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## On the continuity of the walras correspondence for distributional economies with an infinite dimensional commodity space

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**Abstract** Exchange economies are defined by a distribution on the space of characteristics where the commodity space is an ordered separable Banach space. We characterize the continuity of the equilibrium correspondence and a stability concept associated. We provide a positive answer to an open question about the continuity of the Walras correspondence in infinite dimensional spaces. Regarding the stability concept, differentiability assumptions are not required as it is usual in the literature on regularity. Moreover, since distributional economies do not specify a space of agents, our setting encompasses several results in the literature on large economies.

**Keywords** Essential Stability · Walras Correspondence · Infinitely Many Commodities · Distributional Economies

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This is an updated version of Cea-Echenique's Chapter 3 of his Ph.D. dissertation in Applied Mathematics at University Paris 1 - Panthéon Sorbonne.

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## 1 Introduction

The existence of a competitive equilibrium is followed by questions regarding the characterization of the equilibrium set in order to analyze efficiency, uniqueness or regularity properties. For this purpose, it is required to analyze how the set of equilibria responds to perturbations in exogenous parameters that characterize agents and, therefore, economies. This relation between parameters and equilibrium sets has been captured in the literature through *equilibrium correspondences* that associate economies with its equilibria.

Kannai (1970), Hildenbrand (1970) and Hildenbrand and Mertens (1972) introduce the study on the continuity of the *equilibrium correspondence* for pure exchange economies. All these studies, also including Balasko (1975), understand parameters as exogenous characteristics that define the agents (i.e. consumption sets, tastes or endowments). In particular, it turns to be a crucial point the way in which a topology in the space of economies is defined.

Results regarding regularity, are closely related to the finiteness of the equilibrium set. It is the finite property that allows to define a concept of locally stable equilibria. The approach generally consists of proving conditions over the *equilibrium correspondence* in order to conclude that it defines finite sets. This is the aim of the pioneering work of Debreu (1970) assuming differentiability conditions and using the Theorem of Sard (1942).

Mas-Colell (1977) raised the following question regarding the equilibrium set:

Is there a dense set of economies having a finite set of equilibria? We shall see the answer is yes, but this is not by itself a very interesting property; what one wants (for, say, estimation or prediction purposes) is that those equilibria be “essential”, i.e., that they do not disappear by performing an arbitrarily small perturbation of the economy.

This quotation emphasizes that for our purposes the most accurate definition should be regarding essentiality instead of regularity. We remark that every regular equilibrium is essential but the converse is not true. Furthermore, in order to characterize this concept, we need to study the relation between parameters and equilibria instead of the equilibrium set.

In the present work, we study the continuity of the *equilibrium correspondence* for distributional economies with infinite dimensional commodity spaces. Moreover, due to the continuity property, we characterize a concept of stability named *essential stability* that was introduced in the fixed point theory by Fort (1950) and, accordingly to game theory by Wen-Tsun and Jia-He (1962). In particular, the translation from game theory to economies states that an equilibrium is *essentially stable* if it is possible to approximate it by equilibria of “similar” economies, i.e. economies that are close to the economy of reference under a metric in the space of economies that has to be precised. Generally speaking, defining the space of economies by a metric space requires to parameterize the family of economies of interest with respect to the dimensions of similarity. In our case, the dimensions are preference relations and

endowments. It is possible to extend our analysis to other parameterizations of economies, e.g. externalities, tax structures or information, by requiring that the metric space of economies remains complete.

Recently, the continuity of the equilibrium correspondence in general equilibrium theory was stated by Dubey and Ruscitti (2015) and He et al. (2017). We extend their results taking into consideration infinite dimensional commodity spaces and by characterizing stability when the continuity property in the equilibrium correspondence can not be obtained directly. In fact, our results answer the question posed in Dubey and Ruscitti (2015) about the possibility of getting stability results in infinite dimensional economies. In addition, we remark that we have no need to restrict the economies to have the same space of agents as it has typically been done in the literature.

The following section establishes some mathematical preliminaries. Section 3 define the economies and Section 4 studies the Walras correspondence. Section 5 states the essential stability results and Section 6 discusses the representation of economies through measurable functions. Section 7 concludes with some remarks.

## 2 Preliminaries

We recall some properties of topological vector spaces. In order to accomplish that, we define an ordered separable Banach space  $(L, \|\cdot\|)$  whose positive cone has a non-empty interior. The positive cone and the topological dual of  $L$  are denoted by, respectively,  $L_+$  and  $L^*$ .

Endowed with the weak topology  $w$ ,  $(L, w)$  is a complete topological vector space. By  $\|\cdot\|$ -topology we mean the topology induced by the norm  $\|\cdot\|$ . An analogous notation is given to the  $w$ -topology.

For  $Q \subset L_+$ , we define the norm  $\|\cdot\|_Q$  as the one induced from  $\|\cdot\|$ . The weak topology on  $Q$ ,  $w_Q$ , is the relativization to  $Q$  of  $w$ .

**Proposition 1 (Fristedt and Gray (1996), Proposition 3, p. 350)** *If  $Q$  is a closed subset of  $(L, \|\cdot\|)$  then  $(Q, \|\cdot\|_Q)$  is Polish.*

Even though  $(L, w)$  is not metrizable, we can ensure the following.

**Proposition 2** *If  $Q$  is a compact subset of  $(L, w)$  then  $(Q, w_Q)$  is a locally compact Polish space.*

*Proof*  $(Q, w_Q)$  is metrizable since  $Q$  is  $w$ -compact (Dunford and Schwartz (1958), Theorem 3. p. 434). Furthermore,  $(Q, w_Q)$  is separable (Aliprantis and Border (2006), Lemma 3.26, p. 85) and obviously complete.

*Remark 1*  $w$ -compactness of  $Q$  implies  $\|\cdot\|$ -closeness of  $Q$  and, since  $L$  is separable,  $\|\cdot\|$ -boundedness (Diestel (1984), p. 17). If  $Q$  is convex,  $w$ -closeness implies  $\|\cdot\|$ -closeness.

By the separability of  $L$  we can make use of the following result.

**Proposition 3 (Dunford and Schwartz (1958), Theorem 1, p. 426)** *The topology induced on  $S \subset L^*$  compact in the weak\*-topology is metrizable by a translation invariant metric on  $L^*$ .*

**Proposition 4 (Diestel and Uhl (1977), Theorem 2, p. 45)** *Given the measurable space  $(A, \mathcal{A}, \nu)$ , a strong measurable function  $f : A \rightarrow L$  is Bochner integrable if and only if  $\int_A \|f\| d\nu < \infty$ .*

Let  $(\mathcal{M}, \rho)$  be a metric space. For a given  $\mu \in \mathcal{M}$  we denote an open ball in  $(\mathcal{M}, \rho)$  with radius  $\delta > 0$  by  $V(\mu, \delta)$ . Let  $\mathcal{D}W : \mathcal{M}_1 \rightarrow \mathcal{M}_1$  a correspondence between two metric spaces  $(\mathcal{M}_1, \rho_1)$  and  $(\mathcal{M}_2, \rho_2)$ .

**Definition 1 (Yu et al. (2005))**

1. For each  $\mu \in \mathcal{M}_1$  and  $e(\mu) \subset \mathcal{D}W(\mu)$  non-empty and closed, the set  $e(\mu)$  is called an essential set of  $\mathcal{D}W(\mu)$  if for any  $\epsilon > 0$  there is  $\delta > 0$  such that for any  $\mu' \in V_1(\mu, \delta)$  we have  $\mathcal{D}W(\mu') \cap [V_2(e(\mu), \epsilon)] \neq \emptyset$ .
2. For each  $\mu \in \mathcal{M}_1$ , a component of a point  $\tau \in \mathcal{D}W(\mu)$  is the union of all connected subsets of  $\mathcal{D}W(\mu)$  which contain the point  $\tau$ .

Consequently, if for a given  $\mu \in \mathcal{M}_1$  a component of  $\mathcal{D}W(\mu)$  is an essential set, then the component is called essential. Moreover, for  $\mu \in \mathcal{M}_1$ , an essential set  $m(\mu)$  of  $\mathcal{D}W(\mu)$  is minimal if it is a minimal element of the family of essential sets ordered by set inclusion. Regarding set stability we establish the following definition:

**Definition 2 (Correa and Torres-Martínez (2014), Definition 8 (i))**

For  $\mu \in \mathcal{M}_1$ , a subset  $e \subseteq \mathcal{D}W(\mu)$  is stable if for every  $\epsilon > 0$ , there is  $\delta > 0$ , such that given  $\mu' \in \mathcal{M}_1$  with  $\rho_1(\mu, \mu') < \delta$ , there is a minimal essential set  $e' \subset \mathcal{D}W(\mu')$  for which  $e' \subseteq V_2(e, \epsilon)$ .

Finally, we provide a result that was originally proved by Fort (1951) regarding the lower hemicontinuity of correspondences.

**Proposition 5 (Carbonell-Nicolau (2010) Lemmata 5-6)** *Given a nonempty, compact-valued and upper hemi-continuous correspondence from a Baire space into a metric space, there exists a dense residual subset of the domain such that the correspondence is lower hemicontinuous at every point.<sup>1</sup>*

### 3 Model

The commodity space is defined by an ordered separable Banach space  $(L, \|\cdot\|)$  whose positive cone has a non-empty interior. Consequently, the price space is given by the positive cone of the topological dual of  $L$ ,  $L_+^*$ . We endow this space with the weak-star topology  $w^*$ .

<sup>1</sup>Lemma 5 in Carbonell-Nicolau (2010) is due to Theorem 2 Fort (1951). In addition, the fact that there is a dense residual set is also noted by Yu (1999) Theorem 4.2.1.

We consider a convex and  $w$ -compact subset  $Q$  of the space  $L_+$  which includes the vectors  $0$  and  $u$ , where  $u \in \text{int}L_+$  and  $\|u\| = 1$ . Consider the consumption set  $X \subset Q$  which is a norm-closed and convex set containing both the vectors  $0$  and  $u$ . Consequently,  $X$  is a  $w$ -closed subset of  $Q$  whence, a  $w$ -compact one.

Regarding the consideration of having  $w$ -compact subsets of the commodity space  $L$ , we note that similar assumptions are made in large economies. Indeed, the works of Khan and Yannelis (1991) and Noguchi (1997) assume that each consumption set is weak-compact. Recently, Khan and Sagara (2016) assumes the existence of a common weak-compact metrizable consumption set, as well as Bewley (1991) or Suzuki (2013) that assume the existence of a common consumption set.

Let  $(X, \succ)$  be a preference relation where  $\succ \subset X \times X$  is a transitive and irreflexive binary relation on  $X$  such that  $(X, \succ)$  is weak-relatively open in  $X \times X$ . We use  $x \succ y$  to denote  $(x, y) \in \succ$  and  $x \not\succ y$  to denote  $(x, y) \notin \succ$ . Let  $\mathcal{P}$  be the set preference relations. For each  $(X, \succ) \in \mathcal{P}$  we associate the set  $P := \{(x, y) \in X \times X \mid x \not\succ y\}$ . Because of the continuity of preferences, each  $P$  is a  $w \times w$ -closed subset of  $L_+ \times L_+$  and then, a  $w_Q \times w_Q$ -closed subset of  $Q \times Q$ . We denote by  $\mathcal{P}_{mo}$  the set of all monotonic preference relations in  $\mathcal{P}$  which is defined as  $\mathcal{P}_{mo} := \{(X, \succ) \in \mathcal{P}, \text{ such that for all } x, y \text{ in } X, \text{ if } x \geq y \text{ and } x \neq y \text{ then } x \succ y\}$ .

Let  $\mathcal{C}^w(Q \times Q)$  be the set of all  $w_Q \times w_Q$ -closed subsets of  $Q \times Q$ . We denote by  $\tau^C$  the topology of closed convergence on  $\mathcal{C}^w(Q \times Q)$ . Since every  $P$  belongs to  $\mathcal{C}^w(Q \times Q)$ , we can define a mapping  $g : \mathcal{P} \rightarrow \mathcal{C}^w(Q \times Q)$  by  $(X, \succ) \mapsto P$ . It is easily verified that  $g$  is an injection. Indeed, let  $(X, \succ) \neq (X', \succ')$  in  $\mathcal{P}$  and let us assume that  $P = P'$ . If  $X = X'$ , then we have that  $(X \times X) \setminus \succ = (X' \times X') \setminus \succ'$ , whence  $\succ = \succ'$  which contradicts  $(X, \succ) \neq (X', \succ')$ . If  $X \neq X'$  one can assume without loss of generality that  $X \setminus X' \neq \emptyset$ . It follows from  $(X \times X) \setminus \succ = (X' \times X') \setminus \succ'$  that for any  $y \in X \setminus X'$  that  $(y, y) \in \succ$  which contradicts irreflexivity. Consequently, we must have  $P \neq P'$  whenever  $(X, \succ) \neq (X', \succ')$ .

We define the topology  $\tau_{\mathcal{P}}^C$  on  $\mathcal{P}$  by  $\tau_{\mathcal{P}}^C := \{g^{-1}(U) : U \in \tau_C\}$ . Thus  $\tau_{\mathcal{P}}^C$  can be seen as the topology  $\tau^C$  induced on  $\mathcal{P}$ .<sup>2</sup> We characterize the preferences of the space of characteristics in the following lemma which follows and adapts the arguments of Theorem 1 in Hildenbrand (1974).

### Lemma 1

1.  $(\mathcal{P}, \tau_{\mathcal{P}}^C)$  is compact and metrizable (and hence, a Polish space)
2. A sequence of preferences  $\{(X^n, \succ^n)\}_{n \geq 1}$  converges to  $(X, \succ)$  in  $(\mathcal{P}, \tau_{\mathcal{P}}^C)$  if and only if  $Li(P_n) = P = Ls(P_n)$
3. The set  $\{(X, \succ), x, y \in \mathcal{P} \times Q \times Q : x, y \in X \text{ and } x \not\succ y\}$  is closed for the product topology  $\tau_{\mathcal{P}}^C \times w_Q \times w_Q$ . Furthermore,  $\tau_{\mathcal{P}}^C$  is the weakest topology on  $\mathcal{P}$  for which the above set is closed.

<sup>2</sup>See Hervés-Beloso et al. (1999) for another applications of the closed convergence topology for infinite dimensional space of characteristic.

The proof is given in A.1. The following corollary is useful to achieve optimality in the proof of Theorem 1.

**Corollary 1** *Let  $(X, \succ) \in \mathcal{P}$  such that  $x, y \in X$  and  $x \succ y$ . There exists an  $\tau_{\mathcal{P}}^C$ -open neighborhood  $U_{(X, \succ)}$ , a  $w_Q$ -open neighborhood  $V_x$  and a  $w_Q$ -open neighborhood  $V_y$ , such that for all  $(X', \succ') \in U_{(X, \succ)}$  and for all  $(x', y') \in (X' \cap V_x) \times (X' \cap V_y)$  we have  $x' \succ' y'$ .*

*Proof* Since the set  $\{((X, \succ), x, y) \in \mathcal{P} \times Q \times Q : x, y \in X \text{ and } x \neq y\}$  is closed for the product topology  $\tau_{\mathcal{P}}^C \times w_Q \times w_Q$ , then

$$\mathcal{P} \times Q \times Q \setminus \{(X, \succ), x, y) \in \mathcal{P} \times Q \times Q : x, y \in X \text{ and } x \neq y\}$$

is  $\tau_{\mathcal{P}}^C \times w_Q \times w_Q$ -open.

We denote by  $F$  a  $\|\cdot\|$ -closed subset of  $X$ . The space of characteristics is defined by  $T = \mathcal{P}_{mo} \times F$ , i.e., we shall consider economies having characteristics in  $T$ . In particular, it means that the economies under consideration have endowments with values in  $F$ . Consequently, a typical element of  $T$  is  $((X, \succ), e)$ . Naturally,  $T$  is endowed with the product topology  $\tau_{\mathcal{T}}^C \times \|\cdot\|_F$  so that we obtain the following lemma.

**Lemma 2**  *$T$  is a Polish space.*

We defer the proof to A.2.

### 3.1 Space of Economies

We formally define an economy by means of a distribution on the space of characteristics.

**Definition 3** An *economy* is a Borel probability distribution  $\mu$  on  $T$ .

Define,  $\iota$  as the identity map, notice that  $\int_F \iota d\mu_F < \infty$  since elements in  $F$  are norm-bounded. We denote by  $\mathcal{M}(T)$  the set of all Borel probability distributions on the support  $T$ . The price simplex is given by  $S = \{p \in L_+^* : p \cdot u = 1\}$ . By Jameson (1970), Theorem 3.8.6,  $S$  is weak\*-compact and, using Proposition 3, metrizable.

From now on, let  $X$  be the common consumption set of every economy. For  $p \in S$  let us consider the set

$$E_p = \{((X, \succ), e), x) \in T \times X : p \cdot x \leq p \cdot e \text{ and } x' \succ x \text{ implies } p \cdot x' > p \cdot e\}.$$

**Definition 4** A probability measure  $\tau \in \mathcal{M}(T \times X)$  is an *equilibrium* for the economy  $\mu$  if there exists  $p \in S$  such that:

1.  $\tau_T = \mu$ ,
2.  $\int_X \iota d\tau_X = \int_F \iota d\mu_F$
3.  $\tau(E_p) = 1$ ,

where  $\tau_X$  is the distribution of allocations given by the marginal distribution of  $\tau$  on  $X$  over  $T \times X$  and  $\mu_F$  is the distribution of endowments given by the marginal distribution of  $\mu$  on  $F$  over the characteristics' space.

When the distribution  $\tau$  satisfies 1 and 2 of Definition 4 we say that  $\tau$  is *attainable*.

How different are two economies? or how different are two attainable distributions?. The answer to these questions implies endowing the spaces  $\mathcal{M}(T \times X)$  and  $\mathcal{M}(T)$  with a topology. Since  $F$  and  $T \times X$  are separable and complete metric spaces, we know that the set of probability measures on the Borel  $\sigma$ -algebra is also complete and separable with respect to the Prohorov metric denoted by  $\rho$  (Billingsley, 1999, (vi)-(vii) p. 73). In turn, the projection on  $T$ , denoted  $\rho_T$ , induces a separable and complete metric space  $(\mathcal{M}(T), \rho_T)$  that we define as the space of economies.

### 3.2 Examples

The examples presented below belong to the collection of exchange economies. In addition to the examples presented in this section, a linear price may represent situations including uncertainty, information or public goods. Moreover, if the linearity in prices, e.g. Aliprantis et al. (2001), is not required there are more examples to be considered including incomplete markets, ambiguity, among others. Here we concentrate in the classical examples of exchanges economies when there are infinitely many agents and commodities.

1. *Perfectly competitive economies (Rustichini and Yannelis (1991))*. Let us consider the space  $L = C([0, 1])$  of continuous functions on  $[0, 1]$  being  $C_+([0, 1])$  its positive cone. Let  $Q$  be a weakly compact subset of  $C_+([0, 1])$  containing both 0 and  $u$ .  $Q$  is the commodity space. Notice that the existence of a weak compact set in  $C([0, 1])$  is guaranteed since  $[0, 1]$  is Eberlein compact<sup>3</sup> (Fabian et al. (2001), Theorem 12.12, p. 392). By Krein Theorem,  $Q$  is convex. Further, since  $C([0, 1])$  is separable,  $Q$  is metrizable for the weak topology (Dunford and Schwartz (1958), Theorem V. 6.3, p. 434) and hence,  $w$ -Polish. Let  $F = Q \cap \overline{V(0, \alpha)}$  for  $\alpha > 0$ . Thus,  $\mathcal{P} \times F$  is a Polish space. Let  $(A, \mathcal{A}, \nu)$  be an atomless, complete and finite measure space where  $A$  is a compact metric space. Let  $g : A \rightarrow C(Q)$  be a continuous mapping such that for every  $a \in A$  and  $x \in Q$ ,  $g(a)(x) = a\sqrt{\|x\|}$ . Notice that  $\bigcup_{a \in A} g(a)$  is a compact subset of  $C(Q)$  since  $A$  is compact. Let  $e : A \rightarrow F$  be a continuous function. Let  $\beta$  be a real number larger than  $\alpha$  such that  $X(a) = Q \cap [0, \beta]$  for all  $a \in A$ . The restriction of every  $g$  on  $Q \cap [0, \beta]$ ,  $g_X$ , preserves continuity, irreflexivity, transitivity, monotonicity and strong convexity. Consequently, the set  $\{(Q \cap [0, \beta], \cup_{a \in A} e(a))\} \subset \mathcal{P} \times F$  is a characteristic set.

<sup>3</sup>A set is Eberlein compact if it can be embedded as a compact subset of a Banach space with the weak topology.



Furthermore, if  $(A, \mathcal{A}, \nu)$  is seen as an agent space, then every measurable function  $f : A \rightarrow \{(Q \cap [0, \beta], \cup_{a \in A} g_X(a)), \cup_{a \in A} e(a)\}$  such that  $f(a) = \{(X(a), g_X(a)), e(a)\}$  has a distribution  $\nu \circ f^{-1}$  which is an economy with those characteristics. In addition, if  $(A, \mathcal{A}, \nu)$  satisfies the “many more agents than commodities” condition of Rustichini and Yannelis (1991), the distributional economy  $\nu \circ f^{-1}$  is perfectly competitive.

2. *Neoclassical exchange economies.* Let  $L_+ = C(K)_+$  for some  $K$  compact Hausdorff. Let  $Q \subset C(K)_+$  be a convex, bounded and quasi-equicontinuous set<sup>4</sup> which is weak-compact accordingly to Theorem 14.3 in Dunford and Schwartz (1958), p. 269. Let  $\alpha > 0$  such that  $u \in \overline{V(0, \alpha)}$  and  $F = \overline{V(0, \alpha)} \cap Q$ . Let us consider  $X = n\overline{V(0, \alpha)} \cap Q$  for some  $n \in \mathbb{N}$ . Let  $N$  be a set of continuous, strictly increasing and strictly quasi-concave functions, i.e., every  $g \in N \subset C(Q)$  is a neoclassical preference on  $X$ . Let us consider a norm-closed subset  $\hat{N}$  of  $N$ , i.e, a closed set neoclassical preferences. Hence,  $T = \left( n\overline{V(0, \alpha)} \cap Q, \hat{N} \right)$  is a compact metric subset of  $\mathcal{P}$  and  $T \times F$  is a neoclassical space of characteristics. Every economy  $\mu \in \mathcal{M}(T \times F)$  is a neoclassical exchange economy.
3. Let  $Q = \text{co}(\{f_i \in C_+([0, 1]) : i = 1, \dots, n\} \cup \{0\} \cup \{u \in \text{int}C_+([0, 1])\})$  with  $u \in V(0, \alpha)$ . The set of preferences is  $A = \text{co}\{u_1(x) = \int_0^1 tx(t)dt, u_2(x) = \sum_{i=1}^{\infty} 2^{-j} \sqrt{x(r_j)} \text{ with } (r_j)_{j=1}^{\infty} \in \mathbb{Q}^{\infty} \text{ and } u_3(x) = \int_0^1 \sqrt{x(t)dt}\}$  for every  $x \in Q$ . Let us take  $F = X = Q$ . Consequently,  $T = (Q, A) \times Q$  is a characteristic space satisfying the conditions stated through the paper and every measure concentrated on  $T$  will be an economy.

## 4 Walras correspondence

Let  $\mathbb{A}$  be a compact subset of  $\mathcal{M}(T \times X)$  containing all attainable distributions.

**Definition 5** The Walras equilibrium correspondence  $\mathcal{DW} : \mathcal{M}(T) \rightarrow \mathbb{A}$  assigns to every economy  $\mu$  its corresponding walrasian equilibria  $\mathcal{DW}(\mu)$ .

### 4.1 Upper hemi-continuity

**Theorem 1** Let  $\mathbb{E} \subset \mathcal{M}(T)$  such that  $\mathcal{DW}(\mu) \neq \emptyset$  for all  $\mu \in \mathbb{E}$ . Under Assumption E the correspondence  $\mathcal{DW} : \mathbb{E} \rightarrow \mathbb{A}$  is upper hemi-continuous and closed-valued.

The proof is given in A.3.

**Corollary 2** The set  $\mathbb{E}$  is Baire.

<sup>4</sup>The set  $Q$  is quasi equicontinuous if for every sequence  $(t_n)$  such that  $t_n \rightarrow t$  and for every  $\varepsilon > 0$  and  $n_0$ , there exists a finite set of indices  $n_i \geq n_0, i = 1, \dots, m$ , such that for each  $f \in Q, \min_{1 \leq i \leq m} |f(t_{n_i}) - f(t_0)| < \varepsilon$ .

*Proof* Since  $\mathcal{M}(T)$  is a complete metric space it suffices to show that  $\mathbb{E}$  is closed. But it follows since  $\mathcal{DW}$  is u.h.c.

#### 4.2 Lower hemi-continuity

In order to ensure lower hemi-continuity of the Walras correspondence we invoke Theorem 2 in Fort (1951). For being self-contained, we presented Fort's result as stated by Carbonell-Nicolau (2010) in Lemma 5.

A straightforward application of Lemma 5 given the upper hemicontinuity ensured in Theorem 1 gives the following corollary.

**Corollary 3** *The correspondence  $\mathcal{DW}$  is lower hemicontinuous at every point of a dense residual subset  $\mathbb{E}'$  of  $\mathbb{E}$ .*

#### 4.3 Continuity

**Theorem 2** *The correspondence  $\mathcal{DW} : \mathbb{E}' \rightarrow \mathbb{A}$  is continuous.*

*Proof* It follows from Theorem 1 and Corollary 3

### 5 Stability results

We shall now study the stability of economies by analyzing how a Walras equilibrium for an economy  $\mu$  changes when their characteristics are perturbed.

**Definition 6** Let  $\mathbb{E}' \subseteq \mathbb{E}$  and  $\mu \in \mathbb{E}'$ . An equilibrium  $\tau$  of  $\mu$  is essential relative to  $\mathbb{E}'$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every  $\mu' \in V_T(\mu, \delta) \cap \mathbb{E}'$  it follows that  $\mathcal{DW}(\mu') \cap V_{T \times X}(\tau, \varepsilon) \neq \emptyset$ .

An economy  $\mu$  is essential with respect to  $\mathbb{E}' \subseteq \mathbb{E}$  provided that all their associated equilibria are essential with respect to  $\mathbb{E}$ .

Thus, essential stability is equivalent to the lower hemi-continuity of the Walras equilibrium correspondence. By Corollary 3 we already know that the collection of essential economies  $\mathbb{E}'$  is a dense residual subset of  $\mathbb{E}$ .

Additionally, for a given economy  $\mu$  on  $\mathbb{E}$ , we would like to ensure the following:

- (S1) If  $\mathcal{DW}(\mu) = \tau$ , i.e., it is a singleton, then  $\tau$  is essential.
- (S2) There exists a minimal essential subset of  $\mathcal{DW}(\mu)$  and any of such sets is connected.
- (S3) Given an essential and connected set  $m(\mu) \subset \mathcal{DW}(\mu)$ , there exists an essential component of  $\mathcal{DW}(\mu)$  that contains  $m(\mu)$ .
- (S4) Every essential subset of  $\mathcal{DW}(\mu)$  is stable.

**Theorem 3** *Properties (S1)-(S4) hold.*

*Proof* The properties follow from direct application of the results in the literature. More precisely: (S1): Theorem 4.3 in Yu (1999), (S2): Theorem 2.1(1) in Yu et al. (2005), (S3): Theorem 3 (iii) in Correa and Torres-Martínez (2014), and (S4): Theorem 4 in Correa and Torres-Martínez (2014).

## 6 Some results on the representation of economies through measurable functions

### 6.1 Individualized economies and Walras allocations

It is well known that every distributional economy  $\mu$  has a representation through measurable functions. Indeed, by Skorokhod's theorem, for  $\mu \in \mathcal{M}(T)$  there exists a measurable function  $\mathcal{E}$  and a measure space  $(A, \mathcal{A}, \nu)$ , such that  $\mathcal{E} : (A, \mathcal{A}) \rightarrow (T, \mathcal{B}(T))$  and  $\nu \circ \mathcal{E}^{-1} = \mu$ . Since  $T$  is complete,  $(A, \mathcal{A}, \nu)$  may be chosen to be  $(I, \mathcal{B}_I, \lambda)$  for  $I = [0, 1]$ , the unit interval with the Lebesgue measure (Hildenbrand (1974), pp. 50-51). Furthermore, Keisler and Sun (2009), Lemma 2.1 (ii) states that for any atomless measure space  $(A, \mathcal{A}, \nu)$  and a distribution  $\mu$  in  $\mathcal{M}(T)$  there exists a measurable mapping  $\mathcal{E}$  from  $(A, \mathcal{A}, \nu)$  into  $T$  such that  $\nu \circ \mathcal{E}^{-1} = \mu$ . In any of these cases,  $\mathcal{E} : A \rightarrow T$  is said to be an individualized representation of  $\mu$  if  $\int_A e_{\mathcal{E}}(a) d\nu(a) < \infty$  where  $e_{\mathcal{E}} = \text{Proj}_F \mathcal{E}$  is the initial endowment. Notice that there may be more than one representation.

For a given individualized representation,  $\mathcal{E}$ , let  $X_{\mathcal{E}}$  be a correspondence from the agent space  $(A, \mathcal{A}, \nu)$  into  $Q$ . A measurable function  $f : A \rightarrow Q$  is an allocation for the economy  $\mathcal{E}$  if it is Bochner integrable and  $f(a) \in X_{\mathcal{E}}(a)$  almost everywhere. We say that  $f$  is attainable if  $\int_A f d\nu = \int_A e_{\mathcal{E}} d\nu$ . Furthermore, it is a Walrasian allocation if there exists  $p \in L_+^*$  such that  $p \cdot f(a) \leq p \cdot e_{\mathcal{E}}(a)$  and  $p \cdot f'(a) > p \cdot e_{\mathcal{E}}(a)$  whether  $f'(a) \succ_a f(a)$  for almost all  $a \in A$ . It is straightforward to verify that if  $(p, x)$  is a Walrasian equilibrium of  $\mathcal{E}$  whose agent space is  $(A, \mathcal{A}, \nu)$ , then  $\nu \circ (\mathcal{E}, x)^{-1}$  in  $\mathcal{M}(T \times X)$  is a Walras equilibrium distribution for  $\nu \circ \mathcal{E}^{-1}$ .

Let  $\mathcal{E} : A \rightarrow T$  be an individualized economy with  $\tau \in \mathcal{DW}(\nu \circ \mathcal{E}^{-1})$  as distributional equilibrium. We cannot guarantee that there exists a measurable mapping  $x : A \rightarrow X$  such that  $\nu \circ (\mathcal{E}, x)^{-1} = \tau$  unless the agent space  $(A, \mathcal{A}, \nu)$  satisfies a property called *saturation*: for every  $\tau \in \mathcal{M}(T \times X)$  such that for each measurable function  $g : A \rightarrow T$  with  $\tau_T = \nu \circ g^{-1}$  there exists a measurable function  $h : A \rightarrow X$  such that  $\nu \circ (g, h)^{-1} = \tau$ . A probability space is *saturated* if and only if it satisfies the saturation property for every Borel probability measure on the product of any two Polish spaces (Keisler and Sun (2009)). Alternatively one can also ask the  $\sigma$ -algebra  $\mathcal{A}$  to be *relatively saturated* with respect to a sub- $\sigma$ -algebra  $\mathcal{G}$ .<sup>5</sup>

The following result shows that large individualized economies have the closed graph property when the agent space satisfies the saturation property.

<sup>5</sup> $\mathcal{A}$  is relatively saturated with respect to the sub- $\sigma$ -algebra  $\mathcal{G}$  if for every  $\tau \in \mathcal{M}(T \times X)$  such that for each  $\mathcal{G}$ -measurable function  $f : A \rightarrow T$  with  $\tau_T = \nu \circ f^{-1}$  there exists a  $\mathcal{A}$ -measurable function  $g : A \rightarrow X$  such that  $\nu \circ (f, g)^{-1} = \tau$

**Theorem 4** *Let  $\{(A_n, \mathcal{A}_n, \nu_n)\}_{n \in \mathbb{N}}$  be a sequence of probability spaces denoting agent spaces. Let  $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$  be a sequence of economies such that for each  $n \in \mathbb{N}$ ,  $\mathcal{E}_n : A_n \rightarrow T$  and let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence of allocations such that for each  $n \in \mathbb{N}$ ,  $x_n$  is an equilibrium for  $\mathcal{E}_n$ . Let  $(A, \mathcal{A}, \nu)$  be an atomless saturated probability space and let  $\mathcal{E} : A \rightarrow T$  be a measurable function. If  $\{\nu_n \circ \mathcal{E}_n^{-1}\}$  converges weakly to  $\nu \circ \mathcal{E}^{-1}$  and  $\{\nu_n \circ x_n\}_{n \in \mathbb{N}}$  converges in distribution to some measure  $\gamma \in \mathcal{M}(X)$ , then there exists a mapping  $x : A \rightarrow X$  which is an equilibrium for  $\mathcal{E}$  and  $\nu \circ x^{-1} = \gamma$ .*

*Proof* Notice that  $\nu_n \circ (\mathcal{E}_n, x_n)^{-1}$  belongs to  $\mathcal{DW}(\nu_n \circ \mathcal{E}_n^{-1})$  for each  $n \in \mathbb{N}$ ,  $\{\nu_n \circ \mathcal{E}_n^{-1}\}_{n \in \mathbb{N}}$  converges to  $\nu \circ \mathcal{E}^{-1}$  and  $\{\nu_n \circ x_n^{-1}\}_{n \in \mathbb{N}}$  converges to a measure  $\gamma \in \mathcal{M}(X)$ . By Theorem 2.1 (iii) in Keisler and Sun (2009) there exists a subsequence of  $\nu_n \circ (\mathcal{E}_n, x_n)^{-1}$  (say itself) which converges to  $\tau \in \mathcal{M}(T \times X)$  such that the marginals  $\tau_T$  and  $\tau_X$  are  $\nu \circ \mathcal{E}^{-1}$  and  $\gamma$  respectively. By Theorem 1,  $\tau$  belongs to  $\mathcal{DW}(\nu \circ \mathcal{E}^{-1})$  and since  $(A, \mathcal{A}, \nu)$  is saturated, there exists  $x : A \rightarrow X$  such that  $\nu \circ (\mathcal{E}, x)^{-1} = \tau$  and  $\tau_X = \nu \circ x^{-1}$ . Hence,  $x$  is an equilibrium for  $\mathcal{E}$  by Corollary 5 in the proof of Theorem 1.

As pointed out previously, the above Theorem holds true if instead of requiring saturation we specify that  $\mathcal{A}$  is relatively saturated with respect to a sub- $\sigma$ -algebra  $\mathcal{G}$ . Of course, when the sequence  $\{x_n\}_{n \in \mathbb{N}}$  converges in distribution to the (Bochner integrable) function  $x$ , we do not require the space  $(A, \mathcal{A}, \nu)$  to be saturated.

Recently, He et al. (2017) introduced the notion of *nowhere equivalence* to model infinitely many agents. Let  $\mathcal{G}$  be a countably generated sub- $\sigma$ -algebra of the  $\sigma$ -algebra  $\mathcal{A}$ . For any  $A' \in \mathcal{A}$ , such that  $\mu(A') > 0$ , the restricted probability space  $(A', \mathcal{A}^{A'}, \mu^{A'})$  is defined by:  $\mathcal{A}^{A'} = \{A' \cap A'' : A'' \in \mathcal{A}\}$  and  $\mu^{A'}$  is the probability measure rescaled from the restriction of  $\mu$  to  $\mathcal{A}^{A'}$ . We shall say that  $\mathcal{A}$  is nowhere equivalent to  $\mathcal{G}$  if for every  $A' \in \mathcal{A}$  with  $\mu(A') > 0$ , there exists a  $\mathcal{A}$ -measurable subset  $A'_0$  of  $A'$  such that  $\mu(A'_0 \Delta A'_1) > 0$  for any  $A'_1 \in \mathcal{G}^{A'}$ , where  $A'_0 \Delta A'_1$  is the symmetric difference  $(A'_0 \setminus A'_1) \cup (A'_1 \setminus A'_0)$ . He et al. (2017) state the conditions under which saturated spaces and relatively saturated ones are equivalent with the nowhere equivalence condition (Corollary 1, p. 787 and Corollary 3, p. 792). However, Lemma 5 and Theorem 1 in their work allow us to prove Theorem 4 even if the agent probability space is non saturated.

## 6.2 Approximations through large but finitely many agents economies

The objective of several authors has been to prove that every sequence of individualized economies with a nonempty set of Walrasian equilibria in each of them has a limit individualized economy with at least one equilibrium. Usually, the agent space of the limit economy is an atomless measure space which is seen as an idealized limit of a large economy with finitely many agents. The idea is that such idealized economy captures the limiting behavior of large finite models. Our Theorem 1 allows us to go a step forward by considering

an infinite dimensional commodity space. The following result is a Corollary of the above Theorem when considering a sequence of finitely many agents economies and their equilibrium allocations.

**Corollary 4** *Let  $\{(A_n, 2^{|A_n|}, \nu_n)\}_{n \in \mathbb{N}}$  be a sequence of agent spaces with each  $A_n$  finite and directed by set inclusion. Let  $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$  be a sequence of economies such that for each  $n \in \mathbb{N}$ ,  $\mathcal{E}_n : A_n \rightarrow T$  and let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence of allocations such that for each  $n \in \mathbb{N}$ ,  $x_n$  is an equilibrium for  $\mathcal{E}_n$ . Let  $(A, \mathcal{A}, \nu)$  be an atomless and saturated probability space and let  $\mathcal{E} : A \rightarrow T$  be a measurable function. If  $\{\nu_n \circ \mathcal{E}_n^{-1}\}$  converges weakly to  $\nu \circ \mathcal{E}^{-1}$  and  $\{\nu_n \circ x_n\}_{n \in \mathbb{N}}$  converges in distribution to some measure  $\gamma \in \mathcal{M}(X)$ , then there exists a mapping  $x : A \rightarrow X$  which is an equilibrium for  $\mathcal{E}$  and  $\nu \circ x^{-1} = \gamma$ .*

Conversely, given an Walras equilibrium  $(x, p)$  for an individualized economy  $\mathcal{E}$  whose agent space is an atomless probability space  $(A, \mathcal{A}, \nu)$ , one can ask when it is possible to approximate this equilibrium through large economies with finitely many type of agents in order to ensure that the way the continuum has been modeled is not a simply artifact. By Hildenbrand (1974) (29) p. 49,  $\mu \in \mathcal{M}(T)$  is the weak limit of a sequence of economies  $\{\mu_n\}_{n \in \mathbb{N}}$  having finite supports. Even more, Theorem 1 (i) of Hildenbrand (1970), p. 169, says that the sequence  $\{\text{supp}(\mu_n)\}_{n \in \mathbb{N}}$  is directed by set inclusion. If both the economy  $\mu$  and the sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  are concentrated on  $\mathbb{E}'$ , then, because of Corollary 3, for every  $\tau \in \mathcal{DW}(\mu)$  there exists a sequence  $\{\tau_n\}_{n \in \mathbb{N}}$  such that  $\tau_n \in \mathcal{DW}(\mu_n)$  for every  $n \in \mathbb{N}$  and  $\tau_n \rightarrow \tau$ . Since the support of a measure defines the set of types of economic agents participating in the economic system described by the measure, the above results mean that every economy can be approximated by economies with increasing yet finitely many types of agents with their equilibria. Hence, we have the following result

**Proposition 6** *Let  $(A, \mathcal{A}, \nu)$  be an atomless probability space and let  $\mathcal{E} : A \rightarrow T$  and economy whose Walras allocation is the measurable function  $x : A \rightarrow X$ . Let  $\mathcal{M}^{\mathcal{F}}(T)$  be the set of economies with finite support. Assume that there exists a nonempty subset  $\mathbb{E}''$  of  $\mathcal{M}^{\mathcal{F}}(T) \cap \mathbb{E}'$  such that  $\overline{\mathbb{E}''} \subset \mathbb{E}'$ . If the distribution of the economy  $\mathcal{E}$  belongs to  $\overline{\mathbb{E}''}$ , there exists a sequence of individualized economies  $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$  converging in distribution to  $\mathcal{E}$  and a sequence of Walrasian allocations for these economies  $\{x_n\}_{n \in \mathbb{N}}$  converging in distribution to  $x$  such that every economy  $\mathcal{E}_n$  has  $n$  types of agents*

*Proof* By the remarks above, there exists a sequence  $\{\tau_n\}_{n \in \mathbb{N}}$  such that  $\tau_n \in \mathcal{DW}(\mu_n)$  for each  $n \in \mathbb{N}$  and  $\tau_n \rightarrow \nu \circ (\mathcal{E}, x)^{-1}$ . For each  $n$  there exist measurable mappings  $x_n : A \rightarrow X$ ,  $\mathcal{E}_n : A \rightarrow T \times X$  and  $p_n \in L^*$  such that  $\nu \circ (\mathcal{E}_n, x_n)^{-1} = \tau_n$ ,  $\nu \circ \mathcal{E}_n^{-1} = \tau_n|_T = \mu_n$ ,  $\nu \circ x_n^{-1} = \tau_n|_X$  and  $(x_n, p_n) \in WE(\mathcal{E}_n)$  accordingly to Lemma 4. Clearly,  $\{x_n\}_{n \in \mathbb{N}}$  and  $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$  converge in distribution to  $x$  and  $\mathcal{E}$  respectively. Since for each  $n \in \mathbb{N}$ ,  $|\text{supp}(\nu \circ \mathcal{E}_n^{-1})| = n$ , the number of type of agents in  $\mathcal{E}_n$  is  $n$ .

## 7 Concluding remarks

The results of this paper could be improved and/or extended in a number of ways; this seems like a good place to indicate some of these possible cases.

1. Several authors have proved the existence of an equilibrium when the commodity space is an ordered separable Banach space with a nonempty interior. Hence, the set  $\mathbb{E}$  of Theorem 1 is justified. The canonical space, the one we used in the examples, is  $C(K)$  with  $K$  a Hausdorff compact metric space. Furthermore, every weak compact subset of its dual space is metrizable and thus price simplex is a compact metric space. However, we point out that we can extract positive results even though from some non-separable Banach spaces by considering as commodity spaces properly subsets. Indeed, the spaces  $L_\infty$  and  $l_\infty$  of essentially bounded measurable functions and of essentially bounded sequences respectively are not separable but their weak compact subsets are. On the one hand, we have that weakly compact subsets of  $L_\infty$  are norm separable (Diestel and Uhl (1977), Theorem 13, p. 252) with nonempty norm interior and, on the other hand, weak-compact subsets of  $l_\infty$  are metrizable by Proposition 3.29 in Fabian et al. (2001), p. 95. Hence, our main results can be seen as covering these economies provided that prices belong to a compact metric space.
2. Related to previous item, every  $\|\cdot\|$ -bounded set in the non-separable spaces  $L_\infty$  and  $l_\infty$  is  $w^*$ -compact by Alaoglu-Bourbaki Theorem. Furthermore, since  $L_1$  and  $l_1$  are separable, those sets are metrizable. Hence, besides the weak topology, we could also consider the weak\* topology as a way of getting a compact metric space  $Q$ . From an individualized representation of the economy, it implies that Gelfand integrals take place rather than Bochner ones. We shall explore this idea in future studies even for commodity spaces with an empty interior of the positive cone.
3. Throughout the paper we have considered the set of irreflexive, continuous, transitive, symmetric and strictly monotone preferences  $\mathcal{P}_{mo}$  since this is in line with previous works like Hildenbrand (1974), Hart et al. (1974) and He et al. (2017) among many others. However, given a nonempty set  $\mathbb{E}$ , Theorem 1 works even if we consider preferences without a monotonicity assumption,  $\mathcal{P}$ . Thus, the space of characteristics is Polish and consequently  $\mathcal{M}(\mathcal{P} \times F)$  and  $\mathcal{M}(\mathcal{P} \times F \times X)$  are also Polish (Lemma 1). The subset of  $\mathcal{P}$  of all convex preferences,  $\mathcal{P}_{co}$ , can be shown to be Polish too as well as for strongly convex preferences,  $\mathcal{P}_{sco}$ . For definitions about these sets we refer the reader to Hildenbrand (1974). In all these cases, the set of characteristics will have the desired properties for proving closedness of the Walrasian correspondence.
4. The restriction of having a common consumption set may be relaxed by allowing different ones across economies. However, two new assumptions are required:
  - The set of preferences is a closed subset  $\mathcal{T}$  of  $\mathcal{P}_{sco} \cap \mathcal{P}_{mo}$ .

- There exists a norm-closed and convex subset  $X'$  of  $L_+$  such that  $F \subset X'$  and there exists  $\alpha > 0$  such that  $X' \subset V(0, \alpha) \cap L_+ \subset X$  for all  $X$ .

We note that Theorem 1 works for a subset  $\mathbb{E}$  of  $\mathcal{M}(\mathcal{T} \times F)$  and a compact subset  $\mathbb{A}$  of  $\mathcal{M}(\mathcal{T} \times F \times X')$  from which we deduce that our previous stability results work for a closed subset of Walrasian equilibria.

## A Proofs

### A.1 PROOF OF LEMMA 1

We would like to use Theorem 1 p. 96 of Hildenbrand (1974) since it works for locally compact spaces  $Q$  other than  $\mathbb{R}^L$ . For that, we have to ensure that  $(\mathcal{C}^{w_Q}(Q \times Q), \tau_C)$  is compact metrizable. This follows from the application of Theorem 2 p. 19 of Hildenbrand (1974) to  $(Q \times Q, w_Q \times w_Q)$  that is a locally compact Polish space.

1. Let  $\{(X_n, \succ_n)\}_{n \geq 0}$  be a sequence in  $\mathcal{P}$  such that it has a closed limit  $(X, \succ)$ . We shall prove that it belongs to  $\mathcal{P}$ . This is equivalent to  $g(\mathcal{P})$  being closed in  $\mathcal{C}^{w_Q}(Q \times Q)$ . Indeed, let us consider the sequence  $\{g((X_n, \succ_n))\}_{n \in \mathbb{N}} = \{P_n\}_{n \in \mathbb{N}}$  in  $\mathcal{C}^{w_Q}(Q \times Q)$  where  $P_n = \{(x, y) \in X_n \times X_n : x \not\succeq_n y\}$ . We already noted that  $(\mathcal{C}^{w_Q}(Q \times Q), \tau_C)$  is a compact metric space. Then, the sequence  $\{P_n\}_{n \geq 1}$  converges to  $P$  if and only if  $P = \text{Li}(P_n) = \text{Ls}(P_n)$  (Hildenbrand (1974), B.II. Theorem 2, p. 19). Let us define  $X = \text{proj}_Q P$  and  $\succ = X \times X \setminus P$ . We have to prove that  $g((X, \succ)) = P$ . Let us note that for  $x \in X$  it follows that  $(x, x) \in P$ . Indeed, for  $x \in X$  there exists  $x' \in Q$  such that  $(x, x') \in P$ . Since  $P = \text{Li}(P_n) = \text{Ls}(P_n)$ , there exists a sequence  $\{(x_n, x'_n)\}_{n \geq 1}$  belonging to  $P_n$  for each  $n \geq 1$  and  $\lim_{n \rightarrow \infty} (x_n, x'_n) = (x, x')$  for the topology  $w_Q \times w_Q$  (Hildenbrand (1974), p. 15). Since  $\succ_n$  is irreflexive for each  $n \geq 1$  it follows that  $(x_n, x_n) \in P_n$  and then  $(x, x) \in P$ . The argument above implies that  $X$  is the closed limit of the sequence  $\{X_n\}_{n \geq 1}$  and it is nonempty since  $0 \in X_n$  for all  $n \geq 1$ . Following the arguments of Hildenbrand (1974), p. 97, we note that  $X$  is convex and  $\succ$  is irreflexive and transitive. Finally, we only need to show that  $g((X, \succ)) = P$  which is direct since  $g((X, \succ)) = \{(x, y) \in \text{proj}_Q P \times \text{proj}_Q P : (x, y) \in P\} = P$ .
2. and 3. follows from mimicking the proof of Theorem 1(b) of Hildenbrand (1974).  $\square$

### A.2 PROOF OF LEMMA 2

It suffices to show that  $\mathcal{P}_{mo}$  with the metric of the closed convergence is a  $G_\delta$ -set, i.e., a countable intersection of open sets in  $\mathcal{P}$ . We follow the approach given in Lemma of p. 98 by Hildenbrand (1974). Let  $d_{w_Q}$  be the metric for which  $(Q, w_Q)$  is metrizable. For every  $m \in \mathbb{N}$  we define the set  $\mathcal{P}_m = \{(X, \succ) \in \mathcal{P} : \exists x, y \in X, x \geq y, x \not\succeq y \text{ and } d_{w_Q}(x, y) \geq \frac{1}{m}\}$ . Let  $\{(X_n, \succ_n)\}_{n \geq 1}$  be a sequence in  $\mathcal{P}_m$ , then there exists a sequence  $\{(x_n, y_n)\}_{n \geq 1}$  such that  $x_n \geq y_n$ ,  $x_n \not\succeq_n y_n$  and  $d_{w_Q}(x_n, y_n) \geq \frac{1}{m}$ . Since both  $(x_n)$  and  $(y_n)$  belong to  $Q$  which is  $w$ -compact, there are subsequences also denoted by  $(x_n)$  and  $(y_n)$  which  $w$ -converge to  $x$  and  $y$  respectively. Let  $P_n = \{(x', y') \in (X_n, X_n) : x' \not\succeq_n y'\}$  from which we deduce that  $(x_n, y_n) \in P_n$  for each  $n \geq 1$ . By Lemma 1,  $(X, \succ) \in \mathcal{P}$  and  $\text{Li}(P_n) = \text{Ls}(P_n) = P$ . We want to prove that the closed limit  $(X, \succ)$  belongs to  $\mathcal{P}_m$ . It is easily verified that both  $x$  and  $y$  belong to  $\text{Ls}(X_n) = X$ . Notice that  $(x, y) \in \text{Ls}(P_n)$  so that  $x \not\succeq y$ . Since  $Q$  is  $w$ -closed, it follows that  $x \geq y$ . We claim that  $d_{w_Q}(x, y) \geq \frac{1}{m}$ . Otherwise, we would have that there exists  $n_0$  such that for all  $n > n_0$ ,  $d_{w_Q}(x_n, y_n) < \frac{1}{m}$  which is a contradiction. Consequently,  $(X, \succ) \in \mathcal{P}_m$  whence,  $\mathcal{P}_m$  is  $\tau_{\mathcal{P}}^C$ -closed.

Note that  $\mathcal{P}_{mo} = \bigcap_{m \in \mathbb{N}} (\mathcal{P} \setminus \mathcal{P}_m)$  and thus  $\mathcal{P}_{mo}$  is a  $G_\delta$ -set. By the classical Alexandroff lemma (see Aliprantis and Border (2006), Lemma 3.34 p.88), we conclude that  $\mathcal{P}_{mo}$  is completely metrizable. In addition, by Corollary 3.5 p. 73 in Aliprantis and Border (2006) we

know that  $\mathcal{P}_{mo}$  as subset of a separable metric space  $\mathcal{P}$  is separable. Thus,  $\mathcal{P}_{mo}$  is a Polish space.  $\square$

### A.3 PROOF OF THEOREM 1

Since  $\mathbb{A}$  is compact, it suffices to show that the correspondence  $\mathcal{DW}$  has a closed graph (Aliprantis and Border (2006), Theorem 17.11, p. 561). Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a sequence of economies in  $\mathbb{E}$  converging weakly to  $\mu \in \mathbb{E}$ . Let us consider the sequence of distributions  $\{\tau_n\}_{n \in \mathbb{N}}$  such that  $\tau_n \in \mathcal{DW}(\mu_n)$  for all  $n$  and  $\lim_{n \rightarrow \infty} \tau_n = \tau$ . We have to show that  $\tau \in \mathcal{DW}(\mu)$ .

First of all, since the marginal of  $\tau_n$  with respect to the space of characteristics,  $\tau_n|_T$ , is equal to  $\mu_n$  for all  $n$  by definition, we have that  $\tau_T = \mu$ . Second, we know that  $\int_X id\tau_n|_X = \int_F id\mu_n|_F$  for all  $n$ . Since both  $\tau_n|_X$  and  $\mu_n|_F$  converge weakly to  $\tau_X$  and  $\mu_F$  respectively, one deduces that  $\int_X id\tau_X = \int_F id\mu_F$ .

Finally, for each  $n \in \mathbb{N}$ , there exists  $p_n \in S$  such that  $\tau_n(E_{p_n}) = 1$ . Since  $S$  is weak\*-compact, there exists a subnet also denoted  $\{p_n\}_{n \in \mathbb{N}}$  which converges to  $p$ . We proceed to show that  $\tau(E_p) = 1$ . By Skorokhod's Theorem (Billingsley (1999), Theorem 6.7, p. 70, see also, Hildenbrand (1974), p. 50) on the sequence  $\{\tau_n\}_{n \in \mathbb{N}}$ , we know there is a space  $(I, \mathcal{B}_I, \lambda)$  and measurable mappings  $\{(\mathcal{E}_n, x_n)\}_{n \in \mathbb{N}}$  and  $(\mathcal{E}, x)$  from  $I$  into  $T \times X$  such that  $\{(\mathcal{E}_n(i), x_n(i))\}_{n \in \mathbb{N}}$  converges with respect to  $\tau_C^P \times \|\cdot\|_X \times \|\cdot\|_X$  a.e. to  $(\mathcal{E}(i), x(i))$ , and we have that for all  $n$ ,  $\lambda \circ (\mathcal{E}_n, x_n)^{-1} = \tau_n$  and  $\lambda \circ \mathcal{E}_n^{-1} = \mu_n$ . Furthermore,  $\lambda \circ (\mathcal{E}, x)^{-1} = \tau$  and  $\lambda \circ \mathcal{E}^{-1} = \mu$ . Note that  $\lambda \circ (\mathcal{E}_n, x_n)^{-1} \in \mathcal{DW}(\lambda \circ \mathcal{E}_n^{-1})$ .

Notice that  $\{(\mathcal{E}_n, x_n)\}_{n \in \mathbb{N}}$  and  $(\mathcal{E}, x)$  represents a sequence and a limit point of individualized economies whose characteristics' and allocations distributions coincide with the distributional economies in, respectively,  $\{\tau_n\}_{n \in \mathbb{N}}$  and  $\tau$ . Regarding individualized representation of economies, we redirect the reader to section 6.

**CLAIM 1.** *Let  $e$  be the projection of  $\mathcal{E}$  on the space of endowments,  $\int_X id(\lambda \circ x^{-1}) = \int_F id(\lambda \circ e^{-1})$  is equivalent to  $\int_I xd\lambda = \int_I ed\lambda$  provided both  $x$  and  $e$  are Bochner integrable.*

The following lemma provides the proof of the integrability required by the Claim 1 above.

**Lemma 3** *Let  $(A, \mathcal{A}, \nu)$  be a finite measure space, then every  $(\mathcal{A}, \mathcal{B}(X))$ -measurable function  $f : A \rightarrow X \subset L$  is Bochner integrable.*

*Proof* We start by claiming that  $f$  is  $(\mathcal{A}, \mathcal{B}(X))$ -measurable if and only if it is  $(\mathcal{A}, \mathcal{B}(L))$ -measurable. Indeed, suppose that  $f$  is  $(\mathcal{A}, \mathcal{B}(L))$ -measurable. Then, for any  $B \in \mathcal{B}(L)$ ,  $f^{-1}(B) = f^{-1}(B \cap X) \cup f^{-1}(B \cap (L \setminus X))$ . We know that  $f^{-1}(B \cap X) \in \mathcal{A}$  since  $f$  is  $(\mathcal{A}, \mathcal{B}(X))$ -measurable and  $f^{-1}(B \cap (L \setminus X)) = \emptyset$  since  $f$  takes values only in  $X$ . The converse is obvious.

Since  $X$  is separable, there exists a sequence of  $(\mathcal{A}, \mathcal{B}(X))$ -measurable simple functions  $\{f_n\}_{n \in \mathbb{N}}$  from  $A$  into  $X$  which converges in norm to  $f$  a.e. (see Aliprantis and Border (2006) Theorem 4.38 1. p. 145). By the previous claim, the simple functions are  $(\mathcal{A}, \mathcal{B}(L))$ -measurable and  $f$  is Bochner integrable by Proposition 4.

**CLAIM 2.**  *$(x_n, p_n)$  is a Walrasian equilibrium of  $\mathcal{E}_n$  for all  $n$ .*

Recall that  $\mathcal{E}' : A \rightarrow T$  is said to be an individualized representation of  $\mu'$  if  $\int_A e_{\mathcal{E}'}(a) d\nu(a) < \infty$  where  $e_{\mathcal{E}'} = \text{Proj}_F \mathcal{E}'$  is the initial endowment. The next two results show that for any distributional equilibrium there is a individualized representation with their corresponding equilibrium.

**Lemma 4** *Let  $\tau'$  be an equilibrium of the economy  $\mu'$  and let  $(A, \mathcal{A}, \nu)$  be an atomless finite measure space. There exist measurable mappings  $\mathcal{E}' : A \rightarrow T$  and  $x' : A \rightarrow X$  such that  $\mathcal{E}'$  represents  $\mu'$ ,  $x'$  is an equilibrium for  $\mathcal{E}'$  and  $\nu \circ (\mathcal{E}', x')^{-1} = \tau'$*



*Proof* Since  $T \times X$  is separable and  $\tau' \in \mathcal{M}(T \times X)$  one has by Keisler and Sun (2009), Lemma 2.1 (ii), that there exists a measurable mapping  $f : A \rightarrow T \times X$  such that  $\tau' = \nu \circ f^{-1}$ . Let us denote by  $\mathcal{E}'$  and  $x'$  the respective projections of  $f$  on  $T$  and  $X$ . Hence,  $\mu' = \nu \circ \mathcal{E}'^{-1}$  and  $\tau' = \nu \circ (\mathcal{E}', x')^{-1}$ . Notice that  $x'$  as well as  $e' = \text{Proj}_F \mathcal{E}'$  are measurable whence, Bochner integrable by Proposition 3

Next, we prove that  $x'$  is Walrasian equilibrium for  $\mathcal{E}'$ . Indeed, on the one hand, since  $\tau'$  is an equilibrium for  $\mu'$ ,  $\int_X \iota d\tau'_X = \int_F \iota d\mu'_F$ . Hence,  $\int_X \iota d(\nu \circ x'^{-1}) = \int_F \iota d(\nu \circ e'^{-1})$  which implies  $\int_A x' d\nu = \int_A e' d\nu$  by Lemma 8 (f) in Dunford and Schwartz (1958). On the other hand, there exists  $p \in L_+^*$  such that  $\tau(E_p) = 1$ . Then,  $\nu \circ (\mathcal{E}', x')^{-1}(E_p) = 1$ . Thus  $\nu(\{a \in A : (\mathcal{E}'(a), x'(a)) \in E_p\}) = 1$  which implies that  $p \cdot x'(a) \leq p \cdot e(a)$  and  $p \cdot x''(a) > p \cdot e(a)$  whether  $x''(a) \succ_a x'(a)$  a.e. Hence,  $x'$  is a walrasian allocation for  $\mathcal{E}'$ .

**Corollary 5** *Let  $\tau' \in \mathcal{M}(T \times X)$  be an equilibrium of the economy  $\mu'$ . Let  $(A, \mathcal{A}, \nu)$  be an atomless, finite measure space and let  $f : A \rightarrow T \times X$  be a measurable function such that  $\nu \circ f^{-1} = \tau'$  and  $\nu \circ (\text{Proj}_T f)^{-1} = \mu'$ . Then,  $\text{Proj}_X f$  is an equilibrium allocation for  $\text{Proj}_{T \times F} f$ .*

Consequently by the Claim 2 above,  $(x_n, p_n)$  is a Walrasian equilibrium of  $\mathcal{E}_n$  for all  $n$  and since both  $\{x_n\}_{n \in \mathbb{N}}$  and  $\{e_n = \text{Proj}_F \mathcal{E}_n\}_{n \in \mathbb{N}}$  converge in norm to  $x$  and  $e$  respectively, we get  $\lim_n p_n \cdot x_n(i) = p \cdot x(i)$ ,  $\lim_n p_n \cdot e_n(i) = p \cdot e(i)$  and  $p \cdot x(i) = p \cdot e(i)$  a.e. Suppose now that there exists  $\xi \in L_1(\lambda, X)$  such that  $\xi(i) \succ_i x(i)$  and  $p \cdot \xi(i) < p \cdot e(i)$  a.e. By Corollary 1,  $\xi(i) \succ_{n,i} x_n(i)$  for  $n$  large enough. Because of equilibrium conditions in  $\mathcal{E}_n$ , it follows that  $p_n \cdot \xi(i) > p_n \cdot e_n(i)$ . Taking limits we get,  $p \cdot \xi(i) \geq p \cdot e(i)$  a.e. which contradicts the above converse inequality.

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