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# Teaching, learning and assessing in grade 10: an experimental pathway to the culture of theorems

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*This report deals with the design of a teaching and learning pathway that encompasses an assessment method coherent with the educational long term goal chosen by the teacher. More specifically, it is aimed at contributing to promote high school learners' culture of theorems – i.e. mastery of conjecturing and proving supported by meta-mathematical knowledge. The assessment method, which combines formative assessment all during the teaching sequence and final assessment based on student's self-reflective report on her learning trajectory, has been implemented in 10th grade classes. A written essay, in form of letter written by the students to a fictitious schoolmate at the end of the pathway, is analysed in order to evaluate the potential of the implemented course.*

*Keywords: Formative and summative assessment, proving, secondary school.*

## Introduction

In this contribution, we address the following issue, as outlined in the call for papers of TWG21: “The assessment of specific mathematics domains – such as for example algebra or geometry - or capabilities such as problem solving or mathematical modelling”. More specifically, we refer to the teaching and learning of mathematical proof and proving, conceptualized as the development of a specific “culture of theorems” (Boero, 2007). In our research group, we have been working for many years in the development of specific constructs and related experimental activities concerning proof and proving. In particular, we designed methods to foster students' awareness of crucial aspects of proving concerning their inherent strategies, what warrants the truth of the statements, and the logic-linguistic features of proof (Boero, Douek, Morselli & Pedemonte, 2010). Here we focus on the design of a whole teaching and learning pathway that encompasses also formative assessment strategies and final assessment of the individual competencies. From one side, we believe in the efficacy of formative assessment strategies in promoting the development of a culture of theorems. From the other side, we recognize that promoting a culture of theorems, in order to be effective in the institution, requires a coherent method of final assessment. For this reason, we inserted an innovative method of summative assessment in the project. The aim of this report is to present the teaching-learning (and assessing) pathway as it was implemented in two 10th-grade classes, and to illustrate data that provide some initial evidence of its effectiveness.

## The culture of theorems

We refer to Mariotti (2001)'s construct of theorem: the theorem is made up of the statement, together with its proof within a theory. For the same statement, it is possible to consider different

ways of proving it, according to different methods of proof and different theories. In that perspective, competencies in conjecturing and proving concern, in particular, the mastery of meta-mathematical knowledge regarding: the shape of statements (hypothesis, thesis); different methods to validate a statement; the distinct role of previously proved theorems (as synthesizers of deductive chains needed to build up the proof) and axioms (as founding stones of a theory); the distinction between the construction and plausibility of a conjecture (based on different processes: induction, abduction, analogy, etc.) and the construction of a proof (which aims at a deductive enchainment of propositions within a theory). The development of a culture of theorems requires: autonomy in acting as problem solvers in conjecturing and proving; awareness of meta-mathematical issues (like the epistemic constraints and logic-linguistic features of proof – Stylianides & Ball, 2008), which play a crucial role both in understanding a proof produced by others, and in conjecturing and proving by themselves. Promoting a culture of theorems in classroom means to foster students' awareness of these aspects and to make students develop a meta-mathematical knowledge (Boero et al, 2010) that supports the self-regulation of the process of conjecturing and proving.

### **The crucial role of assessment**

We believe that assessment must be consistent with teaching's aims and with task design. As a starting point, we refer to the known distinction between formative assessment and summative assessment. Summative assessment occurs at the end of the learning activity with the goal of estimating the individual student's achievement (McIntosh, 1997), and is generally used as part of the grading process. Formative assessment is performed during the learning activity with the aim of improving learning. It is also called "assessment for learning" and may be considered a method of teaching where "evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction." (Black & Wiliam, 2009, p.7). Wiliam and Thompson (2007) describe five key strategies to promote formative assessment in classroom, involving three agents: the teacher, the students and the peers. The five strategies are: a) clarifying and sharing learning intentions and criteria for success; b) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; c) providing feedback that moves learners forward; d) activating students as instructional resources for one another; e) activating students as the owners of their own learning. Moreover, Nicol and Macfarlane-Dick (2006) argue that formative assessment may promote students' self regulated learning. Recognizing formative assessment as crucial for learning does not mean to deny the crucial role of summative assessment. At the same time, it is important to move beyond a summative assessment that confines itself to a mere "final test", that hardly provides reliable information on the competencies that are acquired. For instance, Taras (2001) describes a research where cycles of summative self-assessment, tutor feedback on self-assessment and final revised self-assessment are carried out. The issue of summative assessment in mathematics becomes crucial when it has to be set up after innovation teaching and learning sequences. Assessment methods based on individual problem solving and answering teacher's questions usually have a negative impact on the development of students' autonomy and responsibility as learners in the classroom social context. In the special case of the teaching and learning of proof, traditional individual assessment tasks (presenting learnt proofs, adapting learnt proofs to similar

situations, only seldom producing and validating conjectures), result in students' focusing on the products and their features.

### **Assessment in a “culture of theorems” perspective**

In our teaching and learning pathway, we aimed at creating a formative and summative assessment that is coherent with the whole educational plan, aimed at promoting a culture of theorems. In the culture of theorems, it is important to make students active and autonomous, willing to take the responsibility of creating their own proofs and critically analyse the proofs produced by others. Accordingly, we created activities where students give constructive feedback to their mates and take profit from mate's feedback, becoming at the same responsible of their own learning and resources for the mates; in this way, formative strategies d) and e) (Wiliam & Thompson, 2007) are promoted. The formative assessment activities are integrated with a summative assessment method based on the student's self-reflective report on the personal learning trajectory, so as to assess whether the student is able to exploit the previous formative assessment activities and act as an autonomous learner. We hypothesize that thanks to the sequence of activities students not only are more free to take intellectual risks in conjecturing and proving, but may be induced to use meta-mathematical aspects of conjecturing and proving as key tools to regulate their productive and reflective activities – given their role (supported by the teacher) in analysing their fellows and their own conjecturing and proving performances. We also hypothesize that a final assessment method, based on their self-reflective reports on their learning trajectory, might offer them the opportunity to get acceptable results and at the same time would offer them (also thanks to the kind of work performed in the classroom) key tools to re-orient and improve their work. Such an assessment method might also be useful for another reason, related to the condition of so many students who enter high school in our country (and in others too) with low self-confidence and scarce willingness to take intellectual risks. Failures in traditional assessment tasks would confirm them in their disposition towards classroom work (Boaler & Greeno, 2000). We note here that such a form of summative assessment may be considered as an integral part of the teaching and learning pathway and, then, the boundary between formative and summative assessment is not strict in the project. A more detailed description of the teaching and learning method, as well as of the assessment strategies, is provided in the subsequent paragraph.

### **An innovative teaching and learning (and assessing) project**

#### **The context**

Our teaching method and assessment choices was implemented in one 10th-grade class of 21 students in the year 2016/17 and in two 10th-grade parallel classes (of 31 and 19 students) in 2017-18. The classes belong to the scientific-oriented high school, which in our country prepares students for all academic courses, especially for the studies in STEM. We focused on the geometry course, that is taught 2 hours each week, from October to May, in parallel with 3 hours devoted to algebra (first and second degree equations and systems of equations, functions, etc.) and probability, taught by the same teacher. The course consists of 3 modules of about 16 hours each; it concerns some content of Euclidean plane geometry, typically taught in our country in the first two years of high school: (first module) to construct a tangent circle to two intersecting straight lines; to construct the

tangent circle inside a given triangle; (second module) to study different cases of triangles inscribed in a circle (rectangle, isosceles, equilateral triangles), with related necessary and sufficient conditions; to study the relations between peripheral angle and central angle; (third module) given a circle, to study the relationships between its tangent and intersecting straight lines; and, given a circle, to deal with some problems concerning inscribed and circumscribed quadrilaterals. The geometry course in grade 10 follows a grade 9 course in Euclidean plane geometry of triangles and quadrilaterals, which, in our case, was taught by the same teacher. All the lessons were given by the teacher (author FT); one researcher (the author PB) was present and acted as participant observer.

**The course: teaching and formative assessment**

In our pathway, a sequence of classroom activities (“module”) usually consists of some cycles, each of them starting with an individual work on a worksheet, which includes one task and some information and framing of it. The task is open, possibly with some general suggestions about how to deal with it; it involves one of the following activities: conjecturing, with the aim of getting a statement conforming to the requirement of Euclidean geometry; constructing a geometric figure (which corresponds to the “building-up” part of Euclid’s constructions); proving of produced conjectures, or justifying produced constructions (after they have been shared and better formulated under the guide of the teacher in classroom discussions); analysing mates’ productions of previous tasks; identifying salient meta-mathematical aspects of produced texts. We note again that peer review is an efficient way of carrying out formative strategies d) and e) (Wiliam & Thompson, 2007). Each initial individual task of the cycle may be followed by an individual (or small group) analysis of some representative students’ solutions to the task, chosen by the teacher; and/or by a direct classroom discussion of those solutions, guided by the teacher. In this way, formative assessment strategies b), d) and e) are activated (Wiliam & Thompson, 2007). Discussion is addressed to identify gaps, mistakes, points of strength, differences of produced strategies, linguistic issues and meta-mathematical aspects. Students are particularly motivated to engage in these activities, because they will turn to be useful for the the preparation of the final personal report (the key material for their summative assessment- see below). Usually, the second phase of the cycle ends with a synthesis, guided by the teacher, about one (or some) different ways of solving the problem – special care being devoted to the organization of the proof text (in the case of tasks asking for a proof of a given statement) according to meta-mathematical constraints. The synthesis is a key moment for clarifying and sharing learning intentions and criteria for success (strategy a), Wiliam and Thompson, 2007). For some crucial steps of the sequence, students are then requested to write the refined, shared solution for the same task; or to identify hypothesis, thesis, theorems or axioms used as warrants, etc. in a proof text written by the teacher. Table 1 summarizes the activities that were carried out, in reference to the activated formative assessment strategies (as described by Wiliam & Thompson, 2007).

Activity	Formative assessment strategy
Class discussion on selected productions	b) engineering effective classroom discussions that elicit evidence of student understanding; c) providing feedback that moves learners forward; d) activating students as instructional resources

	for one another; e) activating students as the owners of their own learning
Peer review of selected productions	d) activating students as instructional resources for one another; e) activating students as the owners of their own learning
Class discussion to synthesize and organize the final product, with a focus on meta-mathematical aspects	a) clarifying and sharing learning intentions and criteria for success; b) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding

**Table 1: Activities and related formative assessment strategies**

### **The final assessment**

Final assessment is made up of two parts: part A is performed at home, part B is done in classroom. In part A students are requested to prepare (at the end of each of the three modules of the geometry course) a written revision of each of the worksheets produced during the module, followed by a synthesis paper. Students' revision work of personal filled worksheets concerns: identification of gaps, mistakes, difficulties and lacking knowledge at the content and at the meta-level; remediating gaps and mistakes, but in strict agreement (when appropriate) with the personal line of thought. According to our hypothesis, students are engaged in using criteria of meta-mathematical type to identify their difficulties and mistakes, with an inner questioning: "Does my conjecture contain a hypothesis and a thesis? (...) Are there some gaps in my enchaining of propositions?". This questioning at the intra-personal level would echo what should have happened at the inter-personal level in the classroom interactions managed by the teacher, and would contribute to enter the culture of theorems. The synthesis paper concerns: difficulties met during the module, and when and how (and if) they have been overcome; and on relevant learning progresses, and how they occurred, taking the collective classroom work into account. In part B, performed in classroom, students are asked to do the written revision of three worksheets, chosen by the teacher (one of them produced by a schoolfellow, the others by the student herself – thus, a second revision for them). Its purpose is to avoid that the student merely relies on help provided by some more competent fellow (or another expert person). There is no other assessment task for the geometry course. The final grade is given to each student taking into account the involvement during all the pathway (participation to the individual and groupwork and to the discussions) and the final assessment (parts A and B). Criteria for assessing parts A and B concern students' ability in: revising mistakes; improving written productions in terms of completeness, correctness and clarity; reflecting on the personal learning pathway. Essential features of the *assessing method* are similar to those implemented at the university level as described in (Guala & Boero, 2017).

### **Evaluation of the project: preliminary results**

Our evaluation of the educational project is aimed at understanding whether the pathway was efficient in promoting students' culture of theorems, with specific reference to autonomy and awareness of meta-mathematical issues. For the evaluation, as we'll detail hereunder, we rely on

direct observation of the students' activities and fieldnotes, as well as on the qualitative analysis of a special reflective task that was carried out by the students at the end of the pathway.

From the direct observation of students' activities and the fieldnotes of the teacher and the observer, we got some preliminary evidences of the fruitfulness of the adopted method. First, students' relationships with their mistakes and pitfalls during the individual work on their worksheets gradually evolved (at the beginning, mistakes and wrong paths to the solution were carefully erased or whitened in order to hide them; gradually students realized that the memory of wrong paths and mistakes is useful to prepare their self-reflective report, but also to better take part in discussions). Second, there was a growth in the attention paid by students to meta-mathematical aspects: gradually, more and more teacher's (or peers') comments on those aspects are reported in students' notebooks. Third, there was an evolution on kind of students' requests and comments concerning meta-mathematical aspects addressed to the teacher (or their schoolfellows) during classroom discussions.

At the end of the three modules, the students were proposed a reflective task: they were asked to write a letter to a fictitious schoolmate, describing the way of working with theorems they lived during the year. The idea of asking students to narrate their experience to a fictitious schoolmate is coherent with the learning-teaching pathway, where students were encouraged to act as resources for the peers; moreover, the request to write a letter was aimed to promote an informal style, where each student could feel free to insert personal reflections on the pathway. Here is the text of the task:

Paolo, a student of your age, will move to our school and will attend the second year course. By now, he did not do any specific activity on theorems (maybe he has some idea of what is a theorem). Since he heard that your class worked a lot on the statement and the proof of theorems, he would like to contact you to get a clearer idea on the theme. Write a letter to Paolo, telling briefly your experience; explain in particular what is the statement of a theorem and what does "proving it" mean.

We analysed the written letters, looking for elements of meta-mathematical knowledge, as well as reflections on the way students lived the pathway. We realized a qualitative analysis of the texts (performed by the authors separately and after compared and discussed so as to reach an agreement), that we summarize in the subsequent part of the paragraph. Concerning mathematical and metamathematical knowledge related to theorems, most students are able to discuss the key features of a theorem. They point out relevant elements of the statement (hypothesis, thesis) and are able to discuss the issue of validity of the theorem.

Ema: the statement is made of two parts: hypothesis and thesis. The hypothesis is often introduced by "IF", and it is the sentence that you suppose to be true or it is necessary to start your proof. The thesis is often introduced by "THEN" and it is that sentence that you proved in your proof and that, thanks to your proof, is true.

Giu: In mathematics, the THEOREM is a statement that, starting from initial assumptions that are taken as true, draws some conclusions, by means of a precise proof. Proving a theorem means to affirm its validity, following precise properties or mathematical laws.

We note that students are also able to describe proving as a process aimed at producing a final product, which is a relevant metamathematical issue.

Lor: there are different ways to think of a proof, I'll propose you some of them. The most common is to see a theorem as a mathematical problem; in both cases [the theorem and the problem] there are starting points and final points, justified by calculations or known theorems, that is to say already proven. But the interpretation [of theorem] I like the most is another one: you must imagine the theorem as a travel. [...] Before starting the travel, but also during the travel, you must keep in mind the final destination (the thesis) and avoid roads that may lead you to a wrong reasoning. Like in a travel, there can be different roads that lead to the same destination, the thesis.

Giu: a good proof must be organized into three parts: exploration, with the construction of the figure, and observation of all possible relations between the parts of the figure; search for relations that are useful for the proof, with the support of mathematical laws and theorems that allow to affirm that a given thing is true; synthesis of all the process with letters, so that it is precise, well formulated and synthetized.

The letters contain also many hints on efficient ways to perform the process. This is again an evidence of awareness of metamathematical issues; moreover, the fact of being able to describe the process and “give advices” to the fictitious mate reveals that the students became responsible of their own learning, real protagonists of the pathway, and also ready to act as resources for the mates.

Giu: In general, during the proof you must observe the figure with careful and “wise” eyes, so as to reach a conclusion. [...] I suggest you to pay attention to the text of the task and to the figure, to write down all your observations and, even if you think they are un-correct, because they will become useful for the subsequent auto-correction activity. I suggest you to carry out the activities with a lot of care, for me it was very useful.

Mar: I suggest you not to focus on a single aspect of the statement, but to have a wider view and try to think of known theorems that could be useful.

The letters contain a description of the learning pathway, often intertwined with personal reflections on the way the students lived such an experience.

Marco: I liked very much this pathway, because it made me appreciate more geometry, that I did not like very much until last year. This system of activities (proving, constructing, experimenting) made some arguments nicer and helped me to learn them in a better way.

Ser: before starting the pathway I felt unsecure and skeptical, but I changed my opinion. I started the pathway with a little knowledge (few theorems and laws), that I had to review and re-discuss. During the pathway I thought again to that period of time when children ask the reason for everything, because by means of the activities we questioned many things that we had always taken for granted. [...] You can't imagine how many times I struggled with wrong ideas that did not lead me anywhere, or we had an idea that we could not develop.

Some students even reveal a good awareness of the rationale of the educational pathway they experienced.

Giu: for me, the activity of correcting our mates' answers was very useful, because we had to understand his reasoning and possibly correct some imperfections.

Mat: do not worry if at the beginning you can't find the good proof or fill the worksheets, because this is a new method for you, as it was for us. Indeed, the teachers do not judge the mistakes but evaluate your ability to recognize and correct mistakes by yourself. Moreover, this pathway will help you to get into another mate's reasoning and complete or correct it.

## **Discussion and further developments**

The preliminary analysis of the letters provided some evidence for the effectiveness of our pathway, with particular reference to the students' development of meta-mathematical knowledge concerning the culture of theorems and growing assumption of responsibility as learners. The research is still ongoing and we are currently analysing the reflective reports provided by the students at the end of the three modules. Some issues need further reflection. One crucial issue is: how to grade self-reflective reports in an objective way? Sharing with students precise criteria might better orient their work and make more explicit the teacher's expectations, but it could lead students to conform their reflections to a model. Another aspect which needs careful consideration is the development of students' awareness about their growing up as learners in the classroom context. It was put into evidence in a few final letters, while it is relevant for the development of competencies.

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