (V_i, p_{i,i+1})$, we get a new neighborhood germ (V_φ, D) of the divisor D , where $\varphi = (\varphi_{i,i+1})_{i \in \mathbb{Z}_4}$, together with a diffeomorphism germ

$$\Pi : (U \setminus C, C) \xrightarrow{\sim} (V_\varphi \setminus D, D)$$

which does not depend on the choice of sectorial normalisations Ψ_i .

More generally, consider a neighborhood (V, D) in which each component $L_i \subset D$ has zero self-intersection. Then after [21], the neighborhood (V, L_i) is trivial (a product $L_i \times \text{disc}$). After identification with our model $\psi_i : (V, L_i) \xrightarrow{\sim} (\mathbb{P}_X^1 \times \mathbb{P}_Y^1, L_i)$, we get that V takes the form V_φ for a convenient 4-uple of diffeomorphisms φ . The gluing data φ is not unique as we can compose each embedding ψ_i by an automorphism germ $\varphi_i \in \text{Diff}(V_i^*, L_i)$. Therefore, it is natural to introduce the following equivalence relation

$$\varphi \sim \varphi' \Leftrightarrow \exists (\varphi_i \in \text{Diff}(V_i^*, L_i))_{i \in \mathbb{Z}_4} \quad \text{such that} \quad \varphi_i \circ \varphi'_{i,i+1} = \varphi_{i,i+1} \circ \varphi_i.$$

Clearly, the moduli space \mathcal{V} of neighborhoods (V, D) up to analytic equivalence identifies with the set of equivalence classes for \sim . Notice that each equivalence classe contains a representative φ such that $\varphi_{1,2}, \varphi_{2,3}, \varphi_{3,4}$ are tangent to the identity, and the linear part

$$\varphi_{4,1}(X, Y) = (aX + \dots, bY + \dots)$$

does not depend on the choice of such representative φ . Therefore, $a, b \in \mathbb{C}^*$ are invariants for the equivalence relation, and we denote by $\mathcal{V}_{a,b}$ the moduli space of those triples. With this in hand, we are able to prove:

Theorem C. *We have a one-to-one correspondance between*

$$\mathcal{U}_{1,0,0} \leftrightarrow \mathcal{V}_{1,1}$$

- the moduli space $\mathcal{U}_{1,0,0}$ of neighborhoods (U, C) formally equivalent to $(U_{1,0,0}, C)$ up to analytic equivalence¹⁴
- the moduli space $\mathcal{V}_{1,1}$ of neighborhoods (V_φ, D) with all $\varphi_{i,i+1}$ tangent to the identity.

Remark 1.1. *The correspondance is analytic in the sense that analytic families of neighborhoods $t \mapsto (U_t, C)$ correspond to analytic families of cocycles $t \mapsto \varphi_t$. As the freedom lie in the choice of one-dimensional diffeomorphisms φ_i , it is quite clear that the moduli space is essentially parametrized by two-dimensional diffeomorphisms, and therefore quite huge.*

¹³The diffeomorphism $\Psi_i : U_i \rightarrow U_{1,0,0}$ admits $\hat{\Psi}_i$ as an asymptotic expansion along C if the entries of its lift $\tilde{\Psi}_i : \tilde{U}_i \rightarrow \mathbb{C}_z^* \times \mathbb{C}_y$ admit the entries of $\hat{\Psi}$ as asymptotic expansion on each open subset $\mathcal{C} \times \mathcal{S}(I_\varepsilon, r)$ (see section 3.1).

¹⁴More precisely, we allow for this statement analytic isomorphisms inducing translations on C ; see, Proposition 4.3 for a more precise statement.

In a similar way, we expect that $\mathcal{U}_{1,\mu,\nu} \simeq \mathcal{V}_{a,b}$ with $a = e^{2i\pi\nu}$ and $b = e^{2i\pi\tau(\nu-\mu)}$. We explain in section 10 how to construct an embedding $\mathcal{V}_{a,b} \hookrightarrow \mathcal{U}_{1,\mu,\nu}$, but the surjectivity needs to adapt our Sectorial Normalization Lemma which seems rather technical. For larger Ueda type $\kappa > 1$, we expect to have κ more sectors with opening $\frac{\pi}{\kappa}$ and the moduli space would be then equivalent to the moduli of neighborhoods of 4κ -cycles of rational curves (the model would be a degree κ cyclic cover of (V, D)).

1.5. Foliations. A neighborhood (U, C) formally conjugated to $(U_{1,0,0}, C)$ admits a pencil of formal foliations $\hat{\mathcal{F}}_t$ (corresponding to \mathcal{F}_t in (1.2) via the formal normalization $\hat{\Psi}$).

Theorem D. *The foliation $\hat{\mathcal{F}}_t$ is convergent if, and only if, there exists a representative φ in the corresponding equivalence class such that each $\varphi_{i,i+1}$ preserves the foliation*

$$(1-t)\tau \frac{dX}{X} + t \frac{dY}{Y} = 0.$$

In that case, these two foliations are conjugated via the isomorphism $U \setminus C \rightarrow V \setminus D$.

When \mathcal{F}_t is not of rational type, i.e. $\tau(1 - \frac{1}{t}) \notin \mathbb{Q} \cup \{\infty\}$, then \mathcal{F}_t is defined by a closed meromorphic 1-form and the logarithmic 1-form of the statement is also preserved by all $\varphi_{i,i+1}$ and defines a global logarithmic 1-form on (V, D) . On the other hand, in the rational case, Ecalle-Voronin moduli of the holonomy provide obstruction to define the foliation by a closed meromorphic 1-form. For instance, when \mathcal{F}_0 is convergent, Martinet-Ramis cocycle are given by the X -coordinate of $\varphi_{1,2} \circ \varphi_{2,3}$ and $\varphi_{3,4} \circ \varphi_{4,1}$ (see section 7.9 for details).

In [11], the two first authors with O. Thom provided the analytic classification of neighborhoods with 2 foliations. In section 7, we provide examples of neighborhoods with only one foliation, and also without foliation¹⁵ which is the generic case.

In section 8, we investigate the automorphism group of neighborhood germs. We prove in Theorem 8.1 that it can be of three types: finite (the generic case), one dimensional and we get an holomorphic vector field (and in particular a convergent foliation), or two dimensional only in the Serre example.

1.6. $\mathrm{SL}_2(\mathbb{Z})$ action. The analytic classification of resonant diffeomorphism germs of one variable is reminiscent in our classification result. However, there are differences like the fact that the sectorial trivialization is not unique in our case. Indeed, our sectorial decomposition $U \setminus C = U_1 \cup U_2 \cup U_3 \cup U_4$ has been imposed by our choice of a basis for the lattice $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$. It comes from the sectorial decomposition of the holonomy maps of the two foliations \mathcal{F}_0 and \mathcal{F}_1 having cyclic holonomy, trivial along 1 and τ respectively. If we change for another basis $(m + \tau n, m' + \tau n')$, with

$$\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

then the change of coordinates

$$x' = \frac{x}{m + \tau n}, \quad \xi' = (m + \tau n)\xi - nx \rightsquigarrow z' = e^{2i\pi x'} = z^{\frac{1}{m + \tau n}}$$

gives (S_0, C) as the quotient of $\mathbb{C}_{z'}^* \times \overline{\mathbb{C}}_{\xi'}$ by the transformation

$$(z', \xi') \mapsto (q' z', \xi' + 1), \quad q' = e^{2i\pi\tau'}, \quad \tau' = \frac{m' + \tau n'}{m + \tau n}.$$

¹⁵An example without foliations has been given by Mishustin in [17].

The new isomorphism is related to the previous one by a monomial transformation

$$(X', Y') = (e^{2i\pi\xi'}, z'^{-1}e^{2i\pi\tau'\xi'}) = (X^m Y^n, X^{m'} Y^{n'}).$$

Using sectorial normalization for a general neighborhood (U, C) with this new basis gives a new compactification (V', D) which is bimeromorphically equivalent to (V, D) .

2. PRELIMINARY REMARK

Recall that $C = \mathbb{C}^* / \langle q \rangle$, and we denote by $\tilde{C} \simeq \mathbb{C}^* \rightarrow C$ the corresponding cyclic cover. Denote $\tilde{U} = \mathbb{C}_z^* \times \mathbb{C}_y$ and $\tilde{C} = \{y = 0\} \subset \tilde{U}$. The following is already mentioned by Arnol'd [1].

Lemma 2.1. *Any germ of neighborhood (U, C) with $C^2 = 0$ is biholomorphic to a germ of the form $(\tilde{U}, \tilde{C}) / \langle F \rangle$ where*

$$(2.1) \quad F(z, y) = (qz + yf(z, y), \lambda(z)y + y^2g(z, y))$$

with f, g holomorphic on a neighborhood of $\{y = 0\}$, where $q = e^{2i\pi\tau}$ and $\lambda \in \mathcal{O}^*(\mathbb{C}_z^*)$.

Proof. Because the self-intersection C^2 determines topologically the neighborhood (more exactly a suitable arbitrary small representative), U is homeomorphic to a product $\mathbb{D} \times C$. So, one can consider the cyclic covering $\tilde{U} \rightarrow U$ extending the cyclic cover $\tilde{C} \rightarrow C$. This gives rise to a neighborhood \tilde{U} of $\tilde{C} \simeq \mathbb{C}^*$. Following Siu [24], the germ of this neighborhood along \tilde{C} is isomorphic to the germ of a neighborhood of the zero section $\{y = 0\}$ in the normal bundle $N_{\tilde{C}} \simeq \mathbb{C}_z^* \times \mathbb{C}_y$. The deck transformation of the (germ of) covering takes the form F of the statement. \square

Definition 2.2. *Any two quotients $(\tilde{U}, \tilde{C}) / \langle F \rangle$ and $(\tilde{U}', \tilde{C}') / \langle F' \rangle$ are analytically (resp. formally) equivalent, and we note*

$$(U, C) \stackrel{\text{an}}{\simeq} (U', C) \quad (\text{resp. } (U, C) \stackrel{\text{for}}{\simeq} (U', C)),$$

if there is a germ of analytic (resp. formal) diffeomorphism

$$(2.2) \quad \Psi(z, y) = \left(z + \sum_{n=1}^{\infty} a_n(x)y^n, \sum_{n=1}^{\infty} b_n(x)y^n \right) \quad \text{such that } \Psi \circ F = F' \circ \Psi.$$

Although the formal classification is already in [11], we need the following formulation and give some basic step.

Proposition 2.3. *A germ of neighborhood (U, C) is formally equivalent to*

$$(U_0, C) = (\tilde{U}, \tilde{C}) / \langle F_0 \rangle, \quad F_0(z, y) = \left(qz, \frac{y}{1-y} \right) = (qz, y + y^2 + y^3 + \dots)$$

if, and only if, it is biholomorphic to a germ of the form $(\tilde{U}, \tilde{C}) / \langle F \rangle$ where

$$(2.3) \quad F(z, y) = (qz + y^2f(z, y), y + y^2 + y^3 + y^4g(z, y)).$$

Precisely, there exists a formal diffeomorphism (tangent to the identity on C)

$$(2.4) \quad \hat{\Psi}(z, y) = \left(z + \sum_{n>0} a_n(z)y^n, y + \sum_{n>1} b_n(z)y^n \right)$$

with $a_n, b_n \in \mathcal{O}(\mathbb{C}_z^*)$ (and no convergence condition on y), such that

$$(2.5) \quad \hat{\Psi} \circ F = F_0 \circ \hat{\Psi}.$$

Moreover, any other formal diffeomorphism $\hat{\Psi}'$ of the form (2.4) satisfying (2.5) writes

$$(2.6) \quad \hat{\Psi}' = \Phi \circ \hat{\Psi} \quad \text{where} \quad \Phi(z, y) = \left(z, \frac{y}{1 - ty}\right) = (z, y + ty^2 + \dots), \quad t \in \mathbb{C}.$$

Proof. Let $g = \sum_{n \in \mathbb{Z}} g_n z^n$ be holomorphic on \mathbb{C}_z^* . The functional equation

$$(2.7) \quad \phi(qz) - \phi(z) = g(z)$$

admits a solution ϕ holomorphic on \mathbb{C}_z^* if, and only if, $g_0 = 0$; then ϕ is unique up to the choice of $\phi(0)$. Indeed, if we write $\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^n$, then equation (2.8) writes $\phi_n(q^n - 1) = g_n$ for all n .

Let f be a holomorphic non vanishing function on \mathbb{C}_z^* . The functional equation

$$(2.8) \quad \varphi(qz)/\varphi(z) = f(z)$$

admits a solution φ holomorphic and non vanishing on \mathbb{C}_z^* if, and only if,

- $f : \mathbb{C}^* \rightarrow \mathbb{C}^*$ has topological index 0 so that $g = \log(f)$ is well-defined,
- the coefficient g_0 of $g = \sum_{n \in \mathbb{Z}} g_n z^n$ vanishes.

Indeed, topological index is multiplicative and those of $\varphi(qz)$ and $\varphi(z)$ are equal and cancel each other. Then we can solve the corresponding equation (2.8) for g and set $\varphi = \exp(\phi)$, which is unique up to a multiplicative constant. Note that, if $g_0 \neq 0$, then we can solve

$$(2.9) \quad \varphi(qz)/\varphi(z) = \frac{f(z)}{a}$$

for $a = \exp(g_0)$.

Let us start with F like in (2.1). The change of coordinate $\Psi_1(z, y) = (z, f(z)y)$ yields

$$\Psi_1^{-1} \circ F \circ \Psi_1(z, y) = (qz + O(y), \frac{\varphi(z)}{\varphi(qz)} f(z)y + O(y^2)).$$

We can easily check that the coefficient f in F defines the normal bundle N_C in the quotient, and its topological index coincides with $\deg(N_C)$ with is zero in our case. Then we can find $\varphi \in \mathcal{O}^*(\mathbb{C}_z^*)$ satisfying (2.9) and get

$$F_1(z, y) = \Psi_1^{-1} \circ F \circ \Psi_1(z, y) = (qz + O(y), ay + O(y^2)).$$

Moreover, φ is unique up to a multiplicative constant. The coefficient a can be interpreted as a flat connection on N_C with trivial monodromy along the loop $1 \in \Gamma$ and monodromy a along the loop $\tau \in \Gamma$. In our case, $N_C = \mathcal{O}_C$ and $a = 1$ and we can write

$$F_1(z, y) = (qz + O(y), y + g(z)y^2 + O(y^3)).$$

Now the change of coordinate $\Psi_2(z, y) = (z, y + \phi(z)y^2)$ gives

$$\Psi_2^{-1} \circ F_1 \circ \Psi_2(z, y) = (qz + O(y), y + [g(z) + \phi(z) - \phi(qz)]y^2 + O(y^3)).$$

Solving equation (2.8), we get

$$F_2(z, y) = \Psi_2^{-1} \circ F_1 \circ \Psi_2(z, y) = (qz + O(y), y + by^2 + O(y^3)).$$

In our case, $b \neq 0$ (i.e. Ueda type $\kappa = 1$). By using a change $(z, \lambda y)$ (freedom in the choice of φ above) we can set $b = 1$ and write

$$F_2(z, y) = (qz + zf(z)y + O(y^2), y + y^2 + g(z)y^3 + O(y^4)).$$

The change of coordinate $\Psi_3(z, y) = (z + \varphi(z)y, y + \phi(z)y^3)$ gives

$$\begin{aligned} F_3(z, y) &= \Psi_3^{-1} \circ F_2 \circ \Psi_3(z, y) = \\ &(qz + z[f(z) + \varphi(z) - \varphi(qz)]y + O(y^2), y + y^2 + [g(z) + \phi(z) - \phi(qz)]y^3 + O(y^4)). \end{aligned}$$

Solving twice equation (2.8), we get

$$F_3(z, y) = (qz + \alpha zy + O(y^2), y + y^2 + \beta y^3 + O(y^4)).$$

Here, we have no freedom and α, β are formal invariants corresponding to μ, ν in the end of section 1.4: in the formal class $U_{1,0,0}$ we get $b = 0$ and $c = 1$. Then, we can kill-out all higher order terms in F by a formal change of coordinate, or better normalize it to $F_{1,0,0}$. Indeed, at the N step, we get

$F_N(z, y) = (qz + zf(z)y^{N-1} + O(y^N), y + y^2 + y^3 + \dots + g(z)y^{N+1} + O(y^{N+2}));$
the coordinate change $\Psi_{N+1}(z, y) = (z + az y^{N-2} + \varphi(z)y^{N-1}, y + by^N + \phi(z)y^{N+1})$ gives

$$\begin{aligned} F_{N+1}(z, y) &= \Psi_{N+1}^{-1} \circ F_N \circ \Psi_{N+1}(z, y) = \\ &= (qz + z[f(z) + \varphi(z) - \varphi(qz) - (N-2)aq]y^{N-1} + O(y^N), \\ &= y + y^2 + y^3 + \dots + [g(z) + \phi(z) - \phi(qz) - (N-4)b]y^{N+1} + O(y^{N+2})). \end{aligned}$$

We can clearly normalize the two coefficients into brackets by a constant, and can even choose the constant by means of a, b .

The composition of all changes of coordinates $\hat{\Psi}^{-1} := \Psi_1 \circ \Psi_2 \circ \Psi_3 \circ \dots$ converges in the formal topology as a formal diffeomorphism satisfying (2.5). For any other formal diffeomorphism $\hat{\Psi}'$ of the form (2.4) satisfying (2.5), we have that $\hat{\Phi} := \hat{\Psi}' \circ \hat{\Psi}^{-1}$ is an automorphism of (U_0, C) inducing the identity on C . As we shall see in Lemma 3.9, $\hat{\Phi}$ is necessarily convergent and of the form (2.6). \square

3. SECTORIAL DECOMPOSITION AND SECTORIAL SYMMETRIES

In this section, we introduce the sectorial decomposition of U by transversely sectorial domains $U_i = \Pi^{-1}(V_i^*)$ and compare spaces of functions on both sides. From now on, we work in the variable $\xi = -1/y$, at the neighborhood of $\xi = \infty$; this is much more convenient for computations. Notations are as in section 1.3.

3.1. Some sheaves of functions on the circle of directions. Let $\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ and I be an open interval of \mathbb{R} (regarded as the universal covering of \mathbb{S}^1).

Definition 3.1. For $c, R > 0$ denote

$$S(I, R; c) = \{(z, \xi) \in \mathbb{C}_z^* \times \mathbb{C}_\xi^* ; \arg(\xi) \subset I, R < |\xi|, e^{-c} < |z| < e^c\}.$$

A sector of aperture I is an open subset $\Sigma_I \subset S(I, 0; \infty)$ such that for all $c \gg 0$, there is a $R_c > 0$ such that

$$S(I, R_c; c) \subset \Sigma_I.$$

Let Σ_I be an open sector as above. Then, $\mathcal{O}(\Sigma_I)$ contains the subalgebra $\mathcal{A}(\Sigma_I)$ of holomorphic functions admitting an asymptotic expansion along \mathbb{C}_z^* :

Definition 3.2. A function $f \in \mathcal{O}(\Sigma_I)$ belongs to $\mathcal{A}(\Sigma_I)$ if there exists a formal power series $\hat{f} = \sum_{k \geq 0} a_k \xi^{-k} \in \mathcal{O}(\mathbb{C}_z^*)[[\xi]]$ such that $\forall c \gg 0$, and $\forall n, \exists C_{c,n}, R_{c,n} > 0$ such that $\forall (z, \xi) \in S(I, R_{c,n}; c) \subset \Sigma_I$, we have

$$(3.1) \quad \left| f(z, \xi) - \sum_{k=0}^n a_k(z) \xi^{-k} \right| \leq \frac{C_{c,n}}{|\xi^{n+1}|}.$$

The asymptotic expansion is unique, and we have a well-defined morphism of \mathbb{C} -algebra

$$\mathcal{A}(\Sigma_I) \rightarrow \mathcal{O}(\mathbb{C}_z^*)[[\xi]] ; f \mapsto \hat{f},$$

whose kernel, denoted $\mathcal{A}^\infty(\Sigma_I)$, consists of flat functions.

When fixing only I and taking inductive limits associated to restriction maps, the collection of algebras of the form $\mathcal{O}(\Sigma_I)$ define an algebra of germs \mathcal{O}_I . The presheaf on \mathbb{S}^1 defined by $I \rightarrow \mathcal{O}_I$ naturally gives rise to a sheaf on \mathbb{S}^1 which we will denote by \mathcal{O} . One can define on the same way the sheaves \mathcal{A} , \mathcal{A}^∞ associated to $I \rightarrow \mathcal{A}_I$, $I \rightarrow \mathcal{A}_I^\infty$ and they are sheaves of differential algebra with respect to ∂_z and ∂_ξ . The stability by derivation is indeed a straightforward consequence of Cauchy's formula. As the asymptotic expansion is independant of the representative, we have a morphism of sheaves

$$\mathcal{A} \rightarrow \mathcal{O}(\mathbb{C}_z^*)[[\xi]] ; f \mapsto \hat{f}$$

whose kernel is \mathcal{A}_I^∞ (here $\mathcal{O}(\mathbb{C}_z^*)[[\xi]]$ is viewed as a constant sheaf over \mathbb{S}^1).

Remark 3.3. *Mind that the inclusion $\mathcal{O}_I \rightarrow \mathcal{O}(I)$ (resp. $\mathcal{A}_I \rightarrow \mathcal{A}(I)$) is strict. For instance, one must think that a section $f \in \mathcal{A}(I)$ can be represented for every interval $J \Subset I$ by a function belonging to $\mathcal{A}(\Sigma_J)$ for suitable sectors of aperture J but does not necessarily admit a representative on a sector of the form Σ_I . In other words, the domain of definition of f is a transversely sectorial open set in the following sense.*

Definition 3.4. *Given an interval $I = [\theta_1, \theta_2] \subset \mathbb{R}$, an open subset $\Sigma \subset \mathbb{C}_z^* \times \mathbb{C}_y$ is said transversely sectorial of opening I if, for arbitrary large $c \gg 0$ and small $\epsilon > 0$, there is a $R_{c,\epsilon}$ such that*

$$\mathcal{S}(I_\epsilon, R_{c,\epsilon}; c) \subset \Sigma, \quad \text{where } I_\epsilon =]\theta_1 + \epsilon, \theta_2 - \epsilon[.$$

Remark 3.5. *The sheaves \mathcal{O} , \mathcal{A} and \mathcal{A}^∞ are invariant under the action of a diffeomorphism F of the form (2.3). Moreover, this action is stalk-preserving due to the fact that F is tangent to the identity along \tilde{C} on the transversal direction y . In particular, they define similar sheaves of sectorial functions on the quotient $(U, C) = (\tilde{U}, \tilde{C}) / \langle F \rangle$ by considering those sections invariant under F . We will denote by $\mathcal{O}[F]$, $\mathcal{A}[F]$ and $\mathcal{A}^\infty[F]$ these latter sheaves. In the next section, we characterize sections of $\mathcal{A}^\infty[F_0](I)$ for special intervals I .*

3.2. Sectorial decomposition. Denote $\varpi = \arg(\tau) \in]0, \pi[$ and let us define¹⁶

$$I_1 =]-\varpi, \pi - \varpi[, \quad I_2 =]-\pi, 0[, \quad I_3 = I_1 + \pi \quad \text{and} \quad I_4 = I_2 + \pi.$$

Denote by V_i a (small enough) neighborhood of $L_i \subset \mathbb{P}_X^1 \times \mathbb{P}_Y^1$ where

$$L_1 : \{Y = 0\}, \quad L_2 : \{X = \infty\}, \quad L_3 : \{Y = \infty\} \quad \text{and} \quad L_4 : \{X = 0\}.$$

Denote $D = L_1 \cup L_2 \cup L_3 \cup L_4$, and $V_i^* = V_i \setminus (V_i \cap D)$. Let $V_{i,i+1} = V_i \cap V_{i+1}$ for $i \in \mathbb{Z}_4$ and $V_{i,i+1}^* = V_{i,i+1} \setminus (V_{i,i+1} \cap D)$. Recall that

$$\Pi : S_0 \setminus C \xrightarrow{\sim} \mathbb{C}_X^* \times \mathbb{C}_Y^* ; (z, \xi) \mapsto (e^{2i\pi\xi}, z^{-1}e^{2i\pi\tau\xi}).$$

Then we have:

Proposition 3.6. *The preimage $U_i = \Pi^{-1}(V_i^*)$ lifts on $\tilde{U} = \mathbb{C}_z^* \times \mathbb{C}_\xi$ as a transversely sectorial open set of aperture I_i (in the sense of Definition 3.4). Moreover, the lift of $\Pi^{-1}(V_{i,i+1}^*) = U_i \cap U_{i+1}$ is transversely sectorial of aperture $I_i \cap I_{i+1}$.*

Proof. For instance, for $a, b, c > 0$, we easily check that

$$U_{4,1} = \{(X, Y) \in \mathbb{C}^* \times \mathbb{C}^* ; |X| < \exp(-a), |Y| < \exp(-b)\}$$

¹⁶Mind that these intervals for $\arg(\xi)$ correspond to those defined in Lemma A for $\arg(y) = \pi - \arg(\xi)$.

contains the sectorial open set

$$\left\{ (z, \xi) \in \mathbb{C}^* \times \mathbb{C} ; e^{-c} < |z| < e^c, \operatorname{Im}(\xi) > \frac{a}{2\pi}, \operatorname{Im}(\tau\xi) > \frac{b+c}{2\pi} \right\}.$$

The remaining cases are similar and straightforward. \square

Denote $p_{i,i+1} = L_i \cap L_{i+1}$. Denote by $\mathcal{O}^0(V_i, L_i)$ (resp. $\mathcal{O}^0(V_{i,i+1}, p_{i,i+1})$) the set of germs of holomorphic functions on (V_i, L_i) (resp. $(V_{i,i+1}, p_{i,i+1})$) vanishing along L_i (resp. at $p_{i,i+1}$). Denote by $\mathcal{A}^\infty[F_0]$ the subsheaf of \mathcal{A}^∞ whose sections f are invariant by $F_0(z, \xi) = (qz, \xi + 1)$.

Proposition 3.7. *A section $f \in \mathcal{O}(I_i)$ (resp. $\mathcal{O}(I_{i,i+1})$) belongs to $\mathcal{A}^\infty[F_0](I_i)$ (resp. $\mathcal{A}^\infty[F_0](I_{i,i+1})$) if, and only if, $f = g \circ \Pi$ with $g \in \mathcal{O}^0(V_i, L_i)$ (resp. $\mathcal{O}^0(V_{i,i+1}, p_{i,i+1})$).*

Proof. As before, we only give the proof for $I_{4,1}$, the other cases are similar. If $f = g \circ \Pi$ with $g \in \mathcal{O}^0(V_i, p_{i,i+1})$, then $g(X, Y) = Xg_1(X, Y) + Yg_2(X, Y)$ with g_k holomorphic at $p_{i,i+1}$ (and therefore bounded), so that $f(z, \xi) = e^{2i\pi\xi}f_1(z, \xi) + e^{2i\pi\tau\xi}f_2(z, \xi)$ with f_k bounded: clearly, f is exponentially flat at $\xi = 0$ in restriction to any sector $S(J, R; c) \subset U_{4,1}$, $J \Subset I_{4,1}$.

Conversely, let $f \in \mathcal{A}^\infty[F_0](I_{4,1})$, defined on a sectorial open set $U_{4,1}$ of aperture $I_{4,1}$. Let $U_{4,1}$ be the domain of definition of f , a transversely sectorial open set of aperture $I_{4,1}$ (see definition 3.4). One can find another one $U'_{4,1} \subset U_{4,1}$ such that

$$\forall (z_0, \xi_0) \in U'_{4,1}, \quad \forall (s, t) \in [0, 1] \times [0, 1], \quad \Rightarrow (z, \xi) = (\xi_0 + s, e^{2i\pi(\tau s - t)}z_0) \in U_{4,1}.$$

If we denote $(X_0, Y_0) = \Pi(z_0, \xi_0)$, then the image of (z, ξ) while (s, t) run over the square is

$$(X, Y) = \Pi(\xi, \xi) = (e^{2i\pi s}X_0, e^{2i\pi t}Y_0)$$

a product of two loops. Therefore, the image $\Pi(U_{4,1})$ contains an open set W' which is saturated by the toric action of $\mathbb{S}^1 \times \mathbb{S}^1$ on $\mathbb{C}_X^* \times \mathbb{C}_Y^*$, i.e. a Reinhardt domain (see [22, Chap.1,sec.2]), and which contains $U'_{4,1}$ (just take W' to be the image of all those (z, ξ) like above when (z_0, ξ_0) runs over $U'_{4,1}$). Since f is invariant under F_0 , $f \circ F_0 = f$, then it factors through Π and, maybe passing to another representative, we have $f = g \circ \Pi$ where $g \in \mathcal{O}(W')$. Mind that W' (as well as $\Pi(U_{4,1})$) might not be of the form $W \setminus (W \cap D)$ for a neighborhood W of $p_{4,1}$, but we will prove that the holomorphic hull of g is such a neighborhood.

As W' is a Reinhardt domain, let us consider the (convergent) Laurent series of g :

$$g(X, Y) = \sum_{m,n \in \mathbb{Z}} a_{m,n} X^m Y^n.$$

The coefficients are given by the integral

$$a_{n,m} = \frac{1}{2i\pi} \int_{\beta_{\xi_0}} \left(\frac{1}{2i\pi} \int_{\alpha_{\xi_0}} g(X, Y) X^{-n-1} Y^{-m-1} dX dY \right)$$

where $\alpha_{\xi_0}(s) = (e^{2i\pi s}X_0, Y_0)$ and $\beta_{\xi_0}(t) = (X_0, e^{2i\pi t}Y_0)$. This can be rewritten as

$$a_{n,m} = \int_{t=0}^1 \left(\int_{s=0}^1 g(X, Y) X_0^{-m} Y_0^{-n} e^{-2i\pi(ms+nt)} dt \right) ds$$

from which we deduce the estimate

$$|a_{n,m}| \leq \int_{t=0}^1 \left(\int_{s=0}^1 |g(X, Y) X_0^{-m} Y_0^{-n}| dt \right) ds$$

$$|a_{n,m}| \leq \|g(X, Y)\|_{W'} |z_0|^n |e^{-2i\pi(m+\tau n)\xi_0}|$$

$$|a_{n,m}| \leq \|f\|_{U_{4,1}} |z_0|^n e^{2\pi\Im\{(m+\tau n)\xi_0\}}.$$

Now, given $m, n \in \mathbb{Z}$, assume that there exists $\theta \in I_{4,1}$ such that $\Im(e^{i\theta}(m + \tau n)) > 0$. The above inequality promptly implies that $a_{n,m} = 0$ by fixing z_0 and making $\xi_0 \rightarrow \infty$ in the direction θ (which is possible in $U'_{4,1}$ as its aperture is $I_{4,1}$). This is possible if, and only if

$$\arg(m + \tau n) + I_{4,1} \text{ intersects }] - \pi, 0[\pmod{2\pi}$$

which, since $I_{4,1} = [0, \pi - \varpi]$, means that

$$\arg(m + \tau n) \in] - \pi, 0[-]0, \pi - \varpi[=] - \pi, 0[+]\varpi - \pi, 0[=]\varpi - 2\pi, 0[.$$

It promptly follows that the only non zero coefficients $a_{m,n}$ occur where

$$\arg(m + \tau n) \in [0, \varpi]$$

which means that $m, n \geq 0$, and g extends holomorphically at $p_{4,1} : X = Y = 0$. Finally, as $f \rightarrow 0$ as $\xi \rightarrow \infty$, we get that $a_{0,0} = 0$ and $g(0, 0) = 0$. \square

Remark 3.8. *The second part of the proof does not use the fact that f is flat (i.e. admits asymptotic expansion zero) along C , but only the fact that it is bounded. As a consequence, any bounded holomorphic function on a transversely sectorial open set U_i or $U_{i,i+1}$ as above automatically admits a constant as asymptotic expansion along C . We note that bounded functions on U_1, U_3 (resp. U_2, U_4) therefore correspond to first integrals of the foliation \mathcal{F}_0 (resp. \mathcal{F}_1).*

3.3. Sheaves of sectorial automorphisms. Denote by $\text{Aut}(S_0)$ the automorphism group of the ruled surface S_0 . It preserves the ruling as well as the section $C \subset S_0$, inducing an action on the neighborhood of C . The subgroup $\text{Aut}^0(S_0, C)$ of elements fixing C point-wise is the one-parameter group generated by the flow of the vector field¹⁷

$$\partial_\xi = 2i\pi(X\partial_X + \tau Y\partial_Y).$$

We have an exact sequence

$$(3.2) \quad 1 \longrightarrow \text{Aut}^0(S_0, C) \longrightarrow \text{Aut}(S_0) \longrightarrow \text{Aut}(C) \longrightarrow 1.$$

The full group $\text{Aut}(S_0)$ is generated by the flows of

$$(3.3) \quad \partial_\xi + 2i\pi\tau z\partial_z = 2i\pi X\partial_X \quad \text{and} \quad -2i\pi z\partial_z = 2i\pi Y\partial_Y$$

and a finite order map which, for a general curve C , is just an involution that can be chosen taking the form $(z, \xi) \rightarrow (\frac{1}{z}, -\xi)$. In fact, specializing $\text{Aut}(S_0)$ to the neighborhood of the curve, we get all analytic, and even formal automorphisms of the neighborhood (S_0, C) :

Lemma 3.9. *Any formal automorphism $\hat{\Phi} : \mathcal{C}(S_0, C)$ fixing C point-wise is actually convergent and belongs to $\text{Aut}^0(S_0, C)$.*

Proof. Recall [11] that the only formal regular foliations on (S_0, C) are those defined by $\omega = 0$ where ω belongs to the vector space of closed 1-forms $E = \mathbb{C}\frac{dz}{z} + \mathbb{C}d\xi$. Moreover, for $\omega \in E \setminus \mathbb{C}\frac{dz}{z}$, \mathcal{F}_ω does not admit non constant formal meromorphic first integral, and

¹⁷See notations of section 1.3.

the only formal closed meromorphic 1-forms defining \mathcal{F}_ω must be a constant multiple of ω , thus belonging to E . If

$$\hat{\Phi}(z, \xi) = \left(z + \sum_{n>0} \frac{a_n(z)}{\xi^n}, \sum_{n \geq 0} \frac{b_n(z)}{\xi^n} \right)$$

is a formal automorphism of (S_0, C) fixing C point-wise, then it must preserves the vector space E . In particular, it must preserves $\mathbb{C} \frac{dz}{z}$ (and z actually as it fixes C point-wise) and send $d\xi$ to some other element $\alpha \frac{dz}{z} + \beta d\xi$. A straightforward computation shows that $\hat{\Phi}$ writes

$$\hat{\Phi}(x, \xi) = (z, \alpha \log(z) + \beta \xi + \gamma), \quad \gamma \in \mathbb{C},$$

and we have $\alpha = 0$. Finally, as $\hat{\Phi}$ must commute with $F_0(z, \xi) = (qz, \xi + 1)$ and we get $\beta = 1$. \square

Corollary 3.10. *Any formal automorphism $\hat{\Phi} : \zeta(S_0, C)$ is actually convergent and belongs to $\text{Aut}(S_0)$.*

Proof. The formal diffeomorphism $\hat{\Phi}$ induces an automorphism of C . Using exact sequence (3.2), after composing $\hat{\Phi}$ by a convenient element of $\text{Aut}(S_0)$, we can assume that it fixes C point-wise, and then apply Lemma 3.9. \square

Let us consider the germs of sectorial biholomorphisms in the direction $\arg(\xi) = \theta$ of (\tilde{U}, \tilde{C}) that are tangent to the identity:

$$\Phi(z, \xi) = (z + \xi^2 f_1(z, \xi), \xi + \xi^2 f_2(z, \xi)), \quad f_1, f_2 \in \mathcal{A}_\theta.$$

The collection of these germs when varying θ naturally gives rise to a sheaf of groups (with respect to the composition law) on \mathbb{S}^1 that will be denoted by \mathcal{G}^1 . We will consider for further use the subsheaf \mathcal{G}^∞ of \mathcal{G}^1 of germs of sectorial biholomorphisms flat to identity, i.e. when $f_1, f_2 \in \mathcal{A}_\theta^\infty$. Denote by $\mathcal{G}^1[F_0]$ (resp. $\mathcal{G}^\infty[F_0]$) the subsheaf of \mathcal{G}^1 (resp. \mathcal{G}^∞) defined by germs of transformations Φ commuting with F_0 : $\Phi \circ F_0 = F_0 \circ \Phi$.

Remark 3.11. *Note that $\Phi \in \mathcal{G}^1[F_0]$ implies that its asymptotic expansion $\hat{\Phi}$ also commutes with F_0 , i.e. $\hat{\Phi} \circ F_0 = F_0 \circ \hat{\Phi}$. According to the description of the formal centralizer of F_0 in Lemma 3.9, it turns out that $\mathcal{G}^1[F_0] = \mathcal{G}^\infty[F_0] \rtimes \text{Aut}^0(S_0, C)$ where $\text{Aut}^0(S_0, C)$ is regarded as a constant sheaf on \mathbb{S}^1 .*

We would like to apply characterization of $\mathcal{A}^\infty[F_0](I)$ obtained in the previous section for our special sectors I_i and $I_{i,i+1}$ to obtain a similar characterization of sections of $\mathcal{G}^\infty[F_0]$. For this, denote by $\text{Diff}(V_i^*, L_i)$ the group of germs of diffeomorphisms of (V_i, L_i) which preserves the germ of divisor $(D \cap V_i, L_i)$, for instance:

$$\text{Diff}(V_1^*, L_1) = \{\varphi(X, Y) = (Xa(Y), Yb(Y)); a, b \in \mathbb{C}\{Y\}, a(0), b(0) \neq 0\},$$

and by $\text{Diff}^1(V_i^*, L_i)$ the subgroup of germs tangent to the identity along V_i , i.e. $a(0) = b(0) = 1$ in the example. In a similar way, denote by $\text{Diff}(V_{i,i+1}^*, p_{i,i+1})$ the group of germs of diffeomorphisms of $(V_{i,i+1}, p_{i,i+1})$ which preserves the germ of divisor $(D \cap V_{i,i+1}, p_{i,i+1})$ and by $\text{Diff}^1(V_{i,i+1}^*, p_{i,i+1})$ the subgroup of germs tangent to the identity at $p_{i,i+1}$. For instance:

$$\text{Diff}(V_{4,1}^*, p_{4,1}) = \{\varphi(X, Y) = (Xa(X, Y), Yb(X, Y)); a, b \in \mathbb{C}\{X, Y\}, a(0), b(0) \neq 0\},$$

and $\text{Diff}^1(V_{4,1}^*, p_{4,1})$ is characterized by $a(0) = b(0) = 1$.

Proposition 3.12. *We have the following characterizations:*

- $\Phi \in \mathcal{G}^\infty[F_0](I_i)$ if and only if $\Pi \circ \Phi = \varphi \circ \Pi$ where $\varphi \in \text{Diff}^1(V_i^*, L_i)$;
- $\Phi \in \mathcal{G}^\infty[F_0](I_{i,i+1})$ if and only if $\Pi \circ \Phi = \varphi \circ \Pi$ where $\varphi \in \text{Diff}^1(V_{i,i+1}^*, p_{i,i+1})$.

Proof. For any interval I , a section Φ of $\mathcal{G}^\infty(I)$ can be written $\Phi(z, \xi) = (z(1+f_1), \xi+f_2)$ with $f_1, f_2 \in \mathcal{A}^\infty(I)$. Then Φ belongs to $\mathcal{G}^\infty[F_0](I)$ if, and only if, f_1, f_2 are invariant by F_0 , i.e. $f_1, f_2 \in \mathcal{A}^\infty[F_0](I)$. Assume now $I = I_{4,1}$, say. Then, by Proposition 3.7, one can write $f_k = g_k \circ \Pi$, i.e. $f_k(z, \xi) = g_k(X, Y)$, with $g_k \in \mathcal{O}^0(V_{4,1}, p_{4,1})$. Therefore, one can write

$$\Pi \circ \Phi = (Xa(X, Y), Yb(X, Y)) \quad \text{with} \quad \begin{cases} a = e^{2i\pi g_2(X, Y)}, \\ b = e^{2i\pi \tau g_2(X, Y)}(1 + g_1(X, Y))^{-1}. \end{cases}$$

Clearly, a, b are holomorphic at $(X, Y) = (0, 0)$ and $a(0, 0) = b(0, 0) = 1$. Conversely, given $\varphi \in \text{Diff}^1(V_{4,1}^*, p_{4,1})$, thus of the form $\varphi(X, Y) = (Xa(X, Y), Yb(X, Y))$, we recover $f_1, f_2 \in \mathcal{A}^\infty[F_0](I_{4,1})$, and $\Phi(z, \xi) = (z(1+f_1), \xi+f_2)$, by setting

$$f_1 = \left(\frac{a^\tau}{b} - 1 \right) \circ \Pi \quad \text{and} \quad f_2 = \frac{\log(a)}{2i\pi} \circ \Pi.$$

The description of elements of $\mathcal{G}^\infty[F_0](I_i)$, $\mathcal{G}^\infty[F_0](I_{i,i+1})$ can be carried out exactly along the same line. \square

4. ANALYTIC CLASSIFICATION: AN OVERVIEW

Here, we would like to detail our main result, namely the analytic classification of all neighborhoods that are formally equivalent to $(U_{1,0,0}, C)$. The most technical ingredient is the sectorial normalization (Lemma A in the introduction) which now reads as follows. Let F be a biholomorphism like in Proposition 2.3

$$F(z, \xi) = \left(qz + \sum_{n \geq 2} \frac{\alpha_n(z)}{\xi^n}, \xi + 1 + \sum_{n \geq 2} \frac{\beta_n(z)}{\xi^n} \right).$$

In particular, there is a formal diffeomorphism $\hat{\Phi}$ conjugating F to $F_0(z, \xi) = (qz, \xi + 1)$, i.e. $F \circ \hat{\Psi} = \hat{\Psi} \circ F_0$.

Lemma 4.1. *Denote $\varpi = \arg \tau$. For each interval*

$$(4.1) \quad I_1 =] - \varpi, \pi - \varpi[, \quad I_2 =] - \pi, 0[, \quad I_3 = I_1 + \pi \quad \text{and} \quad I_4 = I_2 + \pi,$$

there is a section Ψ_i of $\mathcal{G}^1(I_i)$ (see section 3.3) such that

$$\hat{\Psi}_i \circ F = F_0 \circ \Psi_i.$$

Section 9 is devoted to the proof of this lemma. Let us see how to use it in order to provide a complete set of invariants for the neighborhood $(U, C) = (\tilde{U}, \tilde{C}) / \langle F \rangle$. First of all, we note that Ψ_i is unique up to left-composition by a section of $\mathcal{G}^1[F_0](I_i)$, i.e. the composition of an element of the one-parameter group $\text{Aut}^0(S_0, C)$ with a section of $\mathcal{G}^\infty[F_0](I_i)$ (see Remark 3.11). Using this freedom, we may assume that asymptotic expansions coincide:

$$\hat{\Psi}_i = \hat{\Psi}_j.$$

It follows that, on intersections $I_{i,i+1} = I_i \cap I_{i+1}$, we get sections

$$\Phi_{i,i+1} := \Psi_i \circ \Psi_{i+1}^{-1} \in \mathcal{G}^\infty[F_0](I_{i,i+1}).$$

Using Proposition 3.12, we have

$$\Pi \circ \Phi_{i,i+1} = \varphi_{i,i+1} \circ \Pi \quad \text{for some} \quad \varphi_{i,i+1} \in \text{Diff}^1(V_{i,i+1}^*, p_{i,i+1}).$$

In other words, setting $\Pi_i := \Pi \circ \Psi_i$, we get

$$(4.2) \quad \Pi_i = \Pi \circ \Psi_i = \Pi \circ \Phi_{i,i+1} \circ \Psi_{i+1} = \varphi_{i,i+1} \circ \Pi \circ \Psi_{i+1} = \varphi_{i,i+1} \circ \Pi_{i+1}$$

which proves Corollary B. We have therefore associated to each neighborhood (U, C) formally equivalent to $(U_{1,0,0}, C)$ a cocycle $\varphi = (\varphi_{i,i+1})_{i \in \mathbb{Z}_4}$ which is unique up to the freedom for the choice of Ψ_i 's.

Definition 4.2. We say that two cocycles φ and φ' are equivalent if

$$\varphi \approx \varphi' \Leftrightarrow \exists t \in \mathbb{C}, \exists \varphi_i \in \text{Diff}^1(V_i^*, L_i)$$

$$(4.3) \quad \text{such that } \varphi'_{i,i+1} = \phi^t \circ \varphi_i \circ \varphi_{i,i+1} \circ \varphi_{i+1}^{-1} \circ \phi^{-t}$$

where $\phi^t = (e^{2i\pi t}X, e^{2i\pi t}Y)$ is the one-parameter group of the vector field $v_\tau = 2i\pi(X\partial_X + \tau Y\partial_Y)$.

Proposition 4.3. Two neighborhood (U, C) and (U', C) formally equivalent to $(U_{1,0,0}, C)$ are analytically equivalent if, and only if, the corresponding cocycles are equivalent

$$(U, C) \overset{\text{an}}{\sim} (U', C) \Leftrightarrow \varphi \approx \varphi'.$$

Proof. Any biholomorphism germ $(U, C) \rightarrow (U', C)$ lifts-up to a global section $\Psi \in \mathcal{G}^1(\mathbb{S}^1)$ satisfying $\Psi \circ F = F' \circ \Psi$. Let (Ψ_i) and (Ψ'_i) be the sectorial normalizations used to compute the invariants φ and φ' . Clearly, $\Psi'_i \circ \Psi$ provides a new collection of sectorial trivializations for (U, C) . We can write (using Remark 3.11)

$$\Psi'_i \circ \Psi = \exp(t_i \partial_\xi) \circ \Phi_i \circ \Psi_i \quad \text{with } \Phi_i \in \mathcal{G}^\infty[F_0](I_i).$$

However, as $\hat{\Psi}_i = \hat{\Psi}_j$ and $\hat{\Psi}'_i = \hat{\Psi}'_j$, we have $t_i = t_j =: t$ for all i, j . Therefore, we have

$$\begin{aligned} \Phi'_{i,i+1} &= (\Psi'_i \circ \Psi) \circ (\Psi'_{i+1} \circ \Psi)^{-1} \\ &= (\exp(t\partial_\xi) \circ \Phi_i \circ \Psi_i) \circ (\exp(t\partial_\xi) \circ \Phi_{i+1} \circ \Psi_{i+1})^{-1} \\ &= \exp(t\partial_\xi) \circ \Phi_i \circ \Phi_{i+1} \circ \Phi_{i+1}^{-1} \circ \exp(-t\partial_\xi). \end{aligned}$$

After factorization through Π , using (3.3) and Proposition 3.12, we get the expected equivalence relation (4.3) for φ and φ' . Conversely, if $\varphi \overset{\text{an}}{\sim} \varphi'$, then we can trace back the existence of an analytic conjugacy $\Phi : (U, C) \rightarrow (U', C)$ by reversing the above implications. \square

Remark 4.4. We can weaken the notion of analytic equivalence between neighborhoods by considering biholomorphism germs $\Phi : (U, C) \rightarrow (U', C)$ inducing translations on C . This means that, in Definition 2.2, we now allow conjugacies $\Phi(z, y) = (cz + O(y), O(y))$ with $c \in \mathbb{C}^*$ in formula (2.2), i.e. translations on the elliptic curve. In that case, the corresponding cocycles are related by

$$\varphi'_{i,i+1} = \phi \circ \varphi_i \circ \varphi_{i,i+1} \circ \varphi_{i+1}^{-1} \circ \phi^{-1}$$

where $\phi(X, Y) = (aX, bY)$ for arbitrary $a, b \in \mathbb{C}^*$.

To resume, we have just associated to each $(U, C) \overset{\text{for}}{\sim} (U_{1,0,0}, C)$ a cocycle

$$(4.4) \quad \varphi = (\varphi_{i,i+1})_{i \in \mathbb{Z}_4}, \quad \varphi_{i,i+1} \in \text{Diff}^1(V_{i,i+1}^*, p_{i,i+1})$$

and constructed a map from the moduli space $\mathcal{U}_{1,0,0}$ of such neighborhood up to analytic equivalence to the moduli space \mathcal{C} of cocycles φ like (4.4) up to equivalence (4.3):

$$(4.5) \quad \mu : \mathcal{U}_{1,0,0} = \{(U, C) \overset{\text{for}}{\sim} (U_{1,0,0}, C)\} / \overset{\text{an}}{\sim} \longrightarrow \mathcal{C} = \{\varphi\} / \approx$$

which is proved to be injective in Proposition 4.3. In section 5, we prove the surjectivity by constructing an inverse map $\varphi \mapsto U_\varphi$. Before that, we want to reinterpret the cocycle φ as transition maps of an atlas for a neighborhood (V_φ, D) .

5. CONSTRUCTION OF U_φ .

In this section, we construct a large class of non analytically equivalent neighborhoods, all of them formally equivalent to $(U_{1,0,0}, C)$. This is done by sectorial surgery, extending the complex structure along C by means of Newlander-Nirenberg Theorem. In order to do this, we have to work with smooth functions (i.e. of class C^∞).

Definition 5.1. *For any open sector Σ_I , we denote by $\mathcal{E}^\infty(\Sigma_I)$ the \mathbb{C} -algebra of those complex smooth functions $f : \Sigma_I \rightarrow \mathbb{C}$ satisfying the following estimates*

$$\forall \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^4, \forall n \in \mathbb{N}, \forall K \subset \mathbb{C}^* \text{ compact}, \exists C > 0 \text{ such that:}$$

$$\forall (\xi, z) \in \Sigma_I, z \in K, \text{ we have } \left| \frac{\partial^{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} f(z, \xi)}{\partial z^{\alpha_1} \partial \xi^{\alpha_2} \partial \bar{z}^{\alpha_3} \partial \bar{\xi}^{\alpha_4}} \right| \leq \frac{C}{|\xi|^{n+1}}.$$

Passing to inductive limits and sheafification as in section 3.1, we get a sheaf \mathcal{E}^∞ of differential algebra on the circle \mathbb{S}^1 . Like in section 3.3, we can also define the sheaf of groups \mathcal{D}^∞ on the circle, whose sections $\Psi \in \mathcal{D}^\infty(I)$ are smooth sectorial diffeomorphisms asymptotic to the identity, i.e. of the form $\Psi(z, \xi) = (z + h_1, \xi + h_2)$ with $h_1, h_2 \in \mathcal{E}^\infty(I)$. The following property somehow expresses that a cocycle defined by a collection of sectorial biholomorphisms is a coboundary in the C^∞ category.

Lemma 5.2. *Let $(J_i)_{i \in I}$ be a covering of \mathbb{S}^1 by open intervals threewise disjoint. Assume also that there exists on non empty intersections $J_{ij} := J_i \cap J_j$ a family of sectorial biholomorphisms $\Phi_{ij} \in \mathcal{G}^\infty(J_{ij})$ with $\Phi_{ji} = \Phi_{ij}^{-1}$ (in particular $\Phi_{ii} = \text{id}$). Then, there exist smooth sectorial diffeomorphisms flat to identity $\psi_i \in \mathcal{D}^\infty(J_i)$ such that $\Phi_{ij} = \psi_i \circ \psi_j^{-1}$.*

Proof. One can extract from this covering a finite covering $(J_k), k \in \mathbb{Z}_n$ such that only consecutive sectors J_k and J_{k+1} intersect. It clearly suffices to prove the Lemma for this particular subcovering. Let (θ_k) a partition of the unity subordinate to this covering. Write

$$\Phi_{k,k+1}(z, \xi) = (z + h_{k,k+1}^1, \xi + h_{k,k+1}^2) \quad \text{with } h_{k,k+1}^1, h_{k,k+1}^2 \in \mathcal{A}^\infty(J_{k,k+1}).$$

First define $\tilde{\psi}_k \in \mathcal{D}^\infty(J_k)$ for $k \in \mathbb{Z}_n$ by

$$\tilde{\psi}_k = \begin{cases} \text{Id} & \text{when } \arg(\xi) \in J_k \setminus J_{k,k+1}, \\ \text{Id} + \theta_{k+1}(\arg \xi)(h_{k,k+1}^1, h_{k,k+1}^2) & \text{when } \arg(\xi) \in J_{k,k+1} \end{cases}$$

Next, define $\psi_k \in \mathcal{D}^\infty(J_k)$ by

$$\psi_k = \begin{cases} \Phi_{k,k-1} \circ \tilde{\psi}_{k-1} & \text{when } \arg(\xi) \in J_{k-1,k} \\ \tilde{\psi}_k & \text{when } \arg(\xi) \in J_k \setminus J_{k-1,k} \end{cases}$$

One easily check that ψ_k are smooth, equal to the identity outside intersections, and satisfy $\psi_k = \Phi_{k,k-1} \circ \psi_{k-1}$ on intersections as expected. One easily check that $\tilde{\psi}_k$ and ψ_k define diffeomorphisms provided ξ is large enough; certainly this will impose to shrink a little bit the domain of definition of $\Phi_{k,k+1}$, but it will still be transversely sectorial of same aperture. \square

Corollary 5.3. *Notations and assumptions like in Lemma 5.2. Then, there exist sectorial biholomorphisms tangent to identity $\Psi_i \in \mathcal{G}^1(J_i)$ such that $\Phi_{ij} = \Psi_i \circ \Psi_j^{-1}$. In particular, asymptotic expansions coincide $\hat{\Psi}_i = \hat{\Psi}_j$.*

Proof. Let \tilde{U}_i be the sectorial domain of definition of ψ_i and \tilde{U} be their union together with the section \tilde{C} defined by $\xi = \infty$. Lemma 5.2 allows to write $\Phi_{ij} = \psi_i \circ \psi_j^{-1}$ where $\psi_i \in \mathcal{D}^\infty(I_i)$. In particular, denoting by I the standart complex structure on \mathbb{C}^2 , $J := \psi_i^* I = \psi_j^* I$ is a new complex structure on $\tilde{U} \setminus \tilde{C}$ which extends to \tilde{U} as a complex structure by Newlander-Nirenberg's Theorem. In fact, because of flatness of ψ_i to the identity, the almost complex structure J extends at 0 as a C^∞ almost complex structure on \tilde{U} ; by construction, it is integrable on \tilde{U}_i 's and therefore Nijenhuis tensor vanishes identically on $\tilde{U} \setminus \tilde{C}$, and by continuity on \tilde{U} . Then, Newlander-Nirenberg's Theorem tells us that J is integrable. Note that $I = J$ in restriction to \tilde{C} which is then conformally equivalent to \mathbb{C}^* for both structures. Now, we use the fact that two-dimensional germs of neighborhood of \mathbb{C}^* are analytically equivalent as recalled in Section 2. This can be translated into the existence of a smooth diffeomorphism ψ of (\tilde{U}, \tilde{C}) such that $\psi_* I = J$. Up to making a right composition by a biholomorphism of (\tilde{U}, \tilde{C}) with respect to I , one can suppose (exploiting that $I = J$ on $T\tilde{U}_{\tilde{C}}$) that ψ is tangent to the identity along \tilde{C} . This implies that for every i , $\Psi_i := \psi_i \circ \psi \in \mathcal{G}^1(U_i)$ and, because the Φ_{ij} 's are flat to identity, admit an asymptotic expansion $\hat{\Psi}_i$ along $\tilde{C} = \mathbb{C}^*$ independant of i . By construction, we have $\Phi_{ij} = \Psi_i \circ \Psi_j^{-1}$ as desired. Obviously, all along this proof, we might have shrank the domain \tilde{U} of definition without mentioning it. \square

Remark 5.4. *The use of the Newlander-Nirenberg in this context is not new and can be traced back to Malgrange [12] and Martinet-Ramis [13].*

We now specialize to our covering of \mathbb{S}^1 determined by the intervals I_i defined by (4.1) in Lemma 4.1. Let us show how to construct a neighborhood realizing a given cocycle $\varphi = (\varphi_{i,i+1})$ as in (4.4). We first define $\Phi_{i,i+1} \in \mathcal{G}^\infty[F_0](I_{i,i+1})$ satisfying $\Pi \circ \Phi_{i,i+1} = \varphi_{i,i+1} \circ \Pi$. Then use Corollary 5.3 to obtain $\Psi_i \in \mathcal{G}^1(I_i)$ such that $\Phi_{i,i+1} = \Psi_i \circ \Psi_{i+1}^{-1}$. As $\Phi_{i,i+1}$ commute to F_0 , we have on intersections:

$$(\Psi_i \circ \Psi_{i+1}^{-1}) \circ F_0 = F_0 \circ (\Psi_i \circ \Psi_{i+1}^{-1})$$

which rewrites

$$\Psi_{i+1}^{-1} \circ F_0 \circ \Psi_{i+1} = \Psi_i^{-1} \circ F_0 \circ \Psi_i.$$

Therefore, we can define a global diffeomorphism of (\tilde{U}, \tilde{C}) by setting

$$F_\varphi := \Psi_i^{-1} \circ F_0 \circ \Psi_i$$

on U_i 's and extending by continuity as the identity mapping on \tilde{C} . By construction, the quotient

$$(U_\varphi, C) := (\tilde{U}, \tilde{C}) / \langle F_\varphi \rangle$$

has cocycle φ and is formally equivalent to $U_{1,0,0}$. This proves the surjectivity of the map (4.5) whose injectivity has been proved in Proposition 4.3. It remains to prove the Sectorial Normalization Lemma 4.1 (i.e. Lemma A in the introduction), which will be done in section 9. Modulo this technical Lemma, we have achieved the proof of Theorem C.

6. CONSTRUCTION OF V_φ .

In this section, we generalize Serre isomorphism $\Pi : U_{1,0,0} \setminus C \rightarrow \mathbb{C}_X^* \times \mathbb{C}_Y^*$ to the case of a general neighborhood $(U, C) \stackrel{\text{for}}{\sim} (U_{1,0,0}, C)$.

Theorem 6.1. *Given a neighborhood $(U_\varphi, C) \stackrel{for}{\sim} (U_{1,0,0}, C)$, there exists a neighborhood germ (V_φ, C) of D where each L_i have trivial normal bundle, and an isomorphism germ*

$$(6.1) \quad \Pi_\varphi : (U_\varphi \setminus C, C) \xrightarrow{\sim} (V_\varphi \setminus D, D)$$

canonically attached to the analytic class of (U_φ, C) in the following sense: if $(U_{\varphi'}, C)$ is another neighborhood germ, then

$$(6.2) \quad (U_\varphi, C) \stackrel{an}{\sim} (U_{\varphi'}, C) \Leftrightarrow (V_\varphi, D) \stackrel{an}{\sim} (V_{\varphi'}, D)$$

where the analytic equivalence allows translations¹⁸ on C for the left-hand-side, and preserves the numbering of lines L_i on the right-hand-side.

Proof. Given a cocycle not necessarily tangent to the identity

$$\varphi = (\varphi_{i,i+1})_{i \in \mathbb{Z}_4}, \quad \varphi_{i,i+1} \in \text{Diff}(V_{i,i+1}^*, p_{i,i+1}),$$

we define a new germ of analytic neighborhood of D as follows. We consider the disjoint union of neighborhood germs (V_i, L_i) , and patch them together through the transition maps

$$\varphi_{i,i+1} : (V_{i+1}, p_{i,i+1}) \xrightarrow{\sim} (V_i, p_{i,i+1}).$$

The resulting analytic manifold V_φ contains a copy of D , namely the union of lines L_i identified at points $p_{i,i+1}$, and only the germ of neighborhood V_φ makes sense

$$(V_\varphi, D) := \sqcup_i (V_i, L_i) / (\varphi_{i,i+1}).$$

This germ of neighborhood comes with embeddings

$$\psi_i : (V_i, L_i) \hookrightarrow (V_\varphi, D).$$

Conversely, if (V, D) is a germ of neighborhood of D where all lines L_i have zero self-intersection, then there exist trivialization maps

$$\psi_i : (V_i^*, L_i) \xrightarrow{\sim} (V^*, L_i)$$

(where $*$ means that ψ_i preserves the divisor D) in such a way that, near $p_{i,i+1}$ we have

$$\psi_i = \varphi_{i,i+1} \circ \psi_j \quad \text{for some} \quad \varphi_{i,i+1} \in \text{Diff}(V_{i,i+1}^*, p_{i,i+1}).$$

It is clear from above arguments that, for another cocycle φ' , we have

$$(V_\varphi, D) \stackrel{an}{\sim} (V_{\varphi'}, D) \Leftrightarrow \varphi \sim \varphi'$$

where

$$\varphi \sim \varphi' \stackrel{\text{def}}{\Leftrightarrow} \exists \varphi_i \in \text{Diff}(V_i^*, L_i), \quad \varphi_i \circ \varphi'_{i,i+1} = \varphi_{i,i+1} \circ \varphi_i,$$

and in that case, the isomorphism $V_\varphi \xrightarrow{\sim} V_{\varphi'}$ is given by patching

$$(V_\varphi, D) \xleftarrow{\psi_i} (V_i, L_i) \xrightarrow{\varphi_i} (V_i, L_i) \xrightarrow{\psi'_i} (V_{\varphi'}, D).$$

From the linear part of equivalence relation $\varphi \sim \varphi'$, we see that any cocycle φ is equivalent to a cocycle such that

- $\varphi_{1,2}, \varphi_{2,3}, \varphi_{3,4} \in \text{Diff}^1(V_{i,i+1}^*, p_{i,i+1})$ (tangent to the identity),
- $\varphi_{4,1}(X, Y) = (aX + \dots, bY + \dots)$ for $a, b \in \mathbb{C}^*$ independant of the choice.

¹⁸We emphasize that this is not exactly the equivalence relation defined in Definition 2.2.

The pair (a, b) is an invariant of the neighborhood V_φ . Cocycles arising from (U, C) for $(U_{1,0,0}, C)$ have invariants $a = b = 1$. In order to prove the equivalence (6.2), we just have to note that, for equivalent cocycles $\varphi \sim \varphi'$ normalized as above (in particular when all $\varphi_{i,i+1}, \varphi'_{i,i+1}$ are tangent to the identity) then all four conjugating maps φ_i have the same linear part. Then apply Remark 4.4 to show that it corresponds to analytic equivalence of (U_φ, C) and $(U_{\varphi'}, C)$ up to a translation of the curve.

Finally, we construct the isomorphism (6.1) by patching together the sectorial ones

$$U_i \xrightarrow{\Pi_i} V_i^* \xrightarrow{\psi_i} V_\varphi \setminus D$$

using the identity (4.2) $\Pi_i = \varphi_{i,i+1} \circ \Pi_{i+1}$. \square

7. FOLIATIONS

Recall that our model $(U_{1,0,0}, C)$ carries a pencil of foliations

$$\mathcal{F}_t : \{\omega_0 - t\omega_\infty = 0\}, \quad \text{where } \omega_0 = d\xi \quad \text{and} \quad \omega_\infty = \frac{1}{2i\pi\tau} \frac{dz}{z};$$

moreover, there is no other formal foliation on $(U_{1,0,0}, C)$ either tangent, or transversal to C (see [11, section 2.3]). Via the isomorphism $\Pi : U_{1,0,0} \setminus C \rightarrow V_0 \setminus D$, we get the corresponding pencil

$$\Pi_* \mathcal{F}_t : (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y}.$$

The monodromy (or holonomy) of \mathcal{F}_t is given by

$$\pi_1(C) \rightarrow \text{Aut}(\mathbb{C}) ; \begin{cases} 1 \mapsto [\xi \mapsto \xi + \frac{t}{\tau}] \\ \tau \mapsto [\xi \mapsto \xi + t - 1] \end{cases}$$

In particular, for $\frac{m}{n} \in \mathbb{Q}$, are equivalent

- \mathcal{F}_t has trivial monodromy along $m + \tau n \in \Gamma$, viewed as a loop of $\pi_1(C) \simeq \Gamma$;
- $t = \frac{\tau n}{m + \tau n}$, or equivalently $(\frac{1}{t} - 1)\tau = \frac{m}{n}$;
- $\Pi_* \mathcal{F}_t$ admits the rational first integral $X^m Y^n$.

We will say that \mathcal{F}_t is of *rational type* if there is $\frac{m}{n} \in \mathbb{Q} \cup \{\infty\}$ with these properties, and of *irrational type* if not. We note that rational type foliations are characterized by the fact that their holonomy group is cyclic (one generator), and also that the space of leaves (after deleting C) is rational, and not elliptic.

If (U, C) is any analytic neighborhood with a formal conjugacy

$$\hat{\Psi} : (U, C) \xrightarrow{\sim} (U_{1,0,0}, C),$$

then it also carries the pencil of formal foliations $\hat{\mathcal{F}}_t := \hat{\Psi}^* \mathcal{F}_t$. As we shall prove, these foliations are divergent in general. In fact, recall (see [11, Theorem 4])

Theorem 7.1. *Let (U, C) be an analytic neighborhood formally equivalent to $(U_{1,0,0}, C)$. Assume*

- *three elements $\hat{\mathcal{F}}_{t_1}, \hat{\mathcal{F}}_{t_2}, \hat{\mathcal{F}}_{t_3}$ of the pencil are convergent,*
- *or two elements $\hat{\mathcal{F}}_{t_1}, \hat{\mathcal{F}}_{t_2}$ of the pencil are convergent, both of irrational type:*
 $\left(\frac{1}{t_i} - 1\right)\tau \notin \mathbb{Q}$ *for* $i = 1, 2$.

Then the full pencil $\hat{\mathcal{F}}_t$ is convergent, and (U, C) is analytically equivalent to $(U_{1,0,0}, C)$ (in fact $\hat{\Psi}$ is convergent).

In [11, Theorem 5], the two first authors and O. Thom construct infinite dimensional deformations of neighborhoods with two convergent foliations $\hat{\mathcal{F}}_{t_1}$ and $\hat{\mathcal{F}}_{t_2}$, provided that one or two of them is of rational type. In fact, Écalle-Voronin moduli spaces are shown to embed in moduli spaces of neighborhoods through these bifoliated constructions. Now, we know from our main result Theorem C, that the moduli space of neighborhood is larger, comparable with $\mathbb{C}\{X, Y\}$, in contrast with Écalle-Voronin moduli space which is comparable with $\mathbb{C}\{X\}$. The fact is that we missed all neighborhoods with only one, or with no convergent foliation.

7.1. Existence of foliations. We now start examining under which condition on the glueing cocycle $\varphi = (\varphi_{i,i+1})$ the neighborhood (U_φ, C) admits a convergent foliation. Here, we follow notations $\Psi_i, \Pi_i, \Phi_i, \dots$ of section 4.

Lemma 7.2. *Let $t \in \mathbb{P}^1 \setminus \{0, 1\}$. If the formal foliation $\hat{\mathcal{F}}_t$ of (U, C) is convergent, then the induced foliation $\Pi_{i*}\hat{\mathcal{F}}_t$ on V_i^* extends as a singular foliation on V_i and is defined by a closed logarithmic 1-form*

$$(7.1) \quad \theta_i = (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y} + \eta_i$$

with η_i closed holomorphic on V_i .

Remark 7.3. *On V_i with $i = 1, 3$ (resp. $i = 2, 4$), the closed holomorphic 1-form writes*

$$\eta_i = df \quad \text{with } f \in \mathbb{C}\{X\} \quad (\text{resp. } f \in \mathbb{C}\{Y\}).$$

We note that are equivalent:

- θ is a closed logarithmic 1-form on V_i with poles supported by D ,
- $\theta = \alpha \frac{dX}{X} + \beta \frac{dY}{Y} + \eta$ with η holomorphic and closed on V_i ,
- $\theta = \varphi_i^* \left\{ \alpha \frac{dX}{X} + \beta \frac{dY}{Y} \right\}$ where $\varphi_i \in \text{Diff}^1(V_i^*, L_i)$.

For instance, on V_1 , if $\eta_1 = df$ with $f \in \mathbb{C}\{X\}$, then $\varphi_i = (Xe^{f(X)}, Y)$.

Proof. Let us start with the tangent case $t \neq \infty$. Then, by transversality of \mathcal{F}_0 and \mathcal{F}_∞ , and the fact that the ω_t 's are F_0 -invariant, one deduces that $\hat{\mathcal{F}}_t$ is defined by a unique 1-form of the shape $\omega = \omega_0 - u \cdot \omega_\infty$ for a function $u \in \mathcal{A}[F_0](I_i)$, obviously satisfying $\hat{u} = t$. If $i = 1, 3$, then $u = t + f(X)$ with $f \in \mathbb{C}\{X\}$, $f(0) = 0$ (see Proposition 3.7). Then,

$$\frac{t}{u}\omega = t \frac{1-u}{u} \tau \frac{dX}{X} + t \frac{dY}{Y} = (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y} - \frac{f}{t+f} \tau \frac{dX}{X}.$$

If $i = 2, 4$, then $u = t + f(Y)$ with $f \in \mathbb{C}\{Y\}$, $f(0) = 0$, and we arrive at a similar situation

$$\frac{1-t}{1-u}\omega = (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y} + \frac{f}{1-t-f} \tau \frac{dX}{X}.$$

Of course, we have used $t \neq 0, 1$ in order to divide.

Let us end with the case $\hat{\mathcal{F}}_\infty$. The foliation, in that case, can be defined by a closed holomorphic 1-form ω extending the holomorphic 1-form on C . Now, up to a multiplicative constant, we can write $\omega = \omega_t + \eta$ for a closed holomorphic 1-form η . We end the proof as above. \square

Lemma 7.4. *If the formal foliation $\hat{\mathcal{F}}_0$ (resp. $\hat{\mathcal{F}}_1$) of (U, C) is convergent, then the induced foliation $\Pi_{i*}\hat{\mathcal{F}}_t$ on V_i^* extends as a singular foliation on V_i and is defined by a 1-form*

$$\theta_i = \begin{cases} \frac{dX}{X} + f_i(X) \frac{dY}{Y} & \text{if } i = 1, 3 \\ \frac{dX}{X} + \eta_i & \text{if } i = 2, 4 \end{cases} \quad \left(\text{resp. } \theta_i = \begin{cases} \frac{dY}{Y} + \eta_i & \text{if } i = 1, 3 \\ \frac{dY}{Y} + f_i(Y) \frac{dX}{X} & \text{if } i = 2, 4 \end{cases} \right)$$

with $f_i \in \mathbb{C}\{X\}$ (resp. $\mathbb{C}\{Y\}$), $f_i(0) = 0$, and η_i closed holomorphic 1-form on V_i .

Proof. It is similar to the proof of Lemma 7.2. For the case $t = 0$, once we have defined $\hat{\mathcal{F}}_t$ by $\omega = \omega_0 - u \cdot \omega_\infty$ for a function $u \in \mathcal{A}^\infty[F_0](I_i)$, then

$$\frac{1}{1-u}\omega = \tau \frac{dX}{X} + \frac{u}{1-u} \frac{dY}{Y}.$$

Again, by Proposition 3.7, we see that if $i = 2, 4$, then $u = f(Y)$ with $f \in \mathbb{C}\{Y\}$, $f(0) = 0$, and we are done. However, when $i = 1, 3$, then $u = f(X)$, but we cannot divide by $\frac{f}{1-f}$ to get a closed logarithmic 1-form as before: as $f(0) = 0$, the polar locus will increase (see remark 7.5). \square

Remark 7.5. In Lemma 7.4, we can always define the foliation $\Pi_{i*}\hat{\mathcal{F}}_t$ by a closed meromorphic 1-form on V_i provided that we allow non logarithmic poles. For instance, in the case $t = 0$ and $i = 1, 3$, if $f \equiv 0$ (is identically vanishing), there is nothing to do, it is the logarithmic case; if $f \neq 0$, then after division, we get

$$\frac{1}{f}\omega = \tau \frac{1-f}{f} \frac{dX}{X} + \frac{dY}{Y} = \tilde{f}(X) \frac{dX}{X^{k+1}} + \frac{dY}{Y}$$

with $\tilde{f} \in \mathbb{C}\{X\}$, $\tilde{f}(0) \neq 0$, and $k \in \mathbb{Z}_{>0}$. As it is well-known (see [11, section 2.2]), we can write

$$\varphi_i^* \omega = \frac{dX}{X^{k+1}} + \alpha \frac{dX}{X} + \frac{dY}{Y}$$

for some $\alpha \in \mathbb{C}$ (the residue of $\tilde{f}(X) \frac{dX}{X^{k+1}}$) and $\varphi_i \in \text{Diff}^1(V_i^*, L_i)$.

We can now prove Theorem D.

Corollary 7.6. *The formal foliation $\hat{\mathcal{F}}_t$ of (U_φ, C) is convergent if, and only if, there exist η_i closed holomorphic 1-forms on (V_i, L_i) such that*

$$(\varphi_{i,i+1}^* \theta_i) \wedge \theta_{i+1} = 0 \quad \text{where} \quad \theta_i = (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y} + \eta_i.$$

Equivalently, there exists an equivalent cocycle $\varphi' \sim \varphi$ such that

$$(\varphi'_{i,i+1}^* \theta^0) \wedge \theta^0 = 0 \quad \text{where} \quad \theta^0 = (1-t)\tau \frac{dX}{X} + t \frac{dY}{Y}.$$

Proof. When $t \neq 0, 1$, the proof easily follows from Lemma 7.2. Indeed, all $\Pi_{i*}\hat{\mathcal{F}}_t$ are defined by $\theta_i = 0$ and have to patch via the glueing maps $\varphi_{i,i+1}$. Conversely, if $\theta_i = 0$ patch via the glueing maps $\varphi_{i,i+1}$, then this means that we get a foliation \mathcal{F} on $U_\varphi \setminus C$ which is flat to $\hat{\mathcal{F}}_t$ along C , and therefore extends by Riemann. Using Remark 7.3 and Definition 4.2, one easily derive the second (equivalent) assertion. Finally, in the case $t = 0$ for instance, after applying Lemma 7.4 in a very similar way, we note that θ_i defines a regular foliation on V_i for $i = 2, 4$ (as $\eta_i = df$, $f \in \mathbb{C}\{X\}$). On the other hand, on V_i for $i = 1, 3$, θ_i defines a singular foliation as soon as $f_i \neq 0$ (non identically vanishing), i.e. with a saddle-node singular points at the two points $p_{i,i+1}$ and $p_{i-1,i}$; therefore, $f_i \equiv 0$ in the case we have a global foliation and we are back to the logarithmic case. The proof ends-up like before. \square

Remark 7.7. *The statement of Corollary 7.6 can be reformulated as follows. The formal foliation $\hat{\mathcal{F}}_t$ of (U_φ, C) is convergent if, and only if, there exists a foliation \mathcal{G}_t on (V_φ, D) which is locally defined by a closed logarithmic 1-form with poles supported by D and having residues t on L_1 and $(1-t)\tau$ on L_4 (we have automatically opposite residues on opposite sides of D). Indeed, the local foliations $\theta_i = 0$ patch together.*

We can precise Corollary 7.6 for generic t as follows.

Proposition 7.8. *If $\hat{\mathcal{F}}_t$ is not of rational type, i.e. $(\frac{1}{t} - 1)\tau \notin \mathbb{Q}$, then are equivalent*

- (1) $\hat{\mathcal{F}}_t$ is convergent,
- (2) $\hat{\mathcal{F}}_t$ is defined by a closed (convergent) meromorphic 1-form ω ,
- (3) $(\varphi_{i,i+1})^*\theta_i = \theta_{i+1}$ with θ_i like in Corollary 7.6,
- (4) there is a closed logarithmic 1-form θ on (V_φ, D) with poles supported by D and having residues t on L_1 and $(1-t)\tau$ on L_4 .

Obviously, $\omega = \Pi_\varphi^*\omega$ up to a constant.

Proof. When $\hat{\mathcal{F}}_t$ is not of rational type, then we have

$$(\varphi_{i,i+1}^*\theta_i) \wedge \theta_{i+1} = 0 \Leftrightarrow (\varphi_{i,i+1}^*\theta_i) = \theta_{i+1}.$$

Indeed, if $\varphi_{i,i+1}^*\theta_i$ is colinear to θ_{i+1} , then it is proportional to θ_{i+1} , i.e. it writes $f_i \cdot \theta_i$ with f_i meromorphic on V_i (deRham-Saito Lemma). But since it is also closed, we have

$$0 = d(f_i \cdot \theta_i) = df_i \wedge \theta_i + f_i \wedge \underbrace{d\theta_i}_{=0}$$

and f_i is a meromorphic first integral for $\theta_i = 0$, which must be constant in the irrational type. This constant must be $= 1$ as the residues are preserved. As a consequence, all θ_i patch together on V_φ . Finally, note that if $\hat{\mathcal{F}}_t$ is convergent, then its holonomy is not cyclic (because not of rational type) and therefore preserves a meromorphic 1-form on the transversal that we can extend as a closed meromorphic 1-form ω defining the foliation. \square

Let us now illustrate how different is the situation for foliations of rational type by revisiting the classification [11, Theorem 5] of neighborhoods with 2 convergent foliations, in the particular case of $\hat{\mathcal{F}}_0$ and $\hat{\mathcal{F}}_1$, corresponding respectively to vertical and horizontal foliations on V_φ . The proof is a straightforward application of the above criteria.

Proposition 7.9. *The formal foliations $\hat{\mathcal{F}}_0$ and $\hat{\mathcal{F}}_1$ on (U, C) are convergent if, and only if, (U, C) can be defined by a cocycle of the form*

$$\begin{cases} \varphi_{i,i+1}(X, Y) = (\alpha_i(X), Y) & \text{for } i = 1, 3 \\ \varphi_{i,i+1}(X, Y) = (X, \alpha_i(Y)) & \text{for } i = 2, 4 \end{cases}$$

for 1-variable diffeomorphisms α_i tangent to the identity, and the corresponding foliations on V_φ are respectively defined in charts V_i by $dX = 0$ and $dY = 0$. Moreover, this normalization is unique up to conjugacy by $\phi(X, Y) = (e^t X, e^{\tau t} Y)$.

The space of leaves of $\hat{\mathcal{F}}_0$ on $U \setminus C$ corresponds to the space of orbits for its holonomy map, and therefore to Martinet-Ramis' "Chapelet de sphères" (see [14, page 591]). It is given by two copies of \mathbb{C}_X^* patched together by means of diffeomorphism germs $\alpha_1(X)$ at $X = \infty$ and $\alpha_3(X)$ at $X = 0$. A similar description holds for $\hat{\mathcal{F}}_1$ with Martinet-Ramis' cocycle α_2 and α_4 . The invariants found by the third author in [30] are related with the corresponding periodic transformations in variable ξ .

Remark 7.10. *It follows from [11], or from the unicity of the formal pencil $\hat{\mathcal{F}}_t$, that for given $t \in \mathbb{P}^1$, one cannot find two different collections $(\theta_i)_i$ and $(\theta'_i)_i$ defining two global logarithmic foliations \mathcal{G}_t and \mathcal{G}'_t on V , like in Corollary 7.6. One way to see this directly from the point of view of this section is as follows. In the irrational case $(\frac{1}{t} - 1)\tau \notin \mathbb{Q}$, we see from Proposition 7.8 that θ_i 's patch as a global closed logarithmic 1-form.*

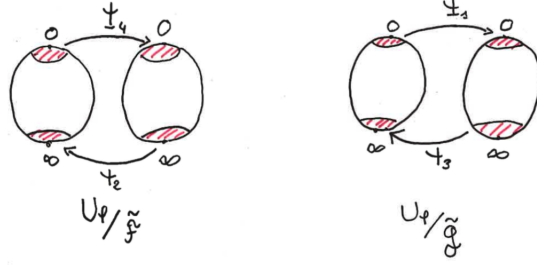


FIGURE 1. Martinet-Ramis moduli

But the difference between two closed logarithmic 1-forms with the same residues is a closed holomorphic 1-form η on (V, C) . Now, η must be zero, even if we restrict on two consecutive line neighborhoods $(V_i, L_i) \cup_{\varphi_{i,i+1}} (V_{i+1}, L_{i+1})$, as it only depends on X or on Y depending on the sector. In the rational case, $(\frac{1}{t} - 1)\tau \notin \mathbb{Q}$, there is also unicity of θ_i 's on two consecutive line neighborhoods whose residues have quotient > 0 ; indeed, after blowing-up, we get a rational fibration which must be unique by Blanchard Lemma.

7.2. Non existence of foliations. For a generic neighborhood (U_φ, C) , there is no convergent foliation. In order to prove this, it is enough to provide a single example without foliation. Such an example has been given quite recently by Mishustin in [17]. With our Corollary 7.6, it is not too difficult to provide an example without foliations.

Theorem 7.11. *Let (U_φ, C) be a neighborhood such that*

$$\varphi_{1,4}(X, Y) = (X(1 + XY), Y(1 + X^2Y)).$$

Then all foliations $\hat{\mathcal{F}}_t$ belonging to the formal pencil $(\hat{\mathcal{F}}_t)_{t \in \mathbb{P}^1}$ are divergent.

Even the transversal fibration $\hat{\mathcal{F}}_\infty$ is divergent in that case.

Proof. Suppose by contradiction that there exists at least one convergent foliation in the pencil. Then, by Corollary 7.6, there exists on each V_i a non trivial logarithmic 1-form

$$\theta_i = \alpha \frac{dX}{X} + \beta \frac{dY}{Y} + \begin{cases} f_i(X)dX & \text{if } i \text{ even} \\ f_i(Y)dY & \text{if } i \text{ odd} \end{cases}$$

with $f_i : (\mathbb{C}, 0) \rightarrow \mathbb{C}$ holomorphic, such that $\theta_{i+1} \wedge (\varphi_{i,i+1})^*\theta_i = 0$. One has

$$\begin{aligned} (\varphi_{1,4})^*\theta_1 &= \alpha \frac{dX}{X} + \alpha \frac{d(XY)}{1+XY} + \beta \frac{dY}{Y} + \beta \frac{d(X^2Y)}{1+X^2Y} \\ &\quad + f_1(Y(1+X^2Y)) \cdot ((1+2X^2Y)dY + 2XY^2dX). \end{aligned}$$

The residual parts of the 2-form $\theta_4 \wedge (\varphi_{1,4})^*\theta_1$ at $X = 0$ and $Y = 0$ respectively write

$$(7.2) \quad \alpha f_1(Y) \frac{dX}{X} \wedge dY \quad \text{and} \quad \beta f_4(X) dX \wedge \frac{dY}{Y}.$$

Both two expressions must be vanishing identically. If $\beta = 0$, then $\alpha \neq 0$ and we deduce from (7.2) that $f_1 \equiv 0$. This imply that $(\varphi_{1,4})^*\theta_1$ only depends on $X(1 + XY)$, while θ_4 only depends on X , contradiction. Assume now $\alpha = 0$ (and $\beta \neq 0$); then, by (7.2), we have $f_4 \equiv 0$. Again, we conclude that $(\varphi_{1,4})^*\theta_1$ only depends on $X(1 + X^2Y)$, while θ_4

only depends on Y , contradiction. Finally, assume that $\alpha \neq 0$ and $\beta \neq 0$; then, by (7.2), we have $f_1, f_2 \equiv 0$ and we obtain

$$\theta_4 \wedge (\varphi_{1,4})^* \theta_1 = \frac{\alpha(\alpha - \beta) + \beta(\alpha - 2\beta)X + (\alpha^2 - 2\beta^2)X^2Y}{(1 + XY)(1 + X^2Y)} dX \wedge dY.$$

Clearly, this expression cannot be zero if α and β are both non zero. \square

7.3. Only one convergent foliation. To complete the picture, it is interesting to provide an example of a foliation having only one convergent foliation in the pencil.

Theorem 7.12. *Let $t_0 = [u_0 : v_0] \in \mathbb{P}^1$, and let (U_φ, C) be the neighborhood such that*

$$\begin{cases} \varphi_{1,4} = (X(1 + XYe^Y)^{-\beta_0}, Y(1 + XYe^Y)^{\alpha_0}) \\ \text{where } \alpha_0 = \tau(v_0 - u_0) \text{ and } \beta_0 = u_0 \end{cases} \quad \text{and} \quad \begin{cases} \varphi_{i,i+1} = Id \\ \text{for } i = 1, 2, 3. \end{cases}$$

Then $\hat{\mathcal{F}}_{t_0}$ is the unique convergent foliation in the formal pencil $(\hat{\mathcal{F}}_t)_{t \in \mathbb{P}^1}$.

Proof. Note that φ preserves the foliation defined the logarithmic form $\alpha_0 \frac{dX}{X} + \beta_0 \frac{dY}{Y}$ which then descends on V_φ , i.e. the formal foliation $\hat{\mathcal{F}}_{t_0}$ is indeed convergent on U_φ . Assuming by contradiction that there is another convergent foliation $\hat{\mathcal{F}}_t$ with $t \neq t_0$, let θ_i be the associated logarithmic 1-form on V_i . As in the proof of Theorem 7.11, we get

$$\theta_1 = \alpha \frac{dX}{X} + \beta \frac{dY}{Y} + f_1(Y)dY \quad \text{and} \quad \theta_4 = \alpha \frac{dX}{X} + \beta \frac{dY}{Y} + f_4(X)dX$$

with $f_i : (\mathbb{C}, 0) \rightarrow \mathbb{C}$ holomorphic, and $[\alpha : \beta] \neq [\alpha_0 : \beta_0]$. We derive

$$\begin{aligned} (\varphi_{1,4})^* \theta_1 &= \alpha \frac{dX}{X} + \beta \frac{dY}{Y} + (\alpha_0\beta - \alpha\beta_0) \frac{d(XYe^Y)}{1 + XYe^Y} \\ &\quad + f_1(Y(1 + XYe^Y)^{\alpha_0}) \cdot d(Y(1 + XYe^Y)). \end{aligned}$$

The residual parts of the 2-form $\theta_4 \wedge (\varphi_{1,4})^* \theta_1$ at $X = 0$ and $Y = 0$ respectively write

$$(7.3) \quad \alpha f_1(Y) \frac{dX}{X} \wedge dY \quad \text{and} \quad \beta f_4(X) dX \wedge \frac{dY}{Y}.$$

We are led to a similar discussion as in the proof of Theorem 7.11. When $\beta = 0$, then $f_1 \equiv 0$ and $(\varphi_{1,4})^* \theta_1$ only depends on $X(1 + XY)^{-\beta_0}$, while θ_4 only depends on X ; we get a contradiction since $\beta_0 \neq 0$ in this case. When $\alpha = 0$, then $f_4 \equiv 0$ and $(\varphi_{1,4})^* \theta_1$ only depends on $Y(1 + XY)^{\alpha_0}$, while θ_4 only depends on Y ; contradiction. Finally, when both $\alpha \neq 0$ and $\beta \neq 0$, then

$$\theta_4 \wedge (\varphi_{1,4})^* \theta_1 = \underbrace{(\alpha_0\beta - \alpha\beta_0)}_{\neq 0} \frac{d(XYe^Y)}{1 + XYe^Y} \wedge \left(\alpha \frac{dX}{X} + \beta \frac{dY}{Y} \right)$$

which cannot be zero, again a contradiction. \square

Remark 7.13. *Given $f \in \mathbb{C}\{X, Y\}$ vanishing along $X = 0$ and $Y = 0$, the same proof shows that the cocycle defined by $\varphi_{1,4} = (Xe^{-\beta_0 f}, Ye^{\alpha_0 f})$ and $\varphi_{i,i+1} = Id$ otherwise also provide a neighborhood (U_φ, C) with only one convergent foliation, namely $\hat{\mathcal{F}}_{t_0}$, provided that $df \wedge d(X^p Y^q) \neq 0$ for all $p, q \in \mathbb{Z}_{>0}$. Moreover, one easily check that two different such f , say f and f' , define non equivalent neighborhoods provided that their difference do not take the form $f' - f \neq g(X) + h(Y)$.*

8. SYMMETRIES

Let (U_φ, C) be a neighborhood formally equivalent to $(U_{1,0,0}, C)$. Formal symmetries (or automorphisms) of (U_φ, C) are those of $(U_{1,0,0}, C)$, i.e. of the form (see Corollary 3.10):

$$\begin{cases} (z, \xi) \mapsto (cz, \xi + t) \\ c \in \mathbb{C}^*, t \in \mathbb{C} \end{cases} \leftrightarrow (X, Y) \mapsto (\underbrace{e^{2i\pi t}}_a X, \underbrace{c^{-1}e^{2i\pi\tau t}}_b Y)$$

The subgroup $\text{Aut}(U_\varphi, C)$ of convergent automorphisms identifies with a subgroup

$$G \subset \text{Aut}(V^0, D) \simeq \mathbb{C}_a^* \times \mathbb{C}_b^*.$$

Theorem 8.1. *Let (U_φ, C) and G be as above. Then the subgroup $G \subset \mathbb{C}_a^* \times \mathbb{C}_b^*$ is algebraic. In particular, we are in one of the following cases:*

- G is finite and $G = \{(a, b) ; a^p b^q = a^{p'} b^{q'} = 1\}$ for some non proportional $(p, q), (p', q') \in \mathbb{Z}^2 \setminus (0, 0)$;
- $G = \{(a, b) ; a^p b^q = 1\}$ for some $(p, q) \in \mathbb{Z}^2 \setminus (0, 0)$; in particular, a finite index subgroup of G is generated by the flow of the rational vector field $pX\partial_X + qY\partial_Y$;
- $G = \mathbb{C}_a^* \times \mathbb{C}_b^*$ and $(U_\varphi, C) \stackrel{\text{an}}{\simeq} (U_{1,0,0}, C)$.

Moreover, in the first two cases, up to equivalence \simeq , the cocycle takes the form

$$\varphi_{i,i+1}(X, Y) = (X \cdot u_{i,i+1}, Y \cdot v_{i,i+1})$$

where $u_{i,i+1}, v_{i,i+1}$ are Laurent series in $X^p Y^q$ and $X^{p'} Y^{q'}$ (resp. in $X^p Y^q$) and the action of G is linear in each chart (V_i, L_i) .

Proof. By similar arguments as in the proof of Lemma 7.2, we see that each automorphism of (U_φ, C) corresponds to a collection of automorphisms of $g_i \in \text{Diff}(V_i, L_i)$ satisfying

$$g_i \circ \varphi_{i,i+1} = \varphi_{i,i+1} \circ g_{i+1}.$$

We first prove that g_i can be linearized in each chart. Indeed, for instance on (V_4, L_4) , g_4 acts by transformations of the form

$$(X, Y) \mapsto (aX \cdot u(X), bY \cdot v(X)), \quad u(0) = v(0) = 1.$$

The gluing condition for $\varphi_{4,1}$ shows that the restriction $\varphi_4|_{L_1} : X \mapsto aX \cdot u(X)$ becomes linear in the chart (V_1, L_1) . Therefore, after changing X coordinates on (V_4, L_4) , we can assume

$$g_4(X, Y) = (aX, bY \cdot v(X)), \quad v(0) = 1.$$

But g_4 must also preserve the fibration $dY_1 = 0$ of (V_1, L_1) which is preserved by g_1 ; that can be normalized to $dY_4 = 0$ in the chart (V_4, L_4) and this implies that $v(X) \equiv 1$ also. We therefore conclude that for each automorphism in $\text{Aut}(U_\varphi, C)$, the corresponding transformation in (V_φ, D) can be linearized in all charts (V_i, L_i) .

A first consequence is that g is the identity whenever $a = b = 1$, proving that the morphism

$$\text{Aut}(U_\varphi, C) \rightarrow \mathbb{C}_a^* \times \mathbb{C}_b^*$$

is injective.

A second consequence is that G contains the Zariski closure of $\langle (a, b) \rangle \subset \mathbb{C}_a^* \times \mathbb{C}_b^*$. Indeed, all $\varphi_{i,i+1}$ have to commute with $g(X, Y) = (aX, bY)$. Writing $\varphi_{i,i+1}(X, Y) = (X \cdot u(X, Y), Y \cdot v(X, Y))$, we see that u, v have to be invariant by g , i.e. $u \circ g = u$ for instance; equivalently, all non zero monomials of u and v are g -invariant. These monomials define an algebraic subgroup $H \subset \mathbb{C}_a^* \times \mathbb{C}_b^*$ which is the group of linear transformations commuting with $\varphi_{i,i+1}$. We conclude that $(a, b) \in H \subset G$.

If g was Zariski dense in $\mathbb{C}_a^* \times \mathbb{C}_b^*$, we are done: the commutation of $\varphi_{i,i+1}$ with all linear transformations shows that $\varphi_{i,i+1}$ is linear and $(U_\varphi, C) \stackrel{\text{an}}{\sim} (U_{1,0,0}, C)$. Now, assuming that G is a strict subgroup of $\mathbb{C}_a^* \times \mathbb{C}_b^*$, we want to prove that it can be linearized globally.

Assume first that G is finite. Then it can be linearized on each line neighborhood (V_i, L_i) . Indeed, for instance on (V_4, L_4) , G acts by transformations of the form

$$g(X, Y) = (aX \cdot u(X), bY \cdot v(X)), \quad u(0) = v(0) = 1.$$

and we denote by $\text{lin}(g)$ its linear part (aX, bY) . Then the transformation

$$\varphi_4 := \frac{1}{\#G} \sum_{g \in G} \text{lin}(g)^{-1} \circ g$$

is of the form $\varphi_4(X, Y) = (X \cdot u(X), Y \cdot v(X))$, $u(0) = v(0) = 1$ and linearizing the group:

$$\varphi_4 \circ g = \text{lin}(g) \circ \varphi_4, \quad \forall g \in G.$$

We can therefore assume that G acts linearly in each chart (V_i, L_i) and the cocycle φ has to commute with all elements. It is well known that the group G is generated by two elements (a_1, b_1) and (a_2, b_2) of finite order; moreover, by duality, G is defined by 2 independant monomial equations $a^p b^q = a^{p'} b^{q'} = 1$. Gluing conditions with $\varphi_{i,i+1}$ show that $u_{i,i+1}, v_{i,i+1}$ must be G -right-invariant and therefore factor through the two monomial equations.

On the other hand, if G contains an element g of infinite order, then we can first linearize this element. The Zariski closure H of its iterates $\langle g \rangle$ in $\mathbb{C}_a^* \times \mathbb{C}_b^*$ is one dimensional, if strictly smaller than $\mathbb{C}_a^* \times \mathbb{C}_b^*$, and defined by a monomial equation $a^p b^q = 0$. If G is larger than H , then it is generated by an element of finite order g' and we can linearize the finite group $\langle g' \rangle$ like above; since g' and its linear part both commute with H , the linearizing transformations also commute with H and G is linearized. It is therefore algebraic, defined by monomial equations of $\varphi_{i,i+1}$, and they all factor into a single monomial. \square

Remark 8.2. We note that the convergence of a non trivial automorphism of (U_φ, C) inducing the identity on C implies that $(U_\varphi, C) \stackrel{\text{an}}{\sim} (U_{1,0,0}, C)$, since the corresponding element $g \in \mathbb{C}_a^* \times \mathbb{C}_b^*$ must be Zariski dense.

Remark 8.3. In Proposition 7.9, the foliation $\hat{\mathcal{F}}_0$ is defined by a holomorphic vector field if , and only if, $\alpha_2(Y) = \alpha_4(Y) = Y$. Equivalently, the Martinet-Ramis invariant of $\hat{\mathcal{F}}_1$ are trivial, i.e. $\hat{\mathcal{F}}_1$ can be defined by a closed 1-form.

9. SECTORIAL NORMALIZATION

We keep notations as before and set $\lambda = 2i\pi\tau$ to simplify formulae, so that $q = e^\lambda$. Let (U, C) be formally equivalent to (U_0, C) . We want to show that (U, C) has the form U_φ .

9.1. Overview of the proofs. One can suppose that $(U, C) = (\tilde{U}, \tilde{C})/F$ where

$$F(z, \xi) = \underbrace{F_0(z, \xi)}_{(qz, \xi-1)} + (\Delta_1, \Delta_2) \quad \text{with} \quad \Delta_i = O(\xi^{-N})$$

where $N \gg 0$ is an arbitrarily large integer, so that there exists a formal diffeomorphism

$$\hat{H}(z, \xi) = (z + \hat{g}, \xi + \hat{h}), \quad \hat{h} = \sum_{n \geq 1} a_n \xi^{-n}, \quad \hat{g} = \sum_{n \geq 1} b_n \xi^{-n}$$

where a_n, b_n are entire functions on $\tilde{C} = \mathbb{C}^*$ such that

$$F \circ \hat{H} = \hat{H} \circ F_0.$$

This can be reformulated as

$$(9.1) \quad \hat{h} \circ F_0 - \hat{h} = \Delta_2 \circ \hat{H}$$

$$(9.2) \quad \hat{g} \circ F_0 - q\hat{g} = \Delta_1 \circ \hat{H}$$

Basically, we will show that there exist *holomorphic* solutions h, g of the previous functional equation defined on "suitable sectorial domains", namely $\bigcup_i \Pi^{-1}(V_i - L_i)$, $i = 1, 3$, and admitting asymptotic expansion along \tilde{C} compatible with the formal conjugacy map. More precisely $h, g \in \mathcal{A}(I_1 \cup I_3)$ with the notations of 3.1 with respective asymptotic expansions \hat{h} and \hat{g} . To this effect, we will first exhibit solutions of the linearized equations satisfying some suitable growth behaviour

$$(9.3) \quad h \circ F_0 - h = \Delta_2$$

$$(9.4) \quad g \circ F_0 - qg = \Delta_1$$

on $\mathcal{O}(I_1 \cup I_3)$.

Remark 9.1. *The equation (9.4) can be reduced to equation (9.3); indeed, after setting*

$$g(z, \xi) = z\tilde{g}(z, \xi) \quad \text{and} \quad \Delta_1(z, \xi) = qz\tilde{\Delta}_1(z, \xi),$$

we get

$$\tilde{g} \circ F_0 - \tilde{g} = \tilde{\Delta}_1.$$

This will enable us to solve by a fairly standard fixed point method the initial functional equations $h \circ F_0 - h = \Delta_2 \circ H$ and $g \circ F_0 - qg = \Delta_1 \circ H$ where $H = \text{Id} + (h, g)$. In order to get rid of the coefficient q on the left hand side, note that both equations can be reformulated as:

$$(9.5) \quad h(qz, \xi - 1) - h(z, \xi) = \Delta_2(z(1 + \tilde{g}(z, \xi)), \xi + h(z, \xi))$$

$$(9.6) \quad \tilde{g}(qz, \xi - 1) - \tilde{g}(z, \xi) = (1 + \tilde{g}(z, \xi))\tilde{\Delta}_1(z(1 + \tilde{g}(z, \xi)), \xi + h(z, \xi))$$

where the symbol $\tilde{}$ stands for the same modification than the one defined in the linear case.

To conclude, it suffices to solve the same homological equations on the union of sectors $I_2 \cup I_4$. This can be done by changing accordingly the presentation of (U, C) , by considering the cyclic covering determined by $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (see Section 1.6). This consists in writing (U_0, C) as the quotient $(\mathbb{C}_{z'}^* \times \mathbb{C}_{\xi'}, \xi' = \infty)$ by the semi-hyperbolic transformation $F'_0(z', \xi') = (q'z', \xi' + \frac{1}{\tau})$ where $q' = \exp \frac{2i\pi}{\tau}$, $z' = \exp \frac{2i\pi}{\tau} x$, $\xi' = \xi + \frac{x}{\tau}$. In these coordinates (U, C) is isomorphic to a quotient of the form $(\mathbb{C}_{z'}^* \times \mathbb{C}_{\xi'} \{ \xi' = \infty \})$ by a bi-holomorphism F' such that $F'(z', \infty) = (z', \infty)$ and we can repeat the process described above and obtain the sought analytic sectorial conjugation on $I_2 \cup I_4$ when coming back to the (z, ξ) coordinates.

9.2. The linearized/homological equation. Our purpose here is to construct some sectorial solution of the linearized functional equations 9.3 and 9.4 belonging to $\mathcal{O}(I_i)$, $i = 1, 3$. We will only detail this construction for $i = 1$ and the first equation, the remaining cases being completely similar (see Remark 9.1).

We are thus going to work in the domain S of the following shape. Consider the annulus $\mathcal{C}_{a,b} = \{a \leq |z^{-1}| \leq b\}$, $a, b > 0$, $b > |q|^{-1}a$. Consider the foliation defined by the level sets $\{J = c\}$ of $J = z^{-1}e^{-\lambda\xi}$ (recall that it corresponds to the fibration $dY = 0$ on a neighborhood of L_1 or L_3 depending on whether c is "small" or "big"). Fix $R \gg 0$ and let $\delta_0 > 0$ such that $\log \frac{1}{\delta} - \log a = R$. For every $0 < \delta \leq \delta_0$. Denote

$$S_{a,b,\delta} = \{(z, \xi) \in \mathbb{C}^2 \mid |J(z, \xi)| < \delta \text{ and } z \in \mathcal{C}_{a,b}\}.$$

Alternatively, this set can be described by the equation $\Re(\lambda\xi) > \frac{|z^{-1}|}{\delta}$, $z \in \mathcal{C}_{a,b}$ so that in particular $\arg \xi \in I_1$. For every complex number c , $0 < |c| < \delta$, consider

$$S_{c,a,b,\delta} = \{J = c\} \cap S_{a,b,\delta} = \{(z, \xi) = (c^{-1}e^{-\lambda\xi}, \xi) ; \xi \in D_{c,a,b}\}$$

where

$$D_{c,a,b} = \{\xi \in \mathbb{C} : (\log \frac{1}{|c|} + \log a) \leq \Re(\lambda\xi) \leq (\log \frac{1}{|c|} + \log b)\}.$$

Note that $\bigcup_{0 < |c| < \delta} S_{c,a,b,\delta} = S_{a,b,\delta}$ and that $F_0(S_{a,b,\delta}) = S_{|q|^{-1}a, |q|^{-1}b, \delta}$. It is thus coherent to investigate the existence of a solution h of 9.3 on the domain $S_{a,|q|^{-1}b,\delta} = S_{a,b,\delta} \cup F_0(S_{a,b,\delta})$ (recall that $|q^{-1}| = e^{-\Re\lambda} > 1$).

In what follows, we will indeed provide a solution of 9.3 with "good estimates" on a domain of the form $S_{a,|q|^{-1}b,\delta}$ using a "leafwise" resolution with respect to the foliation defined by the levels of J . For the sake of notational simplicity we will omit for a while the subscript a, b by setting $S_\delta := S_{a,b,\delta}$ and $S'_\delta := S_{a,|q|^{-1}b,\delta}$. If $\delta_1 \leq \delta_2$, remark that S_{δ_1} and S'_{δ_1} are respectively subdomains of S_{δ_2} and S'_{δ_2} . To state precisely our result, let us fix some additional notations and definitions. Let $m \geq 3$ a positive integer and consider the subspace H_δ^m of $\mathcal{H}(S_\delta)$ defined by the functions Δ such that $\|\Delta\|_m := \sup_{(z,\xi) \in S_\delta} |\Delta(z, \xi)| |\xi|^m < \infty$. We will also introduce the space $H_\delta^{\infty'}$ of holomorphic functions h on S'_δ equipped with the natural norm

$$\|h\|_\infty := \sup_{(z,\xi) \in S'_\delta} |h(z, \xi)| < \infty.$$

Theorem 9.2. *Fix a, b, δ_0 as above with δ_0 small enough. Then there exists a positive constant C enjoying the following properties:*

For every $\delta \leq \delta_0$ and every function Δ_δ in H_δ^m , there exists a unique function $h_\delta \in H_\delta^{\infty'}$ such that

- (1) $h_\delta \circ F_0 - h = \Delta_\delta$
- (2) $\|h\|_\infty \leq C \|\Delta_\delta\|_m$
- (3) For every $(z, \xi) \in S_\delta \cap \{\Im\xi \geq 1\}$, $|h_\delta(z, \xi)| \leq \frac{C \|\Delta_\delta\|_m}{\sqrt{|\Im(\xi)|}}$
- (4) There exists a positive number D depending only on $\theta \in [0, \frac{\pi}{2})$ such that for every $\delta \leq \delta_0$ and $(z, \xi) \in S'_\delta \cap \{\arg \xi + \arg q \leq \theta\}$, we have

$$|h_\delta(z, \xi)| \leq \frac{D \|\Delta_\delta\|_m}{|\zeta|^{m-2}}.$$

In order to prove this result, remark firstly that the equation

$$h \circ F_0 - h = \Delta_2$$

can be then rewritten as

$$(9.7) \quad \varphi_c(\zeta - \lambda) - \varphi_c(\zeta) = \Delta_c(\zeta)$$

where $\zeta = \lambda\xi$, $\varphi_c(\zeta) = h(c^{-1}e^{-\zeta}, \frac{\zeta}{\lambda})$, and $\Delta_c(\zeta) = \Delta_2(c^{-1}e^{-\zeta}, \frac{\zeta}{\lambda})$. We are then reduced to solve a family difference equations (the so-called *homological equations*) in the vertical strip $St_{c,a,b} = \{\zeta \in \mathbb{C} : \log \frac{1}{|c|} + \log a \leq \Re(\zeta) \leq \log \frac{1}{|c|} + \log b\}$ where we impose φ_c to be holomorphic, defined on the larger strip $St_{c,a,|q^{-1}|b} = \{\zeta \in \mathbb{C} : \log \frac{1}{|c|} + \log a \leq \Re(\zeta) \leq \log \frac{1}{|c|} + \log b - \Re\lambda\}$ and to depend analytically on the parameter c .

9.3. Resolution of a difference equation. We know proceed to the construction of φ_c . It is essentially a consequence of the following general result

Theorem 9.3. *Let $A, B \in (1, +\infty)$, $B > A - \Re\lambda$, $(B - \Re\lambda)/A \leq 2$. Let Δ be holomorphic on the strip $S_{A,B} = A \leq \Re \zeta \leq B$. Suppose moreover that $\|\Delta\|_m := \sup_{\zeta \in S_{A,B}} |\Delta(\zeta)| |\zeta|^m < \infty$ for some $m \geq 3$; then there exists a bounded holomorphic function φ on $S_{A,B-\Re\lambda}$ which solves*

$$(9.8) \quad \varphi(\zeta - \lambda) - \varphi(\zeta) = \Delta(\zeta).$$

Moreover φ is unique modulo an additive constant and one can choose φ such that there exists $C = C(B - A) > 0$ ¹⁹ such that the following estimate holds

$$(9.9) \quad \sup_{\zeta \in S_{A,B-\Re\lambda}} |\varphi(\zeta)| \leq C \|\Delta\|_m.$$

Proof. First notice, that if φ_1, φ_2 are two bounded holomorphic functions solving 9.8 together with the fact that $\Re\lambda \neq 0$, the difference $\varphi_1 - \varphi_2$ extends as a bounded λ -periodic entire function, hence constant. For $\zeta \in S_{A,B-\Re\lambda}$, let us define φ by the formula

$$\varphi(\zeta) = \sum_{n=0}^{\infty} F_n(\zeta)$$

where

$$F_n(\zeta) = \frac{1}{2i\pi} \left(\int_{L_+} \frac{\Delta(t)}{t - \zeta - (n+1)\lambda} dt + \int_{L_-} \frac{\Delta(t)}{t - \zeta + n\lambda} dt \right)$$

where $L_- = \{\Re t = A\}$ and $L_+ = \{\Re t = B\}$ both oriented from bottom to top.

Here, in accordance with Cauchy's formula one sets $\frac{1}{2i\pi} \int_{L_-} \frac{\Delta(t)}{t-u} dt := \frac{1}{2i\pi} \int_{L_+} \frac{\Delta(t)}{t-u} dt - \Delta(u)$ whenever $u \in L^-$ and $\frac{1}{2i\pi} \int_{L_+} \frac{\Delta(t)}{t-u} dt := \frac{1}{2i\pi} \int_{L_-} \frac{\Delta(t)}{t-u} dt + \Delta(u)$ if $u \in L^+$. With this convention, and thanks to the fact that $m \geq 2$, F_n is well defined and holomorphic on the strip $S_{A,B-\Re\lambda}$.

Moreover, if $\Re u \geq A - \frac{\Re\lambda}{2}$, one has $|\frac{1}{2i\pi} \int_{L_-} \frac{\Delta(t)}{t-u} dt| \leq \frac{\|\Delta\|_m}{|\Re\lambda|\pi} \int_{\Re t=1} \frac{|dt|}{|t|^2}$. If $A \leq \Re u \leq A - \frac{\Re\lambda}{2}$, Cauchy's formula yields the inequality

$$|\frac{1}{2i\pi} \int_{L_-} \frac{\Delta(t)}{t-u} dt| \leq (1 + \frac{1}{|\Re\lambda|\pi} \int_{\Re t=1} \frac{|dt|}{|t|^2}) \|\Delta\|_m.$$

Following the same principle, we get the inequality

$$|\frac{1}{2i\pi} \int_{L_+} \frac{\Delta(t)}{t-u} dt| \leq C_1 \|\Delta\|_m$$

¹⁹In order to give an unambiguous statement, m, λ are fixed but A, B, Δ are allowed to vary

where $C_1 = (1 + \frac{1}{|\Re\lambda|^\pi} \int_{\Re t=1} \frac{|dt|}{|t|^2})$ and $\Re u \leq B$. This eventually leads to the majoration

$$|F_n(\zeta)| \leq 2C_1 \|\Delta\|_m \quad \forall n \geq 0, \quad \forall \zeta \in S_{A, B-\Re\lambda}.$$

Moreover, when $n > \frac{A-B}{\Re\lambda}$, Cauchy's formula allows to write $\int_{L_+} \frac{\Delta(t)}{t-\zeta-n\lambda} dt = \int_{L_-} \frac{\Delta(t)}{t-\zeta-n\lambda} dt$. Consequently,

$$F_n(\zeta) = \frac{1}{2i\pi} \int_{L_-} \frac{\Delta(t)(2(t-\zeta)-\lambda)}{(t-\zeta+n\lambda)(t-\zeta-(n+1)\lambda)} dt$$

Set $v = \Im(t-\zeta)$. The key point will be to find a suitable upper bound for $A(t, \zeta) = \sum_{n \geq n_0} \left| \frac{1}{(t-\zeta+n\lambda)(t-\zeta-(n+1)\lambda)} \right|$ where $n_0 > 2(\frac{A-B}{\Re\lambda} + 1)$.

Set $v = \Im(t-\zeta)$. For $n \geq n_0$, Cauchy-Schwartz inequality gives
(9.10)

$$A(t, \zeta) \leq \left(\sum_{n \geq n_0} \frac{1}{(v+n\Im\lambda)^2 + (n\Re\lambda)^2} \right)^{\frac{1}{2}} \left(\sum_{n \geq n_0} \frac{1}{(v-(n+1)\Im\lambda)^2 + ((\frac{n}{2}\Re\lambda)^2)} \right)^{\frac{1}{2}}$$

This, together with the fact that $\|\Delta\|_m < \infty$, for some $m \geq 3$, proves immediately that the serie $\sum F_n$ converges uniformly on every compact of $S_{A, B-\Re\lambda}$. The function φ is then well defined and holomorphic on $S_{A, B-\Re\lambda}$ and in addition verifies the difference equation 9.8 as a simple application of Cauchy's Formula.

Moreover, by a simple integral comparison, one then obtains from 9.10 that

$$(9.11) \quad A(t, \zeta) \leq \frac{C_2}{|v|}$$

where C_2 is a positive number depending only on λ . For $n \geq n_0$, consider the decomposition $F_n = G_n + H_n$ where

$$G_n(\zeta) = \frac{1}{2i\pi} \int_{\{t \in L_-, |v| \leq 1\}} \frac{\Delta(t)(2(t-\zeta)-\lambda)}{(t-\zeta+n\lambda)(t-\zeta-(n+1)\lambda)} dt$$

and

$$H_n(\zeta) = \frac{1}{2i\pi} \int_{\{t \in L_-, |v| > 1\}} \frac{\Delta(t)(2(t-\zeta)-\lambda)}{(t-\zeta+n\lambda)(t-\zeta-(n+1)\lambda)} dt.$$

A simple glance at the inequality 9.10 shows that $|G_n(\zeta)| \leq C_3 \|\Delta\|_m$ where $C_3 > 0$ depends only on $B-A$. The upper bound given in 9.11 allows to assert that

$$\sum_{n \geq n_0} |H_n(\zeta)| \leq C_2 \frac{1}{2\pi} \int_{\{t \in L_-, |v| > 1\}} \frac{|\Delta(t)(2(t-\zeta)-\lambda)|}{|v|} |dt|.$$

In particular, there exists a positive number C_4 depending only on $B-A$ such that

$$\sum_{n \geq n_0} |H_n(\zeta)| \leq C_4 \|\Delta\|_m.$$

By putting together the inequalities above involving C_1, \dots, C_4 one obtains that φ satisfies the estimate 9.9 of the Theorem. \square

Remark 9.4. *If one takes a thorough look to the proof, one can observe that the solution constructed above satisfies the little bit more precise estimate:*

$$(9.12) \quad \sup_{\zeta \in S_{A, B-\Re\lambda}} |\varphi(\zeta)| \leq C \|\Delta\|_m \int_{L_-} \frac{|dt|}{|t|^m}$$

where C only depends on $B-A$.

9.4. Additional properties of φ .

Lemma 9.5. *Let φ as in the proof of Theorem 9.3. and set $m_0 = \frac{1}{2\lambda} \int_{L_-} \Delta(t) dt$. Then, there exists a constant $F = F(B - A)$ such that for every $\zeta \in S_{A, B - \Re\lambda}$ such that $\Im(\frac{\zeta}{\lambda}) \geq 1$, one has*

$$(9.13) \quad |\varphi(\zeta) - m_0| \leq \frac{F \|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

Moreover, for every $\theta \in [0, \frac{\pi}{2})$, there exists a constant $D = D(\theta, B - A) > 0$ such that for $\zeta \in K_\theta := S_{A, B - \Re\lambda} \cap \{|\text{Arg}(\zeta)| \leq \theta\}$

$$(9.14) \quad |\varphi(\zeta) - m_0| \leq \frac{D \|\Delta\|_m}{|\zeta|^{m-2}}$$

Proof. Let $\zeta \in S_{A, B - \Re\lambda}$, $\zeta \notin \mathbb{Z}\lambda$. Equivalently $\Im(\frac{\zeta}{\lambda}) \neq 0$. Write

$$\begin{aligned} F_n(\zeta) &= \frac{1}{2i\pi} \left(\int_{L_+} \frac{\Delta(t)}{t - \zeta - (n+1)\lambda} dt + \int_{L_+} \frac{\Delta(t)}{\zeta + (n+1)\lambda} dt - \int_{L_+} \frac{\Delta(t)}{\zeta + (n+1)\lambda} dt \right. \\ &\quad \left. + \int_{L_-} \frac{\Delta(t)}{\zeta - n\lambda} dt - \int_{L_-} \frac{\Delta(t)}{\zeta - n\lambda} dt + \int_{L_-} \frac{\Delta(t)}{t - \zeta + n\lambda} dt \right) \end{aligned}$$

Now using that

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{\zeta + n\lambda} = \frac{\pi}{\lambda} \cot\left(\pi \frac{\zeta}{\lambda}\right)$$

and summing-up the previous equality, one obtains

$$\begin{aligned} \varphi(\zeta) &= \frac{1}{2i\pi} \sum_{n \geq 0} \int_{L_+} \frac{t\Delta(t)}{(\zeta + (n+1)\lambda)(t - \zeta - (n+1)\lambda)} dt \\ &\quad + \frac{1}{2i\pi} \sum_{n \geq 0} \int_{L_-} \frac{t\Delta(t)}{(\zeta - n\lambda)(t - \zeta + n\lambda)} dt - \frac{i}{2\lambda} \cot\left(\pi \frac{\zeta}{\lambda}\right) \int_{L_-} \Delta(t) dt \end{aligned}$$

which makes sense for $\Im(\frac{\zeta}{\lambda}) \neq 0$. As before, consider a positive integer $n_0 \geq 2(\frac{A-B}{\Re\lambda} + 1)$, set $v = \Im(t - \xi)$ and define $B(t, \zeta) = \sum_{n \geq n_0} \frac{1}{(\zeta - n\lambda)(t - \zeta + n\lambda)}$ for $(t, \zeta) \in L^- \times S_{A, B - \Re\lambda}$. Here again, by the successive uses of Cauchy-Schwartz and integral comparison we obtain

$$B(t, \zeta) \leq \frac{F_1}{\sqrt{|\Im(\frac{\zeta}{\lambda})v|}}$$

where F_1 is some positive constant. As before, if one considers separately the case $|v| \leq 1$ and $|v| > 1$ and also $n \leq n_0$, we can easily establish for every $\zeta \in S_{A, B - \Re\lambda}$, $\Im(\frac{\zeta}{\lambda}) \geq 1$ the inequality:

$$\left| \frac{1}{2i\pi} \sum_{n \geq 0} \int_{L_-} \frac{t\Delta(t)}{(\zeta - n\lambda)(t - \zeta + n\lambda)} dt \right| \leq \frac{F_2 \|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

where $F_2 > 0$ only depends on $B - A$. Similarly, one gets

$$\left| \frac{1}{2i\pi} \sum_{n \geq 0} \int_{L_+} \frac{t\Delta(t)}{(\zeta + (n+1)\lambda)(t - \zeta - (n+1)\lambda)} dt \right| \leq \frac{F_3 \|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

where $F_3 > 0$ only depends on $B - A$. Moreover, for ζ lying in the same domain, it is clear that there exists a constant $F_4 > 0$ such that

$$\left| \frac{i}{2\lambda} \cot\left(\pi \frac{\zeta}{\lambda}\right) - \frac{1}{2\lambda} \right| \leq \frac{F_4}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

These different estimates prove 9.13.

Remark 9.6. Note that $|\Im(\frac{\zeta}{\lambda})|$ can be arbitrary big on the domain $S_{A+\Re\lambda, B}$. In particular $\varphi(\zeta) := \sum_{n \geq 0} F_n(\zeta) - m_0$ is the unique solution of the difference equation 9.8 for which there exists a positive number F (depending a posteriori only on $B - A$) such that for every ζ with $\Im(\frac{\zeta}{\lambda}) \geq 1$, one has

$$|\varphi(\zeta)| \leq \frac{F \|\Delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}$$

Now, let us establish the majoration 9.14. We use the same notations as in the proof of Theorem 9.3 by considering the solution $\varphi = \sum_{n \geq 0} F_n$. Now just observe, that on K_θ , $A \leq |\zeta| \leq \frac{B - \Re\lambda}{\cos\theta}$. In particular, one has $|\frac{t}{\zeta}| \geq \frac{\cos\theta}{2}$ for every $t \in L^-$. From Remark 9.4, one deduces that $\sup_{\zeta \in K_\theta} |\varphi(\zeta)| \leq \frac{C \|\Delta\|_m}{|\zeta|^{m-2}} \left(\frac{2}{\cos\theta}\right)^{m-2} \int_{L^-} \frac{|dt|}{|t|^2}$ where $C = C(B - A)$. A similar and straightforward majoration of m_0 gives the sought upper bound 9.14. \square

9.5. Version with parameters and proof of Theorem 9.2. Let $\delta_0 > 0$ and $0 < a < b$ such that $\log b > \log a - \Re\lambda$ where $\lambda \in \mathbb{C}$ has been fixed with $\Re\lambda < 0$. In accordance with the notations introduced in 9.2, for every positive number $\delta \leq \delta_0$ and complex number c such that $0 < |c| < \delta_0$, consider the strips in the complex line defined by

$$St_c = \left\{ \zeta \in \mathbb{C} : \left(\log \frac{1}{|c|} + \log a\right) \leq \Re(\zeta) \leq \left(\log \frac{1}{|c|} + \log b\right) \right\}$$

$$\widetilde{St}_c = St_c \cup (St_c - \Re\lambda)$$

and consider also the domain of \mathbb{C}^2 (with boundary), $\mathcal{S}_{\delta, \log a, \log b} = \bigcup_{c < \delta} \{c\} \times St_c$

Set $A_c = \log \frac{1}{|c|} + \log a$ and $B_c = \log \frac{1}{|c|} + \log b$. It is worth mentioning that $B_c - A_c = \log b - \log a$ does not depend on c . By a suitable choice of δ_0 , one can moreover assume that and that $A_c \geq 1$ and $\frac{B_c - \Re\lambda}{A_c} \leq 2$. Let Δ_δ be a holomorphic function defined on $\mathcal{S}_{\delta, \log a, \log b}$ and assume in addition that

$$\|\Delta_\delta\|_m := \sup_{(c, \zeta) \in \mathcal{S}_{\delta, \log a, \log b}} |\Delta_\delta(c, \zeta)| |\zeta|^m < \infty$$

for some $m \geq 3$. From the results collected in Section 9.3, 9.4, one promptly obtains the following statement:

Proposition 9.7. Let Δ_δ as above, then there exists a unique function $\varphi_\delta \in \mathcal{H}(\mathcal{S}_{\delta, \log a - \Re\lambda, \log b})$ with the following properties

- (1) For every $0 < c < \delta$, for every $\zeta \in \widetilde{St}_c$, one has $\varphi_\delta(c, \zeta - \lambda) - \varphi_\delta(c, \zeta) = \Delta_\delta(c, \zeta)$
- (2) There exists a positive number $C = C(\delta_0, a, b)$ such that

$$\begin{aligned} \sup_{(c, \zeta) \in \mathcal{S}_{\delta, \log a, \log b - \Re\lambda}} |\varphi_\delta(c, \zeta)| &\leq C \|\Delta_\delta\|_m \\ \sup_{(c, \zeta) \in \mathcal{S}_{\delta, \log a, \log b - \Re\lambda} \cap \Im(\frac{\zeta}{\lambda}) \geq 1} |\varphi_\delta(c, \zeta)| &\leq \frac{C \|\Delta_\delta\|_m}{\sqrt{|\Im(\frac{\zeta}{\lambda})|}}. \end{aligned}$$

Proof. Define φ_δ by the formula $\varphi_\delta(c, \zeta) = \sum_{n=0}^{\infty} F_n(c, \zeta) - m_0(c)$ where

$$F_n(c, \zeta) = \frac{1}{2i\pi} \left(\int_{L_+^c} \frac{\Delta_\delta(c, t)}{t - \zeta - (n+1)\lambda} dt + \int_{L_-^c} \frac{\Delta_\delta(c, t)}{t - \zeta + n\lambda} dt \right)$$

setting $L_-^c = \{\Re t = A_c\}$, $L_+^c = \{\Re t = B_c\}$ and $m_0(c) = \frac{1}{2\lambda} \int_{L_-^c} \Delta_\delta(c, t) dt$. From this integral formula, it is clear that for every n , F_n is well defined as a holomorphic function on $\mathcal{S}_{\delta, \log a, \log b - \Re \lambda}$ and that m_0 depends analytically on c . Moreover, one can easily verifies (as in Section 9.3) that the serie $\sum F_n$ converges uniformly on every compact subset of $\mathcal{S}_{\delta, \log a, \log b - \Re \lambda}$. Thus $\varphi \in \mathcal{H}(\mathcal{S}_{\delta, \log a, \log b - \Re \lambda})$ and fullfills the properties stated in the Proposition as a direct application of the construction performed in Section 9.3. \square

Then, the proof of Theorem 9.2 immediately follows when translating this existence and uniqueness result into the original variable (z, ξ) (see Section 9.2) together with the upper bound 9.14 of Lemma 9.5. \square

Once we have solutions to the linearized equation as achieved above, we are in position to obtain a solution to the general functional equations 9.5 and 9.6 by a standard fixed point Theorem. This is detailed in the following section.

9.6. Solving the functional equation. Notations as in 9.2. In particular, we will omit for a while the subscript a, b . As before, $\delta_0 > 0$ has been chosen small enough and δ is any positive number such that $0 < \delta \leq \delta_0$.

Let $m \geq 3$ and $H_\delta^{m, m}$ be the subspace of $\mathcal{H}(S_\delta) \times \mathcal{H}(S_\delta)$ defined by $N_m(\Delta, \Theta) < \infty$ where

$$N_m(\Delta, \Theta) = \sup_{(z, \xi) \in \mathfrak{S}_\delta} |\Delta(z, \xi)| |\xi|^m + \sup_{(z, \xi) \in S_\delta} |\Theta(z, \xi)| |\xi|^m$$

and let $H_\delta^{\infty, \infty'}$ be the subspace of $\mathcal{H}(S'_\delta) \times \mathcal{H}(S'_\delta)$ defined by $N'_\infty(h, \tilde{g}) < \infty$ where

$$N'_\infty(h, \tilde{g}) = \sup_{(z, \xi) \in S'_\delta} |h(z, \xi)| + \sup_{(z, \xi) \in S'_\delta \cap \mathcal{J}(\xi) \geq 1} |h(z, \xi)| \sqrt{|\mathfrak{J}(\xi)|}.$$

$$+ \sup_{(z, \xi) \in S'_\delta} |\tilde{g}(z, \xi)| + \sup_{(z, \xi) \in S'_\delta \cap \mathcal{J}(\xi) \geq 1} |\tilde{g}(z, \xi)| \sqrt{|\mathfrak{J}(\xi)|}.$$

Note that both normed spaces are Banach spaces. For every $M > 0$, set

$$H_\delta^{m, m}(M) = \{(\Delta, \Theta) \in H_\delta^{m, m} | N_m(\Delta, \Theta) \leq M\}$$

and

$$H_\delta^{\infty, \infty'}(M) = \{(h, \tilde{g}) \in H_\delta^{\infty, \infty'} | N'_\infty(h, \tilde{g}) \leq M\}.$$

By Theorem 9.2, one inherits from a continuous linear map \mathcal{L} between the normed spaces $(H_\delta^{m, m}, N_m)$ and $(H_\delta^{\infty, \infty'}, N'_\infty)$ defined by

$$\mathcal{L} : (\Delta, \Theta) \rightarrow (h, \tilde{g}) \quad \text{where} \quad (h \circ F_0 - h = \Delta, \tilde{g} \circ F_0 - \tilde{g} = \Theta).$$

To be more precise, there exists a positive constant C such that for every δ and every $(\Delta, \Theta) \in H_{a, \delta}$, one has

$$N'_\infty(\mathcal{L}(\Delta, \Theta)) \leq CN_m(\Delta, \Theta)$$

9.7. Substitution map. Let us come back to the expression to the of the transformation F defining the formally equivalent neighborhood (U, C') as in 9.1. One will fix an arbitrary integer $N > m$ such that $\Delta_i(z, \xi) = O(\xi^{-N})$. Then up to adjust δ_0 , one can assume that we have a map well defined for every $0 < \delta \leq \delta_0$ by

$$\mathcal{R} : \begin{array}{c} H_\delta^{\infty, \infty'}(1) \rightarrow H_\delta^{m, m} \\ \vec{h} = (h_1, h_2) \rightarrow \vec{\delta}' = (\delta_1, \delta_2) \end{array}$$

where

$$\begin{aligned} \delta_1(z, \xi) &= \Delta_2(z(1 + h_2(z, \xi)), \xi + h_1(z, \xi)) \\ \delta_2(z, \xi) &= (1 + h_2)\widetilde{\Delta}_1(z(1 + h_2(z, \xi)), \xi + h_1(z, \xi)) \end{aligned}$$

Because $\Delta_i = O(\xi^{-N})$, note also that the image of $H_\delta^{\infty, \infty'}(1)$ by \mathcal{R} lies in $H_\delta^{m, m}(R_\delta)$ where $\lim_{\delta \rightarrow 0} R_\delta = 0$.

Lemma 9.8. *Let $\varepsilon > 0$ such that $\varepsilon C < 1$. Then, up to shrink δ_0 , for every $\delta \leq \delta_0$ one has,*

$$\forall \vec{h}, \vec{g} \in H_\delta^{\infty, \infty'}(1) \quad N_m(\mathcal{R}(\vec{h}) - \mathcal{R}(\vec{g})) \leq \varepsilon N'_\infty(\vec{h} - \vec{g})$$

Proof. Exercise. □

The map $\mathcal{L} \circ \mathcal{R}$ then induces a (non linear) contracting operator of the complete metric space $H_\delta^{\infty, \infty'}(1)$. The unique fixed point is a solution to the functional equations 9.5, 9.6. This provides a solution of the original functional equations 9.3 and 9.4, taking into account the renormalization indicated in Remark 9.1. By uniqueness, the solution $(h_\delta, \tilde{g}_\delta)$ attached to δ induces by restriction the solution attached to δ' for $\delta' \leq \delta$.

One can complete this picture by additional properties borrowed from Lemma 9.5. This leads to the following statement where we reintroduce the subscript a, b (with obvious notations) in order to recall that the choice of δ depends of a fixed arbitrary annulus in the z variable.

Proposition 9.9. *Notations as above. Let $\Delta_2, \widetilde{\Delta}_1 = O(\xi^{-N})$ two germs of holomorphic functions in the neighborhood of $\tilde{C} \subset \mathbb{C}^2$ with $N \geq 4$ (as defined from the conjugation equation introduced in 9.1). Let $m < N$. Let $0 < a < b < +\infty$ such that $b > |q|^{-1}a$. Then there exists $\delta(a, b) > 0$ such that for every $0 < \delta \leq \delta(a, b)$, the system of equations 9.5, 9.6 admit a unique solution $(h_{\delta, a, b}, \tilde{g}_{\delta, a, b}) \in H_{\delta, a, b}^{\infty, \infty'}(1) \times H_{\delta, a, b}^{\infty, \infty'}(1)$. Moreover,*

- $(h_{\delta', a, b}, \tilde{g}_{\delta', a, b})$ is the restriction of $(h_{\delta, a, b}, \tilde{g}_{\delta, a, b})$ if $0 < \delta' \leq \delta \leq \delta(a, b)$.
- $\lim_{\delta \rightarrow 0} N'_\infty((h_{\delta, a, b}, \tilde{g}_{\delta, a, b})) = 0$.
- there exists a positive number $D(\theta)$ depending only $\theta \in [0, \frac{\pi}{2})$ such that for every $\delta \leq \delta(a, b)$ and $(z, \xi) \in S'_{\delta, a, b} \cap \{|\arg \xi + \arg \lambda| \leq \theta\}$, we have

$$|h_{\delta, a, b}(z, \xi)| \text{ and } |\tilde{g}_{\delta, a, b}(z, \xi)| \leq \frac{D}{|\xi|^{m-2}}$$

9.8. Asymptotic expansion. We start by fixing $N \geq 4$ and a, b as before. Let a', b' be positive real numbers such that $a' \leq a < b \leq b'$. By Proposition 9.9, note that the unique solution of 9.5, 9.6 lying in $H_{\delta, a', b'}^{\infty, \infty'}(1) \times H_{\delta, a', b'}^{\infty, \infty'}(1)$, $\delta > 0$ small enough, induces by restriction the unique solution of the same functional equation in $H_{\delta, a, b}^{\infty, \infty'}(1) \times H_{\delta, a, b}^{\infty, \infty'}(1)$. In particular, when taking the projective limit with respect to (a, b) , $(a, b) \rightarrow (0, +\infty)$ and exploiting the last asymptotic estimate in the proposition above, one get a solution (h, \tilde{g}) well defined as a flat element of $\mathcal{A}^{m-3}(I_1) \times \mathcal{A}^{m-3}(I_1)$.

Now, consider an integer $p \gg N$ arbitrarily large. Let α a positive integer and consider the truncation $Tr_\alpha(\hat{H})$ of \hat{H} at order α : $Tr_\alpha(\hat{H})(z, \xi) = (z + \sum_{1 \leq n \leq \alpha} b_n \xi^{-n}, \xi + \sum_{0 \leq n \leq \alpha} a_n \xi^{-n})$. If α is large enough, one has

$$(Tr_\alpha \hat{H})^{-1} \circ F \circ Tr_\alpha \hat{H}(\xi, z) = (qz + \Delta_1^\alpha(z, \xi), \xi - 1 + \Delta_2^\alpha(z, \xi))$$

where $\Delta_i^\alpha(z, \xi) = O(\frac{1}{\xi^p})$. One can apply Proposition 9.9 to get existence and uniqueness of $h_\delta^\alpha, \tilde{g}_\delta^\alpha \in H_{\delta, a, b}^{\infty, \infty'}(1)$ (δ small enough)

$$F \circ Tr_\alpha \hat{H} \circ H_\alpha = Tr_\alpha \hat{H} \circ H_\alpha \circ F_0$$

with $H_\alpha = \text{Id} + (g_\delta^\alpha, h_\delta^\alpha)$, $g_\alpha(z, \xi) = z \tilde{g}_\alpha(z, \cdot)$. As before, these solutions are in fact induced by a flat element of $\mathcal{A}^{p-4}(I_1) \times \mathcal{A}^{p-4}(I_1)$. Set $H = \text{Id} + (g, h)$ with $g = z \tilde{g}$ and recall that $F \circ H = H \circ F_0$. Invoking again uniqueness and by simple restrictions considerations, one obtains that $H = Tr_\alpha \hat{H} \circ H_\alpha$. Thus, $H = (z + \sum_{1 \leq n \leq \text{Inf}(\alpha, p-4)} a_n \xi^{-n}, \xi + \sum_{1 \leq n \leq \text{Inf}(\alpha, p-4)} b_n \xi^{-n}) + R_\alpha$ where $R_\alpha \in \mathcal{A}^{p-4}(I_1)$ is flat. As α (hence p) can be chosen arbitrarily large, we eventually get that $H \in G^1(I_1)$ and admits \hat{H} as asymptotic expansion. We have thus obtain the sought normalization on the sector I_1 .

9.9. Construction of other sectorial normalizations.

9.9.1. *On I_3 .* It is just a slight modification of the previous construction for I_1 . The starting domain consists again in the resolution of the linearized equation 9.3 on the corresponding domain. In this situation, it is relevant to deal with the levels $J = c$ of the first integral $J = z^{-1} e^{-\lambda \xi}$ for $|c| \gg 0$. One adapts the notation of Section 9.2 by defining $S_{a, b, \delta} = \{(z, \xi) \in \mathbb{C}^2 \mid |J(z, \xi)| > \delta \text{ and } z \in \mathcal{C}_{a, b}\}$ ($\delta \gg 0$). Here again, this amounts to solve an analytic family of difference equations where the solutions satisfy some estimates which eventually leads by the same fixed point consideration to a normalizing conjugation map. This can be carried out following verbatim the same method.

9.9.2. *On I_2 and I_4 .* As indicated before, one can mimick the construction of normalising sectorial map by changing accordingly the presentation of the neighborhoods. This is roughly explained at the end of Section 9.1. To do this in more details, first remark that (U_0, C) and (U, C) can be presented as the quotients by pair of commuting transformations $(\mathbb{C}_x \times \overline{\mathbb{C}}_\xi, \{\xi = \infty\})/(t, f_0)$ and $(\mathbb{C}_x \times \overline{\mathbb{C}}_\xi, \{\xi = \infty\})/(t, f)$ where $t(x, \xi) = (x + 1, \xi)$, $f_0(x, \xi) = (x + \tau, \xi - 1)$ and $f(x, \xi) = (x + \tau, \xi - 1) + O(\xi^{-N})$ is the expression of F in the uniformizing coordinates (x, ξ) . In those coordinates the formal conjugacy map \hat{H} reads as $\hat{u}(x, \xi) = (x, \xi) + O(\xi^{-1})$ (which then commutes with t and satisfies $\hat{u}^{-1} \circ f \circ \hat{u} = f_0$). The cyclic covering space of both neighborhoods associated to the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is then described by the respective quotients $(\mathbb{C}_x \times \overline{\mathbb{C}}_\xi, \{\xi = \infty\})/(f_0)$ and $(\mathbb{C}_x \times \overline{\mathbb{C}}_\xi, \{\xi = \infty\})/(f)$. These quotients are both germs of analytic neighborhoods of \mathbb{C}^* , hence are isomorphic. More precisely there exists a germ of biholomorphism Ψ of $(\mathbb{C}_x \times \overline{\mathbb{C}}_\xi, \{\xi = \infty\})$ tangent to identity along $\{\xi = \infty\}$ such that $f_0 = \Psi \circ f \circ \Psi^{-1}$. In particular $f' = \Psi \circ t \circ \Psi^{-1}$ commutes with f_0 and then induced a transformation $(z', \xi') \rightarrow F'(z', \xi') = F_0'(z', \xi') + O(\xi'^{-N})$ where $F_0'(z', \xi') = ((q'z', \xi' + \frac{1}{\tau}), q' = \exp \frac{2i\pi}{\tau})$, on the germ of neighborhood $(\mathbb{C}_{z'}^* \times \overline{\mathbb{C}}_{\xi'}, \{\xi' = \infty\})$ equipped with the coordinates $z' = \exp \frac{2i\pi}{\tau} x$, $\xi' = \xi + \frac{x}{\tau}$. For the sake of clarity, recall that F_0' is the transformation induced by t in the (z', ξ') coordinate.

By construction, $\hat{v}^{-1} \circ f' \circ \hat{v} = t$ where \hat{v} is the formal transformation $\Psi \circ \hat{u}$. Moreover \hat{v} commutes with f_0 , hence descends to $(\mathbb{C}_{z'}^* \times \overline{\mathbb{C}}_{\xi'}, \{\xi' = \infty\})$ to a formal conjugacy map of the form $\widehat{H}'(z', \xi')$ between F' and F'_0 . Now, the fundamental biholomorphism

$$\Pi(z, \xi) = (e^{2i\pi\xi}, z^{-1}e^{-\lambda\xi})$$

between the deleted neighborhood $U_0 - C$ and $V - \bigcup_{i \in \mathbb{Z}_4} L_i$ reads in the (z', ξ') coordinates as $\Pi'(z', \xi') = ((z')^{-1}e^{2i\pi\xi'}, e^{-\lambda\xi'})$. This allows to play the same game than before. Indeed, the substitute for the previous first integral J is $J'(z', \xi') = (z')^{-1}e^{2i\pi\xi'}$ (a first integral for the foliation defined by $d\xi = 0$). Small and big values of J' correspond respectively, via the fundamental isomorphism, to the fibers of the vertical fibration $dX = 0$ on the neighborhood of L_4 and L_2 . One can thus produce, exactly by the same process, a normalizing sectorial conjugation map $H' \in G^1(I_2 \cup I_4)$ (with respect to the (z', ξ') coordinate having the same asymptotic expansion than \widehat{H}'). By the process which allows to pass from the (z, ξ) to the (z', ξ') coordinate described above, this sectorial transformation map, lift to sectorial transformation map $H \in G^1(I_2 \cup I_4)$ (relatively to (z, ξ)) conjugating F to F_0 and having \widehat{H} as asymptotic expansion.

This eventually finishes the proof of Lemma A (and more precisely Lemma 4.1).

10. GENERALIZATION

10.1. General formal classification: recollections. We first recall (under a slightly more synthesized form) the results obtained in [11] concerning the formal classification of zero type neighborhood of elliptic curves with torsion normal bundle and finite ueda type. We do not address the case where C fits into a formal fibration (corresponding to infinite Ueda type). According to a result due to Ueda, this fibration is indeed analytic and we fall into the case which satisfies the formal principal: there is no differences between analytic and formal classification and this latter is very easy to describe, see for instance [11]Section 5.1.

Let m be the torsion order of the normal bundle N_C and let $k > 0$ be the Ueda type. Recall that this latter is necessary a multiple of m . The linear monodromy of the corresponding unitary connection along the loops 1 and τ is respectively determined by two roots of unity a_1, a_τ of respective orders m_1 and m_τ such that $\text{lcm}(m_1, m_\tau) = m$. The triple (a_1, a_τ, k) is obviously a formal invariant of the neighborhood. A complete set of invariant is indeed provided in [11, Section 5.2], from which we borrow and adapt the notation.

Set $\varphi_{k,\nu} = \exp(\frac{y^{k+1}}{1+\nu y^k} \frac{\partial}{\partial y})$. Let $-m \leq p < k$ with $p \in m\mathbb{Z}$. Set $k = mk'$ and $p = mp'$. Given $P(z) = \sum_{i=0}^{p'} \lambda_i z^i$ a polynomial of degree p' precisely (with the convention that P vanishes identically if $p' = -1$), define

$$(10.1) \quad \omega_P := P\left(\frac{1}{y^m}\right) \frac{dy}{y}, \quad \text{and} \quad g_{k,\nu,P}(y) := \int_0^y [(a_\tau \varphi_{k,\nu})^* \omega_P - \omega_P].$$

The group $\mathbb{Z}_{k'}$ of k'^{th} roots of unity acts on the set of polynomials P as follows:

$$(10.2) \quad (\mu, P(z)) \mapsto P(\mu^{-1}z)$$

Theorem 10.1. [11] *Notations as above. There exist $\nu \in \mathbb{C}$ and $P \in \mathbb{C}[z]$ of degree p' exactly unique up to the $\mathbb{Z}_{k'}$ -action (10.2) such that (U, C) is formally equivalent to the quotient of $(\tilde{U}, \tilde{C}) := (\mathbb{C}_x \times \mathbb{C}_y, \{y = 0\})$ by the group generated by*

$$(10.3) \quad \begin{cases} \phi_1(x, y) &= (x + 1, a_1 y) \\ \phi_\tau(x, y) &= (x + \tau + g_{k,\nu,P}(y), a_\tau \varphi_{k,\nu}(y)) \end{cases}$$

The pencil $\omega_t = \omega_0 + t\omega_\infty$ of closed 1-forms is generated by

$$(10.4) \quad \omega_0 = \frac{dy}{y^{k+1}} + \nu \frac{dy}{y} \quad \text{and} \quad \omega_\infty = dx - \omega_P.$$

Actually (see [11]), the case $p' = -1$ (i.e. $\omega_P = 0$) corresponds exactly to the existence of a transverse fibration, given at the level of the formal normal forms described above by $\omega_\infty = dx$. If in addition, $k = \nu = 0$ (and necessarily $m = 1$) one recovers Serre's example. The case $p' = 0$, i.e. $P = \frac{1}{c}$, $c \in \mathbb{C}^*$ is the constant polynomial, is covered by Theorem 5.3 in *loc.cit* whereas $p' > 0$ is covered by Theorem 5.4.

When $m = 1$, that is N_C analytically trivial, one observes that ϕ_τ is the flow at time 1 of the holomorphic vector field

$$X_{k,\nu,P} = X_0 + \tau X_\infty$$

where

$$X_0 = \frac{y^{k+1}}{1 + \nu y^k} \frac{\partial}{\partial y} + P\left(\frac{1}{y}\right) \frac{y^k}{1 + \nu y^k} \frac{\partial}{\partial x}.$$

and

$$X_\infty = \frac{\partial}{\partial x}$$

are the dual vector fields of the pair $(\omega_0, \omega_\infty)$. we will denote by $(U_{k,\nu,P}, C)$ the corresponding neighborhood. In the specific situation where $k = 1$, and then $P = \mu$ is constant, we will use the notation $(U_{\nu,\mu}, C)$ and $X_{\nu,\mu} := X_{1,\nu,P}$.

In the coordinates $z = e^{2i\pi x}$ and $\xi = 1/y$, the corresponding presentations are

$$(U_{k,\nu,P}, C) := (\mathbb{C}_z^* \times \overline{\mathbb{C}_\xi}, \{\xi = \infty\}) / \langle F_{k,\nu,P} \rangle$$

where $F_{k,\nu,P} = \exp X_{k,\nu,P}$ and $X_{k,\nu,P} = X_0 + \tau X_\infty$ with $X_0 = \frac{1}{\xi^{k+\nu}} (-\xi \frac{\partial}{\partial \xi} + 2i\pi z P(\xi) \frac{\partial}{\partial z})$, $X_\infty = 2i\pi z \frac{\partial}{\partial z}$. This can be directly borrowed from the presentation given in the (x, y) variable. For notational coherence and simplicity, we will set $F_{\nu,\mu} = \exp X_{\nu,\mu}$. Serre's example is obtained by taking $F_0 := F_{0,0}$. We are now ready to undertake the analytic classification.

10.2. Trivial normal bundle, Uedatype 1. Consider formal models $(U_{\nu,\mu}, C)$ parametrized by $(\nu, \mu) \in \mathbb{C}^2$: $(U_{\nu,\mu}, C) = (\mathbb{C}_z^* \times \overline{\mathbb{C}_\xi}, \xi = \infty) / \langle F_{\nu,\mu}(\xi, z) \rangle$ where $F_{\nu,\mu} = \exp X_{\nu,\mu}$ and $X_{\nu,\mu} = \frac{1}{\xi+\nu} (-\xi \frac{\partial}{\partial \xi} + 2i\pi \mu z \frac{\partial}{\partial z}) + \lambda z \frac{\partial}{\partial z}$. From this description, one easily gets that $F_{\nu,\mu} = \Phi_{\nu,\mu}^{-1} \circ F_0 \circ \Phi_{\nu,\mu}$ where $\Phi_{\nu,\mu}(z, \xi) = (z\xi^{2i\pi\mu}, \xi + \nu \log \xi)$. Note that $\varphi(\xi) = \xi + \nu \log \xi$ makes sense as a transformation of a germ of sector of the form $|\xi| \gg 0$ and $\arg \xi \in]\theta_1, \theta_2[$, $\theta_2 - \theta_1 < 2\pi$ and that $\varphi^{-1}(\xi) = \xi(1 + o(1))$. Then $\Phi_{\nu,\mu}^{-1}(z, \xi) = (z(\varphi^{-1}(\xi))^{-2i\pi\mu}, \varphi^{-1}(\xi))$. Exactly as before, one can express the germ of deleted neighborhood $U_\nu - C$ as the deleted neighborhood of four lines via the multivaluate function $\Pi_{\nu,\mu}(z, \xi) = \Pi \circ \Phi_{\nu,\mu}(z, \xi) = (\xi^{2i\pi\nu} e^{2i\pi\xi}, z^{-1} \xi^{-2i\pi(\mu+\tau\nu)} e^{-\lambda\xi})$.

The multivaluation of $\Pi_{\nu,\mu}$ impose that U_1 and U_4 glue together by the linear cocycle $(X, Y) \rightarrow (e^{-4\pi^2\nu} X, e^{4\pi^2(\tau\nu+\mu)} Y)$. One can adapt without additional difficulty the method detailed in Section 1.3 in the case $\nu = \mu = 0$ to produce infinitely many analytic class constituted (up to equivalence) of elements of the form (V_φ, C) where $\varphi = (\varphi_{i,i+1}, i \in \mathbb{Z}_4) \sim \psi = (\psi_{i,i+1}, i \in \mathbb{Z}_4)$ is defined as before except that $\varphi_{4,1}$ as linear part $(e^{-4\pi^2\nu}, e^{4\pi^2(\tau\nu+\mu)})$. Note that we can realize arbitrary linear part of $\varphi_{1,4}$ and consequently all complex structures on neighborhood of 4-cycle of rational curves (demanding that each rational curve is embedded with vanishing self-intersection).

The construction of normalizing sectorial transformations maps is based the same idea: resolution of an analytic families of difference equations followed by a standard fixed point method which can be carried out using the analytic conjugation induced by $\Phi_{\nu,\mu}$ over each relevant sector $I_i, i = 1, \dots, 4$ between $F_{0,0}$ and $F_{\nu,\mu}$. In particular, the collection of V_φ provides complete set of analytic moduli. This allows as before to describe the full analytic moduli space. Concerning the construction of normalizing maps, we have to take here into account the "wild" behaviour of $\Phi_{\nu,\mu}$ along C represented by the term $z\xi^{2i\pi\mu}$. For notational simplicity, we will set $\Phi := \Phi_{\nu,\mu}$ in the sequel. Let us explicit the modifications needed: let (U, C) be formally equivalent to $(U_{\nu,\mu}, C)$. We want to show that (U, C) has the form U_φ .

One can suppose that $(U, C) = (\tilde{U}, \tilde{C})/F$ where $F(z, \xi) = F_{\nu,\mu}(z, \xi) + (\Delta_1, \Delta_2)$ with $\Delta_i = O(\xi^{-N})$ where $N \gg 0$ is a positive integer sufficiently large and that there exists a formal diffeomorphism of $(\tilde{U}, \tilde{C}), \hat{H}(z, \xi) = (z + \hat{g}, \xi + \hat{h}), \hat{h} = \sum_{n \geq 1} a_n \xi^{-n}, \hat{g} = \sum_{n \geq 1} b_n \xi^{-n}$ where a_n, b_n are entire functions on $\tilde{C} = \mathbb{C}^*$ such that

$$F \circ \hat{H} = \hat{H} \circ F_{\mu,\nu}.$$

One wants as before to replace \hat{H} by an analytic function H , that is

$$(10.5) \quad F \circ H = H \circ F_{\mu,\nu}$$

defined on a suitable domain (analogous to that of Section 9.1) that we proceed to describe now. One adopts the notation of loc.cit.

Consider the annulus $\mathcal{C}_{a,b} = \{a \leq |z^{-1}| \leq b\}, a, b > 0, b > |q|^{-1}a$. Consider the foliation defined by the level sets $\{J_{\nu,\mu} = c\}$ of $J_{\nu,\mu} = J \circ \Phi_{\nu,\mu} = z^{-1}\xi^{-2i\pi(\mu+\tau\nu)}e^{-\lambda\xi}$ (recall that it corresponds to the fibration $dY = 0$ on a neighborhood of L_1 or L_3 depending on whether c is "small" or "big"). Let $\delta_0 > 0$ sufficiently small. For every $0 < \delta \leq \delta_0$. Note $S_{a,b,\delta} = \{(z, \xi) \in \mathbb{C}^2 \mid |J_{\nu,\mu}(z, \xi)| < \delta \text{ and } z \in \mathcal{C}_{a,b}\}$. For every complex number $c, 0 < |c| < \delta$, consider $S_{c,a,b,\delta} = \{J_{\nu,\mu} = c\} \cap S_{a,b,\delta}$.

Note that $\bigcup_{0 < |c| < \delta} S_{c,a,b,\delta} = S_{a,b,\delta}$ and that $F_{\nu,\mu}(S_{a,b,\delta}) \simeq S_{|q^{-1}|a, |q^{-1}|b, \delta}$.

It is thus coherent to investigate the existence of a solution H of 10.5 on the domain $S'_{a,b,\delta} := S_{a,b,\delta} \cup F_{\nu,\mu}(S_{a,b,\delta})$. Remark now that the conjugacy equation 10.5 can be equivalently rewritten

$$F_\Phi \circ H_\Phi = H_\Phi \circ F_0$$

where $F_\Phi = \Phi \circ F \circ \Phi^{-1}, H_\Phi = \Phi \circ H \circ \Phi^{-1}$ (the parameters (ν, μ) have been deliberately omitted for the sake of clarity). This amounts to determine H_Φ on the domain $\Phi(S'_{a,b,\delta})$.

To do this, note that in view of asymptotic behavior of φ and φ^{-1} described below, one has $F_\Phi = F_0 + (\Delta_{1,\Phi}, \Delta_{2,\Phi})$ where one can verifies that $(\widetilde{\Delta_{1,\Phi}}, \Delta_{2,\Phi}) = O(\frac{1}{\xi^N})$ where $\lambda z \widetilde{\Delta_{1,\Phi}} = \Delta_{1,\Phi}$.

We are looking for solution of the form $H_\Phi = \text{Id} + (h_\Phi, g_\Phi)$. Here again, this can be reformulated as

$$(10.6) \quad h_\Phi \circ F_0 - h_\Phi = \Delta_{2,\Phi} \circ H_\Phi$$

$$(10.7) \quad g_\Phi \circ F_0 - qg_\Phi = \Delta_{1,\Phi} \circ H_\Phi$$

First, we will still deal with the linearized equations. This can be reformulated as

$$(10.8) \quad h \circ F_0 - h = \Delta_2$$

$$(10.9) \quad g \circ F_0 - qg = \Delta_1$$

where we have omit the subscript Φ for notational convenience.

In what follows, we will indeed provide a solution of 10.8 (hence also for 10.9 by the usual transform involving $\widetilde{\Delta}_1$) with "good estimates" on a domain of the form $\Phi(S'_{a,b,\delta})$ using a "leafwise" resolution with respect to the foliation defined by the levels of J . This consists as before in solving a family of difference equations parametrized by the leaves space in the variable ξ where the domains are slightly modified as explained now. For every complex number c , $0 < |c| < \delta$, consider $S_{c,a,b,\delta} = \{J_{\nu,\mu} = c\} \cap S_{a,b,\delta}$ and $\Phi(S_{c,a,b,\delta}) = \{(z, \xi) = (c^{-1}e^{-\lambda\xi}, \xi)\}$ where

$$\xi \in D_{c,a,b} = \{\xi \in \mathbb{C} : (\log \frac{1}{|c|} + \log a) \leq \Re(\lambda\xi + 2i\pi\mu \log(\varphi^{-1}(\xi))) \leq (\log \frac{1}{|c|} + \log b)\}$$

where a determination of the logarithm has been chosen in the sector in which we are working, namely $\arg \xi \in I_1$.

The remaining part of the proof then follows *mutatis mutandis* the same line than before by noticing that the equation

$$h \circ F_0 - h = \Delta_2$$

can be then rewritten as

$$(10.10) \quad \varphi_c(\zeta - \lambda) - \varphi_c(\zeta) = \Delta_c(\zeta)$$

where $\zeta = \lambda\xi$, $\varphi_c(\zeta) = h(c^{-1}e^{-\zeta}, \frac{\zeta}{\lambda})$, $\Delta_c(\zeta) = \Delta_2(c^{-1}e^{-\zeta}, \frac{\zeta}{\lambda})$.

We are then reduced to solve a family difference equations (the so-called *homological equations*) in the "quasi" vertical strip

$$St_c = \{\zeta \in \mathbb{C} : (\log \frac{1}{|c|} + \log a) \leq \Re(\zeta + 2i\pi\mu \log(\varphi^{-1}(\frac{\zeta}{\lambda}))) \leq (\log \frac{1}{|c|} + \log b)\}$$

depending analytically on the parameter c and we investigate the existence of a solution φ_c on the domain $St_c \cup St_c - \lambda$. This can be carried out by resolving equation 9.8 using the same method (that is, essentially Cauchy formula) replacing accordingly the integration along the vertical lines L^-, L^+ by $l(\zeta) = A, B$ where $l(\zeta) = \Re(\zeta + 2i\pi\mu \log(\varphi^{-1}(\frac{\zeta}{\lambda})))$. One also obtain in a similar way the same kind of estimates imposing the uniqueness of the solution. This allows by the fixed point method detailed in the previous section to solve the conjugation equation under the forms 10.6 and 10.7 on the relevant domains and finally exhibit a conjugacy sectorial transformation solution of 10.5 $H = \Phi^{-1} \circ H_\Phi \circ \Phi$, well defined as a section of G^1 over I_1 and having \hat{H} as asymptotic expansion. One can analogously construct conjugacy maps on the other sectors.

10.3. Trivial normal bundle, finite Ueda type. Still using the formal classification of [11] as recalled in 10.1, one can investigate, without any further fundamental change, the analytic classification of neighborhood of elliptic curves with arbitrary finite Ueda's type. For the sake of simplicity, we will first focus on the case where the normal bundle N_C of C is analytically trivial. The formal normal forms $(U_{k,\nu,P}, C)$ are parametrized by the triple (ν, k, P) where ν is a complex number, $k \in \mathbb{N}_{>0}$ is the Ueda type P is a polynomial map of one indeterminate of degree $< k$ uniquely defined modulo a certain action of the k^{th} roots of unity on the coefficients of P as described in 10.1.

Following 10.1, one has $F_{k,\nu,P} = \Phi^{-1} \circ F_0 \circ \Phi$ where $\Phi(z, \xi) = (ze^{\int \frac{2i\pi P(\xi)d\xi}{\xi}}, \frac{\xi^k}{k} + \nu \log \xi)$ and consequently $\Phi^{-1}(z, \xi) = (ze^{-Q(\varphi^{-1}(\xi))}, \varphi^{-1}(\xi))$, where $\varphi(\xi) = \frac{\xi^k}{k} + \nu \log \xi$ and $Q(\xi) = \int \frac{2i\pi P(\xi)d\xi}{\xi}$. Note that Φ and Φ^{-1} make sense as univalued functions on sectors (with respect to the ξ variable) of opening $< 2\pi$ and that their expressions depend on the choice of the determination of $\log \xi$. Moreover $\varphi^{-1}(\xi) = k^{\frac{1}{k}} \xi^{\frac{1}{k}} + o(|\xi|^{\frac{1}{k}})$.

Let us choose 4 intervals $I_i^0, i = 1, \dots, 4$ such that $kI_i^0 = I_i$ with non empty overlaps $I_i^0 \cap I_{i+1}^0, i = 1, 2, 3$. Set $I_i^l = I_i^0 + \frac{2l\pi}{k}, l = 0, \dots, 3$. The application $\Pi_{k,\nu,P}(z, \xi) := \Pi \circ \Phi(z, \xi) = (e^{\frac{2i\pi\xi^k}{k}} \xi^{2i\pi\nu}, z^{-1} e^{P_1(\xi)} \xi^{-\lambda\nu - 2i\pi a_0})$, where a_0 is the constant term of P and P_1 is the degree k polynomial $\int \frac{-2i\pi(P(\xi) - a_0)d\xi}{\xi} - \lambda \frac{\xi^k}{k}$, maps the germ of transversely sectorial domain of opening I_i^l (which is $F_{k,\nu,P}$ invariant) onto the germ of deleted neighborhood of L_i . The full picture is thus obtained by taking the successive family of sectors with consecutive overlaps $(I_1^0, \dots, I_4^0, I_1^1, \dots, I_4^1, \dots, I_1^{k-1}, \dots, I_4^{k-1})$ and their corresponding system of transversal sectorial neighborhoods. This defines a cover of the deleted neighborhood $U_{k,\nu,P} - C$ equipped with a system of semi-local charts defined by $\Pi_{k,\nu,P}$. If one follows the cyclic order defined by the succession of overlapping sectors, one conclude that this deleted neighborhood can be represented the deleted neighborhood of a cycle (L_i^l) of $4k$ rational curves of zero type, namely the image under $\Pi_{k,\nu,P}$ of these sectors. The neighborhood of L_i^l is equipped with the standard coordinate system of V_i (neighborhood of L_i in $\mathbb{P}^1 \times \mathbb{P}^1$ and the complex structure of the total neighborhood is determined by trivial glueings along the axis except for the germ of axis determined by $(L_1^0 \cap L_4^{k-1}, L_1^0, L_4^{k-1})$ where identification is made by the monodromy \mathcal{L} of $\Pi_{k,\nu,P}$ which is given by the linear diagonal map $(X, Y) \rightarrow (e^{-4\pi^2\nu} X, e^{4\pi^2(\tau\nu + a_0)} Y)$.

Concerning the construction of normalizing maps, let us explicit as before the modifications needed: let (U, C) be formally equivalent to $(U_{\nu,\mu}, C)$. We want to show that (U, C) has the form U_φ where φ consists in a collection $\varphi_{i,i+1}^l$ of $4k$ germs of glueing biholomorphisms $\varphi_{i,i+1}^l$ localized at the axis $(L_i^l \cap L_{i+1}^{l+\varepsilon(i)}, L_i^l, L_{i+1}^{l+\varepsilon(i)})$, $\varepsilon(i) = \delta_{i4}$, all tangent to identity except the last one $\varphi_{4,1}^{k-1}$ having \mathcal{L} as linear part.

One can suppose that $(U, C) = (\tilde{U}, \tilde{C})/F$ where $F(z, \xi) = F_{k,\nu,P}(z, \xi) + (\Delta_1, \Delta_2)$ with $\Delta_i = O(\xi^{-N})$ where $N \gg 0$ is a positive integer arbitrarily large and that there exists a formal diffeomorphism of (\tilde{U}, \tilde{C}) , $\hat{H}(z, \xi) = (z + \hat{g}, \xi + \hat{h})$, $\hat{h} = \sum_{n \geq 1} a_n \xi^{-n}$, $\hat{g} = \sum_{n \geq 1} b_n \xi^{-n}$ where a_n, b_n are entire functions on $\tilde{C} = \mathbb{C}^*$ such that

$$F \circ \hat{H} = \hat{H} \circ F_{k,\nu,P}.$$

One wants as before to replace \hat{H} by an analytic function H , that is

$$(10.11) \quad F \circ H = H \circ F_{k,\nu,P}$$

defined on a suitable domain (analogous to that of Section 9.1) that we proceed to describe now. One adopts the notation of loc.cit without systematically mentioning the parameters (k, ν, P) .

As before, we will just detail the construction of a normalizing conjugation map on the germ of transversal sectorial domain determined by I_1^l where $l \in \llbracket 0, k-1 \rrbracket$ has been fixed. Consider the annulus $\mathcal{C}_{a,b} = \{a \leq |z^{-1}| \leq b\}$, $a, b > 0, b > |q|^{-1}a$. Consider the foliation defined by the level sets $\{J_{k,\nu,P} = c\} \cap \{\arg \xi \in I_1^l\}$, c "small" of $J_{k,\nu,P} = J \circ \Phi = z^{-1} e^{P_1(\xi)} \xi^{-\lambda\nu - 2i\pi a_0} = z^{-1} e^{-\lambda(\frac{\xi^k}{k} + R(\xi))}$ where $R(\xi) = \frac{2i\pi}{\lambda} (\int \frac{P(\xi)d\xi}{\xi} + \frac{\lambda\nu}{2i\pi} \log \xi) = o(\xi^k)$ is somehow negligible with respect to ξ^k . It corresponds to the fibration $dY = 0$ on a neighborhood of L_1^l . Let $\delta_0 > 0$ sufficiently small For every $0 < \delta \leq \delta_0$. Note $S_{a,b,\delta} = \{(z, \xi) \in \mathbb{C}^2 \mid |J_{k,\nu,P}(z, \xi)| < \delta, z \in \mathcal{C}_{a,b}, \arg \xi \in I_1^l\}$. For every complex number $c, 0 < |c| < \delta$, consider $S_{c,a,b,\delta} = \{J_{k,\nu,P} = c\} \cap S_{a,b,\delta}$

Note that $\bigcup_{0 < |c| < \delta} S_{c,a,b,\delta} = S_{a,b,\delta}$ and that $F_{k,\nu,P}(S_{a,b,\delta}) \simeq S_{|q^{-1}|a, |q^{-1}|b, \delta}$.

It is thus coherent to investigate the existence of a solution H of 9.3 on the domain $S'_{a,b,\delta} = S_{a,b,\delta} \cup F_{\nu,\mu}(S_{a,b,\delta})$. Remark now that the conjugacy equation 10.11 can be

equivalently rewritten

$$F_\Phi \circ H_\Phi = H_\Phi \circ F_0$$

where $F_\Phi = \Phi \circ F \circ \Phi^{-1}$, $H_\Phi = \Phi \circ H \circ \Phi^{-1}$. This amounts to determine H_Φ on the domain $\Phi(S'_{a,b,\delta})$.

To do this, note that in view of asymptotic behavior of φ and φ^{-1} described below, one has $F_\Phi = F_0 + (\Delta_{1,\Phi}, \Delta_{2,\Phi})$ where one can verify that $(\Delta_{1,\Phi}, \Delta_{2,\Phi}) = O(\frac{1}{\xi^N})$ setting $z\widehat{\Delta}_{1,\Phi} = \Delta_{1,\Phi}$.

We are looking for solution of the form $H_\Phi = \text{Id} + (h_\Phi, g_\Phi)$. Here again, this can be reformulated as

$$(10.12) \quad h_\Phi \circ F_0 - h_\Phi = \Delta_{2,\Phi} \circ H_\Phi$$

$$(10.13) \quad \widehat{g}_\Phi \circ F_0 - qg_\Phi = \Delta_{1,\Phi} \circ H_\Phi$$

First, we will still deal with the linearized equations This can be reformulated as

$$(10.14) \quad h \circ F_0 - h = \Delta_2$$

$$(10.15) \quad g \circ F_0 - qg = \Delta_1$$

where we have omit the subscript Φ for notational convenience.

In what follows, we will indeed provide a solution of 10.14 (hence also for 10.15 by the usual transform) with "good estimates" on a domain of the form $\Phi(S'_{a,b,\delta})$ using a "leafwise" resolution with respect to the foliation defined by the levels of J . This consists as before in solving a family of difference equations parametrized by the leaves space in the variable ξ where the domains are slightly modified as explained now. For every complex number c , $0 < |c| < \delta$, consider $S_{c,a,b,\delta} = \{J_{k,\nu,P} = c\} \cap S_{a,b,\delta}$ and $\Phi(S_{c,a,b,\delta}) = \{(z, \xi) = (c^{-1}e^{-\lambda\xi}, \xi)\}$ where

$$\xi \in D_{c,a,b} = \{\xi \in \mathbb{C} : (\log \frac{1}{|c|} + \log a) \leq \Re(\lambda\xi + Q(\varphi^{-1}(\xi))) \leq (\log \frac{1}{|c|} + \log b)\}$$

where a determination of the logarithm has been chosen in the sector in which we are working, namely $\arg \xi \in I_1$. Note also that $Q(\varphi^{-1}(\xi)) = o(\xi)$ and consequently the middle term in the above inequation "behaves" like $\Re(\lambda\xi)$. The remaining part of the proof then follows *mutatis mutandis* the same line than before by noticing that the equation

$$h \circ F_0 - h = \Delta_2$$

can be then rewritten as

$$(10.16) \quad \varphi_c(\zeta - \lambda) - \varphi_c(\zeta) = \Delta_c(\zeta)$$

where $\zeta = \lambda\xi$, $\varphi_c(\zeta) = h(c^{-1}e^{-\zeta}, \frac{\zeta}{\lambda})$, $\Delta_c(\zeta) = \Delta_2(c^{-1}e^{-\zeta}, \frac{\zeta}{\lambda})$.

We are then reduced to solve a family difference equations (the so-called *homological equations*) in the "quasi" vertical strip

$$St_c = \{\zeta \in \mathbb{C} : (\log \frac{1}{|c|} + \log a) \leq \Re(\zeta - R(\varphi^{-1}(\frac{\zeta}{\lambda}))) \leq (\log \frac{1}{|c|} + \log b)\}$$

depending analytically on the parameter c and we investigate the existence of an solution φ_c on the domain $St_c \cup St_c + \lambda$. This can be carried out by resolving equation 9.8 using the same method (that is, essentially Cauchy formula) replacing accordingly the integration along the vertical lines L^-, L^+ by $l(\zeta) = A, B$ where $l(\zeta) = \Re(\zeta - R(\varphi^{-1}(\frac{\zeta}{\lambda})))$. One also obtain in a similar way the same kind of estimates imposing the uniqueness of the solution. This allows by the fixed point method detailed in the previous section to

solve the conjugation equation under the forms 10.6 and 10.7 on the relevant domains and finally exhibit a conjugacy sectorial transformation solution of 10.5 $H = \Phi^{-1} \circ H_{\Phi} \circ \Phi$, well defined as a section of G^1 over I_1 and having \hat{H} as asymptotic expansion. One can analogously construct conjugacy maps on the others sectors.

10.4. Torsion normal bundle, finite Ueda type. Recall that the elliptic curve is regarded as the quotient $C = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. With the notations of 10.1, let $(U_{m_1, m_\tau}, C_{m_1, m_\tau})$ be the minimal covering trivializing the normal bundle. That is

$$(U_{m_1, m_\tau}, C_{m_1, m_\tau}) = (\mathbb{C}_x \times \mathbb{C}_y, \{y = 0\}) / \langle \phi_1^{m_1}, \phi_\tau^{m_\tau} \rangle.$$

Note that, as an effect of this $\mathbb{Z}/m_1 \times \mathbb{Z}/m_\tau$ cover, the modulus of the elliptic curve changes and more precisely C_{m_1, m_τ} is determined by the lattice $\langle m_1, m_\tau \tau \rangle$.

An easy calculus yields

$$(10.17) \quad \begin{cases} \phi_1^{m_1}(x, y) &= (x + m_1, y) \\ \phi_\tau^{m_\tau}(x, y) &= (x + m_\tau \tau + \int_0^y [(\varphi_{k, \nu'} \circ \omega_{P'})^* \omega_P - \omega_P], \varphi_{k, \nu'} \circ \omega_{P'}(y)) \end{cases}$$

And conjugating by the transformation $\alpha : (x, y) \rightarrow (\frac{x}{m_1}, ay)$, where $a = m_\tau \frac{1}{k}$, one can reduce to the simplest and usual normal formal form

$$(U_{m_1, m_\tau}, C_{m_1, m_\tau}) \simeq (\mathbb{C}_x \times \mathbb{C}_y, \{y = 0\}) / (F_1, F_\tau)$$

where $F_1(x, y) = (x + 1, y)$, $F_\tau(x, y) = (x + \frac{m_\tau}{m_1} \tau + h_{k, \nu', P'}(y), \varphi_{k, \nu'}(y))$, $\nu' = \frac{\nu}{m_\tau}$, $P'(z) = \frac{P(z^{m_1})}{m_1}$, $h_{k, \nu', P'}(y) = \int_0^y [\varphi_{k, \nu'}^* \omega_{P'} - \omega_{P'}]$. Note that P' is uniquely defined up to the action described in 10.2, depending of the choice of a .

If one adopts the presentation in the variable (z, ξ) this corresponds to the neighborhood labeled identically in the previous paragraph (except that C is replaced by its cover) which contains the foliation defined by the levels of $J_{k, \nu', P'} = z e^{-\lambda' (\frac{\xi^k}{k} + R(\xi))}$ where $R(\xi) = \frac{2i\pi}{\lambda'} \int \frac{P'(\xi) d\xi}{\xi} + \frac{\lambda' \nu'}{2i\pi} \log \xi$ and $\lambda' = 2i\pi \frac{m_\tau}{m_1} \tau$ and whose presentation as a quotient is given by $(\mathbb{C}_z^* \times \mathbb{C}_\xi, \{\xi = \infty\}) / \langle F_{k, \nu', P'} \rangle$ where $F_{k, \nu', P'} = \exp X_{k, \nu', P'}$ and $X_{k, \nu', P'} = \frac{1}{\xi^k + \nu'} (-\xi \frac{\partial}{\partial \xi} + 2i\pi z P'(\xi) \frac{\partial}{\partial z}) + \lambda' z \frac{\partial}{\partial z}$. This gives a description of the (deleted) neighborhood of the elliptic curve and the (deleted) neighborhood of $4k$ rational curves of zero type. In order to recover the structure of the original neighborhood (U, C) , we have to determine the identifications induced by the deck transformation group of order m of the finite cover generated by ϕ_1 and ϕ_τ in the (x, y) coordinates, which acts by "sectorial permutation". There are explicitly given in the (z, ξ) coordinates by the group generated by $(z, \xi) \rightarrow (e^{-\frac{2i\pi}{m_1}} z, a_1 \xi)$ and $(z, \xi) \rightarrow \exp \frac{X_{k, \nu', P'}}{m_\tau}$. At the level of the neighborhood of rational curves, these symmetries act on the collection of $4k$ single neighborhood by permutations and we eventually end up with a neighborhood of $4k'$ rational curves of zero type.

11. $SL_2(\mathbb{Z})$ ACTION

One could retrieve an analogue description of the analytic moduli space starting from another symplectic parametrisation, changing accordingly the sectorial domain. As it is not fundamental for our purpose, we just indicate heuristically and without entering into more details the effects of this transformation in terms of neighborhood of four lines on some examples. The most simple one is associated to the matrix $M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ associated to the biholomorphism $\varphi_{M_1}(X, Y) = (Y, X)$. In this case, if one starts from a configuration V_φ , the corresponding picture for the reparametrization $\varphi_{M_1} \circ \Pi$ (given in Section ??

consists in permuting the glueing cocycle by φ_{M_1} . Indeed, as one can easily check, the original cocycle $(\varphi_{i,i+1}) \in \widetilde{\mathcal{U}}_4$ is transformed into $(\psi_{i,i+1})$, where $\psi_{4,1} = \varphi_{M_1} \circ \varphi_{1,4} \circ \varphi_{M_1}^{-1}$. The reparametrization associated to the matrix $M' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is more intricate, due to the birational nature of $\varphi_{M'}$. The neighborhood V_φ is modified to give rise to a new neighborhood W of four lines of zero self-intersection E_1, L_2, E_2, L_4 by blowing-up-contraction according to the process described in Fig. 2. Those neighborhood are hence bimeromorphically equivalent but not analytically in general. Following Savelev [21], the neighborhood of each individual line L is a product $L \times (\mathbb{C}, 0)$. Consequently, the analytic structure of W is encoded in glueing cocycles $\varphi'_{i,i+1}$ defined by germs of biholomorphisms at $p_{i,i+1}$. In fact, there is no obvious way to express all these cocycles (corresponding to the M' parametrization) in terms of $\varphi_{i,i+1}$. This explain below where we detail more thoroughly this construction.

The corresponding reparametrization is defined by $\varphi_{M'}(X, Y) = (XY, Y) = (ze^{-\lambda'\xi}, ze^{-\lambda\xi})$ where $\lambda' = 2i\pi(\tau - 1)$. In particular, $\frac{\pi}{2} < \arg \lambda < \arg \lambda' < \frac{3\pi}{2}$. So, the role which was played by the intervals I_{ij}, I_i in 3.2 is played here by I'_{ij}, I'_i which corresponding labels. More explictely, $I'_i = I_i$ for $i = 1, 3$, $I'_2 =]\frac{\pi}{2} - \arg(\lambda'), \frac{3\pi}{2} - \arg \lambda'[, I'_4 =]\frac{3\pi}{2} - \arg(\lambda'), \frac{5\pi}{2} - \arg \lambda'[,$ and $I'_{ij} = I'_i \cap I'_j$. By assumption, there exists F a biholomorphism of $(\widetilde{U}, \widetilde{C})$ and $h_i \in G^1(I_i), h'_i \in G^1(I'_i)$ with $\hat{h}_i = \widehat{h}'_i$ independant of i such that $F = h_i \circ F_0 \circ h_i^{-1} = h'_i \circ F_0 \circ h'_i^{-1}$. This implies that in restriction to $J_i := I_i \cup I'_{i+1}$, one can write $h'_i = h_i \circ g_i$ where $g_i \in G^\infty[F_0](J_i)$. Note that for $i = 1, 3$, one can choose $g_i = Id$. Set $h_{i,i+1} = h_i \circ h_{i+1}^{-1}, h'_{i,i+1} = h'_i \circ h'_{i+1}^{-1}$. Those are cocycles associated repectively to the "trivial" parametrization φ_{Id} and to $\varphi_{M'}$. They are both related through the equality

$$h'_{i,i+1} = g_i^{-1} \circ h_{i,i+1} \circ g_{i+1}$$

which holds on the intersections $I_{i,i+1} \cap I'_{i,i+1}$. Let $V_\varphi, V_{\varphi'}$ the respective neighborhoods of the cycle of four line associated to these parametrizations. According to Fig.??, one pass to V_φ to $V_{\varphi'}$ by first blowing-up $p_{1,2}$ and then contracting L_1 and L_3 . Consider the corresponding cocycles, also denoted by φ, φ' at the level of the blow-up manifold and let G_i the local transformation induced by g_i . We have represented their domain of definition on the picture below (Fig.2).

We may have in mind that each transformation preserves normal crossing whenever defined and are tangent to identity at the intersection points. Conversely, these picture can be helpful to recover the cocycle φ' from φ . Indeed, the neighborhood of $L_1 \cup L_2$ (these curves being considered as -1 rational curves) is analytically rigid. This is for instance due to the fact that the contraction of L_1 gives rise to a 0-type neighborhood of L_4 which is isomorphic to $\mathbb{P}^1 \times (\mathbb{C}, 0)$. This rigidity allows to claim the existence of $\varphi'_{1,4}$ and G_4 such that $\varphi'_{4,1} = G_4^{-1} \circ \varphi_{4,1}$ (adjusting if necessary by the automorphism group of $\mathbb{P}^1 \times (\mathbb{C}, 0)$ to impose the prescribed behavior at normal crossings). One can exhibit similarly $\varphi'_{2,3}$ and G_2 such that $\varphi'_{2,3} = G_2^{-1} \circ \varphi_{2,3}$. Hence $\varphi'_{1,2} = \varphi_{1,2} \circ G_2$ and $\varphi'_{3,4} = \varphi_{3,4} \circ G_4$ are automatically defined on the right domains.

Note that by combining these "elementary" operations associated to M_1 and M' , one can recover geometrically the corresponding analytic class (in terms of neighborhood of four lines) for any symplectic parametrization.

We thus inherit in this way from an action of $GL(2, \mathbb{Z})$ on $\widetilde{\mathcal{U}}_4$. For this action, Every g belonging to the stabilizer of V_φ gives rise to a bimeromorphic transformation $\rho(g)$

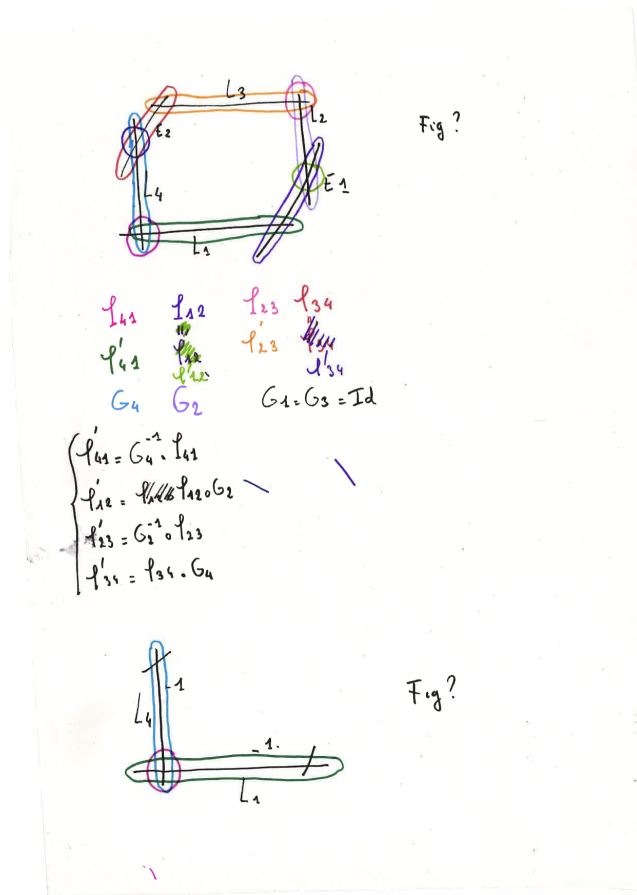


FIGURE 2. Base change M'

of V_φ uniquely defined modulo the group $Z(\varphi)$. We will denote by $\text{Bir}(V_\varphi)$ the group generated by the $\rho(g)$'s $g \in \text{Stab}(V_\varphi)$ and $Z(\varphi)$. It is isomorphic to a subgroup of $\text{Bir}(V_{id}) = \widetilde{\text{Iso}}_0(V) \rtimes GL(2, \mathbb{Z})$ (where Id is the trivial cocycle). Moreover $Z(\varphi)$ is a normal subgroup in it and the quotient $\text{Bir}(V_\varphi)/Z(\varphi)$ is isomorphic to $\text{Stab}(V_\varphi)$. It seems difficult to describe the possible values of $\text{Bir}(V_\varphi)$. It is very likely that this group is trivial in general.

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UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE

E-mail address: frank.loray@univ-rennes1.fr

E-mail address: frederic.touzet@univ-rennes1.fr

DEPARTMENT OF MATHEMATICS, CHELYABINSK STATE UNIVERSITY, CHELYABINSK, RUSSIA

E-mail address: voron@csu.ru