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Λ -buildings associated to quasi-split groups over Λ -valued fields

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Abstract

Let \mathbf{G} be a quasi-split reductive group and \mathbb{K} be a Henselian field equipped with a valuation $\omega : \mathbb{K}^\times \rightarrow \Lambda$, where Λ is a totally ordered abelian group. In 1972, Bruhat and Tits constructed a building on which the group $\mathbf{G}(\mathbb{K})$ acts provided that Λ is a subgroup of \mathbb{R} . In this paper, we deal with the general case where there are no assumptions on Λ and we construct a set on which $\mathbf{G}(\mathbb{K})$ acts. We then prove that it is a Λ -building, in the sense of Bennett.

Contents

1	Introduction	2
1.1	Bennett's Λ -buildings	3
1.2	The building $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$	4
1.3	Affine structure of the standard apartment	5
1.4	Main results	6
1.5	Structure of the paper	6
2	Abstract definition of \mathfrak{R}^S-buildings and statement of the main theorem	7
2.1	Definition of the standard apartment	7
2.1.1	Root system, vectorial apartment over \mathbb{R} and Weyl group	7
2.1.2	Topology over a totally ordered commutative pseudo-ring	8
2.1.3	R -affine spaces	11
2.1.4	R -metrics over affine spaces	14
2.1.5	Filters	14
2.1.6	Enclosure map	15
2.1.7	Germs, faces and local faces	16
2.1.8	The pseudo-ring \mathfrak{R}^S	16
2.1.9	Topology and metric in \mathfrak{R}^S	18
2.2	Definition of \mathfrak{R}^S -buildings	19
2.2.1	Bennett's definition of \mathfrak{R}^S -buildings	19
2.2.2	Equivalent definition of \mathfrak{R}^S -buildings	19
2.3	Main theorem	20

3	<i>R</i>-valued root group datum	21
3.1	Abstract groups and axioms of a root group datum	21
3.2	<i>R</i> -valuation of a root group datum	24
3.3	Action of <i>N</i> on an <i>R</i> -affine space	27
3.4	Rank-one Levi subgroups	29
3.5	Subgroups generated by unipotent elements	35
3.6	Local root systems	39
4	Parahoric subgroups and Bruhat decomposition	44
4.1	Parahoric subgroups	44
4.2	Action of <i>N</i> on parahoric subgroups	45
4.3	Parahoric subgroups as intersections over their fixed points	47
4.4	Subgroups associated to a filter	50
4.5	Iwasawa decomposition	53
4.6	Bruhat decomposition	54
5	Building associated to a valued root group datum	59
6	Valuation for quasi-split reductive groups	63
6.1	Notations and recalls for quasi-split reductive groups	63
6.2	Recalls on root groups and their parametrizations	63
6.2.1	Definition of root groups	63
6.2.2	The Galois action on the absolute root system	64
6.2.3	Parametrization of root groups	64
6.3	\mathfrak{R}^S -valuation of a root groups datum	68
6.4	Action of <i>N</i> on an affine space	70
6.5	The \mathfrak{R}^S -building of a quasi-split reductive group	73
7	Projection maps	73
7.1	Construction and explicit description of the fibers	73
7.2	The root group data axioms for the fibers	75
7.3	The valuation axioms for the fibers	80
7.4	Compatibility axioms for the <i>N</i> -action	82
7.5	Conclusion	82
7.6	Further notations related to the projection maps	84
8	The axiom (CO)	85
8.1	A sufficient condition for (CO) for \mathbb{R} -buildings	85
8.2	Preservation of the opposition	86
8.2.1	Preliminaries on enclosed sets	86
8.2.2	Projection of an intersection of apartments	88
8.2.3	The exchange condition	88
8.2.4	Germ of a gallery of local chambers and conclusion	89
8.3	Proof of (CO) when <i>S</i> admits a minimum	91
8.4	Proof of (CO) in the general case	92
8.5	Proof of (CO) for the fibers of the projection maps	92
	Bibliographie	93

1 Introduction

Reductive groups over non-Archimedean local fields have been extensively studied for the past sixty years. To study such a group \mathbf{G} and the group of its rational points G ,

Bruhat and Tits associated in [BT72] and [BT84] a space \mathcal{I} - called a **Bruhat-Tits building** - on which G acts. The space \mathcal{I} encapsulates significant information about the group G .

In the 1970's, Kato ([Kat78]) and Parshin ([Par75]) introduced **higher-dimensional local fields**, a natural generalization of the usual local fields. A 0-local field is by definition a finite field, and for $d > 1$, a d -dimensional field is a complete discrete valuation field whose residue field is a $(d - 1)$ -local field. For instance, 1-local fields coincide with usual non-Archimedean local fields. The equicharacteristic 2-local fields are the fields of the form $k((t))$ with k a 1-local field, but there are other 2-local fields that have mixed characteristic.

Higher-dimensional local fields play an important role both in algebraic geometry and in number theory. On the one hand, just as p -adic fields encode local information on arithmetic schemes with relative dimension 0 such as $\text{Spec}(\mathbb{Z})$, 2-local fields encode local information on arithmetic curves such as $\mathbb{A}_{\mathbb{Z}}^1$, and d -local fields encode local information on arithmetic schemes with relative dimension $d - 1$ over \mathbb{Z} . On the other hand, by Kato's work, higher-dimensional local fields also provide the good framework to generalize local class field theory, which is a crucial step in the understanding of p -adic fields. Taking into account that class field theory is the most basic example of Langlands correspondence, one may ask whether Langlands' ideas can be studied in a higher-dimensional setting, and then it seems natural to study reductive groups over higher-dimensional local fields.

In this article, we will work over a field \mathbb{K} that is endowed with a valuation $\omega : \mathbb{K}^\times \rightarrow \Lambda$, where Λ is any non-zero totally ordered abelian group. This covers the case of d -local fields for $d > 0$ since they are endowed with a \mathbb{Z}^d -valuation, where \mathbb{Z}^d is equipped with the lexicographical order. It also allows us to work with a field of the form $\mathbb{C}((t_1)) \cdots ((t_d))$, which is the geometric counterpart of higher-dimensional local fields, since it encodes local information in higher-dimensional complex varieties. Note that Λ does not need to be discrete and could for instance be the group $\mathbb{R} \times \mathbb{R}$ with the lexicographical order. It could also have infinite rank.

In the case where \mathbb{K} is a higher-dimensional local field, Parshin constructed in [Par94], [Par00b] a "higher Bruhat-Tits building" on which $\text{PGL}_n(\mathbb{K})$ acts, for $n \in \mathbb{N}^*$. Independently of this work, Bennett defined in [Ben94] a notion of Λ -building for Λ a totally ordered abelian group and, for any field \mathbb{K} equipped with a valuation $\omega : \mathbb{K} \rightarrow \Lambda \cup \{+\infty\}$, he constructed such a Λ -building on which $\text{SL}_n(\mathbb{K})$ acts. As sets (when we forget the topological structures), the buildings of Bennett and Parshin are very close.

Given any totally ordered abelian group Λ , any Λ -valued field \mathbb{K} and any quasi-split reductive group \mathbf{G} over \mathbb{K} , the goal of this article is to construct a Λ -building $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ endowed with a suitable $\mathbf{G}(\mathbb{K})$ -action. This partially answers [Par94, Problem 2 p 187].

1.1 Bennett's Λ -buildings

When \mathbb{K} is a field equipped with a nontrivial valuation $\omega : \mathbb{K} \rightarrow \mathbb{R}$ and \mathbf{G} is a reductive \mathbb{K} -group, Bruhat and Tits associated to $\mathbf{G}(\mathbb{K})$ its **Bruhat-Tits building** $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ on which $\mathbf{G}(\mathbb{K})$ acts. When $\mathbf{G} = \text{SL}_2$ and ω is discrete for example, this space is a simplicial tree.

The Bruhat-Tits building $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ is covered by subsets called **apartments**, which are Euclidean spaces equipped with an arrangement of hyperplanes called **walls**. These apartments are all obtained by translation by an element of $\mathbf{G}(\mathbb{K})$ from a standard apartment $\mathbb{A}(\mathbf{G}, \mathbb{K}, \omega)$. The hyperplane arrangement of $\mathbb{A}(\mathbf{G}, \mathbb{K}, \omega)$ depends on the root system of \mathbf{G} and on the set of values of ω . It naturally defines the notion of a face on $\mathbb{A}(\mathbf{G}, \mathbb{K}, \omega)$ and by translation, on $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$. Then we have the following important properties:

- (I1) for every two faces, there exists an apartment containing them;

- (I2) for every two apartments A and B , there exists an isomorphism of affine spaces from A to B fixing $A \cap B$ and preserving the hyperplane arrangement.

These properties motivate the definition of an **abstract building**: as a first approximation (see Definition 2.22 for a precise definition), it is a set covered with subspaces called apartments, which satisfy (I1) and (I2) and which are isomorphic to a standard apartment \mathbb{A} depending on a root system.

Let now Λ be any totally ordered abelian group. Since every totally ordered abelian group can be embedded in an ordered real vector space, let's assume for simplicity that Λ itself is a real vector space. Bennett then defined in [Ben94] the notion of a Λ -**building**: it is a set covered by subsets called apartments, all isomorphic to a standard apartment $\mathbb{A} \otimes_{\mathbb{R}} \Lambda$, and satisfying axioms similar to those of a Bruhat-Tits building.

Examples of Λ -buildings When $\Lambda \subset \mathbb{R}$ and \mathbb{K} is a field equipped with a valuation $\omega : \mathbb{K} \rightarrow \Lambda \cup \{\infty\}$, the Bruhat-Tits building of $(\mathbf{G}, \mathbb{K}, \omega)$ is a Λ -building. In the case that Λ cannot be embedded as an ordered abelian group in \mathbb{R} , there are three main previously known classes of examples of Λ -buildings:

1. Let \mathbb{K} be a field that is equipped with a valuation $\omega : \mathbb{K} \rightarrow \Lambda$. In [MS84], inspired by Serre's construction of the tree of $\mathrm{SL}_2(\mathbb{K})$ when $\Lambda = \mathbb{Z}$, Morgan and Shalen define the notion of a Λ -tree and construct a Λ -tree on which $\mathrm{SL}_2(\mathbb{K})$ acts. Generalizing these works, Bennett define in [Ben94, Example 3.2] a Λ -building on which $\mathrm{SL}_n(\mathbb{K})$ acts, for $n \in \mathbb{Z}_{\geq 2}$.
2. Let Λ, Λ' be totally ordered abelian groups and $e : \Lambda \rightarrow \Lambda'$ be a morphism of ordered groups. Then Schwer and Struyve construct a functor from the category of Λ -buildings to the category of Λ' -buildings, compatible with e (see [SS12]). Using this and using ultraproducts, they construct nontrivial examples of Λ -buildings, for $\Lambda \not\subset \mathbb{R}$. They in particular construct ultracones and asymptotic cones of buildings (see [SS12, Section 6]).
3. Let \mathcal{G} be a semi-simple Lie group. Let $\mathbb{K}_{\mathrm{real}}$ be a real closed nonarchimedean field and $\mathcal{O}_{\mathrm{real}} \subset \mathbb{K}_{\mathrm{real}}$ be an o-convex valuation ring. Then Kramer and Tent equip $\mathcal{G}(\mathbb{K}_{\mathrm{real}})/\mathcal{G}(\mathcal{O}_{\mathrm{real}})$ with the structure of a Λ_{real} -building, where $\Lambda_{\mathrm{real}} = \mathbb{K}_{\mathrm{real}}^*/\mathcal{O}_{\mathrm{real}}^*$; see [KT04, Theorem 4.3] and [KT09]. They deduce that the asymptotic cone of $\mathcal{G}(\mathbb{K}_{\mathrm{real}})$ and the ultracone of $\mathcal{G}(\mathbb{K}_{\mathrm{real}})$ are Λ -buildings, for some Λ (see [KT04, Corollary 4.4]). Using these results, they give a new proof of Margulis's conjecture (see [KT04, § 5]).

Our result yields a new class of examples of Λ -buildings.

1.2 The building $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$

In differential geometry, when one works with some real Lie group G , it is often useful to study the action of G on a symmetric space. For instance, if $G = \mathrm{SL}_n(\mathbb{R})$, then one may consider the action of $\mathrm{SL}_n(\mathbb{R})$ on $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. The manifold X is then the quotient of a real Lie group by a maximal compact subgroup. According to Goldman-Iwahori ([GI63]), the space X can be identified with the space of norms on \mathbb{R}^n up to homothety.

Now, when one works over the p -adic field \mathbb{Q}_p instead of \mathbb{R} , by analogy with Goldman-Iwahori's result, one can associate to the group $\mathrm{SL}_n(\mathbb{Q}_p)$ the space $\mathcal{I}(\mathrm{SL}_n, \mathbb{Q}_p)$ of ultrametric norms over \mathbb{Q}_p^n up to homothety. This is a particular case of a more general construction given by Bruhat-Tits in [BT72, 10.2]. In order to generalize the previous

constructions and study in this context other classical groups of Lie type over Henselian valued fields, Tits introduced the definition of buildings in the 1960's.

When \mathbb{K} is a field endowed with a valuation $\omega : \mathbb{K}^\times \rightarrow \mathbb{Z}$, several approaches have been developed to construct a Bruhat-Tits building $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ associated to a reductive \mathbb{K} -group \mathbf{G} . The most elementary construction relies on lattices. For instance, when \mathbf{G} is split and has type A_n (e.g. $\mathbf{G} = \mathrm{GL}_n, \mathrm{SL}_n$ or PGL_n), one may define $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ as the set of \mathbb{O} -lattices contained in \mathbb{K}^n up to homothety, where \mathbb{O} stands for the ring of integers of \mathbb{K} . The action of $\mathbf{G}(\mathbb{K})$ on $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ is then induced by the action of $\mathbf{G}(\mathbb{K})$ on the \mathbb{O} -lattices of \mathbb{K}^n (see [Ser77, Chapter II] for the case where $n = 2$). Note that this construction depends substantially on the Lie type of the group \mathbf{G} and a case by case definition has to be settled.¹ Parshin [Par94] and Bennett [Ben94] both use this approach with lattices to construct a space analogous to $\mathcal{I}(\mathrm{SL}_n, \mathbb{K}, \omega)$ when the valuation ω takes values in a totally ordered abelian group that might be different from \mathbb{Z} .

A more general approach due to Bruhat and Tits ([BT72], [BT84]) mainly consists in generalizing the construction as a quotient of $\mathbf{G}(\mathbb{K})$ by a maximal compact subgroup. Unfortunately, maximal compact subgroups need not be pairwise conjugate in general. That is why, for an arbitrary reductive group \mathbf{G} over a Λ -valued field with $\Lambda \subset \mathbb{R}$, Bruhat and Tits' construction relies on the use of parahoric subgroups. More precisely, they start by defining the standard apartment \mathbb{A} of their building as an affine space endowed with an action by affine transformations of the group N of rational points of the normalizer \mathbf{N} of a maximal \mathbb{K} -split torus of \mathbf{G} . The root system Φ of \mathbf{G} can be regarded as a set of affine maps on \mathbb{A} . Then, for each element x of \mathbb{A} , Bruhat and Tits define a parahoric subgroup $P_x \subset G$, which depends on the values $\alpha(x)$, for $\alpha \in \Phi$. They finally define $\mathcal{I} = \mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ as the set $\mathbf{G}(\mathbb{K}) \times \mathbb{A} / \sim$, where \sim is an equivalence relation on $\mathbf{G}(\mathbb{K}) \times \mathbb{A}$ whose definition involves the parahoric subgroups P_x for $x \in \mathbb{A}$. The group $\mathbf{G}(\mathbb{K})$ acts on \mathcal{I} by $g \cdot [g', x] = [gg', x]$, for $g, g' \in \mathbf{G}(\mathbb{K})$ and $x \in \mathbb{A}$, so that P_x is the stabilizer of x in G for each x . This is the approach we follow in this paper to deal with the case when Λ is not necessarily contained in \mathbb{R} .

For that purpose, we start from a Chevalley-Steinberg system of the quasi-split group \mathbf{G} (i.e. a parametrization of the root groups \mathbf{U}_α of \mathbf{G} taking into account the Galois extension $\tilde{\mathbb{K}}/\mathbb{K}$ that splits \mathbf{G}). To such a pining, we associate a space \mathcal{I} . We then need to prove that it is a Λ -building. A part of the proof consists in proving that G satisfies certain decompositions, namely the Iwasawa decomposition and the Bruhat decomposition. To prove them, we generalize the proof by Bruhat and Tits to our framework. After proving these decompositions, the main issue is to prove that certain retractions are 1-Lipschitz continuous. When $\Lambda \subset \mathbb{R}$, a standard proof of this property uses the fact that the segments of \mathbb{R} are compacts. This is no longer true in our framework and we thus need some additional work to prove this property.

1.3 Affine structure of the standard apartment

An important step in our construction consists in understanding the geometry of the apartments of our building. Roughly speaking, the apartment will be a tensor product $Y \otimes_{\mathbb{Z}} R$ where Y is the finitely generated free \mathbb{Z} -module of cocharacters of \mathbf{G} and R is some ordered ring that contains Λ as a linearly ordered subgroup.

Consider, for instance, a Henselian valued field \mathbb{K} with a discrete valuation $\omega : \mathbb{K}^* \rightarrow \Lambda = \mathbb{Z}$. There is a natural structure of ring on \mathbb{Z} but, since we have to consider field extensions \mathbb{L}/\mathbb{K} , the group $\Lambda' = \omega(\mathbb{L}^*)$ is not anymore naturally equipped with a ring structure extending that of Λ . Bruhat-Tits's idea consists in seeing both Λ and Λ' in the ring $R = \mathbb{R}$.

¹ Note that in the remarkable case of type A_n , there are other similar approaches such as using maximal orders (see [Vig80] and [Ser77] for $n = 2$).

Now, assume for instance that $\Lambda = \mathbb{Z}^d$ equipped with the lexicographical order and let $Y = \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. A natural way to generalize the previous construction is to consider the ring $R = \mathbb{R}[t]/(t^d)$ equipped with the lexicographical order induced by the degree of monomials. On this example, the apartment $\mathbb{A} = Y \otimes_{\mathbb{Z}} R$ will be a dn -dimensional \mathbb{R} -vector space and a free R -module of rank n . Thus, there are two viewpoints for an element $x \in \mathbb{A}$: one can write either $x = \sum_{i=1}^n \lambda_i e_i$ with $\lambda_i \in R$ (structure of R -module) or $x = \sum_{s=0}^{d-1} x_s t^s$ with $x_s \in Y \otimes \mathbb{R}$. The first viewpoint allows to endow the apartment \mathbb{A} with a geometric and combinatorial structure (it is an R -affine space together with combinatorial data such as chambers, faces, sectors...). The second viewpoint allows to endow the apartment with a fibration $\mathbb{A} \rightarrow Y \otimes_{\mathbb{R}} \mathbb{R}[t]/(t^d)$ that we will extend to the whole building for each $d' \leq d$.

In the situation of a general totally ordered abelian group Λ , we will define the apartment \mathbb{A} by introducing a totally ordered (non-unital) ring \mathfrak{R}^S together with an increasing embedding $\Lambda \rightarrow \mathfrak{R}^S$.

1.4 Main results

We now briefly describe the main results of this paper, see Theorem 2.25 for a more precise statement.

Let Λ be a totally ordered abelian group. Define the rank $S := \text{rk}(\Lambda)$ of Λ as the (totally ordered) set of Archimedean equivalence classes of Λ . For instance, if $\Lambda = \mathbb{Z}^n$ for some $n \geq 1$, then $S = \{1, 2, \dots, n\}$. By Hahn's embedding theorem, Λ can then be regarded as a subgroup of:

$$\mathfrak{R}^S := \{(x_s)_{s \in S} \in \mathbb{R}^S \mid \text{the support of } (x_s)_{s \in S} \text{ is a well-ordered subset of } S\}.$$

Let now \mathbb{K} be a field with a valuation $\omega : \mathbb{K} \rightarrow \Lambda \cup \{\infty\}$, fix a quasi-split (connected) reductive \mathbb{K} -group \mathbf{G} and let \mathbf{S} be a maximal split torus in \mathbf{G} with cocharacter module $X_*(\mathbf{S})$. If \mathbf{G} is not split, assume that \mathbb{K} is Henselian. Our results can then be summarized as follows:

- (i) We construct a set $\mathcal{I} = \mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$, a Λ -distance $d : \mathcal{I} \times \mathcal{I} \rightarrow \Lambda_{\geq 0} := \{\lambda \in \Lambda \mid \lambda \geq 0\}$ and we equip (\mathcal{I}, d) with the structure of an \mathfrak{R}^S -building whose apartments are modelled on some quotient of $X_*(\mathbf{S}) \otimes \mathfrak{R}^S$. The group $G = \mathbf{G}(\mathbb{K})$ acts isometrically on \mathcal{I} and the induced action on the set of apartments is transitive.
- (ii) Let $s \in S$, let $S_{\leq s} = \{t \in S \mid t \leq s\}$ and let $\pi_{\mathfrak{R}^S, \leq s} : \mathfrak{R}^S \rightarrow \mathfrak{R}^{S_{\leq s}}$ be the natural projection. Consider the valuation $\omega_{\leq s} = \pi_{\mathfrak{R}^S, \leq s} \circ \omega$. We construct an (explicit) surjective map:

$$\pi_{\leq s} : \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \rightarrow \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$$

compatible with the $\mathbf{G}(\mathbb{K})$ -action such that, for each $X \in \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$, the fiber $\pi_{\leq s}^{-1}(X)$ is a product:

$$\mathcal{I}_X \times \tilde{V},$$

where \tilde{V} is an $\ker(\pi_{\mathfrak{R}^S, \leq s})$ -module and \mathcal{I}_X is a $\ker(\pi_{\mathfrak{R}^S, \leq s})$ -building.

1.5 Structure of the paper

In section 2, we provide all useful definitions concerning the construction of Λ -buildings. In subsection 2.1, we introduce all preliminary definitions that are necessary to define Λ -buildings. These definitions are used all along the article. In subsection 2.2, we provide the definition of Λ -buildings themselves. This allows us to state the main Theorem in subsection 2.3.

In sections 3 and 4, we follow the strategy of Bruhat and Tits in order to provide a generalization of the Iwasawa decomposition and the Bruhat decomposition. Doing this, we introduce some abstract subgroups that generalize parahoric subgroups and get a better understanding of the action of G on the Λ -building.

In section 5, we glue up the apartments via an equivalence relation similar to the one introduced by Bruhat and Tits in order to provide a space $\mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ which will be the Λ -building associated to \mathbf{G} . We then prove that it satisfies all the Λ -building axioms except the axiom (CO).

In section 6, we use the classical reductive group theory over a field (Chevalley-Steinberg systems, Borel-Tits theory) in order to construct data that satisfy the axioms of sections 3 and 4.

In section 7, inspired by work of Parshin ([Par94], [Par00b]), we prove that a surjective morphism of totally ordered abelian groups $f : \Lambda \rightarrow \Lambda'$ induces a projection map from $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \rightarrow \mathcal{I}(\mathbb{K}, f \circ \omega, \mathbf{G})$ which is surjective and compatible with the action of G . We then give a detailed description of the fibers of this projection.

Section 8 is dedicated to the proof of axiom (CO) and completes the proof of the main Theorem.

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2 Abstract definition of \mathfrak{R}^S -buildings and statement of the main theorem

In this section we recall the definition of Λ -buildings, as defined by Bennett in [Ben94].

2.1 Definition of the standard apartment

2.1.1 Root system, vectorial apartment over \mathbb{R} and Weyl group

Let $V_{\mathbb{R}}$ be a finite dimensional vector space over \mathbb{R} . Let $\Phi \subset (V_{\mathbb{R}})^*$ be a root system in the definition of [Bou81, 6.1.1]. Let $\Phi^{\vee} \subset V_{\mathbb{R}}$ be the dual root system of Φ . In particular, Φ (resp. Φ^{\vee}) is a finite subset of $(V_{\mathbb{R}}^*) \setminus \{0\}$ (resp. $V_{\mathbb{R}} \setminus \{0\}$) and there is a bijection $\vee : \Phi \rightarrow \Phi^{\vee}$ such that for all $\alpha \in \Phi$, $\alpha(\alpha^{\vee}) = 2$.

Because we have to deal with non-reduced root systems, we recall the following definitions and facts [Bou81, VI §1.3 & §1.4]. A root $\alpha \in \Phi$ is said to be multipliable if $2\alpha \in \Phi$; otherwise, it is said to be non-multipliable. A root $\alpha \in \Phi$ is said to be divisible if $\frac{1}{2}\alpha \in \Phi$; otherwise, it is said to be non-divisible. The set of non-divisible roots, denoted by Φ_{nd} , is a root system.

2.1 Notation. For $\alpha, \beta \in \Phi$ and $\alpha^{\vee}, \beta^{\vee} \in \Phi^{\vee}$ the coroots of α and β respectively, we denote by:

- $\Phi(\alpha, \beta) = \{r\alpha + s\beta \in \Phi, (r, s) \in \mathbb{Z}^2\}$;
- $(\alpha, \beta) = \{r\alpha + s\beta \in \Phi, (r, s) \in (\mathbb{Z}_{>0})^2\}$;
- $r_{\alpha}(\beta) = \beta - \beta(\alpha^{\vee})\alpha$;

- $r_\alpha(\beta^\vee) = \beta^\vee - \alpha(\beta^\vee)\alpha^\vee$.

Note that $\Phi(\alpha, \beta)$ is a root system of rank 1 or 2 depending on the fact that α and β are, or not, collinear. The subset (α, β) is a positively closed subset of $\Phi(\alpha, \beta)$ when $\beta \notin -\mathbb{R}_{>0}\alpha$.

The map $r_\alpha : \Phi \rightarrow \Phi$ (resp. $r_\alpha : \Phi^\vee \rightarrow \Phi^\vee$) extends linearly onto a map $r_\alpha \in \text{GL}(V_{\mathbb{R}}^*)$ (resp. $r_\alpha \in \text{GL}(V_{\mathbb{R}})$). The reflection r_α satisfies $r_\alpha^2 = \text{id}$ and $r_{2\alpha} = r_\alpha$ when $2\alpha \in \Phi$. We denote by $W(\Phi)$ the subgroup of $\text{GL}(V_{\mathbb{R}}^*)$ (resp. $W(\Phi^\vee)$) generated by the r_α , it is a finite subgroup called the Weyl group associated to Φ (resp. Φ^\vee). Moreover, for any basis Δ of Φ , the Weyl group $W(\Phi)$ of Φ is generated by the r_α for $\alpha \in \Delta$. In particular, there is a natural isomorphism of finite groups $W(\Phi) \rightarrow W(\Phi^\vee)$ sending $r_\alpha \in W(\Phi)$ onto $r_\alpha \in W(\Phi^\vee)$ for any $\alpha \in \Phi$. By abuse of notations, for any $w \in W(\Phi)$ we still denote by w^{-1} its image in $W(\Phi^\vee)$ by this isomorphism and we denote $W^v = W(\Phi) = W(\Phi^\vee)$. Note that for any $\alpha \in \Phi$ and any $w \in W(\Phi)$, one has $w(\alpha) = \alpha \circ w^{-1}$.

A **vector chamber** $C_{\mathbb{R}}^v$ is a connected component of $V_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Phi} \alpha^{-1}(\{0\})$. The associated **basis** $\Delta_{C_{\mathbb{R}}^v}$ of Φ is the set of roots $\alpha \in \Phi$ such that $\alpha(C_{\mathbb{R}}^v) > 0$ and such that $\alpha^{-1}(\{0\}) \cap \overline{C_{\mathbb{R}}^v}$ spans a hyperplane of $V_{\mathbb{R}}$ (where $\overline{C_{\mathbb{R}}^v}$ denotes the closure of $C_{\mathbb{R}}^v$ for the topology of finite dimensional vector space on $V_{\mathbb{R}}$). Let $\Phi_{C_{\mathbb{R}}^v} = \{\alpha \in \Phi \mid \alpha(C_{\mathbb{R}}^v) > 0\}$ be the set of positive roots for $C_{\mathbb{R}}^v$. Then $\Phi = \Phi_{C_{\mathbb{R}}^v} \sqcup -\Phi_{C_{\mathbb{R}}^v}$ and $\Phi_{C_{\mathbb{R}}^v} \subset \bigoplus_{\alpha \in \Delta_{C_{\mathbb{R}}^v}} \mathbb{Z}_{\geq 0}\alpha$.

Let $W^v = W(\Phi^\vee)$ be the **vectorial Weyl group**. Then W^v is finite and for every vector chamber $C_{\mathbb{R}}^v$ of $V_{\mathbb{R}}$, $(W^v, \{r_\alpha \mid \alpha \in \Delta_{C_{\mathbb{R}}^v}\})$ is a Coxeter system.

We now fix a vector chamber $C_{f,\mathbb{R}}^v$ of $V_{\mathbb{R}}$, that we call the **fundamental chamber** and we set $\Delta_f = \Delta_{C_{f,\mathbb{R}}^v}$. We denote by ℓ the length on W^v associated to $\{r_\alpha \mid \alpha \in \Delta_{C_{f,\mathbb{R}}^v}\}$.

2.1.2 Topology over a totally ordered commutative pseudo-ring

We call **pseudo-ring** an algebraic structure satisfying the same axioms as a ring, without assuming the existence of a multiplicative identity. For instance, an ideal of a commutative ring is a commutative pseudo-ring. If Λ is an abelian group (e.g. a vector space), it can be equipped with the trivial pseudo-ring structure given by $\lambda\mu = 0$ for $\lambda, \mu \in \Lambda$.

A commutative pseudo-ring R is said to be totally ordered if it is equipped with a total order $<$ such that:

$$\forall a, b, c \in R, a < b \implies a + c < b + c$$

and

$$\forall a, b \in R, a > 0 \text{ and } b > 0 \implies ab \geq 0.$$

Note that we do not assume that R is an integral domain but the first condition implies that R is a torsion-free \mathbb{Z} -module. Thus, if we denote by $R_{\mathbb{Q}} = R \otimes_{\mathbb{Z}} \mathbb{Q}$, there is a natural embedding of R in $R_{\mathbb{Q}}$ and the total order on R extends naturally to a total order on $R_{\mathbb{Q}}$. Useful examples of such a totally ordered commutative pseudo-ring R are \mathbb{Z} , \mathbb{R} or the non-unital rings \mathfrak{R}^S defined in 2.18.

We introduce a symbol ∞ and we extend the total ordering over R to a total ordering of the set $R \cup \{\infty\}$ by setting $\lambda < \infty$ for any $\lambda \in R$. For any $\lambda \in R$, we will denote by:

- $[\lambda, \infty] = \{\mu \in R, \lambda \leq \mu\} \cup \{\infty\}$ and $]\lambda, \infty[= \{\mu \in R, \lambda < \mu\} \cup \{\infty\}$;
- $R_{>\lambda} = \{\mu \in R, \lambda < \mu\} =]\lambda, \infty[$ and $R_{\geq \lambda} = \{\mu \in R, \lambda \leq \mu\} = [\lambda, \infty[$;
- $R_{<\lambda} = \{\mu \in R, \lambda > \mu\} =]-\infty, \lambda[$ and $R_{\leq \lambda} = \{\mu \in R, \lambda \geq \mu\} =]-\infty, \lambda]$.

We equip R with the order topology for which a base is given by the sets $R_{>\lambda}$ and $R_{<\lambda}$ for $\lambda \in R$. Thus, R is a completely normal Hausdorff space.

At some point, we will need that the order topology is dense at 0, so that it is sufficient, for instance, to ask that R is equipped with a \mathbb{Q} -module structure, i.e. $R = R_{\mathbb{Q}}$. For instance, the totally ordered pseudo-ring \mathfrak{R}^S defined in 2.18 satisfy these properties and, moreover, Λ naturally embeds into \mathfrak{R}^S .

Let Δ^\vee be a basis of Φ_{nd}^\vee . We denote by $V_{\mathbb{Z}}$ the free \mathbb{Z} -submodule of $V_{\mathbb{R}}$ spanned by the α^\vee for $\alpha^\vee \in \Delta^\vee$. Note that since W^v acts transitively on the set of basis of Φ_{nd}^\vee [Bou81, VI.1.5], this does not depend on the choice of Δ^\vee .

For any commutative pseudo-ring R , we denote by $V_R = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. Let $\alpha \in \Phi$ be any root. By definition of root systems, $\alpha(\Phi^\vee) \subset \mathbb{Z}$. Thus α induces a \mathbb{Z} -linear map on $V_{\mathbb{Z}}$ and therefore an R -linear map on V_R uniquely determined by $\alpha(x \otimes \lambda) = \alpha(x)\lambda$ for $x \in V_{\mathbb{Z}}$ and $\lambda \in R$ since $\alpha(x) \in \mathbb{Z}$.

For $\alpha \in \Phi$ and any $\lambda \in R$, denote by $H_{R,\alpha,\lambda} = \alpha^{-1}(\{\lambda\}) \subset V_R$, by $\mathring{D}_{R,\alpha,\lambda} = \alpha^{-1}(R_{>\lambda})$ and by $D_{R,\alpha,\lambda} = \alpha^{-1}(R_{\geq\lambda}) = H_{R,\alpha,\lambda} \sqcup \mathring{D}_{R,\alpha,\lambda}$. If R is clear in the context, one can denote those sets by $H_{\alpha,\lambda}$, $\mathring{D}_{\alpha,\lambda}$ and $D_{\alpha,\lambda}$ respectively. If $\lambda = 0$, one can denote those sets by H_α , \mathring{D}_α and D_α respectively. We equip V_R with the topology generated by the sets $\mathring{D}_{\alpha,\lambda}$ for $\alpha \in \Phi$ and $\lambda \in R$. Thus the linear maps $\alpha : V_R \rightarrow V_R$ are automatically continuous.

For any basis Δ of Φ and any subset $\Delta_P \subset \Delta$, we set:

$$F_R^v(\Delta, \Delta_P) = \bigcap_{\alpha \in \Delta_P} H_{R,\alpha} \cap \bigcap_{\alpha \in \Delta \setminus \Delta_P} D_{R,\alpha} = \left\{ v \in V_R, \forall \alpha \in \Delta, \begin{array}{l} \alpha(v) = 0 \quad \text{if } \alpha \in \Delta_P \\ \alpha(v) > 0 \quad \text{if } \alpha \notin \Delta_P \end{array} \right\}$$

and

$$\overline{F_R^v(\Delta, \Delta_P)} = \bigcap_{\alpha \in \Delta_P} H_{R,\alpha} \cap \bigcap_{\alpha \in \Delta \setminus \Delta_P} \overline{D}_{R,\alpha} = \left\{ v \in V_R, \forall \alpha \in \Delta, \begin{array}{l} \alpha(v) = 0 \quad \text{if } \alpha \in \Delta_P \\ \alpha(v) \geq 0 \quad \text{if } \alpha \notin \Delta_P \end{array} \right\}.$$

If Δ is any basis of Φ and $\Delta_P = \emptyset$, we set $C_{R,\Delta}^v = F_R^v(\Delta, \emptyset)$ and $\overline{C}_{R,\Delta}^v = \overline{F_R^v(\Delta, \emptyset)}$. As before, if R is clear in the context, we may omit to mention it in the notation.

2.2 Lemma. *For any basis Δ of Φ , we have $C_\Delta^v = \{v \in V_R, \forall \alpha \in \Phi_\Delta^+, \alpha(v) > 0\}$ and $\overline{C}_\Delta^v = \{v \in V_R, \forall \alpha \in \Phi_\Delta^+, \alpha(v) \geq 0\}$.*

If Δ' is another basis of Φ , we have $C_\Delta^v \cap C_{\Delta'}^v \neq \emptyset \iff \Delta = \Delta'$.

Proof. Let $\alpha \in \Phi_\Delta^+$ and $v \in C_\Delta^v$ (resp. \overline{C}_Δ^v). By [Bou81, VI.1.6], there exist positive integers n_β such that $\alpha = \sum_{\beta \in \Delta} n_\beta \beta$. Since $\beta(v) > 0$ (resp. ≥ 0) for any $\beta \in \Delta$, we get that $\sum_{\beta \in \Delta} n_\beta \beta(v) > 0$ (resp. ≥ 0). The converse is immediate since $\Delta \subset \Phi_\Delta^+$. Hence $C_\Delta^v = \{v \in V_R, \forall \alpha \in \Phi_\Delta^+, \alpha(v) > 0\}$ and $\overline{C}_\Delta^v = \{v \in V_R, \forall \alpha \in \Phi_\Delta^+, \alpha(v) \geq 0\}$.

Suppose $\Delta' \neq \Delta$, then $\Phi_{\Delta'}^+ \neq \Phi_\Delta^+$ and let $\alpha \in \Phi_{\Delta'}^+ \setminus \Phi_\Delta^+$. Hence $-\alpha \in \Phi_\Delta^+$. For any $v \in C_{\Delta'}^v$, we have $-\alpha(v) > 0$ so that $v \notin C_\Delta^v$. \square

2.3 Definition. For any $v \in V_{\mathbb{Z}}$, denote by $\delta_v = \{\lambda v, \lambda \in R_{>0}\}$. This is called the ‘‘open’’ half-line in V_R with direction $v \in V_{\mathbb{Z}}$.

2.4 Lemma. *For any basis Δ of Φ and any $\Delta_P \subset \Delta$, the \mathbb{Z} -vector facet $F_{\mathbb{Z}}^v(\Delta, \Delta_P)$ is non-empty.*

Proof. It suffices to prove it inside the reduced root system Φ_{nd} , so that we assume in this proof that $\Phi = \Phi_{\text{nd}}$ is reduced. Since $W(\Phi^\vee)$ acts simply transitively on the set of bases of Φ^\vee , we know that the free \mathbb{Z} -module $V_{\mathbb{Z}}$ is the set of elements $x = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$ such that $n_\alpha \in \mathbb{Z}$ for any $\alpha \in \Delta$. Define values $p_\alpha \in \{0, 1\}$ by $p_\alpha = 0$ if $\alpha \in \Delta_P$ and $p_\alpha = 1$ if $\alpha \in \Delta \setminus \Delta_P$. Consider the Cartan matrix $C = (\beta(\alpha^\vee))_{\alpha, \beta \in \Delta}$ and the matrix $P = (p_\alpha)_{\alpha \in \Delta}$. According to [Bou81, VI.1.10], C is an invertible matrix over \mathbb{Q} . Hence there exist $m \in \mathbb{Z}_{>0}$ and a matrix $N = (n_\alpha)_{\alpha \in \Delta}$ with coefficients $n_\alpha \in \mathbb{Z}$ such that $CN = mP$. Define $x = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$. Then, for any $\beta \in \Delta$, we get that $\beta(x) = \sum_{\alpha \in \Delta} n_\alpha \beta(\alpha^\vee) = mp_\alpha$. If $\beta \in \Delta_P$, we have $\beta(x) = 0$ and if $\beta \notin \Delta_P$, we have $\beta(x) = m \in \mathbb{Z}_{>0}$. Thus $x \in F_{\mathbb{Z}}^v(\Delta, \Delta_P)$. \square

2.5 Lemma. For any $v \in F_{\mathbb{Z}}^v(\Delta, \Delta_P)$, we have $\delta_v \subset F_R^v(\Delta, \Delta_P)$.

Proof. For any $\alpha \in \Delta$, any $v \in C_{\mathbb{Z}, \Delta}^v$ and any $\lambda \in R_{>0}$, we have $\alpha(\lambda v) = \alpha(v)\lambda$ by R -linearity of α . If $\alpha \in \Delta_P$, then $\alpha(v\lambda) = 0$. If $\alpha \notin \Delta_P$, then $\alpha(v) \in \mathbb{Z}_{>0}$ by assumption on v . Since $\lambda > 0$ and $\alpha(v) \in \mathbb{Z}$, we have $\alpha(v)\lambda > 0$. Thus $\alpha(\lambda v) > 0$ for any $\alpha \in \Delta \setminus \Delta_P$. Therefore $\lambda v \in F_R^v(\Delta, \Delta_P)$ for any $\lambda \in R_{>0}$. \square

2.6 Remark. Note that the set $\{\lambda x, \lambda \in R_{>0}\}$ for $x \in C_{R, \Delta}^v$ is not contained in $C_{R, \Delta}^v$ in general, even if x is R -torsion-free. For instance, take $R = \mathbb{R}[[t]]/(t^2)$ with the lexicographical order $a_1 + tb_1 < a_2 + tb_2 \iff a_1 < a_2$ or $a_1 = a_2$ and $b_1 < b_2$. Take $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ of type A_2 . Take $x = 2\alpha^\vee + (1+t)\beta^\vee$. Then $\alpha(x) = 4 - (1+t) = 3 - t > 0$ and $\beta(x) = 2(1+t) - 2 = 2t > 0$, so that $x \in C_{R, \{\alpha, \beta\}}^v$. Thus for any $\lambda \in R \setminus \{0\}$, we have $\alpha(\lambda x) = (3 - t)\lambda \neq 0$ since $3 - t$ is invertible in R so that, in particular, x is torsion-free. But $\beta(tx) = t(2t) = 0$ so that $tx \notin C_{R, \{\alpha, \beta\}}^v$ with $t > 0$.

The following Proposition generalizes [BT72, 7.3.5]. Since the topology of the R -module V_R is not easy to manipulate (for instance, it is not necessarily a connected space so that vector chambers cannot be defined as some connected components), we prove it in a combinatorial way instead of a topological way.

2.7 Proposition. Assume that $R = R_{\mathbb{Q}}$. Let $x \in V_R$ and Δ, Δ' be two bases of Φ . There exists a unique $w \in W(\Phi)$ such that $x \in \overline{C}_{w(\Delta)}^v$ and $C_{w(\Delta)}^v \cap (x + C_{\Delta'}^v) \neq \emptyset$.

Proof. Since any basis of a root system is contained in Φ_{nd} , we can assume that Φ is reduced (i.e. $\Phi = \Phi_{\text{nd}}$).

For uniqueness, consider $w, w' \in W(\Phi)$ such that $x \in \overline{C}_{w(\Delta)}^v \cap \overline{C}_{w'(\Delta)}^v$ and $C_{w(\Delta)}^v \cap (x + C_{\Delta'}^v) \neq \emptyset$ and $C_{w'(\Delta)}^v \cap (x + C_{\Delta'}^v) \neq \emptyset$. Let $y \in C_{w(\Delta)}^v \cap (x + C_{\Delta'}^v)$ and $z \in C_{w'(\Delta)}^v \cap (x + C_{\Delta'}^v)$. Let $\alpha \in \Phi_{w(\Delta)}^+$. Then $\alpha(y) > 0$ and $\alpha(x) \geq 0$ by Lemma 2.2. If $\alpha \notin \Phi_{w'(\Delta)}^+$, then $\alpha(z) < 0$ and $\alpha(x) \leq 0$. Thus $\alpha(x) = 0$ so that $\alpha(y - x) > 0$ and $\alpha(z - x) < 0$ by linearity of α . Hence the sign of α is non-constant on $C_{\Delta'}^v$, which is a contradiction. Thus $\Phi_{w(\Delta)}^+ = \Phi_{w'(\Delta)}^+$ which gives $w(\Delta) = w'(\Delta)$ and therefore $w = w'$ according to [Bou81, VI.1.5 Thm.2].

For existence, we proceed as follows: let $v \in C_{\mathbb{Z}, \Delta'}^v$ so that $\delta_v \subset C_{R, \Delta'}^v$ according to Lemma 2.5 and for any $\alpha \in \Phi$, we have $\alpha(v) \in \mathbb{Z} \setminus \{0\}$. We denote by Ψ the set of roots $\alpha \in \Phi$ such that $\alpha(x) < 0$ or $\forall \eta \in R_{>0}, \exists \varepsilon \in]0, \eta[, \alpha(x + \varepsilon v) \leq 0$. We denote by $n(\Delta)$ the cardinality of $\Psi \cap \Phi_{\Delta}^+$. We prove, by induction on n , that for any basis Δ of Φ such that $n(\Delta) \leq n$, there exists $w \in W(\Phi)$ such that $x \in \overline{C}_{w(\Delta)}^v$ and $C_{w(\Delta)}^v \cap (x + C_{\Delta'}^v) \neq \emptyset$.

Basis: suppose that $\Delta \cap \Psi = \emptyset$. For any $\alpha \in \Delta$, we have $\alpha(x) \geq 0$ and $\exists \eta_{\alpha} > 0, \forall \varepsilon \in]0, \eta_{\alpha}[, \alpha(x + \varepsilon v) > 0$. Denote by $\eta = \min\{\eta_{\alpha}, \alpha \in \Delta\} > 0$. Since any $\beta \in \Phi_{\Delta}^+$ is a linear combination with positive integer coefficients of the simple roots $\alpha \in \Delta$, we have $\beta(x) \geq 0$ and for any $\varepsilon \in]0, \eta[, \beta(x + \varepsilon v) > 0$ since $\varepsilon < \eta_{\alpha}$ for any α . Hence $\Psi \cap \Phi_{\Delta}^+ = \emptyset$ and, therefore, $n(\Delta) = 0$. Moreover $x \in \overline{C}_{\Delta}^v$. Let $\varepsilon \in]0, \eta[$ and $y = x + \varepsilon v$. Then $y \in C_{\Delta}^v \cap (x + \delta_v) \subset C_{\Delta}^v \cap (x + C_{\Delta'}^v)$ according to Lemma 2.5. In particular, if Δ is any basis of Φ such that $n(\Delta) = 0$, we have, in particular, $\Delta \cap \Psi = \emptyset$ and we have seen that the element $w = \text{id}$ provides the basis of the induction.

Induction step: Let Δ be any basis of Φ such that $n(\Delta) = n > 0$. Thus $\Delta \cap \Psi \neq \emptyset$ and let $\alpha \in \Delta \cap \Psi$, that means we have either $\alpha(x) < 0$ or $\forall \eta > 0, \exists \varepsilon \in]0, \eta[, \alpha(x + \varepsilon v) \leq 0$. We prove that $-\alpha \notin \Psi$.

Suppose that $\alpha(x) < 0$. Then $-\alpha(x) > 0$. If $\alpha(v) \in \mathbb{Z}_{<0}$, let η be any element in $R_{>0}$. Then for any $\varepsilon \in]0, \eta[,$ we have $-\alpha(x + \varepsilon v) > -\alpha(v)\varepsilon > 0$. If $\alpha(v) \in \mathbb{Z}_{>0}$, let $\eta = -\frac{1}{\alpha(v)}\alpha(x) > 0$. Then for any $\varepsilon \in]0, \eta[,$ we have $-\alpha(x + \varepsilon v) = -\alpha(x) - \alpha(v)\varepsilon > -\alpha(x) - \alpha(v)\eta = 0$. Hence, in both cases, we get $-\alpha \notin \Psi$.

Otherwise, for any $\eta > 0$ there exists $\rho \in]0, \eta[$ such that $\alpha(x + \rho v) \leq 0$.

• If $\alpha(v) \in \mathbb{Z}_{>0}$, let $\rho = \frac{\alpha(x)}{\alpha(v)} > 0$. Then $-\alpha(x) \geq \alpha(v)\rho > 0$. Moreover, for any $\varepsilon \in]0, \rho[,$ we have $-\alpha(x + \varepsilon v) \geq \alpha(v)(\rho - \varepsilon) > 0$. Hence $-\alpha \notin \Psi$.

• If $\alpha(v) \in \mathbb{Z}_{<0}$, suppose by contradiction that $-\alpha(x) < 0$. Then $-\alpha(x) \geq \alpha(v)\rho > \alpha(v)\eta$. Thus, for $\eta = -\frac{1}{2\alpha(v)}\alpha(x) > 0$, we get $-\alpha(x) > -\frac{1}{2}\alpha(x)$ which is a contradiction. Hence $-\alpha(x) \geq 0$. Now, for any $\varepsilon \in]0, \rho[$, we have $-\alpha(x + \varepsilon v) \geq -\alpha(v)\varepsilon > 0$. Thus $-\alpha \notin \Psi$.

As a consequence, in all cases, we get $-\alpha \notin \Psi$. According to [Bou81, VI.1.6 Cor. 1 of Prop. 17], we know that r_α stabilizes $\Phi_\Delta^+ \setminus \{\alpha\} = \Phi_{r_\alpha(\Delta)}^+ \setminus \{-\alpha\}$. Thus, $\Phi_{r_\alpha(\Delta)}^+ \cap \Psi = (\Phi_\Delta^+ \setminus \{\alpha\}) \cap \Psi$ so that $n(r_\alpha(\Delta)) = n(\Delta) - 1$ since $\alpha \in \Psi$. By induction, since $r_\alpha(\Delta)$ is a basis of Φ , we know that there exists $w \in W(\Phi)$ such that $x \in \overline{C}_{w \circ r_\alpha(\Delta)}^v$ and $C_{w \circ r_\alpha(\Delta)}^v \cap (x + C_{\Delta'}^v) \neq \emptyset$. \square

We fix a basis Δ_f of Φ and we denote by $F_R^v(\Delta_P)$ instead of $F_R^v(\Delta_f, \Delta_P)$. The **fundamental chamber** is the set $C_{f,R}^v = F_R^v(\emptyset)$. A **vector face** (resp. **vector chamber**) is a set of the form $w \cdot F_R^v(\Delta_P)$ (resp. $w \cdot C_{f,R}^v$), for some $w \in W^v$ and $\Delta_P \subset \Delta_f$.

2.8 Lemma. *Let Δ be a basis of Φ . For any $\Delta_P \subset \Delta$, we have $w \cdot F_R^v(\Delta, \Delta_P) = F_R^v(w(\Delta), w(\Delta_P))$ for every $w \in W(\Phi)$. Moreover $V_R = \bigcup_{w \in W(\Phi)} w \cdot \overline{C}_\Delta^v$.*

Proof. For any $\alpha \in \Delta$ and any $x \in V_R$, we have

$$w(\alpha)(w(x)) = ww^{-1}\alpha(x) = \alpha(x)$$

since $w^{-1}(\alpha) = \alpha \circ w$. Thus $x \in F_R^v(\Delta, \Delta_P) \iff w(x) \in w \cdot F_R^v(\Delta, \Delta_P) \iff w(x) \in F_R^v(w(\Delta), w(\Delta_P))$.

By Proposition 2.7, for any $x \in V_R$, there exists $w \in W^v$ such that $x \in \overline{C}_{R, w(\Delta)}^v = w \cdot \overline{C}_R^v$. Hence we get the second equality. \square

2.9 Lemma. *Let $\beta \in \Phi$ and F^v be a vector face of V_R . Then either $\beta(F^v) \subset R_{>0}$ or $\beta(F^v) = \{0\}$ or $\beta(F^v) \subset R_{<0}$.*

Proof. Write $F^v = w \cdot F_R^v(\Delta_P)$, where $\Delta_P \subset \Delta_f$ and $w \in W^v$. Let $\beta' = w^{-1} \cdot \beta$. Then $\beta'(F_S^v(\Delta_P)) = \beta(F_S^v)$. Maybe considering $-\beta'$, we may assume that $\beta' \in \Phi_+$. Write $\beta' = \sum_{\alpha \in \Delta_f} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta_f$. Then if $\{\alpha \in \Delta_f | n_\alpha \neq 0\} \subset \Delta_P$, $\beta'(F_S^v) = \{0\}$ and else $\beta'(F_S^v) \subset R_{>0}$. Lemma follows. \square

Given a vector face F^v , we denote by $\Phi_{F^v}^+ = \{\beta \in \Phi, \beta(F^v) \subset R_{>0}\}$, by $\Phi_{F^v}^- = \{\beta \in \Phi, \beta(F^v) \subset R_{<0}\}$ and by $\Phi_{F^v}^0 = \{\beta \in \Phi, \beta(F^v) = \{0\}\}$.

2.1.3 R -affine spaces

An R -affine space is defined in the same way as an affine space over a field in which we replace the underlying vector space by an R -module.

Let Z be any domain and R be a pseudo-ring equipped with a Z -module structure such that:

$$\forall z \in Z \setminus \{0\}, \exists r \in R, zr \neq 0.$$

The group $\text{GL}_n(Z)$ canonically acts onto R^n by $(g_{i,j})_{i,j} \cdot (x_k)_k = \left(\sum_{j=1}^n g_{k,j} x_j \right)_k$. We can therefore introduce the group:

$$\text{Aff}_Z(n, R) = R^n \rtimes \text{GL}_n(Z).$$

Similarly, if V_Z is a free Z -module, then $\text{GL}(V_Z)$ naturally acts on $V_R := V_Z \otimes_Z R$. We can therefore define the Z -affine group of V_R as

$$\text{Aff}_Z(V_R) = V_R \rtimes \text{GL}(V_Z).$$

If \mathbb{A} is an affine space over V_R with some origin o , we define a natural action of $\text{Aff}_Z(V_R)$ on \mathbb{A} by:

$$(v, g) \cdot x = o + g(x - o) + v \quad \forall (v, g) \in V_R \rtimes \text{GL}(V_Z), \forall x \in \mathbb{A}.$$

This action is faithful and its image is called the Z -affine group of the affine space \mathbb{A} and denoted by $\text{Aff}_Z(\mathbb{A})$. We define the linear part of an element $h \in \text{Aff}_Z(\mathbb{A})$, denoted by \overrightarrow{h} , as the image of h by the quotient morphism $\text{Aff}_Z(\mathbb{A}) \simeq \text{Aff}_Z(V_R) \rightarrow \text{GL}(V_Z)$.

2.10 Notation. Let \mathbb{A}_R be an R -affine space with some origin o and underlying R -module V_R defined as in 2.1.2. Note that $V_R = 0$ and $\mathbb{A}_R = \{o\}$ when $\Phi = \emptyset$ so that any action of any group on \mathbb{A}_R is trivial. In the rest of this section, we assume that $\Phi \neq \emptyset$ but any result can obviously be extended to the case of an empty root system.

Any root $\alpha \in \Phi$ induces canonically a continuous R -linear form on V_R so that for any $\alpha \in \Phi$ and any $\lambda \in R$, one can define:

- an affine map $\theta_{\alpha, \lambda} : \mathbb{A}_R \rightarrow R$ by $\theta_{\alpha, \lambda}(x) = \alpha(x - o) + \lambda$;
- an affine hyperplane $H_{\alpha, \lambda} = \theta_{\alpha, \lambda}^{-1}(\{0\})$;
- an open half-affine space $\mathring{D}_{\alpha, \lambda} = \theta_{\alpha, \lambda}^{-1}(R_{>0})$ (resp. close $D_{\alpha, \lambda} = \theta_{\alpha, \lambda}^{-1}(R_{\geq 0})$);
- an affine reflection $r_{\alpha, \lambda} : \mathbb{A}_R \rightarrow \mathbb{A}_R$ with respect to $H_{\alpha, \lambda}$ by $r_{\alpha, \lambda}(x) = x - \theta_{\alpha, \lambda}(x)\alpha^\vee$.

By abuse of notation, for any root $\alpha \in \Phi$ and any point $x \in \mathbb{A}_R$, we will often denote by $\alpha(x)$ instead of $\alpha(x - o)$.

We denote, by convention, $D_{\alpha, \infty} = \mathring{D}_{\alpha, \infty} = \mathbb{A}_R$ for any $\alpha \in \Phi$ so that $x \in D_{\alpha, \infty} \iff \infty \geq -\alpha(x)$ extends the definition of the $D_{\alpha, \lambda} = \{x \in \mathbb{A}_R, \lambda \geq -\alpha(x)\}$ to any $\lambda \in R \cup \{\infty\}$.

2.11 Fact. For any $\alpha \in \Phi$ and any $\lambda \in R$, the element $r_{\alpha, \lambda} \in \text{Aff}_Z(\mathbb{A})$ is identified with $(-\lambda\alpha^\vee, r_\alpha) \in \text{Aff}_Z(V_R)$. In particular, $\overrightarrow{r_{\alpha, \lambda}} = r_\alpha \in \text{GL}(V_Z)$.

Proof. For $x \in \mathbb{A}$, we have

$$\begin{aligned} (-\lambda\alpha^\vee, r_\alpha)(x) &= o + r_\alpha(x - o) - \lambda\alpha^\vee \\ &= o + (x - o) - \alpha(x - o)\alpha^\vee - \lambda\alpha^\vee \\ &= x + \theta_{\alpha, \lambda}(x)\alpha^\vee \\ &= r_{\alpha, \lambda}(x) \end{aligned}$$

We get the identification by faithfulness of the action. □

2.12 Fact. For $(\alpha, \lambda) \in \Phi \times R$, the map $r_{\alpha, \lambda}$ is well-defined since $\theta_{\alpha, \lambda}(x)\alpha^\vee$ belongs to the R -module V_R for any $x \in \mathbb{A}_R$. It is an affine reflection in the sense that it satisfies:

- $r_{\alpha, \lambda}^2 = \text{id}_{\mathbb{A}_R}$;
- $r_{\alpha, \lambda}(x) = x \iff x \in H_{\alpha, \lambda}$;
- $r_{\alpha, \lambda}(D_{\alpha, \lambda}) = D_{-\alpha, -\lambda}$.

Proof. The second statement is immediate from the formula since $\lambda\alpha^\vee = 0 \iff \lambda = 0$.

Let $y = r_{\alpha, \lambda}(x) = x - \theta_{\alpha, \lambda}(x)\alpha^\vee$. Then

$$\begin{aligned} \theta_{\alpha, \lambda}(y) &= \alpha(y) - \lambda \\ &= \alpha(x) - \alpha(\alpha^\vee)\theta_{\alpha, \lambda}(x) - \lambda \\ &= (\alpha(x) - \lambda) - 2\theta_{\alpha, \lambda}(x) \\ &= -\theta_{\alpha, \lambda}(x) \end{aligned}$$

Thus we get the first statement:

$$\begin{aligned} r_{\alpha,\lambda}(y) &= y - \theta_{\alpha,\lambda}(y)\alpha^\vee \\ &= y + \theta_{\alpha,\lambda}(x)\alpha^\vee \\ &= x \end{aligned}$$

Finally, the third statement is given by Fact 2.13. \square

2.13 Fact. For any $(\alpha, \lambda), (\beta, \mu) \in \Phi \times R$, there is a unique $(\gamma, \rho) \in \Phi \times R$ such that:

$$r_{\alpha,\lambda}(D_{\beta,\mu}) = D_{\gamma,\rho}.$$

More precisely, we have

$$\gamma = r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha \qquad \rho = \mu - \beta(\alpha^\vee)\lambda$$

which make sense since R is a \mathbb{Z} -module.

Proof. These are exactly the same results as in [Bou81, VI§2]. We recall that W^{aff} is generated by the $r_\alpha : \Phi \rightarrow \Phi$ for $\alpha \in \Phi$ given by $r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$ so that $r_\alpha^2 = \text{id}$.

We firstly prove that $r_{\alpha,\lambda}(\mathring{D}_{\beta,\mu}) \subset \mathring{D}_{\gamma,\rho}$. For any $(\alpha, \lambda), (\beta, \mu) \in \Phi \times R$, let $\gamma = r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha \in \Phi$. Note that $\gamma(\alpha^\vee) = \beta(\alpha^\vee) - \beta(\alpha^\vee)\alpha(\alpha^\vee) = -\beta(\alpha^\vee)$. For any $x \in \mathring{D}_{\beta,\mu}$, let $y = r_{\alpha,\lambda}(x)$. Then

$$\begin{aligned} \gamma(y) &= \gamma(x) - \gamma(\alpha^\vee)\theta_{\alpha,\lambda}(x) \\ &= \left(\beta(x) - \beta(\alpha^\vee)\alpha(x) \right) + \beta(\alpha^\vee)\theta_{\alpha,\lambda}(x) \\ &= \beta(x) - \beta(\alpha^\vee)\alpha(x) + \beta(\alpha^\vee)(\alpha(x) + \lambda) \\ &= \beta(x) + \beta(\alpha^\vee)\lambda \\ &> -\mu + \beta(\alpha^\vee)\lambda = -\rho \end{aligned}$$

Thus $y \in \mathring{D}_{\gamma,\rho}$.

Conversely, if $y \in \mathring{D}_{\gamma,\rho}$, let $x = r_{\alpha,\lambda}(y)$. Then $y = r_{\alpha,\lambda}(x)$ since $r_{\alpha,\lambda}^2 = \text{id}$. Thus $x \in r_{\alpha,\lambda}(\mathring{D}_{\gamma,\rho}) \subset \mathring{D}_{r_\alpha(\gamma), \rho - \gamma(\alpha^\vee)\lambda}$. Since $r_\alpha(\gamma) = r_\alpha^2(\beta) = \beta$ and $\rho - \gamma(\alpha^\vee)\lambda = \rho + \beta(\alpha^\vee)\lambda = \mu$, we get that $r_{\alpha,\lambda}(\mathring{D}_{\gamma,\rho}) \subset \mathring{D}_{\beta,\mu}$. Thus $\mathring{D}_{\gamma,\rho} = r_{\alpha,\lambda}(\mathring{D}_{\beta,\mu})$.

The same equality holds for affine hyperplanes. \square

A **sector-face** Q (resp. a **sector** Q) is a set of the form $x + F^v$ (resp. $x + C^v$) for some $x \in \mathbb{A}_R$ and some vector face F^v (resp. vector chamber C^v) of V_R . The **direction** of Q is F^v and its **base point** is x .

2.14 Definition. An **affine apartment** over R is a 4-tuple

$$\underline{\mathbb{A}}_R = (\mathbb{A}_R, V_{\mathbb{R}}, \Phi, (\Gamma_\alpha)_{\alpha \in \Phi}) \text{ such that:}$$

1. Φ is a root system over $(V_{\mathbb{R}})^*$;
2. $(\Gamma_\alpha)_{\alpha \in \Phi}$ is a family of non-bounded subsets of R containing 0, satisfying the following property. Let $\mathcal{H} = \{H_{\alpha,\lambda} | \alpha \in \Phi, \lambda \in \Gamma_\alpha\}$. For $H = H_{\alpha,\lambda} \in \mathcal{H}$ we denote by $r_H = r_{\alpha,\lambda}$ the (affine) reflexion of \mathbb{A}_S fixing H and whose vectorial part is r_α . We assume that r_H stabilizes \mathcal{H} for every $H \in \mathcal{H}$.
3. \mathbb{A}_R is the affine space over R with underlying vector-space V_R .

A set of the form $D_{\alpha,\lambda}$ (resp. $\overset{\circ}{D}_{\alpha,\lambda}$, resp. $H_{\alpha,\lambda}$) for $\lambda \in \Gamma_\alpha$ and $\alpha \in \Phi$ is called a **half-apartment** (resp. an **open half-apartment**, resp. a **wall**) of \mathbb{A}_R .

A set of the form $D_{\alpha,\lambda}$ (resp. $\overset{\circ}{D}_{\alpha,\lambda}$, resp. $H_{\alpha,\lambda}$) for $\lambda \in R$ and $\alpha \in \Phi$ is called a **phantom half-apartment** (resp. an **phantom open half-apartment**, resp. a **phantom wall**) of \mathbb{A}_R . This phantom part terminology is there to indicate that the objects are carried by the directions induced by the roots but not by the values prescribed by the valuation. When in section 6 we make a use of algebraic groups over a valued field, these phantom parts may appear after some extension of the base field.

The **affine weyl group** W^{aff} of \mathbb{A}_R is the subgroup of the group of affine automorphisms of \mathbb{A}_R generated by the r_H , for $H \in \mathcal{H}$. It is a subgroup of $W^v \times V_R$. Let $\alpha \in \Phi$, $\lambda \in \Gamma_\alpha$ and $M = H_{\alpha,-\gamma}$. By condition (2), if $\alpha \in \Phi$ and $\lambda \in \Gamma_\alpha$, then $r_\alpha(H_{\alpha,\lambda}) = H_{\alpha,-\lambda}$ and thus $-\Gamma_\alpha = \Gamma_\alpha$. If $\alpha \in \Phi$, $w \in W^v$ and $\lambda \in \Gamma_\alpha$, then $w \cdot H_{\alpha,\lambda} = H_{w \cdot \alpha,\lambda}$ and thus $\Gamma_\alpha = \Gamma_{w \cdot \alpha}$.

2.1.4 R -metrics over affine spaces

For $\lambda \in R$, we denote by $|\lambda| = -\lambda$ if $\lambda \in R_{<0}$ and $|\lambda| = \lambda$ if $\lambda \in R_{\geq 0}$ the absolute value of λ . It satisfies $|\lambda + \mu| \leq |\lambda| + |\mu|$ and $|\lambda\mu| = |\lambda||\mu|$ for any $\lambda, \mu \in R$. The finitely generated free R -module $V_R = \bigoplus_{\alpha^\vee \in \Delta^\vee} R\alpha^\vee$ can be equipped with a $W(\Phi^\vee)$ -invariant R -norm as follows:

Let Φ^+ be any choice of a subset of positive roots and set $\|x\|_R = \sum_{\alpha \in \Phi^+} |\alpha(x)|$. Then $2\|x\|_R = \sum_{\alpha \in \Phi} |\alpha(x)|$. Since R is a torsion-free \mathbb{Z} -module, by definition of root systems, this defines a $W(\Phi^\vee)$ -invariant map $\|\cdot\|_R : V_R \rightarrow R_{\geq 0}$ that does not depend on the choice of Φ^+ . One says that $\|\cdot\|_R$ is an R -norm, that is a map $\|\cdot\|_R : V_R \rightarrow R_{\geq 0}$ such that for $x, y \in V_R$, one has:

1. $\|x\|_R = 0 \iff x = 0$;
2. $\|\lambda x\|_R = |\lambda|\|x\|_R$;
3. $\|x + y\|_R \leq \|x\|_R + \|y\|_R$.

Moreover, it satisfies $\|\lambda v\|_R = \|v\|_{\mathbb{Z}}|\lambda|$ for any $v \in V_{\mathbb{Z}}$ and any $\lambda \in R$ so that $\lambda v \in V_R$.

Let \mathbb{A}_R be an R -affine space with V_R as underlying R -module. Then, the map $d_R^{\text{std}} : \mathbb{A}_R \times \mathbb{A}_R \rightarrow R_{\geq 0}$ defined by $d_R^{\text{std}}(x, y) = \|y - x\|_R$ defines an R -distance. That is a map $d : \mathbb{A}_R \times \mathbb{A}_R \rightarrow R$ such that for all $x, y, z \in \mathbb{A}_R$, one has:

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(z, y)$.

The map d_R^{std} is the **standard R -metric** considered by Bennett in [Ben94].

For $x \in \mathbb{A}_R$ and $\varepsilon \in R_{>0}$, we denote by $B_R(x, \varepsilon)$ the set $\{y \in \mathbb{A}_R \mid d(x, y) < \varepsilon\}$. We equip \mathbb{A}_R with the topology whose a base is given by the $B_R(x, \varepsilon)$ for $x \in \mathbb{A}_R$ and $\varepsilon > 0$.

2.15 Remark. It is easy to check that it coincides with the topology on V_R defined in Subsubsection 2.1.2.

2.1.5 Filters

A filter on a set \mathcal{E} is a nonempty set \mathcal{F} of nonempty subsets of \mathcal{E} such that, for all subsets E, E' of \mathcal{E} , one has:

- $E, E' \in \mathcal{F}$ implies $E \cap E' \in \mathcal{F}$

- $E' \subset E$ and $E' \in \mathcal{F}$ implies $E \in \mathcal{F}$.

If \mathcal{E} is a set and $\mathcal{F}, \mathcal{F}'$ are filters on \mathcal{E} , we define $\mathcal{F} \cup \mathcal{F}'$ to be the filter $\{E \cup E' \mid (E, E') \in \mathcal{F} \times \mathcal{F}'\}$.

If $\mathcal{E}, \mathcal{E}'$ be sets, $f : \mathcal{E} \rightarrow \mathcal{E}'$ be a map and \mathcal{F} be a filter on \mathcal{E} . Then $f(\mathcal{F}) := \{f(E) \mid E \in \mathcal{F}\}$ is a filter on $f(\mathcal{E}')$. We say that a map fixes a filter if it fixes at least one element of this filter.

If \mathcal{F} is a filter on a set \mathcal{E} , and E is a subset of \mathcal{E} , one says that \mathcal{F} contains E if every element of \mathcal{F} contains E . We denote it $\mathcal{F} \ni E$. If E is nonempty, the **principal filter** on \mathcal{E} associated with E is the filter $\mathcal{F}_{E, \mathcal{E}}$ of subsets of \mathcal{E} containing E .

A filter \mathcal{F} is said to be contained in another filter \mathcal{F}' : $\mathcal{F} \subseteq \mathcal{F}'$ (resp. in a subset Z in \mathcal{E} : $\mathcal{F} \subseteq Z$) if every set in \mathcal{F}' is in \mathcal{F} (resp. if $Z \in \mathcal{F}$).

These definitions of containment are inspired by the following facts. Let \mathcal{E} be a set, \mathcal{F} be a filter on \mathcal{E} and $E, E' \subset \mathcal{E}$. Then :

- $E \subset E'$ if and only if $\mathcal{F}_{E, \mathcal{E}} \subseteq \mathcal{F}_{E', \mathcal{E}}$,
- $E \in \mathcal{F}$ if and only if $\mathcal{F}_{E, \mathcal{E}} \subseteq \mathcal{F}$,
- $E \ni \mathcal{F}$ if and only if $\mathcal{F}_{E, \mathcal{E}} \ni \mathcal{F}$.

2.1.6 Enclosure map

We now define the enclosure of a filter on \mathbb{A}_R . This definition is motivated by the fact that the intersection of two apartments is a finite intersection of half-apartments. We assume that we are given a families of sets Γ_α .

2.16 Definition. Let $\underline{\mathbb{A}}_R = (\mathbb{A}_R, V_{\mathbb{R}}, \Phi, (\Gamma_\alpha)_{\alpha \in \Phi})$ be an apartment. Let \mathcal{V} be a filter on \mathbb{A}_R . Then we define the **enclosure** $\text{cl}(\mathcal{V})$ as:

$$\text{cl}(\mathcal{V}) = \{X \subset \mathbb{A}_R \mid \exists (\lambda_\alpha) \in \prod_{\alpha \in \Phi} (\Gamma_\alpha \cup \{\infty\}) \mid X \supset \bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_\alpha} \ni \mathcal{V}\}.$$

If Ω is a subset of \mathbb{A}_R , we set $\text{cl}(\Omega) = \text{cl}(\mathcal{F}_{\Omega, \mathbb{A}_R})$. A subset Ω of \mathbb{A}_R is said to be **enclosed** if it is an element of $\text{cl}(\Omega)$, that is, if it is a finite intersection of half-apartments.

Our definition of enclosure is inspired by [GR08, 2.2.2]. It is different from the enclosure cl_{BT} defined in [BT72, 7.1.2].

- If $\Omega \subset \mathbb{A}_R$, then $\text{cl}(\Omega)$ is a filter whereas $\text{cl}_{\text{BT}}(\Omega)$ is a set. But even if we identify a set with the principal filter associated, the notions differ. Indeed, suppose for example that $\mathbb{A} = \mathbb{R}$ and that $\Gamma_\alpha = \mathbb{Q}$, for all $\alpha \in \Phi$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $\text{cl}(\{x\})$ is the set of subsets of \mathbb{R} containing a neighborhood of x , whereas $\text{cl}_{\text{BT}}(\{x\}) = \{x\}$ and $\mathcal{F}_{\{x\}, \mathbb{A}_R} \neq \text{cl}(\{x\})$.
- One has $\text{cl}(\Omega) \ni \text{cl}_{\text{BT}}(\Omega)$ for every subset Ω of \mathbb{A}_R : our enclosure is bigger.

Note that when Λ is a discrete subset of \mathbb{R} , if we work with $R = \mathbb{R}$, then the enclosure of each subset Ω of \mathbb{A} is the principal filter on $\mathbb{A}_{\mathbb{R}}$ associated to $\text{cl}_{\text{BT}}(\Omega)$ and thus we can avoid the use of filters. When $|S| \geq 2$ however, this property is no longer true, even in the discrete case. Indeed, suppose that $\Lambda = \mathbb{Z}^2$ and that $\Phi = \{\text{Id}, -\text{Id}\}$ so that $\mathbb{A}_{\mathbb{R}} = \mathbb{R}$. In this case, we work with $R = \mathfrak{R}^2 \simeq \mathbb{R}[t]/(t^2)$. Let $\Omega = \{(0, x) \mid x \in \mathbb{R}\}$. Then

$$\text{cl}(\Omega) = \{X \subset \mathfrak{R}^2 \mid \exists a, b \in \mathbb{R}, X \supset [(-1, a), (1, b)]_{\mathfrak{R}^2}\},$$

and this filter is not principal.

2.1.7 Germs, faces and local faces

Let $Q = x + F^v$ be a sector-face of \mathbb{A}_R , where $x \in \mathbb{A}_R$ and F^v is a vector face in V_R .

The **germ of Q at x** is the filter $\text{germ}_x(Q) = \{\Omega \cap Q \mid \Omega \subset \mathbb{A} \text{ is a neighborhood of } x\}$. A **local face** (resp. a **local chamber**) is a filter of the form $\text{germ}_x(Q)$ for some sector-face Q (resp. sector Q) based at x . Two local sector-germs $\text{germ}_x(Q)$ and $\text{germ}_x(Q')$ are said to be opposite if the direction of Q is opposite to the direction of Q' , i.e if $Q = x + C^v$ and $Q' = x + w_0 \cdot C^v$, where C^v is the direction of Q and w_0 is the longest element of W^v .

Alternatively, we denote by $F^\ell(x, F^v)$, as done in [Rou11], or by \mathcal{F}_{x, F^v} , as done in [BT72, 7.2], instead of $\text{germ}_x(Q)$. This will be a useful notation for Bruhat decomposition 4.38.

The **face $F(x, F^v)$** is the filter on \mathbb{A}_R generated by the sets of the form

$$\mathcal{X} = \bigcap_{\alpha \in \Psi} \overset{\circ}{D}_{\alpha, \lambda_\alpha} \cap \bigcap_{\alpha \in \Phi \setminus \Psi} D_{\alpha, -\lambda_\alpha},$$

where $\Psi \subset \Phi$ and $\lambda \in R \cup \{\infty\}$ for $(\lambda_\alpha) \in (\mathfrak{R}^S \cup \{\infty\})^\Phi$ such that $\mathcal{X} \ni F^\ell(x, F^v)$.

The **germ of Q at infinity** is the filter $\text{germ}_\infty(Q) = \{\Omega \subset \mathbb{A} \mid \exists \xi \in F^v, \Omega \supset x + \xi + F^v\}$. If Q is a sector, then a subset \mathcal{X} of \mathbb{A} is in $\text{germ}_\infty(Q)$ if and only if \mathcal{X} contains a subsector of Q .

2.17 Lemma. *Let Q be a sector of \mathbb{A}_R , x be its basis and $y \in Q$. Then $\text{cl}(\{x, y\}) \ni \text{germ}_x(Q)$.*

Proof. Let C^v be the vector chamber of V_R such that $Q = x + C^v$. Let Δ be the base of Φ associated with C^v . Let $\Omega \in \text{cl}(\{x, y\})$. Let $(\lambda_\alpha)_{\alpha \in \Phi} \in (R \cup \{\infty\})^\Phi$ be such that $\Omega \supset \{x, y\}$. Then for all $\alpha \in \Phi_\Delta^-$, $\alpha(x) > \alpha(y) \geq -\lambda_\alpha$ and for all $\alpha \in \Phi_\Delta^+$, $\alpha(x) \geq -\lambda_\alpha$. Set $\mathcal{X}_R = \bigcap_{\alpha \in \Phi_\Delta^-} \overset{\circ}{D}_{\alpha, \lambda_\alpha} \ni y$. Then

$$\Omega \supset \bigcap_{\alpha \in \Phi_\Delta^-} D_{\alpha, \lambda_\alpha} \cap \bigcap_{\alpha \in \Phi_\Delta^+} \overset{\circ}{D}_{\alpha, \lambda_\alpha} \supset Q \cap \mathcal{X}_R$$

and thus $\Omega \in \text{germ}_x(Q)$, which proves the lemma. \square

2.1.8 The pseudo-ring \mathfrak{R}^S

In the previous sections, we have defined the notion of an affine apartment over a totally ordered commutative pseudo-ring R . The current section is dedicated to the definition of the pseudo-rings R that will be considered in the sequel.

In the classical Bruhat-Tits theory, when one works over a field \mathbb{K} endowed with a valuation $\omega : \mathbb{K}^\times \rightarrow \mathbb{Z}$, one usually chooses the ring $R = \mathbb{R}$, in which the valuation group \mathbb{Z} is embedded. In our case, we will work over a field \mathbb{K} that is endowed with a valuation $\omega : \mathbb{K}^\times \rightarrow \Lambda$, where Λ can be any totally ordered abelian group. The rank $\text{rk}(\Lambda)$ of Λ is then defined as the (totally ordered) set of Archimedean equivalence classes of Λ . For instance, if $\Lambda = \mathbb{Z}^n$ for some $n \geq 1$, then $\text{rk}(\Lambda) = \{1, 2, \dots, n\}$. By Hahn's embedding theorem, Λ can always be embedded as an ordered subgroup of the lexicographically ordered real vector subspace of $\mathbb{R}^{\text{rk}(\Lambda)}$ given by families $(x_s)_{s \in \text{rk}(\Lambda)}$ with well-ordered support. This motivates the following more general definition:

2.18 Definition. Given S a totally ordered set, \mathfrak{R}^S is the real vector subspace of \mathbb{R}^S whose elements are given by families $(x_s)_{s \in S} \in \mathbb{R}^S$ with well-ordered support. It is endowed with the lexicographical order.

In order to be able to choose $R = \mathfrak{R}^{\text{rk}(\Lambda)}$ and use the theory that has been developed in the previous sections, it is now necessary to endow \mathfrak{R}^S with a pseudo-ring structure. The choice of this pseudo-ring structure does not play a crucial role in this article and we could just formally decide that the product of two elements in \mathfrak{R}^S is always 0. However, as we are going to explain in the rest of this paragraph, it is sometimes possible to endow \mathfrak{R}^S with other pseudo-ring structures. This might play an important role in future articles, for instance in order to endow the buildings we construct in the present article with a Euclidean distance.

A simple method to endow \mathfrak{R}^S with various pseudo-ring structures consists in considering a totally ordered commutative monoid Γ and a non-decreasing embedding:

$$\iota : S \hookrightarrow \Gamma$$

such that:

- (i) for any $s \in S$, $\iota(s) \geq 0$;
- (ii) for any $s \in S$ and $\gamma \in \Gamma$, if $0 < \gamma < \iota(s)$, then $\gamma \in \iota(S)$;
- (iii) if S has a minimal element s_0 , then $\iota(s_0) = 0$.

Let $\Gamma_{\geq 0}$ (resp. $\Gamma_{> 0}$) be the monoid of non-negative (resp. positive) elements in Γ and assume first that S has no minimal element. Consider the non-unital \mathbb{R} -algebra $\mathbb{R}[[t^{\Gamma_{> 0}}]]$ given by formal power series with real coefficients $f = \sum_{\gamma \in \Gamma_{> 0}} a_{\gamma} t^{\gamma}$ such that the support of the family $(a_{\gamma})_{\gamma \in \Gamma_{> 0}}$ is a well-ordered subset of $\Gamma_{> 0}$. It is endowed with the lexicographical order and the ideal I_S of $\mathbb{R}[[t^{\Gamma_{> 0}}]]$ spanned by the $\mathbb{R}t^{\gamma}$ with $\gamma \in \Gamma_{\geq 0} \setminus (\{0\} \cup \iota(S))$ is the kernel of the \mathbb{R} -linear order-preserving surjective morphism:

$$\begin{aligned} \pi : \mathbb{R}[[t^{\Gamma_{> 0}}]] &\rightarrow \mathfrak{R}^S \\ \sum_{\gamma \in \Gamma_{> 0}} a_{\gamma} t^{\gamma} &\mapsto (a_{\iota(s)})_{s \in S}. \end{aligned}$$

The non-unital ordered commutative \mathbb{R} -algebra structure on $\mathbb{R}[[t^{\Gamma_{> 0}}]]$ then induces a non-unital ordered commutative \mathbb{R} -algebra structure on \mathfrak{R}^S .

Assume now that S has a minimal element. Let $\mathbb{R}[[t^{\Gamma_{\geq 0}}]]$ be the unital \mathbb{R} -algebra of formal power series with real coefficients $f = \sum_{\gamma \in \Gamma_{\geq 0}} a_{\gamma} t^{\gamma}$ such that the support of the family $(a_{\gamma})_{\gamma \in \Gamma_{\geq 0}}$ is a well-ordered subset of $\Gamma_{\geq 0}$. It is endowed with the lexicographical order and the ideal I_S of $\mathbb{R}[[t^{\Gamma_{\geq 0}}]]$ spanned by the $\mathbb{R}t^{\gamma}$ with $\gamma \in \Gamma_{\geq 0} \setminus \iota(S)$ is the kernel of the \mathbb{R} -linear order-preserving surjective morphism:

$$\begin{aligned} \pi : \mathbb{R}[[t^{\Gamma_{\geq 0}}]] &\rightarrow \mathfrak{R}^S \\ \sum_{\gamma \in \Gamma_{\geq 0}} a_{\gamma} t^{\gamma} &\mapsto (a_{\iota(s)})_{s \in S}. \end{aligned}$$

This allows us to endow \mathfrak{R}^S with the structure of a unital ordered commutative \mathbb{R} -algebra.

2.19 Remark. Note that one can always find a monoid Γ and an embedding ι satisfying the desired conditions. Indeed:

- (i) if S does not have a minimal element, then Γ can be chosen to be the monoid of non-increasing maps $S \rightarrow \mathbb{N}$ together with the lexicographical order and ι the embedding that sends $s \in S$ to the element of Γ :

$$\begin{aligned} \iota(s) : S &\rightarrow \mathbb{N} \\ t &\mapsto \begin{cases} 1 & \text{if } t \leq s \\ 0 & \text{if } t > s. \end{cases} \end{aligned}$$

The product of two elements of \mathfrak{R}^S is then always 0, so that we recover the trivial pseudo-ring structure on \mathfrak{R}^S .

- (ii) if S has a minimal element s_0 , then Γ can be chosen to be the monoid of non-increasing maps $S \setminus \{s_0\} \rightarrow \mathbb{N}$ together with the lexicographical order and ι the embedding that sends $s \in S$ to the element of Γ :

$$\begin{aligned} \iota(s) : S \setminus \{s_0\} &\rightarrow \mathbb{N} \\ t &\mapsto \begin{cases} 1 & \text{if } t \leq s \\ 0 & \text{if } t > s. \end{cases} \end{aligned}$$

The product of two elements $(a_s)_{s \in S}$ and $(b_s)_{s \in S}$ of \mathfrak{R}^S is then $(a_{s_0}b_s + b_{s_0}a_s)_{s \in S}$.

2.20 Example. If S is well-ordered, it can be identified with an ordinal α . The Hessenberg sum endows the ordinal ω^α with the structure of a totally ordered commutative cancellative monoid. The Grothendieck group $K(\omega^\alpha)$ of ω^α is therefore a totally ordered abelian group. We can then set $\Gamma := K(\omega^\alpha)$ and consider the natural embedding:

$$\iota : S = \alpha \hookrightarrow \Gamma.$$

For example, when S is finite and has n elements, the \mathbb{R} -algebra \mathfrak{R}^S is then none other than $\mathbb{R}[[t]]/(t^n)$.

2.1.9 Topology and metric in \mathfrak{R}^S

In this part, we state some specificities over the totally ordered commutative pseudo-ring $R = \mathfrak{R}^S$ satisfying $\mathfrak{R}_{\mathbb{R}}^S = \mathfrak{R}_{\mathbb{Q}}^S = \mathfrak{R}^S$. In order to simplify the notation, we denote by $V_S, \mathbb{A}_S, F_S^v(\Delta_P)$, etc. instead of $V_{\mathfrak{R}^S}, \mathbb{A}_{\mathfrak{R}^S}, F_{\mathfrak{R}^S}^v(\Delta_P)$, etc.

If $s \in S$ and \top is a binary relation on S (for example $\leq, <, >, \dots$). We denote by $\mathbb{A}_{\top s}$ the space $\mathbb{A}_{\{t \in S \mid t \top s\}}$ and we define $\pi_{\top s} : \mathbb{A}_S \rightarrow \mathbb{A}_{\top s}$ by $\pi_{\top s}((x_t)_{t \in S}) = (x_t)_{t \top s}$, for $(x_s) \in \mathbb{A}_S$.

2.21 Lemma. *The distance $d_S^{\text{std}} : \mathbb{A}_S \times \mathbb{A}_S \rightarrow \mathfrak{R}^S$ satisfies the following properties:*

1. *For all $\varepsilon \in \mathfrak{R}_{>0}^S$, for all $s \in S$, there exists $t \in [s, +\infty[$ and an open neighborhood $\mathcal{X}_{\mathbb{R}}$ of 0 in $\mathbb{A}_{\mathbb{R}}$ such that $\pi_{\leq t}(B(0, \varepsilon)) \supset \{0_{\mathfrak{R} < t}\} \times \mathcal{X}_{\mathbb{R}}$.*
2. *For all $s \in S$, for every open subset $\mathcal{X}_{\mathbb{R}}$ of $\mathbb{A}_{\mathbb{R}}$ containing $\{0\}$, there exists $\epsilon \in \mathfrak{R}_{>0}^S$ such that $B_S(0, \epsilon) \subset \{0_{\mathfrak{R} < s}\} \times \mathcal{X}_{\mathbb{R}} \times \mathbb{A}_{> s}$.*
3. *It is invariant under translation, that is: for all $x, y, z \in \mathbb{A}_S$, $d_S^{\text{std}}(x, y) = d_S^{\text{std}}(x + z, y + z)$.*
4. *It is Weyl-compatible (see [BS14, Definition 3.1]).*

Proof. Point (3) is clearly satisfied and point (4) is [BS14, Lemma 10.1]). Let us prove (1). Let $\epsilon \in \mathfrak{R}_{>0}^S$ and $s \in S$. Let $s_0 = \min\{s' \in \text{supp}(\epsilon) \mid \epsilon_{s'} > 0\}$. Suppose $s \leq s_0$. Let $\mathcal{X}_{\mathbb{R}} = \{x \in \mathbb{A}_{\mathbb{R}} \mid d_{\mathbb{R}}^{\text{std}}(0, x) < \frac{1}{2}\epsilon_{s_0}\}$. Then $\pi_{\leq s_0}(B(0, \epsilon)) \supset \{0_{\mathfrak{R} < s_0}\} \times \mathcal{X}_{\mathbb{R}}$. Suppose now $s > s_0$. Then $B(0, \epsilon) \supset \{0_{\mathfrak{R} < s}\} \times \mathbb{A}_{[s, +\infty[}$ and thus $\pi_{\leq s}(B(0, \epsilon)) \supset \{0_{\mathfrak{R} < s}\} \times \mathbb{A}_{\mathbb{R}}$, which proves (1).

Let us prove (2). Let $s \in S$ and let $\mathcal{X}_{\mathbb{R}}$ be an open neighborhood of 0 in $\mathbb{A}_{\mathbb{R}}$. Let $\epsilon_{\mathbb{R}} \in \mathbb{R}_{>0}$ be such that $\{x \in \mathbb{A}_{\mathbb{R}} \mid d(0, x) < \epsilon_{\mathbb{R}}\} \subset \mathcal{X}_{\mathbb{R}}$. Let $\epsilon = (\frac{1}{2}\delta_{t,s}\epsilon_{\mathbb{R}})_{t \in S} \in \mathfrak{R}^S$. Then $B(0, \epsilon) \subset \{0\} \times \mathcal{X}_{\mathbb{R}} \times \mathbb{A}_{[s, +\infty[}$, which proves (2). \square

2.2 Definition of \mathfrak{R}^S -buildings

2.2.1 Bennett's definition of \mathfrak{R}^S -buildings

Let $\underline{\mathbb{A}}_S = (\mathbb{A}_S, V_{\mathbb{R}}, \Phi, (\Gamma_{\alpha})_{\alpha \in \Phi})$ be an affine apartment over \mathfrak{R}^S (see Definition 2.14). An **apartment of type $\underline{\mathbb{A}}_S$** is a set A equipped with a nonempty set $\text{Isom}(\underline{\mathbb{A}}_S, A)$ of bijections $f : \mathbb{A}_S \rightarrow A$ such that if $f_0 \in \text{Isom}(\underline{\mathbb{A}}_S, A)$, then $\text{Isom}(\underline{\mathbb{A}}_S, A) = \{f_0 \circ w \mid w \in W^{\text{aff}}\}$. An **isomorphism** between two apartments A, A' is a bijection $\phi : A \rightarrow A'$ such that there exists $f_0 \in \text{Isom}(A, \underline{\mathbb{A}}_S)$ such that $\phi \circ f_0 \in \text{Isom}(\underline{\mathbb{A}}_S, A')$.

Each apartment A of type $\underline{\mathbb{A}}_S$ can be equipped with the structure of an affine space by using an isomorphism of apartments $\phi : \underline{\mathbb{A}}_S \rightarrow A$.

We extend all the notions that are preserved by W^{aff} to each apartment. In particular half-apartments, walls, enclosure, sector-faces, local germs, germs at infinity, ... are well defined in each apartment of type $\underline{\mathbb{A}}_S$.

We say that an apartment contains a filter if it contains at least one element of this filter. Recall that we say that a map fixes a filter if it fixes at least one element of this filter.

2.22 Definition. An \mathfrak{R}^S -building is a set \mathcal{I} equipped with a covering \mathcal{A} by subsets called **apartments** such that:

- (A1) Each $A \in \mathcal{A}$ is equipped with the structure of an apartment of type $\underline{\mathbb{A}}_S$.
- (A2) If A, A' are two apartments, then $A \cap A'$ is enclosed in A and there exists an isomorphism $\phi : A \rightarrow A'$ fixing $A \cap A'$.
- (A3) For any pair of points in \mathcal{I} , there is an apartment containing both.

Given a W^{aff} -invariant \mathfrak{R}^S -metric d on the model space \mathbb{A}_S , axioms (A1)–(A3) imply the existence of a \mathfrak{R}^S -valued distance function on \mathcal{I} , that is a function $d : \mathcal{I} \times \mathcal{I} \rightarrow \mathfrak{R}^S$ satisfying all conditions of the definition of a \mathfrak{R}^S -metric except possibly the triangle inequality. The distance is defined as follows. Let $x, y \in \mathcal{I}$ and A be an apartment containing them. Then the distance x and y is the distance the distance between their images under any isomorphism of apartments from A to \mathbb{A} .

- (A4) For any pair of sector-germs in \mathcal{I} , there is an apartment containing both.
- (A5) For any apartment A and all $x \in A$, there exists a retraction $\rho_{A,x} : A \rightarrow \mathcal{I}$ such that $\rho_{A,x}$ does not increase distances and $\rho_{A,x}^{-1}(\{x\}) = \{x\}$.
- (A6) Let A_1, A_2, A_3 be apartments such that $A_1 \cap A_2$, $A_2 \cap A_3$ and $A_3 \cap A_1$ are half-apartments. Then $A_1 \cap A_2 \cap A_3$ is nonempty.

2.23 Remark. Suppose that S is reduced to a single element (thus $\mathfrak{R}^S \simeq \mathbb{R}$). The axioms (A1) to (A4) correspond to the axioms (A1) to (A4) of [Par00a, 1.2]. Axiom (A5) corresponds to axiom (A5') of [Par00a] and axiom (A6) corresponds to axiom (A5) of [Par00a, 1.4]. Note that under this assumption, (A6) is a consequence of the axioms (A1) to (A5).

2.2.2 Equivalent definition of \mathfrak{R}^S -buildings

We recall that we \mathbb{A}_S is equipped with d_S^{std} (see 2.1.4 and 2.1.9), which is Weyl-compatible ([BS14, Definition 3.1]).

In [BS14], Bennett and Schwer prove that the definition of \mathfrak{R}^S -buildings is equivalent to many other ones, see [BS14, Theorem 3.3]. In order to prove that the “building” associated with a quasi-split reductive group over a valued field is indeed a \mathfrak{R}^S -building, we will prove that it satisfies one of the equivalent set of axioms. We first need to define two other axioms: (GG) and (CO). Let \mathcal{I} be a set satisfying axioms (A1), (A2) and (A3).

(GG) Any two local sector-germs based at the same vertex are contained in a common apartment.

Two sectors Q, Q' based at the same point x are called **opposite at x** if there exists an apartment A containing $\text{germ}_x(Q), \text{germ}_x(Q')$ such that $\text{germ}_x(Q)$ and $\text{germ}_x(Q')$ are opposite in A .

(CO) If Q and Q' are two sectors which are opposite at their base points, then there exists a unique apartment containing Q and Q' .

As a particular case of [BS14, Theorem 3.3], we have:

2.24 Theorem. *Let (\mathcal{I}, d) be a set satisfying (A1), (A2) and (A3), where d is a Weyl compatible \mathfrak{R}^S metric on \mathbb{A}_S . Then \mathcal{I} is a Λ -building if and only if \mathcal{I} satisfies (GG) and (CO).*

2.3 Main theorem

2.25 Theorem. *Let Λ be a totally ordered abelian group with rank S , so that Λ can be seen as a totally ordered subgroup of \mathfrak{R}^S . Let \mathbb{K} be a field with a valuation $\omega : \mathbb{K} \rightarrow \Lambda \cup \{\infty\}$, let \mathbf{G} be a quasi-split (connected) reductive \mathbb{K} -group and let \mathbf{S} be a maximal split torus in \mathbf{G} with cocharacter module $X_*(\mathbf{S})$. If \mathbf{G} is not split, we assume that \mathbb{K} is Henselian.*

(i) *The set $\mathcal{I}(\mathbf{G}) = \mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ defined in section 6.5 and endowed with the distance introduced in 2.1.4 is an \mathfrak{R}^S -building whose apartments have type:*

$$\underline{\mathbb{A}}_S = (\mathbb{A}_S, V_{\mathbb{R}}, \Phi, (\Gamma_{\alpha})_{\alpha \in \Phi})$$

where $V_{\mathbb{R}}$ is the quotient of the real vector space $X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ by the orthogonal of the roots of \mathbf{G} , \mathbb{A}_S is an affine space over \mathfrak{R}^S whose underlying \mathfrak{R}^S -module is $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathfrak{R}^S$, Φ is the root system associated to \mathbf{G} in $V_{\mathbb{R}}^*$ and, for $\alpha \in \Phi$, Γ_{α} is a subset of \mathfrak{R}^S that generates a subgroup in which Λ has finite index. The group $\mathbf{G}(\mathbb{K})$ acts on $\mathcal{I}(\mathbf{G})$ by isometries and the induced action on the set of apartments is transitive.

(ii) *Let $s \in S$, let $S_{\leq s} = \{t \in S \mid t \leq s\}$ and let $\pi_{\mathfrak{R}^S, \leq s} : \mathfrak{R}^S \rightarrow \mathfrak{R}^{S_{\leq s}}$ be the natural projection. Consider the valuation $\omega_{\leq s} = \pi_{\mathfrak{R}^S, \leq s} \circ \omega$. There exists an (explicit) surjective map:*

$$\pi_{\leq s} : \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \rightarrow \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$$

compatible with the $\mathbf{G}(\mathbb{K})$ -action such that, for each $X \in \mathcal{I}(\mathbb{K}, \omega_{\leq s}, \mathbf{G})$, the fiber $\pi_{\leq s}^{-1}(X)$ is a product:

$$\mathcal{I}_X \times (\langle \Phi_X \rangle^{\perp} \otimes_{\mathbb{R}} \ker(\pi_{\mathfrak{R}^S, \leq s}))$$

where Φ_X is a root system contained in Φ , $\langle \Phi_X \rangle^{\perp}$ is the orthogonal of Φ_X in $V_{\mathbb{R}}$, and \mathcal{I}_X is a $\ker(\pi_{\mathfrak{R}^S, \leq s})$ -building. The apartments of \mathcal{I}_X have type:

$$\underline{\mathbb{A}}_X = (\mathbb{A}_X, V_{\mathbb{R}} / \langle \Phi_X \rangle^{\perp}, \Phi_X, (\Gamma_{X, \alpha})_{\alpha \in \Phi})$$

where \mathbb{A}_X is an affine space over $\ker(\pi_{\mathfrak{R}^S, \leq s})$ whose underlying $\ker(\pi_{\mathfrak{R}^S, \leq s})$ -module is $(V_{\mathbb{R}} / \langle \Phi_X \rangle^{\perp}) \otimes_{\mathbb{R}} \ker(\pi_{\mathfrak{R}^S, \leq s})$ and for $\alpha \in \Phi_X$, $\Gamma_{X, \alpha}$ is some subset of $\ker(\pi_{\mathfrak{R}^S, \leq s})$.

The construction of the building $\mathcal{I} := \mathcal{I}(\mathbf{G}, \mathbb{K}, \omega)$ is very close to the construction of Bruhat-Tits buildings by Bruhat and Tits.

We start from the datum of a quasi-split reductive \mathbb{K} -group \mathbf{G} and a maximal \mathbb{K} -split torus \mathbf{S} of \mathbf{G} . We let \mathbf{T} and \mathbf{N} be respectively the centralizer and the normalizer of \mathbf{S} in \mathbf{G} . The standard apartment \mathbb{A}_S is an \mathfrak{R}^S -affine space over the \mathfrak{R}^S -module given by scalar extension to \mathfrak{R}^S of the \mathbb{Z} -module of cocharacter defined over \mathbb{K} of \mathbf{T} . We construct an action of $N = \mathbf{N}(\mathbb{K})$ on \mathbb{A}_S . Using a Chevalley-Steinberg system $(x_{\alpha})_{\alpha \in \Phi}$

of \mathbf{G} (i.e. a parametrisation of root groups U_α compatible with Galois action of the splitting extension $\tilde{\mathbb{K}}/\mathbb{K}$ of \mathbf{G}), where Φ denotes the \mathbb{K} -root system of (\mathbf{G}, \mathbf{S}) , we define the parahoric subgroups $P_{\mathcal{V}}$ of $G = \mathbf{G}(\mathbb{K})$, for every filter \mathcal{V} on \mathbb{A}_S . We then define \mathcal{I} as in Bruhat-Tits as $G \times \mathbb{A}_S / \sim$, where \sim is an equivalence relation defined in section 5.

We then need to prove that \mathcal{I} satisfies the axioms (A1) to (A6). For this, we use Theorem 2.24 and we prove that \mathcal{I} satisfies the axioms (A1), (A2), (A3), (GG) and (CO). The fact that \mathcal{I} satisfies (A1) follows immediately from the definitions. The axiom (A2) is obtained similarly as in [BT72]. In order to prove (GG) and (A3) we generalize the Bruhat decomposition (see Theorem 4.38). Before proving this decomposition, we prove that G satisfies the Iwasawa decomposition (see Theorem 4.36). Restated in terms of buildings, this decomposition asserts that if F is a face of \mathcal{I} and C_∞ is a sector-germ at infinity of \mathcal{I} , then there exists an apartment containing F and C_∞ . We also prove (A4), which is actually a consequence of the Bruhat decomposition.

Section 7 is dedicated to the projection map $\pi_{\leq s}$ that has been introduced in part (ii) of theorem 2.25. We first construct the map $\pi_{\leq s}$ itself, and we give an explicit description of its fibers. We then prove that those fibers are associated to an \mathfrak{R}^S -valued root group datum together with a compatible N -action: in other words, we check all the axioms (RGD1-6), (V0-5) and (CA1-2). By the general theory developed in sections 3, 4 and 5, we deduce the decomposition of theorem 2.25:

$$\pi_{\leq s}^{-1}(X) = \mathcal{I}_X \times (\langle \Phi_X \rangle^\perp \otimes_{\mathbb{R}} \mathfrak{R}^S)$$

for some \mathcal{I}_X that satisfies axioms (A1), (A2), (A3), (A4) and (GG).

In section 8, we finish the proof of theorem 2.25 by establishing axiom (CO). This axiom is more geometric in nature. In order to prove it we first give a sufficient condition for an “ \mathbb{R} -building” to satisfy (CO) (see Lemma 8.2). Using this criterion and the projection maps defined in section 7, we prove that our building satisfies (CO).

3 R -valued root group datum

In this section, if G is a group and X, Y are subsets of G , we denote by:

- 1 the identity element of G ;
- $XY = \{xy, x \in X, y \in Y\}$ the subset of G obtained as image of the map $X \times Y \rightarrow G$ given by multiplication in G ;
- $\langle X, Y \rangle$ the subgroup of G generated by $X \cup Y$;
- $[X, Y]$ the subgroup of G generated by the set of commutators $[x, y]$ for $x \in X$ and $y \in Y$.

3.1 Abstract groups and axioms of a root group datum

We recall the following definition from [BT72, 6.1.1].

3.1 Definition. Let G be a group and Φ be a root system. A root group datum of G of type Φ is a system $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ satisfying the following axioms:

- (RGD1) T is a subgroup of G and, for any root $\alpha \in \Phi$, the set U_α is a nontrivial subgroup of G , called the root group of G associated to α ;
- (RGD2) for any roots $\alpha, \beta \in \Phi$ such that $\beta \notin \mathbb{R}_{<0}\alpha$, the commutator subgroup $[U_\alpha, U_\beta]$ is contained in the subgroup generated by the root groups U_γ for $\gamma \in (\alpha, \beta)$;

(RGD3) if α is a multipliable root, we have $U_{2\alpha} \subset U_\alpha$ and $U_{2\alpha} \neq U_\alpha$;

(RGD4) for any root $\alpha \in \Phi$, the set M_α is a right coset of T in G and we have $U_{-\alpha} \setminus \{1\} \subset U_\alpha M_\alpha U_\alpha$;

(RGD5) for any roots $\alpha, \beta \in \Phi$ and any $m \in M_\alpha$, we have $mU_\beta m^{-1} = U_{r_\alpha(\beta)}$;

(RGD6) for any choice of positive roots Φ^+ on Φ , we have $TU^+ \cap U^- = \{1\}$ where U^+ (resp. U^-) denotes the subgroup generated by the U_α for $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^- = -\Phi^+$).

A root group datum is said generating if G is generated by the subgroups T and the U_α for $\alpha \in \Phi$. As in [BT72, 6.1.2(10)], we denote by N the subgroup of G generated by the M_α for $\alpha \in \Phi$ if $\Phi \neq \emptyset$ and by $N = T$ otherwise.

We recall that, according to [BT72, 6.1.2(10)], axiom (RGD5) defines an epimorphism $\nu : N \rightarrow W(\Phi)$ such that $\nu(m) = r_\alpha$ for any $m \in M_\alpha$, any $\alpha \in \Phi$. Thus, for any $\alpha \in \Phi$ and any $n \in N$, we have $nU_\alpha n^{-1} = U_{\nu(\alpha)}$.

3.2 Example. Let \mathbb{K} be any field and \mathbf{G} be a reductive \mathbb{K} -group, \mathbf{S} a maximal \mathbb{K} -split torus of \mathbf{G} and $\mathbf{Z} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$. According to [BT84, 4.1.19], there exist right cosets M_α such that $\mathbf{G}(\mathbb{K})$ admits a generating root group datum $(\mathbf{Z}(\mathbb{K}), (U_\alpha(\mathbb{K}), M_\alpha)_{\alpha \in \Phi})$ of type Φ which is the \mathbb{K} -root system of \mathbf{G} with respect to \mathbf{S} . In particular, for such a root group datum, one can apply any result of section [BT72, 6.1]. Moreover, $N = \mathcal{N}_{\mathbf{G}}(\mathbf{S})(\mathbb{K})$ in this example.

In Bruhat-Tits theory, it appears to be useful to consider some groups Z generated by some well-chosen subgroups X_α of the root groups U_α . A first result is given by Proposition [BT72, 6.1.6], for a group generated over a positively closed subset $\Psi \subset \Phi^+$ of roots, assuming a "condition (i)". A second result is given by Proposition [BT72, 6.4.9], for a group generated over the whole root system, for some specific groups X_α denoted by $U_{\alpha,f}$ in [BT72, §6]. In fact, we observe that the proof of this Proposition only relies on two axioms of "quasi-concavity" (QC1) and (QC2) (see [BT72, 6.4.7]) that are satisfied by a quasi-concave map f , and that we can translate those conditions onto conditions over the groups X_α . Nevertheless, according to addendum [BT84, E2], condition (QC2) is a bit too weak for some general results, so that it is useful to assume that the X_α also satisfies a condition (QC0). In our definition, the condition (QC2) takes into account simultaneously both conditions (QC0) and (QC2) of [BT84]. Moreover, it appears to be useful to extend conditions over the X_α with some subgroup that normalizes the X_α . Thus, we will use the following definition:

3.3 Definition. Let $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ be a generating root group datum of a group G and N be the subgroup of G generated by T and the M_α for $\alpha \in \Phi$. Let $(X_\alpha)_{\alpha \in \Phi}$ be a family of subgroups $X_\alpha \subset U_\alpha$ for $\alpha \in \Phi$ and Y be a subgroup of T .

For $\alpha \in \Phi_{\text{nd}}$, denote by:

- $X_{2\alpha}$ the trivial subgroup if $\alpha \in \Phi$ and $2\alpha \notin \Phi$;
- L_α the subgroup generated by $X_\alpha, X_{2\alpha}, X_{-\alpha}, X_{-2\alpha}$ and Y ;
- $N_\alpha = L_\alpha \cap N$.

We say that the family $((X_\alpha)_{\alpha \in \Phi}, Y)$ is quasi-concave if it satisfies the axioms:

(QC1) $L_\alpha = X_\alpha X_{2\alpha} X_{-\alpha} X_{-2\alpha} N_\alpha = X_{-\alpha} X_{-2\alpha} X_\alpha X_{2\alpha} N_\alpha$ for any $\alpha \in \Phi_{\text{nd}}$;

(QC2) for every $\alpha, \beta \in \Phi$ with $\beta \notin -\mathbb{R}_{>0}\alpha$, the commutator group $[X_\alpha, X_\beta]$ is contained in the group $X_{(\alpha,\beta)}$ generated by the X_γ for $\gamma \in (\alpha, \beta)$;

(QC3) Y normalizes X_α for every $\alpha \in \Phi$.

If Y is trivial, by abuse of language, the family of groups $(X_\alpha)_{\alpha \in \Phi}$ is said quasi-concave.

Note that condition (QC2) implies that X_α normalizes $X_{2\alpha}$ so that one can also write $L_\alpha = X_{2\alpha}X_\alpha X_{-\alpha}X_{-2\alpha}N_\alpha$ in (QC1) for instance.

Because axiom (QC2) does not depend on Y in this definition, we will say by abuse of language that the family $(X_\alpha)_{\alpha \in \Phi}$ satisfies (QC2) when this condition is satisfied.

With this definition, we get the following Proposition analogous to [BT72, 6.4.9].

3.4 Proposition. *Let $((X_\alpha)_{\alpha \in \Phi}, Y)$ be a quasi-concave family of groups. Denote by X the group generated by Y and by the X_α for $\alpha \in \Phi$. Suppose that Φ is non-empty. Then for any choice of a subset of positive roots Φ^+ of Φ :*

- (1) $U_\alpha \cap X = X_\alpha X_{2\alpha}$ for any $\alpha \in \Phi_{\text{nd}}$;
- (2) the product map $\prod_{\alpha \in \Phi_{\text{nd}}^+} (X_\alpha X_{2\alpha}) \rightarrow X \cap U^+$ (resp. $\prod_{\alpha \in \Phi_{\text{nd}}^+} (X_{-\alpha} X_{-2\alpha}) \rightarrow X \cap U^-$) induced by multiplication in G is a bijection for any ordering on the product;
- (3) we have $X = (X \cap U^+)(X \cap U^-)(X \cap N)$ for any choice of Φ^+ in Φ ;
- (4) the group $X \cap N$ is generated by the N_α for $\alpha \in \Phi_{\text{nd}}$.

Note that, by definition of N_α , even if $Y = 1$, it may happen that $X_\alpha = X_{-\alpha} = 1$ but $N_\alpha \neq 1$ for a multipliable root α . Moreover, $N_{2\alpha} \subset N_\alpha$ for any multipliable root α . Thus $X \cap N$ is also generated by the N_α for $\alpha \in \Phi$.

Proof. We observe that, since the family $((X_\alpha X_{2\alpha})_{\alpha \in \Phi}, Y)$ is also a quasi-concave family of groups, we can, and do, assume that $X_\alpha = X_\alpha X_{2\alpha}$ in the proof.

Consider an arbitrary ordering on Φ_{nd}^+ (resp. Φ_{nd}^-). Let $f_+ : \prod_{\alpha \in \Phi_{\text{nd}}^+} U_\alpha \rightarrow G$ (resp. $f_- : \prod_{\alpha \in \Phi_{\text{nd}}^-} U_\alpha \rightarrow G$) the map induced by multiplication. Denote by $X^+ = f_+ \left(\prod_{\alpha \in \Phi_{\text{nd}}^+} X_\alpha \right)$ and by $X^- = f_- \left(\prod_{\alpha \in \Phi_{\text{nd}}^-} X_\alpha \right)$. According to axiom (QC2) and [BT72, 6.1.6], we know that the restriction of the map f_+ (resp. f_-) to $\prod_{\alpha \in \Phi_{\text{nd}}^+} X_\alpha$ (resp. $\prod_{\alpha \in \Phi_{\text{nd}}^-} X_\alpha$) induces a bijection onto X^+ (resp. X^-) and that X^+ (resp. X^-) is a subgroup of G . In fact, $X^+ \subset U^+ \cap X$ and $X^- \subset U^- \cap X$. To prove (2), it suffices to prove that these inclusions are equalities.

Denote L_α and N_α as in Definition 3.3. Denote by Z the group generated by the N_α for $\alpha \in \Phi$. Note that $Y \subset N_\alpha$ for every $\alpha \in \Phi_{\text{nd}}$, and therefore $Y \subset Z$ since Φ_{nd} is non-empty. By definition, $X^- X^+ Z$ is a subset of X as product of subgroups.

We will prove that this subset $X^- X^+ Z$ does not depend on the basis (or Weyl chamber) Δ defining the choice of positive roots Φ^+ in the root system Φ . More precisely, we prove that it is the same set if we replace Δ by $r_\alpha(\Delta)$ for $\alpha \in \Delta$ a simple root and we are done since the r_α for $\alpha \in \Delta$ generate the Weyl group of Φ . Let $\alpha \in \Delta \subset \Phi_{\text{nd}}$ and denote by \widehat{X}_α (resp. $\widehat{X}_{-\alpha}$) the product of the X_β (resp. $X_{-\beta}$) for $\beta \in \Phi_{\text{nd}}^+ \setminus \{\alpha\}$. According to axiom (QC2), one can apply [BT72, 6.1.6] to the family of groups X_α (resp. $X_{-\alpha}$) and X_β for $\beta \in \Phi_{\text{nd}}^+ \setminus \{\alpha\}$ to get that X_α (resp. $X_{-\alpha}$) normalizes the group \widehat{X}_α . By the same way, $\widehat{X}_{-\alpha}$ is normalized by X_α and $X_{-\alpha}$. Moreover, \widehat{X}_α and $\widehat{X}_{-\alpha}$ are normalized by Y since Y normalizes the X_α by (QC3). Therefore, L_α normalizes \widehat{X}_α and $\widehat{X}_{-\alpha}$, and so does N_α .

Moreover $L_\alpha = X_\alpha X_{-\alpha} N_\alpha = X_{-\alpha} X_\alpha N_\alpha$ by (QC1). Hence we have

$$\begin{aligned}
X^- X^+ Z &= \left(\widehat{X}_{-\alpha} X_{-\alpha} \right) \left(X_\alpha \widehat{X}_\alpha \right) (N_\alpha Z) \\
&= \widehat{X}_{-\alpha} X_{-\alpha} X_\alpha N_\alpha \widehat{X}_\alpha Z \\
&= \widehat{X}_{-\alpha} L_\alpha \widehat{X}_\alpha Z \\
&= \widehat{X}_{-\alpha} X_\alpha X_{-\alpha} N_\alpha \widehat{X}_\alpha Z \\
&= \left(\widehat{X}_{-\alpha} X_\alpha \right) \left(X_{-\alpha} \widehat{X}_\alpha \right) N_\alpha Z \\
&= \left(\prod_{\beta \in r_\alpha(\Phi_{\text{nd}}^-)} X_\beta \right) \left(\prod_{\beta \in r_\alpha(\Phi_{\text{nd}}^+)} X_\beta \right) Z
\end{aligned}$$

As a consequence, the set $X^- X^+ Z$ does not depend on the choice of Δ and, therefore, is stable by left multiplication by elements in X_α for any $\alpha \in \Phi_{\text{nd}}$. Moreover, it is stable by left multiplication by elements in Y since Y normalizes X^- and X^+ . Thus, $X^- X^+ Z = X$.

Now, let $g \in X \cap U^-$ and write it as $g = x^- x^+ z$ with $x^- \in X^-$, $x^+ \in X^+$ and $z \in Z$. Then $x^+ z = (x^-)^{-1} g \in U^-$. By Bruhat decomposition [BT72, 6.1.15 (c)], we have $z = 1$ since $N \rightarrow U^+ \backslash G / U^-$ is a bijection. Hence $x^+ \in U^+ \cap U^- = \{1\}$ by axiom (RGD6). Thus we get $X \cap U^- = X^-$. This proves surjectivity of the map $X^- \rightarrow X \cap U^-$ and therefore (2). By the same way, we get the bijectivity of maps onto $X \cap U^+$. We deduce (1) from (2) by intersection with U_α for $\alpha \in \Phi_{\text{nd}}$.

If $n \in X \cap N$, write it as $n = x^- x^+ z$. Then $n = z$ by Bruhat decomposition [BT72, 6.1.15 (c)]. This proves $Z = X \cap N$ which is (4) and, therefore, we deduce (3). \square

Thanks to a valuation of a root group datum, we will apply later this proposition to various examples of quasi-concave families of groups such as given in Example 3.44 or Proposition 3.58.

3.2 R -valuation of a root group datum

In the following, we will assume that R is a totally ordered commutative pseudo-ring.

When $R \neq \mathbb{R}$, there is no reason for R to satisfy the least-upper-bound property. In particular, in this work, we avoid to introduce a notion of infimum and supremum in R . If we did this, we would have to talk in terms of the totally ordered monoid of the convex subsets of R containing ∞ which is not easy to manipulate. Obviously, for $R = \mathbb{R}$, such a monoid has been introduced in [BT72, 6.4.1]. This difference firstly appears in the definition of the subsets of values Γ'_α (see Notation 3.8).

The definition of a valuation of a root group datum, given by [BT72, 6.2.1], can be naturally extended as follows:

3.5 Definition. Let Φ be a root system and $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ be a root group datum. An R -valuation of the root group datum is a family $(\varphi_\alpha)_{\alpha \in \Phi}$ of maps $\varphi_\alpha : U_\alpha \rightarrow R \cup \{\infty\}$ satisfying the following axioms:

- (V0) for any $\alpha \in \Phi$, the set $\varphi_\alpha(U_\alpha)$ contains at least 3 elements;
- (V1) for any $\alpha \in \Phi$ and $\lambda \in R \cup \{\infty\}$, the set $U_{\alpha, \lambda} = \varphi_\alpha^{-1}([\lambda, \infty])$ is a subgroup of U_α and $U_{\alpha, \infty} = \{1\}$;
- (V2) for any $\alpha \in \Phi$ and $m \in M_\alpha$, the map $U_{-\alpha} \setminus \{1\} \rightarrow R$ defined by $u \mapsto \varphi_{-\alpha}(u) - \varphi_\alpha(mum^{-1})$ is constant;

(V3) for any $\alpha, \beta \in \Phi$ such that $\beta \notin \mathbb{R}_{\leq 0}\alpha$ and any $\lambda, \mu \in R$, the commutator group $[U_{\alpha, \lambda}, U_{\beta, \mu}]$ is contained in the group generated by the $U_{r\alpha + s\beta, r\lambda + s\mu}$ for $r, s \in \mathbb{Z}_{>0}$ such that $r\alpha + s\beta \in \Phi$;

(V4) for any multipliable root $\alpha \in \Phi$, the map $\varphi_{2\alpha}$ is the restriction of the map $2\varphi_\alpha$ to $U_{2\alpha}$;

(V5) for any $\alpha \in \Phi$ and $u \in U_\alpha$, for any $u', u'' \in U_{-\alpha}$ such that $u'uu'' \in M_\alpha$, we have $\varphi_{-\alpha}(u') = -\varphi_\alpha(u)$.

It is convenient to introduce notation of the trivial subgroup $U_{2\alpha, \lambda} = \{1\}$ for any $\alpha \in \Phi$ such that $2\alpha \notin \Phi$ and $\lambda \in R \cup \{\infty\}$.

As in Bruhat-Tits theory, in section 6, we will make a use of Chevalley-Steinberg systems in order to provide such a valuation. Namely, R will be the commutative pseudo-ring \mathfrak{R}^S so that $\Lambda = \omega(\mathbb{K})$ will be canonically identified to a subset of R . For instance, if \mathbf{G} is split, then the root groups \mathbf{U}_α are isomorphic to \mathbb{G}_a . Thus, the pinnings of these groups give isomorphisms $x_\alpha : \mathbb{K} \rightarrow \mathbf{U}_\alpha(\mathbb{K})$ and one can define $\varphi_\alpha : \mathbf{U}_\alpha(\mathbb{K}) \rightarrow \Lambda \subset \mathfrak{R}^S$ by $\varphi_\alpha \circ x_\alpha = \omega$.

3.6 Lemma. *Axiom (V1) is equivalent to the following axiom:*

(V1bis) for any $\alpha \in \Phi$ and any $u, v \in U_\alpha$, we have $\varphi_\alpha(uv^{-1}) \geq \min(\varphi_\alpha(u), \varphi_\alpha(v))$ and $\varphi_\alpha^{-1}(\{\infty\}) = \{1\}$.

In particular, for any $\alpha \in \Phi$,

(1) for any $u \in U_\alpha$, we have $\varphi_\alpha(u^{-1}) = \varphi_\alpha(u)$;

(2) for any $u, v \in U_\alpha$ such that $\varphi_\alpha(v) > \varphi_\alpha(u)$, we have $\varphi_\alpha(uv) = \varphi_\alpha(u)$.

Proof. Consider $\alpha \in \Phi$. By definition, $U_{\alpha, \infty} = \{1\}$ is equivalent to $\varphi_\alpha^{-1}(\{\infty\}) = \{1\}$.

Suppose axiom (V1). Consider $u, v \in U_\alpha$ and let $\lambda = \min(\varphi_\alpha(u), \varphi_\alpha(v))$. Hence $U_{\alpha, \lambda}$ is a subgroup containing u, v since $\varphi_\alpha(u) \geq \lambda$ and $\varphi_\alpha(v) \geq \lambda$. Therefore, $uv^{-1} \in U_{\alpha, \lambda}$ gives us $\varphi_\alpha(uv^{-1}) \geq \lambda$.

Conversely, suppose axiom (V1bis). Consider $\lambda \in R$ and pick $u, v \in U_{\alpha, \lambda}$. Then $\varphi_\alpha(uv^{-1}) \geq \min(\varphi_\alpha(u), \varphi_\alpha(v)) \geq \lambda$. Hence $uv^{-1} \in U_{\alpha, \lambda}$ and this proves that $U_{\alpha, \lambda}$ is a subgroup of U_α .

(1) Hence, for any $v \in U_\alpha$, if we take $u = 1$, then $\varphi_\alpha(v^{-1}) = \varphi_\alpha(uv^{-1}) \geq \min(\varphi_\alpha(1), \varphi_\alpha(v)) = \varphi_\alpha(v)$ and this inequality is also true for v^{-1} instead of v .

(2) We have $\varphi_\alpha(u) = \varphi_\alpha((uv)v^{-1}) \geq \min(\varphi_\alpha(uv), \varphi_\alpha(v^{-1})) \geq \min(\varphi_\alpha(u), \varphi_\alpha(v), \varphi_\alpha(v^{-1})) = \varphi_\alpha(u)$ whenever $\varphi_\alpha(v^{-1}) = \varphi_\alpha(v) > \varphi_\alpha(u)$. \square

3.7 Lemma. *Under the assumption of axiom (V1bis), axiom (V5) is equivalent to the following axiom:*

(V5bis) for any $\alpha \in \Phi$ and $u \in U_\alpha$, for any $u', u'' \in U_{-\alpha}$ such that $u'uu'' \in M_\alpha$, we have $\varphi_{-\alpha}(u'') = -\varphi_\alpha(u)$.

Proof. Let $u \in U_\alpha$ and $u', u'' \in U_{-\alpha}$ such that $u'uu'' \in M_\alpha$. Then, by [BT72, 6.1.2(4)], we know that $(u'')^{-1}u^{-1}(u')^{-1} \in M_\alpha$. Hence by axiom (V5), we have $\varphi_{-\alpha}((u'')^{-1}) = \varphi_\alpha(u^{-1})$. By Lemma 3.6, we have $\varphi_{-\alpha}((u'')^{-1}) = \varphi_{-\alpha}(u'')$ and $\varphi_\alpha((u')^{-1}) = \varphi_\alpha(u)$. Hence $\varphi_{-\alpha}(u'') = \varphi_\alpha(u)$ which gives us axiom (V5bis). The converse is the same argument. \square

In all the following, we assume that a root group datum $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ and an R -valuation $(\varphi_\alpha)_{\alpha \in \Phi}$ are given. When $\alpha \in \Phi$ is such that $2\alpha \notin \Phi$, we define $U_{2\alpha} = \{1\}$.

The valuation enables us to introduce the following sets of values:

3.8 Notation. For any root $\alpha \in \Phi$, we define the following subsets of R :

- $\Gamma_\alpha = \varphi_\alpha(U_\alpha \setminus \{1\})$;
- $\Gamma'_\alpha = \left\{ \varphi_\alpha(u), u \in U_\alpha \setminus \{1\} \text{ and } U_{\alpha, \varphi_\alpha(u)} = \bigcap_{v \in U_{2\alpha}} U_{\alpha, \varphi_\alpha(uv)} \right\} \subset \Gamma_\alpha$.

3.9 Fact. From axioms (V1) and (V4), we deduce $2\Gamma_\alpha = 2\Gamma'_\alpha \cup \Gamma_{2\alpha}$.

Proof. By definition $2\Gamma'_\alpha \subset 2\Gamma_\alpha$ and, by axiom (V4), we have $\Gamma_{2\alpha} \subset 2\Gamma_\alpha$.

Conversely, let $\lambda \in 2\Gamma_\alpha \setminus \Gamma_{2\alpha}$. Let $u \in U_\alpha$ such that $2\varphi_\alpha(u) = \lambda$. Then for any $v \in U_{2\alpha}$, we have $\lambda \neq \varphi_{2\alpha}(v)$ by definition. Thus $2\varphi_\alpha(u) = \lambda \neq 2\varphi_\alpha(v)$ by axiom (V4) and therefore $\varphi_\alpha(u) \neq \varphi_\alpha(v)$ since R is \mathbb{Z} -torsion free. If $\varphi_\alpha(v) > \varphi_\alpha(u)$, we have $\varphi_\alpha(uv) = \varphi_\alpha(u)$ by Lemma 3.6(2). Thus $U_{\alpha, \varphi_\alpha(u)} = U_{\alpha, \varphi_\alpha(uv)}$ by definition. If $\varphi_\alpha(v) < \varphi_\alpha(u)$, we have $\varphi_\alpha(uv) = \varphi_\alpha(v) < \varphi_\alpha(u)$ by Lemma 3.6(2). Thus $U_{\alpha, \varphi_\alpha(u)} \subset U_{\alpha, \varphi_\alpha(uv)}$ by definition. Hence we have $U_{\alpha, \varphi_\alpha(u)} \subset \bigcap_{v \in U_{2\alpha}} U_{\alpha, \varphi_\alpha(uv)}$ and this is, in fact, an equality by considering $v = 1 \in U_{2\alpha}$. Thus $\varphi_\alpha(u) \in \Gamma'_\alpha$ by definition. Therefore $\lambda = 2\varphi_\alpha(u) \in 2\Gamma'_\alpha$. \square

3.10 Remarks.

1. From axiom (V5) and [BT72, 6.1.2(2)], we deduce $\Gamma_{-\alpha} = -\Gamma_\alpha$.
2. By definition, if $2\alpha \notin \Phi$, we have $\Gamma'_\alpha = \Gamma_\alpha$.
3. As in the Bruhat-Tits theory, when α is multipliable, it may happen that Γ'_α is empty (dense valuation); it may happen that the intersection $2\Gamma'_\alpha \cap \Gamma_{2\alpha}$ is non-empty (discrete valuation with unramified splitting extension $\tilde{\mathbb{K}}/\mathbb{K}$).

3.11 Notation. For $\alpha \in \Phi$ and $\lambda \in R$, we denote:

$$M_{\alpha, \lambda} = M_\alpha \cap U_{-\alpha} \varphi_\alpha^{-1}(\{\lambda\}) U_{-\alpha}.$$

We provide some details of [BT72, 6.2.2]:

3.12 Proposition. Let $\alpha \in \Phi$ and $\lambda \in R$.

- (1) $M_{\alpha, \lambda}$ is non-empty if, and only if, $\lambda \in \varphi_\alpha(U_\alpha \setminus \{1\}) = \Gamma_\alpha$;
- (2) $\varphi_\alpha^{-1}(\{\lambda\}) \subset U_{-\alpha, -\lambda} M_{\alpha, \lambda} U_{-\alpha, -\lambda}$;
- (3) $M_{\alpha, \lambda} \subset \varphi_{-\alpha}^{-1}(\{-\lambda\}) \varphi_\alpha^{-1}(\{\lambda\}) \varphi_{-\alpha}^{-1}(\{-\lambda\}) \subset U_{-\alpha, -\lambda} U_{\alpha, \lambda} U_{-\alpha, -\lambda}$;
- (4) $M_{-\alpha, -\lambda} = M_{\alpha, \lambda}$;
- (5) $M_{2\alpha, 2\lambda} \subset M_{\alpha, \lambda}$.

Proof. (1) is a consequence of [BT72, 6.1.2(2)], since $\lambda \neq \infty$.

(2) For any $u \in \varphi_\alpha^{-1}(\{\lambda\})$, axioms (RGD1) and (RGD4) provide elements $u', u'' \in U_{-\alpha}$ such that $m := u'uu'' \in M_{\alpha, \lambda}$. By axioms (V1bis), (V5) and (V5bis), we have $\varphi_{-\alpha}((u')^{-1}) = -\varphi_\alpha(u) = \varphi_{-\alpha}((u'')^{-1}) = -\lambda$.

(3) For any $m \in M_{\alpha, \lambda}$, by definition, there exist $u', u'' \in U_{-\alpha}$ and $u \in U_\alpha$ such that $m = u'uu''$ and $\varphi_\alpha(u) = \lambda$. By axioms (V5) and (V5bis), we have $\varphi_{-\alpha}(u') = -\varphi_\alpha(u) = \varphi_{-\alpha}(u'') = -\lambda$.

(4) For any $\alpha \in \Phi$ and $\lambda \in R$, consider $m \in M_{-\alpha, -\lambda}$. By (3) and axiom (V1bis), there exist $u', u'' \in U_\alpha$ and $u \in U_{-\alpha}$ such that $\varphi_{-\alpha}(u) = -\lambda = -\varphi_\alpha(u') = -\varphi_\alpha(u'')$ and $m = u'uu''$. Consider $v = m(u'')^{-1}m^{-1}$ so that $u' = vmu^{-1}$. By [BT72, 6.2.1(2)] $v \in U_{-\alpha}$ and by [BT72, 6.2.1(4)], $m \in M_\alpha$. Hence $m = v^{-1}u'u^{-1} \in M_\alpha \cap U_{-\alpha} \varphi_\alpha^{-1}(\{\lambda\}) U_{-\alpha} = M_{\alpha, \lambda}$. Hence $M_{-\alpha, -\lambda} \subset M_{\alpha, \lambda}$ and we get the converse inclusion by exchanging (α, λ) with $(-\alpha, -\lambda)$.

(5) If α is multipliable, then $U_{2\alpha} \subset U_\alpha$ and $U_{-2\alpha} \subset U_{-\alpha}$ by axiom (RGD3) and $M_{2\alpha} = M_\alpha$ by [BT72, 6.1.2(4)]. For $u \in \varphi_{2\alpha}^{-1}(\{2\lambda\})$, by axiom (V4), we have $2\lambda = \varphi_{2\alpha}(u) = 2\varphi_\alpha(u)$. Since R is \mathbb{Z} -torsion free, it gives $u \in \varphi_\alpha^{-1}(\{\lambda\})$. \square

Here, we follow a different strategy than in [BT84, 6.2] and we do not work with the notion of “valuations équipollentes”.

We introduce the following useful Lemma from [Lan96, 7.5] with a different proof since we do not define integral models here:

3.13 Lemma. *Let $\alpha \in \Phi_{\text{nd}}$, $\lambda \in \Gamma_\alpha$, $\beta \in \Phi$ and $\mu \in R$. For any $m \in M_{\alpha,\lambda}$, we have*

$$mU_{\beta,\mu}m^{-1} = U_{r_\alpha(\beta),\mu-\beta(\alpha^\vee)\lambda}.$$

In particular, we have $mU_{\alpha,\lambda}m^{-1} = U_{-\alpha,-\lambda}$.

Proof. We distinguish three cases on β .

First case: $\beta \in \Phi_{\text{nd}} \setminus \mathbb{R}\alpha$. For $\gamma = r\alpha + s\beta \in \Phi(\alpha, \beta)$ with $r, s \in \mathbb{Z}$, define a map $t : \Phi(\alpha, \beta) \rightarrow R$ by $t(\gamma) = r\lambda + s\mu$ if $s > 0$ and $t(\gamma) = \infty$ if $s \leq 0$. Denote by $X_\gamma = U_{\gamma,t(\gamma)}$. Then the family of groups $(X_\gamma)_{\gamma \in \Phi(\alpha, \beta)}$ is quasi-concave. Indeed, for every $\gamma \in \Phi(\alpha, \beta)$ we have either $X_{2\gamma} = X_\gamma = 1$ or $X_{-2\gamma} = X_{-\gamma} = 1$ so that axiom (QC1) is satisfied. Let $r_1, s_1, r_2, s_2 \in \mathbb{Z}$ such that $\gamma_1 = r_1\alpha + s_1\beta$ and $\gamma_2 = r_2\alpha + s_2\beta$ are in $\Phi(\alpha, \beta)$ with $\gamma_2 \notin \mathbb{R}_{<0}\gamma_1$. Then for any $r, s \in \mathbb{Z}$, if $s_1 \leq 0$ or $s_2 \leq 0$ we have $[X_{\gamma_1}, X_{\gamma_2}] = 1$ and if $s_1, s_2 > 0$, we have $rt(\gamma_1) + st(\gamma_2) = t(r\gamma_1 + s\gamma_2)$ so that axiom (QC2) is satisfied according to axiom (V3). Let Y be the group generated by the X_γ for $\gamma \in \Phi(\alpha, \beta)$. For any $\gamma \in \Phi(\alpha, \beta)$, we have $X_\gamma = 1$ if $\gamma \in \mathbb{R}\alpha$ and $[U_{\alpha,\lambda}, X_\gamma] \subset Y$ by axiom (V3) otherwise. Thus $U_{\alpha,\lambda}$ normalizes Y . By the same way, $U_{-\alpha,-\lambda}$ normalizes Y so that $M_{\alpha,\lambda}$ normalizes Y . Moreover, for every $m \in M_{\alpha,\lambda}$, we have $mU_\alpha m^{-1} = U_{r_\alpha(\beta)}$ by [BT72, 6.1.2(10)]. Thus $mU_{\beta,\mu}m^{-1} \subset U_{r_\alpha(\beta)} \cap Y = U_{r_\alpha(\beta),t(r_\alpha(\beta))}U_{2r_\alpha(\beta),2t(r_\alpha(\beta))}$ by Proposition 3.4(1). Finally, $U_{2r_\alpha(\beta),2t(r_\alpha(\beta))} \subset U_{r_\alpha(\beta),t(r_\alpha(\beta))}$ by definition of t since $t(2r_\alpha(\beta)) = 2t(r_\alpha(\beta))$.

Second case: $\beta \in \mathbb{R}\alpha \cap \Phi_{\text{nd}} = \{\pm\alpha\}$. Since $M_{\alpha,\lambda} = M_{-\alpha,-\lambda}$, $r_\alpha = r_{-\alpha}$ and $(-\alpha)^\vee = -\alpha^\vee$ it suffices to do it for $\beta = -\alpha$. Let $m \in M_{\alpha,\lambda}$ and write it as $m = u'uu''$ with $u', u'' \in \varphi_{-\alpha}^{-1}(\{-\lambda\})$ and $u \in \varphi_\alpha^{-1}(\{\lambda\})$. Then $u' = m(u'')^{-1}m^{-1}mu^{-1}$ with $m(u'')^{-1}m^{-1} \in U_\alpha$. Thus, axiom (V5) applied to $(mu''m^{-1})u'u = m \in M_{-\alpha}$ gives us $\varphi_\alpha(mu''m^{-1}) = \varphi_{-\alpha}(u') = -\lambda$. Let $v \in U_{-\alpha,\mu} \setminus \{1\}$. Then axiom (V2) gives us $\varphi_{-\alpha}(v) - \varphi_\alpha(mvm^{-1}) = \varphi_{-\alpha}(u'') - \varphi_\alpha(mu''m^{-1}) = -2\lambda$. Hence $\varphi_\alpha(mvm^{-1}) = \mu + 2\lambda = \mu - \beta(\alpha^\vee)\lambda$. Hence $mU_{\beta,\mu}m^{-1} \subset U_{r_\alpha(\beta),\mu-\beta(\alpha^\vee)\lambda}$.

Third case: $\beta \in \Phi \setminus \Phi_{\text{nd}}$: For $m \in M_{\alpha,\lambda}$ and β a multipliable root, we have $mU_{2\beta,\mu}m^{-1} \subset mU_{\beta,\frac{1}{2}\mu}m^{-1} \cap U_{r_\alpha(2\beta)} = U_{2r_\alpha(\beta),\mu-2\beta(\alpha^\vee)\lambda} = U_{r_\alpha(2\beta),\mu-(2\beta)(\alpha^\vee)\lambda}$. Since the inclusion $mU_{\beta,\mu}m^{-1} \subset U_{r_\alpha(\beta),\mu-\beta(\alpha^\vee)\lambda}$ holds for every $\alpha \in \Phi_{\text{nd}}$, $\beta \in \Phi$, $\lambda \in \Gamma_\alpha$, $\mu \in R$ and $m \in M_{\alpha,\lambda}$, it is in fact an equality. \square

3.3 Action of N on an R -affine space

We use the notations introduced in section 2.1.3. In the rest of this section, we assume that $\Phi \neq \emptyset$ but any result can obviously be extended to the case of an empty root system.

For $\alpha \in \Phi$ and $u \in U_\alpha \setminus \{1\}$, we denote by $m(u)$ the unique element in M_α given by [BT72, 6.1.2(2)].

3.14 Definition. Let $\nu : N \rightarrow \text{Aff}_{\mathbb{Z}}(\mathbb{A}_R)$ be an action of N onto \mathbb{A}_R by \mathbb{Z} -affine transformations. We say that the action of N onto \mathbb{A}_R is compatible with the valuation $(\varphi_\alpha)_{\alpha \in \Phi}$ if:

(CA1) the linear part of this action is equal to $\nu : N \rightarrow W(\Phi)$ defined in [BT72, 6.1.2(10)];

(CA2) for any $\alpha \in \Phi$ and any $u \in U_\alpha \setminus \{1\}$, we have $2\varphi_\alpha(u) + \alpha(\nu(m(u))(o) - o) = 0$.

In the rest of this section, we assume that an action $\nu : N \rightarrow \text{Aff}_{\mathbb{Z}}(\mathbb{A}_R)$ satisfying (CA1) and (CA2) is given.

3.15 Lemma. *For any $\alpha \in \Phi$ and any $u \in U_{\alpha} \setminus \{1\}$, we have $\nu(m(u)) = r_{\alpha, \varphi_{\alpha}(u)}$.*

In particular, for any $m \in M_{\alpha, \lambda}$, we have $\nu(m) = r_{\alpha, \lambda}$.

Proof. Since $m(u) \in M_{\alpha}$, we have ${}^u\nu(m(u)) = r_{\alpha}$. By (CA1), there is a value $\lambda \in R$ such that $\nu(m(u)) = r_{\alpha, \lambda}$. Since $r_{\alpha, \lambda}(o) = o - \theta_{\alpha, \lambda}(o)\alpha^{\vee}$, we have $\alpha(\nu(m(u))(o) - o) = -2\lambda = -2\varphi_{\alpha}(u)$ by axiom (CA2). Thus $\lambda = \varphi_{\alpha}(u)$ since R is \mathbb{Z} -torsion free.

Let $m \in M_{\alpha, \lambda}$. By definition (see Notation 3.11), there exist $u \in \varphi_{\alpha}^{-1}(\{\lambda\}) \subset U_{\alpha, \lambda}$, $u', u'' \in U_{-\alpha}$ such that $m = u'uu''$. Thus $m = m(u)$ and $\nu(m) = \nu(m(u)) = r_{\alpha, \lambda}$. \square

We want to understand how N acts by conjugation on the set of groups $U_{\alpha, \lambda}$ for $\alpha \in \Phi$ and $\lambda \in R$. To do this, we introduce a set of affine maps Θ containing the $\theta_{\alpha, \lambda}$ and an action of N onto this set.

3.16 Notation. Consider $\Theta = \{\theta_{\alpha, \lambda}, \alpha \in \Phi, \lambda \in R\}$. We endow Θ with the natural partial ordering given by:

$$\theta \geq \theta' \iff \forall x \in \mathbb{A}_R, \theta(x) \geq \theta'(x).$$

3.17 Lemma. *The group N acts onto Θ via*

$$\forall \alpha \in \Phi, \forall \lambda \in R, n \cdot \theta_{\alpha, \lambda} = \theta_{\alpha, \lambda} \circ \nu(n^{-1}) = \theta({}^n\nu(\alpha), \lambda + \alpha(\nu(n^{-1})(o) - o)).$$

In particular, for any $\alpha, \beta \in \Phi$ and any $\lambda, \mu \in R$, we have

$$\theta_{\alpha, \lambda} \circ r_{\beta, \mu} = \theta(r_{\beta}(\alpha), \lambda - \alpha(\beta^{\vee})\mu).$$

Proof. Let $\theta_{\alpha, \lambda} \in \Theta$. It suffices to prove that the map $\theta_{\alpha, \lambda} \circ \nu(n^{-1})$ belongs to Θ . Since N is generated by the M_{β} for $\beta \in \Phi$, it suffices to prove it for any $\beta \in \Phi$ and any $m \in M_{\beta}$. Since ${}^u\nu(m) = r_{\beta}$ by [BT72, 6.1.2(10)], by (CA1), there is a constant μ depending on m such that $\nu(m) = r_{\beta, \mu}$. Then $\theta_{\alpha, \lambda}(r_{\beta, \mu}(x)) = (\alpha - \alpha(\beta^{\vee})\beta)(x) + \lambda - \alpha(\beta^{\vee})\mu = r_{\beta}(\alpha)(x) + \lambda - \alpha(\beta^{\vee})\mu$. Thus $\theta \circ \nu(m) \in \Theta$. Hence the formula $n \cdot \theta = \theta \circ \nu(n^{-1})$ defines an action of N on Θ .

Let $n \in N$. Since the map ${}^u\nu : N \rightarrow \text{GL}(V^*)$ induces an action of N on Φ , for any $\alpha \in \Phi$ and any $\lambda \in R$, there is a value $\mu \in R$ such that $\theta_{\alpha, \lambda} \circ \nu(n^{-1}) = \theta({}^n\nu(n^{-1})(\alpha), \mu)$. Evaluating this map in the origin o , we get $\mu = \theta({}^n\nu(n^{-1})(\alpha), \mu)(o) = \theta_{\alpha, \lambda} \circ \nu(n^{-1})(o) = \alpha(\nu(n^{-1})(o) - o) + \lambda$. \square

3.18 Definition. For $\theta = \theta_{\alpha, \lambda} \in \Theta$, define $U_{\theta} = \{u \in U_{\alpha}, \theta \circ \nu(m(u)) \leq -\theta\}$.

We recall the following result from [Lan96, 7.3]:

3.19 Lemma. *For any $\theta \in \Theta$, any $n \in N$, we have $nU_{\theta}n^{-1} = U_{n \cdot \theta}$.*

Proof. Let $u \in U_{\theta} \subset U_{\alpha}$ for some $\alpha \in \Phi$. Consider $n \in N$ and denote $\beta = {}^n\nu(n)(\alpha) \in \Phi$.

Suppose that $u \neq 1$ and consider the element $m(u) = u'uu'' \in U_{-\alpha}U_{\alpha}U_{-\alpha} \cap M_{\alpha}$, with $u', u'' \in U_{-\alpha}$ uniquely determined by u . For any $n \in N$, we have $nm(u)n^{-1} = (nu'n^{-1})(nun^{-1})(nu''n^{-1})$. By [BT72, 6.1.2(10)], $(nu'n^{-1}), (nu''n^{-1}) \in U_{-\beta}$ and $nun^{-1} \in U_{\beta}$. Thus, by uniqueness in [BT72, 6.1.2(2)], we have $m(nun^{-1}) = nm(u)n^{-1} \in M_{\beta}$.

For any $x \in \mathbb{A}_R$, we have

$$\begin{aligned} \theta \circ \nu(n^{-1}) (\nu(m(nun^{-1}))(x)) &= \theta \circ \nu(n^{-1}) \circ (\nu(n) \circ \nu(m(u)) \circ \nu(n^{-1})) (x) \\ &= \theta \circ \nu(m(u)) (\nu(n^{-1})(x)) \\ &\leq -\theta (\nu(n^{-1})(x)) \end{aligned}$$

Thus $\theta \circ \nu(n^{-1}) \circ \nu(m(nun^{-1})) \leq -\theta \circ \nu(n^{-1})$ which means that $nun^{-1} \in U_{\theta \circ \nu(n^{-1})} = U_{n \cdot \theta}$.

Hence $nU_{\theta}n^{-1} \subset U_{n \cdot \theta}$ for any $n \in N$ and any $\theta \in \Theta$. Thus, by applying it to $(n^{-1}, n \cdot \theta)$ we get $n^{-1}U_{n \cdot \theta}n \subset U_{\theta}$ which gives the equality $nU_{\theta}n^{-1} = U_{n \cdot \theta}$. \square

We get the following consequence as in [Lan96, 7.7]:

3.20 Proposition. *For any $\alpha \in \Phi$ and any $\lambda \in R$, we have $U_{\alpha,\lambda} = U_{\theta_{\alpha,\lambda}}$.*

Proof. Let $u \in U_{\alpha}$ and $\mu = \varphi_{\alpha}(u)$. By Lemma 3.15, we have $\nu(m(u)) = r_{\alpha,\mu}$. For any $x \in \mathbb{A}_R$, we have

$$\begin{aligned}\theta_{\alpha,\lambda}(\nu(m(u))(x)) &= \theta_{\alpha,\lambda}(x - (\alpha(x) + \mu)\alpha^{\vee}) \\ &= -\alpha(x) - 2\mu + \lambda \\ &= -\theta_{\alpha,\lambda}(x) + 2(\lambda - \mu)\end{aligned}$$

Thus $u \in U_{\theta_{\alpha,\lambda}} \iff \theta_{\alpha,\lambda} \circ \nu(m(u)) \leq -\theta_{\alpha,\lambda} \iff \mu \geq \lambda \iff u \in U_{\alpha,\lambda}$. \square

3.21 Corollary. *For any $\alpha \in \Phi$, any $\lambda \in R$ and any $n \in N$, we have*

$$nU_{\alpha,\lambda}n^{-1} = U_{\nu(n)(\alpha),\lambda+\alpha(\nu(n^{-1})(o)-o)}.$$

In particular, $T_b = \ker \nu$ normalizes $U_{\alpha,\lambda}$.

Proof. This is a consequence of Proposition 3.20 and Lemma 3.17. \square

In section 6, we will see that, for $R = \mathfrak{R}^S$ containing the Γ_{α} for $\alpha \in \Phi$, the datum of a root group datum together with an \mathfrak{R}^S -valuation is sufficient to define a structure of \mathfrak{R}^S -building. Because it may be simpler to provide an explicit affine action of N onto $\mathbb{A}_{\mathfrak{R}^S}$, we will make the following assumption:

3.22 Hypothesis. It is given a group G and a non-empty root system Φ . It is given a generated root group datum $(T, (U_{\alpha}, M_{\alpha})_{\alpha \in \Phi})$ of G . It is given an R -valuation $(\varphi_{\alpha})_{\alpha \in \Phi}$ of the root group datum. Let N be the group generated by the M_{α} for $\alpha \in \Phi$. It is given an action $\nu : N \rightarrow \text{Aff}_{\mathbb{Z}}(\mathbb{A}_R)$ compatible with the valuation.

3.4 Rank-one Levi subgroups

In this section, we work under data and notations of assumption 3.22.

3.23 Notation. For any $\alpha \in \Phi$, any $\lambda \in R$ and any $\varepsilon \in R_{\geq 0}$, we define the subgroups

- $U'_{\alpha,\lambda} = \varphi_{\alpha}^{-1}([\lambda, +\infty]) = \bigcup_{\mu > \lambda} U_{\alpha,\mu}$;
- $L_{\alpha,\lambda}^{\varepsilon}$ (resp. $L'_{\alpha,\lambda}$) the subgroup generated by $U_{\alpha,\lambda}$ and $U_{-\alpha,-\lambda+\varepsilon}$ (resp. $U_{\alpha,\lambda}$ and $U'_{-\alpha,-\lambda}$);
- $N_{\alpha,\lambda}^{\varepsilon} = N \cap L_{\alpha,\lambda}^{\varepsilon}$ and $N'_{\alpha,\lambda} = N \cap L'_{\alpha,\lambda}$;
- $T_{\alpha,\lambda}^{\varepsilon} = T \cap L_{\alpha,\lambda}^{\varepsilon}$ and $T'_{\alpha,\lambda} = T \cap L'_{\alpha,\lambda}$.

If $\varepsilon = 0$, we will denote $L_{\alpha,\lambda}, N_{\alpha,\lambda}, T_{\alpha,\lambda}$ instead of $L_{\alpha,\lambda}^0, N_{\alpha,\lambda}^0, T_{\alpha,\lambda}^0$.

3.24 Remark. Note that, $L_{\alpha,\lambda} = L_{-\alpha,-\lambda}$ by definition, but $L'_{-\alpha,-\lambda}$ may differ from $L_{\alpha,\lambda}$ since the groups $U_{\alpha,\lambda}$ and $U'_{\alpha,\lambda}$ are distinct for $\lambda \in \Gamma_{\alpha}$.

We have $U_{\alpha,\lambda} \supseteq U'_{\alpha,\lambda}$ with equality if, and only if, $\lambda \notin \Gamma_{\alpha}$.

Indeed, if $\exists u \in U_{\alpha,\lambda} \setminus U'_{\alpha,\lambda}$, then $\forall \mu > \lambda$, we have $\lambda \leq \varphi_{\alpha}(u) < \mu$ since $u \notin U_{\alpha,\mu} \subset U'_{\alpha,\lambda}$. Thus $\varphi_{\alpha}(u) = \lambda$ and so $\lambda \in \Gamma_{\alpha}$. Conversely, if $\lambda \in \Gamma_{\alpha}$, there exists $u \in U_{\alpha}$ such that $\varphi_{\alpha}(u) = \lambda$. Hence $u \notin U_{\alpha,\mu}$ for any $\mu > \lambda$ and therefore $u \notin \bigcup_{\mu > \lambda} U_{\alpha,\mu} = U'_{\alpha,\lambda}$.

3.25 Remark. Since $L_{\alpha,\lambda}^{\varepsilon} = \langle U_{\alpha,\lambda}, U_{-\alpha,-\lambda+\varepsilon} \rangle$, taking $\beta = -\alpha$ and $\mu = -\lambda + \varepsilon$, we have $L_{\alpha,\lambda}^{\varepsilon} = \langle U_{\beta,\mu}, U_{-\beta,-\mu+\varepsilon} \rangle$. Hence $L_{\alpha,\lambda}^{\varepsilon} = L_{-\alpha,-\lambda+\varepsilon}^{\varepsilon}$. By intersection with T , we deduce that $T_{\alpha,\lambda}^{\varepsilon} = T_{-\alpha,-\lambda+\varepsilon}^{\varepsilon}$.

For $\varepsilon' \geq 0$, we have $U_{\alpha,\lambda+\varepsilon'} \subset U_{\alpha,\lambda}$ and $U_{-\alpha,-\lambda+\varepsilon} = U_{-\alpha,-(\lambda+\varepsilon')+(\varepsilon+\varepsilon')}$. Hence we have $L_{\alpha,\lambda+\varepsilon'}^{\varepsilon+\varepsilon'} \subset L_{\alpha,\lambda}^{\varepsilon}$ and thus $T_{\alpha,\lambda+\varepsilon'}^{\varepsilon+\varepsilon'} \subset T_{\alpha,\lambda}^{\varepsilon}$. By the same way $L_{\alpha,\lambda}^{\varepsilon+\varepsilon'} \subset L_{\alpha,\lambda}^{\varepsilon}$ and $T_{\alpha,\lambda}^{\varepsilon+\varepsilon'} \subset T_{\alpha,\lambda}^{\varepsilon}$.

We state the following Lemma, which can be proven exactly as in [BT72, 6.3.1].

3.26 Lemma. *Let $\alpha \in \Phi$.*

(1) *For any $u \in U_\alpha$ and $v \in U_{-\alpha}$ such that $\varphi_\alpha(u) + \varphi_{-\alpha}(v) > 0$, there is a unique triple $(u', t, v') \in U_\alpha \times T \times U_{-\alpha}$ such that $vu = u'tv'$.*

(2) *Moreover, we have $t \in T_b$, $\varphi_\alpha(u') = \varphi_\alpha(u)$ and $\varphi_\alpha(v') = \varphi_\alpha(v)$.*

Proof. (1) By [BT72, 6.1.2 (4) & (7)], we know that $L_{-\alpha} = M_\alpha U_{-\alpha} \cup U_\alpha T U_{-\alpha}$. If $vu \in M_\alpha U_{-\alpha}$, then $\exists u'' \in U_{-\alpha}$ such that $vu u'' \in M_\alpha$. By axiom (V5), we have $\varphi_{-\alpha}(v) = -\varphi_\alpha(u)$ which contradicts the assumption. Hence $vu \in U_\alpha T U_{-\alpha}$ and we get the existence. The uniqueness is an immediate consequence of [BT72, 6.1.2(3)] and axiom (RGD6).

(2) By uniqueness, it is obvious if $u = 1$ or $v = 1$. Assume that $u \neq 1$ and $v \neq 1$. Then, applying axiom (RGD6), we have $u' \neq 1$. By axiom (RGD4), there exist $w, w' \in U_{-\alpha}$ and $m \in M_{-\alpha} = M_\alpha$ such that $u' = w'mw''$. Moreover $u = v^{-1}u'tv' = (v^{-1}w')(mt)(t^{-1}w''tv')$ where $(v^{-1}w') \in U_{-\alpha}$, $(mt) \in M_\alpha$ and $(t^{-1}w''tv') \in U_{-\alpha}$. Hence by axioms (V5), (V5bis) and Lemma 3.6(1) we have

$$\varphi_{-\alpha}(w') = -\varphi_\alpha(u') = \varphi_{-\alpha}(w'') \quad (1)$$

and

$$\varphi_{-\alpha}(v^{-1}w') = -\varphi_\alpha(u) = \varphi_{-\alpha}(t^{-1}w''tv'). \quad (2)$$

The assumption on u and v and equation 2 give us

$$\varphi_{-\alpha}(v) > -\varphi_\alpha(u) = \varphi_{-\alpha}(v^{-1}w'). \quad (3)$$

Hence, equations 1, 2, inequation 3 and Lemma 3.6(2) give

$$-\varphi_\alpha(u') = \varphi_{-\alpha}(w') = \varphi_{-\alpha}(vv^{-1}w') = \varphi_{-\alpha}(v^{-1}w') = -\varphi_\alpha(u). \quad (4)$$

Since $(vu)^{-1} = u^{-1}v^{-1} = (v')^{-1}t^{-1}(u')^{-1}$, one can replace α by $-\alpha$, u by v^{-1} and u' by $(v')^{-1}$ in order to have $\varphi_{-\alpha}(v^{-1}) = \varphi_{-\alpha}((v')^{-1})$. Applying Lemma 3.6(1), we get $\varphi_{-\alpha}(v) = \varphi_{-\alpha}(v')$.

Denote $\lambda = \varphi_\alpha(u) = \varphi_\alpha(u')$. We know that $m = (w')^{-1}u'(w'')^{-1} \in M_\alpha \cap U_{-\alpha} \varphi_\alpha^{-1}(\{\lambda\}) U_{-\alpha} = M_{\alpha, \lambda}$. In the same way, we have $tm \in M_{\alpha, \lambda}$. Hence $\nu(t) = \nu(tm)\nu(m)^{-1} = r_{\alpha, \lambda} r_{\alpha, \lambda}^{-1} = 1$. \square

The following Proposition is similar to [BT72, 6.3.2, 6.3.3] and [Lan96, 8.1–8.6], including considerations on the groups $L_{\alpha, \lambda}^\varepsilon$ we have introduced.

3.27 Proposition. *Consider any $\alpha \in \Phi$, any $\lambda \in R$ and any $\varepsilon \in R_{>0}$. Under the notations 3.23, we have the following equalities.*

(1) *We have $L'_{\alpha, \lambda} = U_{\alpha, \lambda} U'_{-\alpha, -\lambda} T'_{\alpha, \lambda}$ and $L_{\alpha, \lambda}^\varepsilon = U_{\alpha, \lambda} U_{-\alpha, -\lambda + \varepsilon} T_{\alpha, \lambda}^\varepsilon$ for any ordering on factors.*

(2) *Moreover $N'_{\alpha, \lambda} = T'_{\alpha, \lambda}$ and $N_{\alpha, \lambda}^\varepsilon = T_{\alpha, \lambda}^\varepsilon$.*

(3) *If $\lambda \in \Gamma_\alpha$, then for any $m \in M_{\alpha, \lambda}$, we have $L_{\alpha, \lambda} = (U_{\alpha, \lambda} T_{\alpha, \lambda} U'_{-\alpha, -\lambda}) \sqcup (U_{\alpha, \lambda} m T_{\alpha, \lambda} U_{\alpha, \lambda})$ and $N_{\alpha, \lambda} = T_{\alpha, \lambda} \{1, m\}$.*

(4) *If $\lambda \notin \Gamma_\alpha$, we have $L_{\alpha, \lambda} = U_{\alpha, \lambda} T_{\alpha, \lambda} U_{-\alpha, -\lambda}$ and $N_{\alpha, \lambda} = T_{\alpha, \lambda} = T'_{\alpha, \lambda}$.*

(5) *The groups $T_{\alpha, \lambda}$, $T'_{\alpha, \lambda}$ and $T_{\alpha, \lambda}^\varepsilon$ are subgroups of T_b .*

Proof. (1) Consider $\varepsilon \in R_{>0}$ and $H_\varepsilon = L_{\alpha,\lambda}^\varepsilon \cap T_b$. Then H_ε normalizes $U_{\alpha,\lambda}$ and $U_{-\alpha,-\lambda+\varepsilon}$ since so does T_b . Consider $X_\varepsilon = U_{\alpha,\lambda}U_{-\alpha,-\lambda+\varepsilon}H_\varepsilon$. Then $X_\varepsilon \subset L_{\alpha,\lambda}^\varepsilon$ by definition and is a subset stable by multiplication on the left by elements in H_ε and $U_{\alpha,\lambda}$. Moreover, $U_{-\alpha,-\lambda+\varepsilon}U_{\alpha,\lambda} \subset U_{\alpha,\lambda}(T_b \cap L_{\alpha,\lambda}^\varepsilon)U_{-\alpha,-\lambda+\varepsilon}$ by Lemma 3.26. Hence X_ε is stable by multiplication on the left by elements in $U_{-\alpha,-\lambda+\varepsilon}$ since H_ε normalizes the subgroup $U_{-\alpha,-\lambda+\varepsilon}$. Hence $X_\varepsilon = L_{\alpha,\lambda}^\varepsilon$. Moreover, by uniqueness of $t \in T$ in Lemma 3.26, we get $T_{\alpha,\lambda}^\varepsilon = H_\varepsilon$ and therefore $T_{\alpha,\lambda}^\varepsilon \subset T_b$.

Since $U'_{-\alpha,-\lambda} = \bigcup_{\varepsilon>0} U_{-\alpha,-\lambda+\varepsilon}$ is an increasing union, we have $L'_{\alpha,\lambda} = \bigcup_{\varepsilon>0} L_{\alpha,\lambda}^\varepsilon$ as increasing union so that the equality $L'_{\alpha,\lambda} = \bigcup_{\varepsilon>0} U_{\alpha,\lambda}T_{\alpha,\lambda}^\varepsilon U_{-\alpha,-\lambda+\varepsilon} = U_{\alpha,\lambda}T'_{\alpha,\lambda}U'_{-\alpha,-\lambda}$ holds.

Finally, since $U_{\alpha,\lambda}$ and $U_{-\alpha,-\lambda+\varepsilon}$ (resp. $U'_{-\alpha,-\lambda}$) are subgroups normalized by $T_{\alpha,\lambda}^\varepsilon$ (resp. $T'_{\alpha,\lambda}$), we get the equality for any ordering by applying the inverse map.

(2) If $n \in N_{\alpha,\lambda}^\varepsilon$, then $n \in U^+TU^-$ and by spherical Bruhat decomposition [BT72, 6.1.15 (c)], we get $n \in T$. Thus $N_{\alpha,\lambda}^\varepsilon \subset T_{\alpha,\lambda}^\varepsilon$. The same holds in $L'_{\alpha,\lambda} \subset U^+TU^-$.

(3) We know that $\emptyset \neq M_{\alpha,\lambda} \subset U_{\alpha,\lambda}U_{-\alpha,-\lambda}U_{\alpha,\lambda} \subset L_{\alpha,\lambda}$ by Proposition 3.12(3) and definitions. Consider $H = L_{\alpha,\lambda} \cap T_b \subset T_{\alpha,\lambda}$ and any $m \in M_{\alpha,\lambda}$. Define $X = (U_{\alpha,\lambda}HU'_{-\alpha,-\lambda}) \cup (U_{\alpha,\lambda}mHU_{\alpha,\lambda}) \subset L_{\alpha,\lambda}$. By Lemma 3.15, we know that $\nu(m) = r_{\alpha,\lambda}$. Hence, we have $\nu(m^2) = r_{\alpha,\lambda}^2 = \text{id}$. Thus $m^2 \in T_b \cap L_{\alpha,\lambda} = H$ and $m^{-1} \in Hm = mH$. Since $U_{-\alpha,-\lambda} = m^{-1}U_{\alpha,\lambda}m$ by Lemma 3.13, we deduce that $L_{\alpha,\lambda}$ is generated by $U_{\alpha,\lambda}$ and m . Thus, it suffices to prove that X is stable by right multiplication by m and elements in the groups H and $U_{\alpha,\lambda}$.

Since $H \subset T_b$ normalizes $U'_{-\alpha,-\lambda}$ and $U_{\alpha,\lambda}$ and $T'_{\alpha,\lambda} = L'_{\alpha,\lambda} \cap T_b \subset H$, we deduce from (1) that $L'_{\alpha,\lambda}H = U_{\alpha,\lambda}HU'_{-\alpha,-\lambda}$ is a group. Hence $XH = XU_{\alpha,\lambda} = X$.

On the one hand, we have $U_{\alpha,\lambda}HU'_{-\alpha,-\lambda}m \subset U_{\alpha,\lambda}HmU_{\alpha,\lambda}$. On the other hand, we have $U_{\alpha,\lambda}HmU_{\alpha,\lambda}m \subset U_{\alpha,\lambda}HU_{-\alpha,-\lambda}m^2 = U_{\alpha,\lambda}HU'_{-\alpha,-\lambda} \cup U_{\alpha,\lambda}H\varphi_{-\alpha}^{-1}(\{-\lambda\})$. Let $u \in \varphi_{-\alpha}^{-1}(\{-\lambda\})$. By Proposition 3.12(2) and (4), there exist $u', u'' \in U_{\alpha,\lambda}$ and $m' \in M_{-\alpha,-\lambda} = M_{\alpha,\lambda}$ such that $m' = u'uu''$, so that $m' \in L_{\alpha,\lambda}$. But $\nu(mm') = \nu(m)\nu(m') = r_{\alpha,\lambda}^2 = \text{id}$ by Lemma 3.15 since $m, m' \in M_{\alpha,\lambda}$. Thus $mm' \in T_b \cap L_{\alpha,\lambda} = H$. Hence $u \in U_{\alpha,\lambda}HmU_{\alpha,\lambda}$. As a consequence, $U_{\alpha,\lambda}H\varphi_{-\alpha}^{-1}(\{-\lambda\}) \subset U_{\alpha,\lambda}HmU_{\alpha,\lambda}$ since $H \subset T_b$ normalizes $U_{\alpha,\lambda}$. This proves that $Xm \subset X$ and therefore $L_{\alpha,\lambda} = X$. Since $H \subset T_{\alpha,\lambda} \subset L_{\alpha,\lambda}$, we deduce $L_{\alpha,\lambda} = (U_{\alpha,\lambda}T_{\alpha,\lambda}U'_{-\alpha,-\lambda}) \cup (U_{\alpha,\lambda}mT_{\alpha,\lambda}U_{\alpha,\lambda})$. Finally, $(U_{\alpha,\lambda}T_{\alpha,\lambda}U'_{-\alpha,-\lambda}) \cap (U_{\alpha,\lambda}mT_{\alpha,\lambda}U_{\alpha,\lambda}) = \emptyset$ because the existence of an element in this intersection would imply that $m \in L'_{\alpha,\lambda}$, which is not possible by (2). By uniqueness in Lemma 3.26, we deduce $H = T_{\alpha,\lambda} \subset T_b$.

Let $n \in N_{\alpha,\lambda}$. If $n \in U_{\alpha,\lambda}T_{\alpha,\lambda}U'_{-\alpha,-\lambda} \subset U^-TU^+$, then $n \in T$ by Bruhat decomposition [BT72, 6.1.15(c)]. Hence $n \in T \cap L_{\alpha,\lambda} = T_{\alpha,\lambda}$. Otherwise, $n \in U_{\alpha,\lambda}mT_{\alpha,\lambda}U_{\alpha,\lambda}$. Hence $nm \in U_{\alpha,\lambda}T_{\alpha,\lambda}U_{-\alpha,-\lambda}$ because $m^2 \in H \subset T_{\alpha,\lambda}$ and Lemma 3.13. Thus $nm \in T$ and therefore $n \in T_{\alpha,\lambda}m$. Hence $N_{\alpha,\lambda} = T_{\alpha,\lambda}\{1, m\}$.

(4) If $\lambda \notin \Gamma_\alpha$, we know that $-\lambda \notin -\Gamma_\alpha = \Gamma_{-\alpha}$ by Fact 3.9. Hence we have $U_{-\alpha,-\lambda} = U'_{-\alpha,-\lambda}$ by Remark 3.24 and therefore $L'_{\alpha,\lambda} = L_{\alpha,\lambda}$ by definition. Hence $N_{\alpha,\lambda} = N'_{\alpha,\lambda} = T'_{\alpha,\lambda}$. Since $L'_{\alpha,\lambda} = U_{\alpha,\lambda}T'_{\alpha,\lambda}U_{-\alpha,-\lambda}$, we deduce that $T_{\alpha,\lambda} = T'_{\alpha,\lambda}$ by uniqueness in Lemma 3.26.

(5) has been shown among the proof. \square

3.28 Corollary. *For any $\varepsilon \geq 0$, any $\alpha \in \Phi$ and any $\lambda \in R$, we have*

$$L_{\alpha,\lambda}^\varepsilon = U_{\alpha,\lambda}U_{-\alpha,-\lambda+\varepsilon}N_{\alpha,\lambda}^\varepsilon = U_{-\alpha,-\lambda+\varepsilon}U_{\alpha,\lambda}N_{\alpha,\lambda}^\varepsilon.$$

Proof. If $\varepsilon > 0$, it is a consequence of (1), (2) and (5) since T_b normalizes $U_{\alpha,\lambda}$ and $U_{-\alpha,-\lambda+\varepsilon}$.

If $\varepsilon = 0$, the first equality is a consequence of (3), (4) and (5) since T_b normalizes $U_{-\alpha,-\lambda}$, $U'_{-\alpha,-\lambda}$ and $U_{\alpha,\lambda}$ and since we have $mU_{\alpha,\lambda}m^{-1} = U_{-\alpha,-\lambda}$ by Lemma 3.13. The last equality is obtained in the same way by exchanging (α, λ) with $(-\alpha, -\lambda)$ since $L_{\alpha,\lambda} = L_{-\alpha,-\lambda}$ by definition and, therefore, $N_{\alpha,\lambda} = N_{-\alpha,-\lambda}$. \square

Technical lemmas of computation of some commutators

We want to estimate some commutators in terms of the valuation of root groups. Let us firstly recall immediate consequences of axioms.

The following Lemma is [BT72, 6.3.5] with ε denoting the $r + s$ of Bruhat-Tits.

3.29 Lemma. *Let $\lambda \in R$ and $\varepsilon > 0$. For any $u \in U_{2\alpha, 2\lambda}$ and any $v \in U_{-\alpha, -\lambda+\varepsilon}$, we have*

$$[u, v] \in U_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha, -\lambda+2\varepsilon}.$$

Proof. We can assume that $u \neq 1$ and $v \neq 1$. Let $\mu = \varphi_\alpha(u) \in \Gamma_\alpha$ and $\rho = \varphi_{-\alpha}(v) \in \Gamma_{-\alpha}$. Let $m \in M_{\alpha, \mu}$ and $n \in M_{-\alpha, \rho}$. Let $v', v'' \in U_{-\alpha, -\mu}$ such that $u = v' m v''$. Let $u', u'' \in U_{\alpha, -\rho}$ such that $v = u' n u''$.

On the one hand, one can write

$$\begin{aligned} [u, v] &= u(u' n u'') u^{-1} (u' n u'')^{-1} \\ &= (u u') n u'' u^{-1} (u'')^{-1} u u^{-1} n^{-1} (u')^{-1} \\ &= (u u') (n [u'', u^{-1}] n^{-1}) (n u^{-1} n^{-1}) (u')^{-1} \end{aligned} \quad (5)$$

By axiom (RGD2), we have $[u'', u^{-1}] = 1$. By Lemma 3.13, we have $n u^{-1} n^{-1} \in n U_{\alpha, \mu} n^{-1} = U_{-\alpha, \mu+2\rho}$. Moreover $\mu + 2\rho \geq -\lambda + 2\varepsilon$. Hence $[u, v] \in U_\alpha U_{-\alpha, -\lambda+2\varepsilon} U_{\alpha, \lambda-\varepsilon}$. By Proposition 3.27, we have $U_{-\alpha, -\lambda+2\varepsilon} U_{\alpha, \lambda-\varepsilon} \subset U_{\alpha, \lambda-\varepsilon} T_b U_{-\alpha, -\lambda+2\varepsilon}$. Hence $[u, v] \in U_\alpha T_b U_{-\alpha, -\lambda+2\varepsilon}$.

On the other hand, an analogous writing gives

$$[u, v] = v'(m[v'', v]m^{-1})(mvm^{-1})(v')^{-1}v^{-1} \quad (6)$$

By axiom (RGD2), we have $[v'', v] = 1$. By Lemma 3.13, we have $mvm^{-1} \in mU_{-\alpha, \rho}m^{-1} = U_{\alpha, \rho+2\mu}$. Moreover $\rho + 2\mu \geq \lambda + \varepsilon$. Hence $[u, v] \in U_{-\alpha, -\lambda+\varepsilon} U_{\alpha, \lambda+\varepsilon} U_{-\alpha}$. By Proposition 3.27, we have $U_{-\alpha, -\lambda+\varepsilon} U_{\alpha, \lambda+\varepsilon} \subset U_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha, -\lambda+\varepsilon}$. Hence $[u, v] \in U_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha}$.

By uniqueness of the writing in $U^+ T U^-$ (axiom (RGD6)), equations (5) and (6) give

$$[u, v] \in U_\alpha T_b U_{-\alpha, -\lambda+2\varepsilon} \cap U_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha} \subset U_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha, -\lambda+2\varepsilon}.$$

□

The following Lemma is [BT72, 6.3.6] with ε denoting the $k + \ell$ of Bruhat-Tits.

3.30 Lemma. *Let $\lambda \in R$ and $\varepsilon > 0$. Then the product*

$$U_{2\alpha, 2\lambda-\varepsilon} U_{\alpha, \lambda} T_b U_{-\alpha, -\lambda+\varepsilon}$$

is a group.

Proof. According to Proposition 3.27, this subset is stable by right multiplication by elements in $U_{\alpha, \lambda}$, T_b and $U_{-\alpha+\varepsilon}$. If $u \in U_{2\alpha, 2\lambda-\varepsilon}$. Then for any $v \in U_{-\alpha, -\lambda+\varepsilon}$, we have

$$vu = u[u^{-1}, v]v \in uU_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha, -\lambda+2\varepsilon} v$$

by Lemma 3.29. Thus $U_{2\alpha, 2\lambda-\varepsilon} U_{\alpha, \lambda} T_b v u \subset U_{2\alpha, 2\lambda-\varepsilon} U_{\alpha, \lambda} T_b U_{2\alpha, 2\lambda-\varepsilon} U_{\alpha, \lambda+\varepsilon} T_b U_{-\alpha, -\lambda+2\varepsilon} U_{-\alpha, -\lambda+\varepsilon}$. Since $U_{2\alpha}$ normalizes U_α by axiom (RGD2) and T_b , and since $U_{-\alpha, -\lambda+2\varepsilon}$ is a subgroup of $U_{-\alpha, -\lambda+\varepsilon}$, we are done. □

The following Lemma is [BT72, 6.3.7] with ε denoting the $k + \ell$ and λ denoting the k of Bruhat-Tits.

3.31 Lemma. *Let $\lambda \in R$ and $\varepsilon > 0$. For any $u \in U_{\alpha, \lambda}$ and $v \in U_{-\alpha, -\lambda + \varepsilon}$, we have*

$$[u, v] \in U_{2\alpha, 2\lambda + \varepsilon} U_{\alpha, \lambda + \varepsilon} T_b U_{-\alpha, -\lambda + 2\varepsilon} U_{-2\alpha, -2\lambda + 3\varepsilon}. \quad (7)$$

If, moreover, $u \in U_{2\alpha}$, that is $u \in U_{2\alpha, 2\lambda}$, then

$$[u, v] \in U_{2\alpha, 2\lambda + 2\varepsilon} U_{\alpha, \lambda + \varepsilon} T_b U_{-\alpha, -\lambda + 3\varepsilon} U_{-2\alpha, -2\lambda + 4\varepsilon}. \quad (8)$$

Note that $U_{2\alpha, 2\lambda + 2\varepsilon} U_{\alpha, \lambda + \varepsilon} = U_{\alpha, \lambda + \varepsilon}$.

Proof. We can assume that $u \neq 1$ and $v \neq 1$. We keep the same notations as in the proof of Lemma 3.29: $\lambda \leq \mu = \varphi_\alpha(u) \in \Gamma_\alpha$ and $-\lambda + \varepsilon \leq \rho = \varphi_{-\alpha}(v) \in \Gamma_{-\alpha}$; $m \in M_{\alpha, \mu}$ and $n \in M_{-\alpha, -\rho}$; $v', v'' \in U_{-\alpha, -\mu}$ such that $u = v'mv''$; $u', u'' \in U_{\alpha, -\rho}$ such that $v = u'nu''$.

By axiom (V3), we have $[u'', u^{-1}] \in U_{2\alpha, \mu - \rho}$. Hence, by Lemma 3.13, we have

$$nU_{2\alpha, \mu - \rho}n^{-1} \subset U_{-2\alpha} \cap nU_{\alpha, \frac{\mu - \rho}{2}}n^{-1} = U_{-2\alpha} \cap U_{-\alpha, \frac{\mu - \rho}{2} + 2\rho} = U_{-2\alpha, \mu + 3\rho} \subset U_{-2\alpha, -2\lambda + 3\varepsilon}.$$

Thus, formula (5) gives

$$[u, v] \in U_\alpha U_{-2\alpha, -2\lambda + 3\varepsilon} U_{-\alpha, -\lambda + 2\varepsilon} U_{\alpha, \lambda - \varepsilon}.$$

Applying Lemma 3.30, we get

$$[u, v] \in U_\alpha T_b U_{-\alpha, -\lambda + 2\varepsilon} U_{-2\alpha, -2\lambda + 3\varepsilon}.$$

By axiom (V3), we have $[v'', v] \in U_{-2\alpha, \rho - \mu}$. Hence, by Lemma 3.13, we have

$$mU_{-2\alpha, \rho - \mu}m^{-1} \subset U_{2\alpha} \cap mU_{-\alpha, \frac{\rho - \mu}{2}}m^{-1} = U_{2\alpha} \cap U_{\alpha, \frac{\rho - \mu}{2} + 2\mu} = U_{2\alpha, 3\mu + \rho} \subset U_{2\alpha, 2\lambda + \varepsilon}.$$

Thus, formula (6) gives

$$[u, v] \in U_{-\alpha, \lambda - \varepsilon} U_{2\alpha, 2\lambda + \varepsilon} U_{\alpha, \lambda + \varepsilon} U_{-\alpha}.$$

Applying Lemma 3.30, we get

$$[u, v] \in U_{2\alpha, 2\lambda + \varepsilon} U_{\alpha, \lambda + \varepsilon} T_b U_{-\alpha}.$$

Uniqueness of the writting in U^+TU^- (axiom (RGD6)) gives (7).

If, moreover, $u \in U_{2\alpha}$, then we have $[u'', u^{-1}] = 1$ by axiom (RGD2) and $nu^{-1}n^{-1} \in U_{-2\alpha} \cap U_{-\alpha, -\lambda + 2\varepsilon} = U_{-2\alpha, -2\lambda + 4\varepsilon}$. Thus

$$[u, v] \in U_\alpha U_{-2\alpha, -2\lambda + 4\varepsilon} U_{\alpha, \lambda - \varepsilon}.$$

Applying Lemma 3.30, we get

$$[u, v] \in U_\alpha T_b U_{-\alpha, -\lambda + 3\varepsilon} U_{-2\alpha, -2\lambda + 4\varepsilon}.$$

Hence, by Lemma 3.29 and uniqueness of the writting in U^+TU^- (axiom (RGD6)), we get (8). \square

3.32 Notation. Let $\lambda \in R$, $\varepsilon > 0$ and $\alpha \in \Phi$. Consider $u \in U_{\alpha, \lambda}$ and $v \in U_{-\alpha, -\lambda + \varepsilon}$. Then, according to Lemma 3.26 and axiom (RGD6), there is a unique $t(u, v) \in T_b$ such that $[u, v] \in U_{\alpha, \lambda} t(u, v) U_{-\alpha, -\lambda + \varepsilon}$. The element $t(u, v)$ is then called the T -component of $[u, v]$.

The following Proposition details [BT72, 6.3.9]. This Proposition becomes really important in order to prove that fibres of the \mathfrak{R}^S -building among projection maps also are buildings, see Proposition 7.4.

3.33 Proposition. Let $\lambda \in R$, $\varepsilon > 0$ and $\alpha \in \Phi$. $T_{\alpha,\lambda}^\varepsilon$ is the group generated by the T -components $t(u, v)$ of commutators $[u, v]$ for $u \in U_{\alpha,\lambda}$ and $v \in U_{-\alpha,-\lambda+\varepsilon}$.

In particular, $T'_{\alpha,\lambda}$ is the group generated by the T -components $t(u, v)$ of commutators $[u, v]$ for $u \in U_{\alpha,\lambda}$ and $v \in U'_{-\alpha,-\lambda}$.

Proof. Denote by X the subgroup of T_b generated by $\{t(u, v), u \in U_{\alpha,\lambda}, v \in U_{-\alpha,-\lambda+\varepsilon}\}$. Since $[u, v] \in L_{\alpha,\lambda}^\varepsilon$, we have $X \subset L_{\alpha,\lambda}^\varepsilon \cap T_b$. We prove that $Y = U_{\alpha,\lambda}XU_{-\alpha,-\lambda+\varepsilon}$ is a group. It is stable by left multiplication by elements in $U_{\alpha,\lambda}$ and X since the subgroup $X \subset T_b$ normalizes $U_{\alpha,\lambda}$. For $v \in U_{-\alpha,-\lambda+\varepsilon}$ and $uxw \in Y$, we have $vuwxw = u[u^{-1}, v]vwxw$. Let $u' \in U_{\alpha,\lambda}$ and $v' \in U_{-\alpha,-\lambda+\varepsilon}$ such that $[u^{-1}, v]u't(u^{-1}, v)v'$. Then $vuwxw = (uu')(t(u^{-1}, v)x)(x^{-1}v'vx)w$. We have $uu' \in U_{\alpha,\lambda}$ and $t(u^{-1}, v)x \in X$. Moreover, since $x \in X$ normalizes $U_{-\alpha,-\lambda+\varepsilon}$, we have $(x^{-1}v'vx)w \in U_{-\alpha,-\lambda+\varepsilon}$. This proves that Y is a subgroup of $L_{\alpha,\lambda}^\varepsilon$ containing $U_{\alpha,\lambda}$ and $U_{-\alpha,-\lambda+\varepsilon}$. Hence $Y = L_{\alpha,\lambda}^\varepsilon$. By Proposition 3.27 (1) and uniqueness in axiom (RGD6), we get $X = T_{\alpha,\lambda}^\varepsilon$.

Since $U'_{-\alpha,-\lambda}$ is the union $\bigcup_{\varepsilon>0} U_{-\alpha,-\lambda+\varepsilon}$, we get that $T'_{\alpha,\lambda} = \bigcup_{\varepsilon>0} T_{\alpha,\lambda}^\varepsilon$ by intersection of $L'_{\alpha,\lambda} = \bigcup_{\varepsilon>0} U_{\alpha,\lambda}U_{-\alpha,-\lambda+\varepsilon}T_{\alpha,\lambda}^\varepsilon$ with T . This gives the second assertion of the Proposition. \square

We obtain the following Corollary, corresponding to [BT72, 6.4.25 (ii) and (iii)] which does not use infimum and supremum.

3.34 Corollary. Let $\lambda \in R$, $\varepsilon \geq 0$ and $\alpha \in \Phi$. Then the following commutator subgroups satisfy

$$[T_{\alpha,\lambda}^\varepsilon, U_{\alpha,\lambda}] \subset U_{\alpha,\lambda+\varepsilon}U_{2\alpha,2\lambda+\varepsilon} \quad (9)$$

and

$$[T_{\alpha,\lambda}^\varepsilon, U_{-\alpha,-\lambda}] \subset U_{-\alpha,-\lambda+\varepsilon}U_{-2\alpha,-2\lambda+\varepsilon}. \quad (10)$$

Moreover,

$$[T'_{\alpha,\lambda}, U_{\alpha,\lambda}] \subset U'_{\alpha,\lambda} \quad (11)$$

and

$$[T'_{\alpha,\lambda}, U_{-\alpha,-\lambda}] \subset U'_{-\alpha,-\lambda} \quad (12)$$

Proof. If $\varepsilon = 0$, the inclusions are immediate since $T_{\alpha,\lambda} \subset T_b$ normalizes $U_{\alpha,\lambda}$ and $U_{2\alpha,2\lambda} \subset U_{\alpha,\lambda}$. Assume that $\varepsilon > 0$ and consider the group $Z = U_{2\alpha,2\lambda+\varepsilon}U_{\alpha,\lambda+\varepsilon}T_bU_{-\alpha,-\lambda+\varepsilon}U_{-2\alpha,-2\lambda+\varepsilon}$.

We prove inclusion (9) on the set of generators $\{[t(u, v), x], x, u \in U_{\alpha,\lambda}, v \in U_{-\alpha,-\lambda+\varepsilon}\}$. We will show that $[t(u, v), x] \in Z$. By Lemma 3.31, we have $[u, v] \in Z$ and we can write it as $[u, v] = u_2u_1t(u, v)v_1v_2$. By Lemma 3.31 and axiom (V3), all the commutators $[v^{-1}, xu]$, $[v^{-1}, x]$, $[x, u_1]$, $[x, u_2]$, $[x, v_1]$ and $[x, v_2]$ are in Z . We have that $x[u, v]x^{-1} \in Z$. Indeed, $xuv = v[v^{-1}, xu]xu \in Zxu$ and $xvu = v[v^{-1}, x]xu \in Zxu$ since $v \in Z$. Moreover $xu_i x^{-1} = [x, u_i]u_i \in Z$ and $xv_i x^{-1} = [x, v_i]v_i \in Z$ for $i \in \{1, 2\}$ since $u_i, v_i \in Z$. Hence $xt(u, v)x^{-1} \in Z$ and we get $[t(u, v), x] \in Z$ since $t(u, v) \in T_b \subset Z$. Since T_b normalizes $U_{\alpha,\lambda}$, we have $[t(u, v), x] \in U_\alpha$. Thus $[t(u, v), x] \in Z \cap U_\alpha = U_{2\alpha,2\lambda+\varepsilon}U_{\alpha,\lambda+\varepsilon}$.

We prove inclusion (10) on the set of generators $[t(u, v), x]$ for $x \in U_{-\alpha,-\lambda}$, $u \in U_{\alpha,\lambda}$ and $v \in U_{-\alpha,-\lambda+\varepsilon}$. Let $\mu = \varphi_{-\alpha}(x) \in \Gamma_{-\alpha}$. If $\mu \geq -\lambda + \varepsilon$, then $x \in Z$ and therefore $[t(u, v), x] \in Z$. Otherwise, denote $\varepsilon' = \lambda + \mu$ so that $0 \leq \varepsilon' < \varepsilon$. By Lemma 3.30 applied to the root $-\alpha$ and the parameters $-\lambda + \varepsilon \in R$ and $\varepsilon - \varepsilon' > 0$, it gives a group $Z' = U_{\alpha,-(-\lambda+\varepsilon)+(\varepsilon-\varepsilon')}T_bU_{-\alpha,-\lambda+\varepsilon}U_{-2\alpha,2(-\lambda+\varepsilon)-(\varepsilon-\varepsilon')}$ which one can rewrite as $Z' = U_{\alpha,-\mu}T_bU_{-\alpha,-\lambda+\varepsilon}U_{-2\alpha,-2\lambda+\varepsilon+\varepsilon'}$. Let $m \in M_{-\alpha,\mu} = M_{\alpha,-\mu}$ and write $x = ymz$ with $y, z \in U_{\alpha,-\mu}$. Denote $t = t(u, v)$. Then $[t, x] = tymzt^{-1}z^{-1}m^{-1}y^{-1} = [t, y]y[t, m]m[t, z]m^{-1}y^{-1}$. We have $[t, m] \in T_b \subset Z'$ since $\nu([t, m]) = [1, r_{-\alpha,\mu}] = \text{id}$. By inclusion (9), we have $[t, y], [t, z] \in U_{\alpha,-\mu+\varepsilon}U_{2\alpha,-2\mu+\varepsilon} \subset U_{\alpha,-\mu} \subset Z'$. By Lemma 3.13, we have $mU_{\alpha,-\mu+\varepsilon}U_{2\alpha,-2\mu+\varepsilon}m^{-1} = U_{-\alpha,-\mu+\varepsilon+2\mu}U_{-2\alpha,-2\mu+\varepsilon+4\mu}$. Since $\mu + \varepsilon \geq -\lambda + \varepsilon$ and $2\mu + \varepsilon \geq -2\lambda + \varepsilon + \varepsilon'$, we have $m[t, z]m^{-1} \in U_{-\alpha,-\lambda+\varepsilon}U_{-2\alpha,-2\lambda+\varepsilon+\varepsilon'} \subset Z'$. Since $y \in Z'$,

we get that $[t, x] \in Z'$ as a product of such elements. Moreover, $[t, x] \in U_{-\alpha}$ since T normalizes $U_{-\alpha}$. Hence, we have either $[t, x] \in Z \cap U_{-\alpha}$ or $[t, x] \in Z' \cap U_{-\alpha}$. By uniqueness in axiom (RGD6), we get $U_{-\alpha, -\lambda+\varepsilon} U_{-2\alpha, -2\lambda+\varepsilon+\varepsilon'} = Z' \cap U_{-\alpha} \subset Z \cap U_{-\alpha} = U_{-\alpha, -\lambda+\varepsilon} U_{-2\alpha, -2\lambda+\varepsilon}$ and we are done.

Since $L'_{\alpha, \lambda}$ is the increasing union $L'_{\alpha, \lambda} = \bigcup_{\varepsilon>0} L_{\alpha, \lambda}^\varepsilon$, and also is $T'_{\alpha, \lambda} = \bigcup_{\varepsilon>0} T_{\alpha, \lambda}^\varepsilon$, we get that the commutator subgroup $[T'_{\alpha, \lambda}, U_{\alpha, \lambda}]$ (resp. $[T'_{\alpha, \lambda}, U_{-\alpha, -\lambda}]$) is the increasing union of commutator subgroups $\bigcup_{\varepsilon>0} [T_{\alpha, \lambda}^\varepsilon, U_{\alpha, \lambda}] = \bigcup_{\varepsilon>0} U_{\alpha, \lambda+\varepsilon} U_{2\alpha, 2\lambda+\varepsilon}$ which is $U'_{\alpha, \lambda}$ (resp. $U'_{-\alpha, -\lambda}$) since $U_{2\alpha, 2\lambda+\varepsilon} \subset U_{\alpha, \lambda+\frac{\varepsilon}{2}}$ for every $\varepsilon > 0$. \square

3.5 Subgroups generated by unipotent elements

In this section, we work under data and notations of 3.22. Moreover, we fix a non-empty subset $\Omega \subset \mathbb{A}_R$. For simplicity, we will forget the chosen origin o of \mathbb{A}_R in this section, i.e. we will denote by $\alpha(x)$ the quantity $\alpha(x - o)$ by abuse of notation.

3.35 Remark. In Bruhat-Tits theory, some results on groups are expressed in terms of concave functions f [BT72, §6.4] associated to abstract groups endowed with the structure of a valued root group datum. In fact, only some well-chosen concave functions are considered for reductive groups over local fields. For instance, the optimized function f' associated to a concave function f is defined in [BT84, 4.5.2] as:

$$f'(\alpha) = \inf\{\lambda \in \Gamma'_\alpha, \lambda \geq f(\alpha) \text{ or, when } \frac{\alpha}{2} \in \Phi, \frac{\lambda}{2} \geq f(\frac{\alpha}{2})\}.$$

In Bruhat-Tits theory, since the valuation group Λ is contained in \mathbb{R} , $f'(\alpha)$ is well defined and is an element of Γ'_α so that it is relevant to work with the group $U_{\alpha, f'(\alpha)}$. More specifically, for a subset $\Omega \in \mathbb{A}_\mathbb{R}$, some concave functions f_Ω can be considered in order to define parahoric subgroups $\widehat{P}_\Omega \subset G$. Once the building \mathcal{I} has been constructed, this subgroup \widehat{P}_Ω turns out to be the pointwise stabilizer of Ω .

Recall that the quasi-concave maps were defined by:

$$f_\Omega(\alpha) = \inf\{\lambda \in R, \forall x \in \Omega, \alpha(x) + \lambda \geq 0\} = \inf \bigcap_{x \in \Omega} [-\alpha(x), +\infty[= \sup\{-\alpha(x), x \in \Omega\}$$

Here, we do not use infimum and supremum, so that we replace this definition by some intersections or unions of groups.

Note that in the Bruhat-Tits theory, such a function f_Ω satisfies $f_\Omega(2\alpha) = 2f_\Omega(\alpha)$ which is the case of equality of the concave inequality $2f(\alpha) \geq f(2\alpha)$. This equality induces an equality of groups $U_{\alpha, f} = U_{\alpha, f} U_{2\alpha, f}$ in [BT72, §6.4].

3.36 Notation. We denote by:

$$U_{\alpha, \Omega} = \bigcap_{x \in \Omega} U_{\alpha, -\alpha(x)} \qquad U'_{\alpha, \Omega} = \bigcap_{x \in \Omega} U'_{\alpha, -\alpha(x)}$$

We denote by U_Ω (resp. U'_Ω) the subgroup of G generated by the union of subgroups $U_{\alpha, \Omega}$ (resp. $U'_{\alpha, \Omega}$) for $\alpha \in \Phi$.

It is convenient to introduce the notation $U_{2\alpha, \Omega}$ (resp. $U'_{2\alpha, \Omega}$) as being the trivial group when $\alpha \in \Phi$ and $2\alpha \notin \Phi$.

If it is given a choice of positive roots Φ^+ , we denote U^+ and U^- as in (RGD6). We denote by:

$$\begin{aligned} U_\Omega^+ &= U_\Omega \cap U^+ & U_\Omega'^+ &= U'_\Omega \cap U^+ \\ U_\Omega^- &= U_\Omega \cap U^- & U_\Omega'^- &= U'_\Omega \cap U^- \\ T_\Omega &= U_\Omega \cap T & T'_\Omega &= U'_\Omega \cap T \\ \widetilde{N}_\Omega &= U_\Omega \cap N & \widetilde{N}'_\Omega &= U'_\Omega \cap N \end{aligned}$$

If $\Omega = \{x\}$, we denote $U_{\alpha,x}$, U_x , etc. instead of $U_{\alpha,\{x\}}$, $U_{\{x\}}$, etc.

Note that the notation \tilde{N}_Ω was not introduced in [BT72]. In the context of algebraic groups, under favourable assumptions (typically a simply-connectedness assumption), the rational points of a group are generated by the unipotent elements. That is why we denote with a \tilde{N}_Ω the N -component of the group U_Ω generated by some unipotent elements.

3.37 Fact. *By definition, we have the following equalities:*

$$U_{\alpha,\Omega} = \bigcap_{x \in \Omega} U_{\alpha,x} = \varphi_\alpha^{-1} \left(\bigcap_{x \in \Omega} [-\alpha(x), +\infty] \right)$$

and

$$U'_{\alpha,\Omega} = \bigcap_{x \in \Omega} U'_{\alpha,x} = \varphi_\alpha^{-1} \left(\bigcap_{x \in \Omega}]-\alpha(x), +\infty] \right).$$

Note that the intersections $\bigcap_{x \in \Omega} [-\alpha(x), +\infty]$ and $\bigcap_{x \in \Omega}]-\alpha(x), +\infty]$ are convex subsets of R that may not be intervals of R when $R \neq \mathbb{R}$.

3.38 Fact. *The group T_b normalizes $U_{\alpha,\Omega}$ and $U'_{\alpha,\Omega}$ for any $\alpha \in \Phi$.*

Proof. For any $x \in \Omega$ and any $\varepsilon \geq 0$, we know that T_b normalizes $U_{\alpha,-\alpha(x)+\varepsilon}$. It remains true by taking increasing unions and intersections of these groups. \square

3.39 Lemma. *For any $\alpha \in \Phi_{\text{nd}}$, we have $U_{2\alpha,\Omega} \subset U_{\alpha,\Omega}$.*

Proof. By axiom (V4), we have $U_{2\alpha,2\lambda} \subset U_{\alpha,\lambda}$. Hence

$$U_{2\alpha,\Omega} = \bigcap_{x \in \Omega} U_{2\alpha,x} = \bigcap_{x \in \Omega} U_{2\alpha,-2\alpha(x)} \subset \bigcap_{x \in \Omega} U_{\alpha,-\alpha(x)} = \bigcap_{x \in \Omega} U_{\alpha,x} = U_{\alpha,\Omega}.$$

\square

3.40 Notation. Let $\alpha \in \Phi$. We denote by $L_{\alpha,\Omega}$ (resp. $L'_{\alpha,\Omega}$) the subgroup of G generated by $U_{\alpha,\Omega}$ and $U_{-\alpha,\Omega}$ (resp. $U_{\alpha,\Omega}$ and $U'_{-\alpha,\Omega}$). We denote

$$\begin{aligned} T_{\alpha,\Omega} &= L_{\alpha,\Omega} \cap T & T'_{\alpha,\Omega} &= L'_{\alpha,\Omega} \cap T \\ \tilde{N}_{\alpha,\Omega} &= L_{\alpha,\Omega} \cap N & \tilde{N}'_{\alpha,\Omega} &= L'_{\alpha,\Omega} \cap N. \end{aligned}$$

Note that since Ω is not empty, there exists some point $x \in \Omega$ so that we have $L_{\alpha,x} \supset L_{\alpha,\Omega}$ since $U_{\alpha,x} \supset U_{\alpha,\Omega}$ and $U_{-\alpha,x} \supset U_{-\alpha,\Omega}$. Therefore $T_{\alpha,\Omega}$ and $T'_{\alpha,\Omega}$ are subgroups of T_b according to Proposition 3.27(5) applied to $L_{\alpha,x} = L_{\alpha,-\alpha(x)}$.

Note that, $L_{\alpha,\Omega} = L_{-\alpha,\Omega}$ by definition, but $L'_{-\alpha,\Omega}$ may differ from $L'_{\alpha,\Omega}$.

The following Lemma corresponds to [BT72, 6.4.7 (QC1)] with a different proof since we do not use concave maps.

3.41 Lemma. *For any $\alpha \in \Phi$, we have:*

$$L_{\alpha,\Omega} = U_{-\alpha,\Omega} U_{\alpha,\Omega} \tilde{N}_{\alpha,\Omega} = U_{\alpha,\Omega} U_{-\alpha,\Omega} \tilde{N}_{\alpha,\Omega} \quad (\text{QC1})$$

$$\tilde{N}'_{\alpha,\Omega} = T'_{\alpha,\Omega} \subset T_b \quad \text{and} \quad L'_{\alpha,\Omega} = U_{\alpha,\Omega} T'_{\alpha,\Omega} U'_{-\alpha,\Omega} = U'_{-\alpha,\Omega} T'_{\alpha,\Omega} U_{\alpha,\Omega}. \quad (\text{QC1}')$$

Note that the group U_α may not be commutative, when α is multipliable for instance.

Proof. Obviously, the set $U_{\alpha,\Omega}U_{-\alpha,\Omega}\tilde{N}_{\alpha,\Omega}$ is contained in $L_{\alpha,\Omega}$ and the set $U_{\alpha,\Omega}U'_{-\alpha,\Omega}\tilde{N}'_{\alpha,\Omega}$ is contained in $L'_{\alpha,\Omega}$ by definition. Assume that $g \in L_{\alpha,\Omega}$ (resp. $L'_{\alpha,\Omega}$) and write it as a product $g = \prod_{i=1}^m h_i$ with $h_i \in U_{\alpha,\Omega} \cup U_{-\alpha,\Omega}$ (resp. $h_i \in U_{\alpha,\Omega} \cup U'_{-\alpha,\Omega}$). For $1 \leq i \leq m$, denote

$$\lambda_i = \begin{cases} \varphi_\alpha(h_i) & \text{if } h_i \in U_{\alpha,\Omega}, \\ \infty & \text{if } h_i \in U_{-\alpha,\Omega}, \end{cases} \quad \mu_i = \begin{cases} \infty & \text{if } h_i \in U_{\alpha,\Omega}, \\ \varphi_{-\alpha}(h_i) & \text{if } h_i \in U_{-\alpha,\Omega}. \end{cases}$$

Hence for any $x \in \Omega$, we have $\lambda_i \geq -\alpha(x)$ and $\mu_i \geq -(-\alpha)(x) = \alpha(x)$ (resp. $\mu_i > \alpha(x)$). Let $\lambda = \min_{1 \leq i \leq m} \lambda_i$ and $\mu = \min_{1 \leq i \leq m} \mu_i$. Then $\lambda = \lambda_i$ for some i so that $U_{\alpha,\lambda} = U_{\alpha,\lambda_i} \subset U_{\alpha,x}$ and $\mu = \mu_j$ for some j so that $U_{-\alpha,\mu} = U_{-\alpha,\mu_j} \subset U_{-\alpha,x}$. Moreover $h_i \in U_{\alpha,\lambda} \cup U_{-\alpha,\mu}$ for any $1 \leq i \leq m$ by definition of λ and μ . Let $\varepsilon = \lambda + \mu \geq -\alpha(x) + \alpha(x) = 0$ (resp. $\varepsilon > 0$). Thus g is in the group generated by $U_{\alpha,\lambda}$ and $U_{-\alpha,\mu} = U_{-\alpha,-\lambda+\varepsilon}$ which is $L_{\alpha,\lambda}^\varepsilon = U_{\alpha,\lambda}U_{-\alpha,-\lambda+\varepsilon}N_{\alpha,\lambda}^\varepsilon$ by Corollary 3.28. Since $U_{\alpha,\lambda} \subset U_{\alpha,x}$ for every $x \in \Omega$, we have $U_{\alpha,\lambda} \subset U_{\alpha,\Omega}$. By the same way, $U_{-\alpha,-\lambda+\varepsilon} \subset U_{-\alpha,\Omega}$. Hence $L_{\alpha,\lambda}^\varepsilon \subset L_{\alpha,\Omega}$. Thus $N_{\alpha,\lambda}^\varepsilon = N \cap L_{\alpha,\lambda}^\varepsilon \subset N \cap L_{\alpha,\Omega} = \tilde{N}_{\alpha,\Omega}$. Moreover, $N_{\alpha,\lambda}^\varepsilon = T_{\alpha,\lambda}^\varepsilon \subset T'_{\alpha,\Omega}$ when $\varepsilon > 0$ by Proposition 3.27(2). This gives $g \in U_{\alpha,\Omega}U_{-\alpha,\Omega}\tilde{N}_{\alpha,\Omega}$ for any $g \in L_{\alpha,\Omega}$ (resp. $g \in U_{\alpha,\Omega}U'_{-\alpha,\Omega}T'_{\alpha,\Omega}$ for any $g \in L'_{\alpha,\Omega}$). The two other equalities in (QC1) are obtained in the same way.

Moreover, by Bruhat decomposition [BT72, 6.1.15(c)], it gives $L'_{\alpha,\Omega} \cap N = \tilde{N}'_{\alpha,\Omega} \subset T$, which gives the first part of (QC1'). Since $\tilde{N}'_{\alpha,\Omega} = T'_{\alpha,\Omega} \subset T_b$ normalizes $U_{\alpha,\Omega}$, we get the first equality of the second part of (QC1'). Finally, last equality can be obtained in the same way replacing $L_{\alpha,\lambda}^\varepsilon$ by $L_{-\alpha,\mu}^\varepsilon = U_{-\alpha,\mu}N_{-\alpha,\mu}^\varepsilon U_{\alpha,-\mu+\varepsilon} = U_{-\alpha,-\lambda+\varepsilon}T_{-\alpha,\varepsilon-\lambda}^\varepsilon U_{\alpha,\lambda}$. \square

3.42 Lemma. *The following commutator subgroups satisfy*

$$[U_{\alpha,\Omega}, U'_{\alpha,\Omega}] \subset U'_{2\alpha,\Omega} \quad [U_{\alpha,\Omega}, U'_{-\alpha,\Omega}] \subset U'_{\alpha,\Omega}U'_{-\alpha,\Omega}T'_{\alpha,\Omega}$$

Proof. We check it on a set of generators.

Consider $u \in U_{\alpha,\Omega}$ and $u' \in U'_{\alpha,\Omega}$. For any $x \in \Omega$, let $\lambda = -\alpha(x) \in R$. Let $\mu = \varphi_\alpha(u) = \lambda + \varepsilon$ and $\mu' = \varphi_\alpha(u') = \lambda + \varepsilon'$ with $\varepsilon \geq 0$ and $\varepsilon' > 0$. Then, by axiom (V3), we have $[u, u'] \in U_{2\alpha, 2\lambda+\varepsilon+\varepsilon'} \subset U_{2\alpha, 2\lambda} = U'_{2\alpha, x}$. Hence $[u, u'] \in \bigcap_{x \in \Omega} U'_{2\alpha, x} = U'_{2\alpha, \Omega}$.

Consider $u \in U_{\alpha,\Omega}$ and $v \in U'_{-\alpha,\Omega}$. For any $x \in \Omega$, let $\lambda = -\alpha(x) \in R$. Then $u \in U_{\alpha,\lambda}$ and there exists $\varepsilon > 0$ such that $v \in U_{-\alpha,-\lambda+\varepsilon}$. By Lemma 3.31, we have $[u, v] \in U_{2\alpha, 2\lambda+\varepsilon}U_{\alpha,\lambda+\varepsilon}t(u, v)U_{-\alpha,-\lambda+2\varepsilon}U_{-2\alpha,-2\lambda+3\varepsilon}$ where $t(u, v)$ is the T -component of $[u, v]$. Hence, by Lemma 3.26, there are $u' \in U_\alpha$, $v' \in U_{-\alpha}$ uniquely determined such that $[u, v] = u't(u, v)v'$. Moreover, $\varphi_\alpha(u') \geq \lambda + \frac{\varepsilon}{2} > -\alpha(x)$ and $\varphi_{-\alpha}(v') \geq -\lambda + \frac{3\varepsilon}{2} > \alpha(x)$. Hence $u' \in U'_{\alpha, x}$ and $v' \in U'_{-\alpha, x}$ for any $x \in \Omega$. Hence $u' \in U'_{\alpha,\Omega}$ and $v' \in U'_{-\alpha,\Omega}$. Thus $t(u, v) \in L'_{\alpha,\Omega} \cap T = T'_{\alpha,\Omega}$. Hence $[u, v] \in U'_{\alpha,\Omega}T'_{\alpha,\Omega}U'_{-\alpha,\Omega} = U'_{\alpha,\Omega}U'_{-\alpha,\Omega}T'_{\alpha,\Omega}$ since $T'_{\alpha,\Omega} \subset T_b$ normalizes $U'_{-\alpha,\Omega}$. \square

The following Lemma corresponds to [BT72, 6.4.7 (QC2)]. It is an immediate consequence of axiom (V3) and of the definitions.

3.43 Lemma. *Let $\alpha, \beta \in \Phi$ be such that $\beta \notin -\mathbb{R}_{>0}\alpha$. Let $U_{(\alpha,\beta),\Omega}$ (resp. $U'_{(\alpha,\beta),\Omega}$) be the subgroup of G generated by the $U_{\gamma,\Omega}$ (resp. $U'_{\gamma,\Omega}$) for $\gamma \in (\alpha, \beta)$ (see Notation 2.1). Then the commutator subgroups satisfy:*

$$[U_{\alpha,\Omega}, U_{\beta,\Omega}] \subset U_{(\alpha,\beta),\Omega} \quad (\text{QC2})$$

$$[U_{\alpha,\Omega}, U'_{\beta,\Omega}] \subset U'_{(\alpha,\beta),\Omega} \quad (\text{QC2}')$$

Proof. Let $u \in U_{\alpha,\Omega}$, $v \in U_{\beta,\Omega}$ and $v' \in U'_{\beta,\Omega}$. Let $\lambda = \varphi_\alpha(u)$, $\mu = \varphi_\beta(v)$ and $\mu' = \varphi_\beta(v')$. Let $r, s \in \mathbb{Z}_{>0}$ such that $\gamma = r\alpha + s\beta \in \Phi$. Then for any $x \in \Omega$, we have $\lambda \geq -\alpha(x)$, $\mu \geq -\beta(x)$ and $\mu' > -\beta(x)$. Hence $r\lambda + s\mu \geq -r\alpha(x) - s\beta(x) = -\gamma(x)$ and $r\lambda + s\mu' > -r\alpha(x) - s\beta(x) = -\gamma(x)$. Thus $U_{\gamma, r\lambda+s\mu} \subset U_{\gamma, x}$ and $U_{\gamma, r\lambda+s\mu'} \subset U'_{\gamma, x}$. Because this

inclusion holds for any $x \in \Omega$, we have, by definition of $U_{\gamma,\Omega}$ and $U'_{\gamma,\Omega}$, that $U_{\gamma,r\lambda+s\mu} \subset U_{\gamma,\Omega}$ and $U_{\gamma,r\lambda+s\mu'} \subset U'_{\gamma,\Omega}$. By axiom (V3), the commutator $[u, v]$ (resp. $[u, v']$) is contained in the group generated by the $U_{r\alpha+s\beta,r\lambda+s\mu} \subset U_{r\alpha+s\beta,\Omega}$ (resp. $U_{r\alpha+s\beta,r\lambda+s\mu'} \subset U'_{r\alpha+s\beta,\Omega}$). \square

3.44 Example. According to Lemmas 3.41(QC1) and 3.43(QC2), the family $((U_{\alpha,\Omega})_{\alpha \in \Phi}, Y)$ is quasi-concave for every subgroup Y of T_b since T_b normalizes each $U_{\alpha,\Omega}$ for $\alpha \in \Phi$ and $U_{\alpha,\Omega} = U_{\alpha,\Omega}U_{2\alpha,\Omega}$ by Lemma 3.39. In particular, for $Y = 1$, we get from Proposition 3.4:

- (1) $U_\alpha \cap U_\Omega = U_{\alpha,\Omega}$ and $U_{2\alpha} \cap U_\Omega = U_{2\alpha} \cap U_{\alpha,\Omega} = U_{2\alpha,\Omega}$ for any $\alpha \in \Phi_{\text{nd}}$.
- (2) The product map $\prod_{\alpha \in \Phi_{\text{nd}}^+} U_{\alpha,\Omega} \rightarrow U_\Omega^+ = U_\Omega \cap U^+$ (resp. $\prod_{\alpha \in \Phi_{\text{nd}}^+} U_{-\alpha,\Omega} \rightarrow U_\Omega^- = U_\Omega \cap U^-$) is a bijection for any ordering on the product.
- (3) We have $U_\Omega = U_\Omega^+ U_\Omega^- \tilde{N}_\Omega = U_\Omega^- U_\Omega^+ \tilde{N}_\Omega$.
- (4) The group \tilde{N}_Ω is generated by the $\tilde{N}_{\alpha,\Omega}$ for $\alpha \in \Phi_{\text{nd}}$.

Since for every $\alpha \in \Phi$, the group $T'_{\alpha,\Omega} \subset T_b$ normalizes $U'_{\alpha,\Omega}$ and $U'_{-\alpha,\Omega}$, we deduce from Lemmas 3.42 and 3.41(QC1') that $U'_{\alpha,\Omega} U'_{-\alpha,\Omega} (T \cap U'_\Omega)$ is a group for any ordering of the product. Since $U'_{\alpha,\Omega}$ is a subgroup of $U_{\alpha,\Omega}$, we deduce from Lemma 3.43(QC2') that the family $((U'_{\alpha,\Omega})_{\alpha \in \Phi}, Y)$ is quasi-concave for any subgroup Y of T_b since T_b normalizes the $U'_{\alpha,\Omega}$ for $\alpha \in \Phi$.

We have, as in [BT72, 6.4.25(i)], the following:

3.45 Proposition. *Let $\alpha, \beta \in \Phi$ be two roots such that $\beta \notin \mathbb{R}\alpha$. Let $\lambda, \mu \in R$ and $\varepsilon \geq 0$. Then the following commutator subgroup satisfies*

$$[T_{\beta,\mu}^\varepsilon, U_{\alpha,\lambda}] \subset U_{\alpha,\lambda+\varepsilon} U_{2\alpha,2\lambda+\varepsilon}.$$

Moreover,

$$[T'_{\beta,\mu}, U_{\alpha,\lambda}] \subset U'_{\alpha,\lambda}.$$

Proof. There is nothing to prove for $\varepsilon = 0$. Assume $\varepsilon > 0$ and consider the group X generated by the subsets $U_{\beta,\mu}$, $U_{-\beta,-\mu+\varepsilon}$ and $U_{-2\beta,-2\mu+\varepsilon}$. Then X contains $T_{\beta,\mu}^\varepsilon$ by definition.

For $s \in \mathbb{Z}$, denote $\varepsilon(s) = \begin{cases} 0 & \text{if } s > 0 \\ \varepsilon & \text{if } s \leq 0 \end{cases}$. Consider the group Y generated by the $U_{r\alpha+s\beta,r\lambda+s\mu+\varepsilon(s)}$ for $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ such that $r\alpha + s\beta \in \Phi$.

By axiom (V3), we observe that the commutator $[x, y]$ for $x \in U_{\beta,\mu} \cup U_{-\beta,-\mu+\varepsilon} \cup U_{-2\beta,-2\mu+\varepsilon}$ and $y \in \bigcup_{(r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}} U_{r\alpha+s\beta,r\lambda+s\mu+\varepsilon(s)}$ belongs to Y . Thus X normalizes Y and $[X, U_{\alpha,\lambda}] \subset Y$. Hence $[T_{\beta,\mu}^\varepsilon, U_{\alpha,\lambda}] \subset Y \cap U_\alpha$ since $T_{\beta,\mu}^\varepsilon \subset X \cap T$ normalizes U_α .

Consider the root system $\Phi(\alpha, \beta) = \Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ and the family of groups defined by $X_{r\alpha+s\beta} = U_{r\alpha+s\beta,r\lambda+s\mu+\varepsilon(s)}$ for $r > 0$ and $r\alpha + s\beta \in \Phi$ and by $X_{r\alpha+s\beta} = 1$ otherwise. Then the family of groups $(X_\gamma)_{\gamma \in \Phi(\alpha,\beta)}$ is quasi-concave (axiom (QC1) is obvious for this set since either $X_{-2\gamma} = X_{-\gamma} = 1$ or $X_{2\gamma} = X_\gamma = 1$ and axiom (QC2) is a consequence of axiom (V3)). Thus by Proposition 3.4, we get that $Y \cap U_\alpha = X_\alpha X_{2\alpha} = U_{\alpha,\lambda+\varepsilon} U_{2\alpha,2\lambda+\varepsilon}$.

Since $T'_{\beta,\mu} = \bigcup_{\varepsilon > 0} T_{\beta,\mu}^\varepsilon$, we get that any element in the commutator subgroup $[T'_{\beta,\mu}, U_{\alpha,\lambda}]$ belongs to the commutator subgroup $[T_{\beta,\mu}^\varepsilon, U_{\alpha,\lambda}]$ for some $\varepsilon > 0$. Thus $[T'_{\beta,\mu}, U_{\alpha,\lambda}] \subset \bigcup_{\varepsilon > 0} [T_{\beta,\mu}^\varepsilon, U_{\alpha,\lambda}] = \bigcup_{\varepsilon > 0} U_{\alpha,\lambda+\varepsilon} U_{2\alpha,2\lambda+\varepsilon} = U'_{\alpha,\lambda}$. \square

3.6 Local root systems

In this section, we work under data and notations of 3.22. Moreover, we fix a non-empty subset $\Omega \subset \mathbb{A}_R$. For simplicity, we will forget the chosen origin o of \mathbb{A}_R in this section, i.e. we will denote by $\alpha(x)$ the quantity $\alpha(x - o)$ by abuse of notation. The goal of this section is to define a root system Φ_Ω^* depending on the local geometry of Ω inside \mathbb{A}_R with respect to some hyperplanes $H_{\alpha,\lambda}$ for $\alpha \in \Phi$ and $\lambda \in \Gamma'_\alpha$ that will be the walls of \mathbb{A}_R .

3.46 Notation. We define the following subset of roots:

$$\Phi_\Omega^* = \{\alpha \in \Phi, \exists \lambda_\alpha \in R, \forall x \in \Omega, -\alpha(x) = \lambda_\alpha\}$$

$$\Phi_\Omega = \{\alpha \in \Phi, \exists \lambda_\alpha \in \Gamma'_\alpha, \forall x \in \Omega, -\alpha(x) = \lambda_\alpha\}.$$

Note that, by definition, we have $\Phi_\Omega \subset \Phi_\Omega^* \subset \Phi$.

3.47 Lemma. *Let $\alpha \in \Phi$. The following are equivalent:*

- (i) $\alpha \in \Phi_\Omega$;
- (ii) $\exists u \in U_\alpha$ such that $\forall x \in \Omega, \varphi_\alpha(u) = -\alpha(x)$ and $\forall v \in U_{2\alpha}, \varphi_\alpha(u) \geq \varphi_\alpha(uv)$;
- (iii) $\alpha \in \Phi_\Omega^*$ and $\exists u \in U_{\alpha,\Omega}$ such that $\forall v \in U_{2\alpha}, uv \notin U'_{\alpha,\Omega}$.

Proof.

(i) \Rightarrow (ii): If $\alpha \in \Phi_\Omega$, let $\lambda_\alpha \in \Gamma'_\alpha$ such that $\forall x \in \Omega, -\alpha(x) = \lambda_\alpha$. By definition, there is $u \in U_\alpha$ such that $-\alpha(x) = \varphi_\alpha(u)$ and $U_{\alpha,\varphi_\alpha(u)} = \bigcap_{v \in U_{2\alpha}} U_{\alpha,\varphi_\alpha(uv)}$. Thus $\forall v \in U_{2\alpha}$, we have $\varphi_\alpha^{-1}([\varphi_\alpha(u), +\infty]) \subset \varphi_\alpha^{-1}([\varphi_\alpha(uv), +\infty])$ and therefore $\varphi_\alpha(u) \geq \varphi_\alpha(uv)$.

(ii) \Rightarrow (iii): Let $u \in U_\alpha$ such that $\forall x \in \Omega, \varphi_\alpha(u) = -\alpha(x)$ and $\forall v \in U_{2\alpha}, \varphi_\alpha(u) \geq \varphi_\alpha(uv)$. For any $x, y \in \Omega$, we have $-\alpha(x) = \varphi_\alpha(u) = -\alpha(y)$ so that $u \in \bigcap_{x \in \Omega} U_{\alpha,x} = U_{\alpha,\Omega}$. Suppose by contradiction that there is a $v \in U_{2\alpha}$ such that $uv \in U'_{\alpha,\Omega}$. Then for any $x \in \Omega$, we have $uv \in U'_{\alpha,x}$ and therefore $\varphi_\alpha(uv) > -\alpha(x) = \varphi_\alpha(u)$ which contradicts the assumption.

(iii) \Rightarrow (i): Assume that there exists a $u \in U_{\alpha,\Omega}$ such that $\forall v \in U_{2\alpha}, uv \notin U'_{\alpha,\Omega}$. Let $\lambda_\alpha = \varphi_\alpha(u)$. For $x \in \Omega$, since $u \in U_{\alpha,\Omega} \subset U_{\alpha,x}$, we have $\lambda_\alpha = \varphi_\alpha(u) \geq -\alpha(x)$. Let $v \in U_{2\alpha}$. Then $uv \notin U'_{\alpha,\Omega}$ and there is $x_v \in \Omega$ such that $uv \notin U'_{\alpha,x_v}$. Thus $\varphi_\alpha(uv) \leq -\alpha(x_v) \leq \varphi_\alpha(u)$. Hence $U_{\alpha,\varphi_\alpha(u)} \subset U_{\alpha,\varphi_\alpha(uv)}$ for every $v \in U_{2\alpha}$ and we get $U_{\alpha,\varphi_\alpha(u)} = \bigcap_{v \in U_{2\alpha}} U_{\alpha,\varphi_\alpha(uv)}$ by taking $v = 1$, so that $\lambda_\alpha = \varphi_\alpha(u) \in \Gamma'_\alpha$. If, moreover, $\alpha \in \Phi_\Omega^*$, then $\forall y \in \Omega$, we have $-\alpha(y) = -\alpha(x) = \lambda_\alpha$. \square

Without infimum and supremum, we provide a different proof and a slightly different statement of [BT72, 6.4.11]:

3.48 Lemma. *For any $\alpha \in \Phi_{\text{nd}}$, we have ${}^v\nu(\tilde{N}_{\alpha,\Omega}) \subset \{1, r_\alpha\}$ with equality if, and only if, either α or 2α belongs to Φ_Ω .*

Proof. According to [BT72, 6.1.2(7) & (10)], since $\tilde{N}_{\alpha,\Omega} \subset N \cap \langle U_\alpha, U_{-\alpha}, T \rangle = T \sqcup M_\alpha$, we have ${}^v\nu(\tilde{N}_{\alpha,\Omega}) \subset \{1, r_\alpha\}$.

Suppose that $\alpha \in \Phi_\Omega$. Let $u \in U_\alpha$ given by Lemma 3.47(ii) and $\lambda = \varphi_\alpha(u) \in \Gamma'_\alpha \subset \Gamma_\alpha$. Then, by Lemma 3.15, we have ${}^v\nu(M_{\alpha,\lambda}) = \{r_\alpha\}$. Moreover, by Proposition 3.12(3), the set $M_{\alpha,\lambda}$ is contained in $L_{\alpha,\lambda}$. Moreover, for any $x \in \Omega$, we have $-\alpha(x) = \lambda$ which gives $U_{\alpha,x} = U_{\alpha,\Omega}$ and $U_{-\alpha,x} = U_{-\alpha,\Omega}$ and therefore $M_{\alpha,\lambda} \subset \tilde{N}_{\alpha,\Omega}$. Thus $r_\alpha \in {}^v\nu(\tilde{N}_{\alpha,\Omega})$.

Suppose that $2\alpha \in \Phi_\Omega$, then $r_\alpha \in {}^\nu\mathcal{N}(\tilde{N}_{\alpha,\Omega})$ since $\tilde{N}_{2\alpha,\Omega} \subset \tilde{N}_{\alpha,\Omega}$ by definition.

Conversely, suppose that $r_\alpha \in {}^\nu\mathcal{N}(\tilde{N}_{\alpha,\Omega})$. Consider any $n \in {}^\nu\mathcal{N}^{-1}(\{r_\alpha\}) \cap \tilde{N}_{\alpha,\Omega}$. Note that $n \notin T$ since ${}^\nu\mathcal{N}(T) = \text{id}$ by [BT72, 6.1.11(ii)]. Write n as a product $n = \prod_{i=1}^r u_i$ for $r \in \mathbb{Z}_{>0}$ and $u_i \in U_{\alpha,\Omega} \cup U_{-\alpha,\Omega}$ with $u_i \neq 1$ which is a generating set of the group $L_{\alpha,\Omega}$ containing $\tilde{N}_{\alpha,\Omega}$. For any $i \in \llbracket 1, r \rrbracket$, denote

$$\lambda_i = \begin{cases} \varphi_\alpha(u_i) & \text{if } u_i \in U_\alpha, \\ \infty & \text{if } u_i \in U_{-\alpha}, \end{cases} \quad \mu_i = \begin{cases} \varphi_{-\alpha}(u_i) & \text{if } u_i \in U_{-\alpha}, \\ \infty & \text{if } u_i \in U_\alpha, \end{cases}$$

and consider $\lambda = \min\{\lambda_i, i \in \llbracket 1, r \rrbracket\} \in R \cup \{\infty\}$ and $\mu = \min\{\mu_i, i \in \llbracket 1, r \rrbracket\} \in R \cup \{\infty\}$. For any $i \in \llbracket 1, r \rrbracket$, we have either $\lambda_i = \infty$ or $u_i \in U_{\alpha,\Omega} = \bigcap_{x \in \Omega} U_{\alpha,x}$ which gives $\varphi_\alpha(u_i) = \lambda_i \geq -\alpha(x)$ for every $x \in \Omega$. Thus $U_{\alpha,\lambda_i} \subset U_{\alpha,\Omega}$. By the same way, $U_{-\alpha,\mu_i} \subset U_{-\alpha,\Omega}$. Thus $U_{\alpha,\lambda} \subset U_{\alpha,\Omega}$ and $U_{-\alpha,\mu} \subset U_{-\alpha,\Omega}$. Hence, for $x \in \Omega$, we have $\lambda \geq -\alpha(x)$ and $\mu \geq \alpha(x)$. Denote $\varepsilon = \lambda + \mu \geq 0$. Then $n \in L_{\alpha,\lambda}^\varepsilon$ and thus $n \in N_{\alpha,\Omega}^\varepsilon \setminus T$ which gives $\varepsilon = 0$ and $\lambda \in \Gamma_\alpha$ by Proposition 3.27. Hence $\forall x \in \Omega, \lambda \geq -\alpha(x)$ and $\mu = -\lambda \geq \alpha(x)$. Thus $\forall x \in \Omega, -\alpha(x) = \lambda \in \Gamma_\alpha$. If $\lambda \in \Gamma'_\alpha$, then we have $\alpha \in \Phi_\Omega$. Otherwise, by Fact 3.9, $\lambda \in \frac{1}{2}\Gamma'_{2\alpha}$ and thus $\forall x \in \Omega, -(2\alpha)(x) = 2\lambda \in \Gamma'_{2\alpha}$ which means $2\alpha \in \Phi_\Omega$. \square

3.49 Proposition. *The group \tilde{N}_Ω normalizes U'_Ω .*

Proof. Let $\alpha \in \Phi_{\text{nd}}$ and $n \in \tilde{N}_{\alpha,\Omega}$. Let $\beta \in \Phi$ and $u \in U'_{\beta,\Omega}$.

If ${}^\nu\mathcal{N}(n) = \text{id}$, then $n \in T$ by [BT72, 6.1.11(ii)] and therefore $n \in T_{\alpha,\Omega} \subset T_b$. Thus $nun^{-1} \in U'_{\beta,\Omega}$.

Otherwise α or $2\alpha \in \Phi_\Omega$ by Lemma 3.48, so that for any $x \in \Omega$, we have $-\alpha(x) = \lambda \in \Gamma_\alpha$. Denote by $\gamma = {}^\nu\mathcal{N}(n)(\beta)$. Then $n \in L_{\alpha,x}$ since $U_{\pm\alpha,\Omega} \subset U_{\pm\alpha,x}$ and therefore $n \in T_b m$ for any $m \in M_{\alpha,\lambda}$ by Proposition 3.27.

Let $x \in \Omega$ and $\varepsilon = \varphi_\beta(u) + \beta(x) > 0$. Then $nun^{-1} \in U_{\gamma,-\beta(x)+\varepsilon+\beta(\alpha^\vee)\lambda} = U_{\gamma,-\gamma(x)+\varepsilon}$ with $\gamma = r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$ according to Lemma 3.13. Thus $nun^{-1} \in U'_{\gamma,x}$ for every $x \in \Omega$ and therefore $nun^{-1} \in U'_\Omega$. Hence n normalizes U'_Ω since it is generated by the $U'_{\beta,\Omega}$.

Thus \tilde{N}_Ω normalizes U'_Ω since it is generated by those elements n according to Example 3.44(4). \square

In the following Proposition, we detail the proof of [BT72, 6.4.10] with some changes since we do not use infimum here.

3.50 Proposition. *The subset Φ_Ω is a sub-root system of Φ , i.e. a root system in the \mathbb{R} -subspace V_Ω^* generated by Φ_Ω .*

Moreover, the group homomorphism ${}^\nu\mathcal{N} : N \rightarrow \text{GL}(V^)$ induces a group homomorphism $\tilde{N}_\Omega \rightarrow \text{GL}(V_\Omega^*)$ sending n onto the restriction of ${}^\nu\mathcal{N}(n)$ to V_Ω^* with image the Weyl group of Φ_Ω and kernel T_Ω .*

Proof. Let $W = {}^\nu\mathcal{N}(\tilde{N}_\Omega)$ which is a subgroup of the Weyl group of Φ .

We firstly prove that Φ_Ω is stable by W^{aff} . Let $n \in \tilde{N}_\Omega$ and $w = {}^\nu\mathcal{N}(n) \in W(\Phi)$. Consider any $\alpha \in \Phi_\Omega$ and denote $\beta = w(\alpha)$. Consider an element $u \in U_\alpha$ such that $\forall x \in \Omega, \varphi_\alpha(u) = -\alpha(x)$ and $\forall v \in U_{2\alpha}, \varphi_\alpha(u) \geq \varphi_\alpha(uv)$ given by 3.47(ii). Let $\tilde{u} = nun^{-1}$. By [BT72, 6.1.2 (10)], we have $\tilde{u} \in U_\beta$. Since $n \in \tilde{N}_\Omega \subset U_\Omega$, we have $\tilde{u} \in U_\beta \cap U_\Omega = U_{\beta,\Omega}$ by Example 3.44(1). Let $\tilde{v} \in U_{2\beta}$ and denote $v = n^{-1}\tilde{v}n \in U_{w^{-1}(2\beta)} = U_{2\alpha}$. Suppose that $\tilde{u}\tilde{v} \in U'_{\beta,\Omega}$. Since \tilde{N}_Ω normalizes U'_Ω by Proposition 3.49, we have $uv = n^{-1}\tilde{u}\tilde{v}n \in U'_\Omega \cap U_\alpha = U'_{\alpha,\Omega}$ by Proposition 3.4(1) applied to U'_Ω (this is possible according to Example 3.44). This contradicts the assumption on u since for any $x \in \Omega$, we have $\varphi_\alpha(uv) > -\alpha(x) = \varphi_\alpha(u)$. Thus $\tilde{u}\tilde{v} \notin U'_{\beta,\Omega}$.

Finally, for $x \in \Omega$, we have $U'_{\beta,x} = U'_x \cap U_\beta = n(U'_x \cap U_\alpha)n^{-1} = nU'_{\alpha,x}n^{-1}$ since $n \in \tilde{N}_\Omega \subset N_x$ normalizes U'_x by Proposition 3.49. Thus for $x, y \in \Omega$, we have $U'_{\beta,x} = nU'_{\alpha,x}n^{-1} = nU'_{\alpha,y}n^{-1} = U'_{\beta,y}$. Thus, for $\tilde{v} = 1$, we have $\tilde{u} \notin U'_{\beta,\Omega} = U'_{\beta,x}$ for any $x \in \Omega$. Hence, for any $x \in \Omega$, we get $\varphi_\beta(\tilde{u}) \leq -\beta(x) \leq \varphi_\beta(u)$. Thus, by characterisation in Lemma 3.47(iii), we have $\beta \in \Phi_\Omega$.

Secondly, since \tilde{N}_Ω is generated by the $\tilde{N}_{\alpha,\Omega}$ according to Example 3.44(4), we get from Lemma 3.48 that W^{aff} is the group generated by the reflections r_α for $\alpha \in \Phi_\Omega$. Hence Φ_Ω is a root system inside V_Ω^* by definition and the restriction of elements in W^{aff} to V_Ω^* is the Weyl group of Φ_Ω .

Finally, the kernel of $\tilde{N}_\Omega \rightarrow \text{GL}(V_\Omega^*)$ is contained in $\ker \nu \cap \tilde{N}_\Omega = T \cap (N \cap U_\Omega) = T_\Omega$ by definition. \square

3.51 Example. Consider $G = \text{SL}_2(\mathbb{K})$, let T be the subgroup of diagonal matrices and U_α (resp. $U_{-\alpha}$) the subgroup of upper (resp. lower) unitriangular matrices, inducing a generating root group datum of G if we take $m = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $M_\alpha = M_{-\alpha} = mT$. There are parametrizations $\hat{\alpha} : \mathbb{K}^* \rightarrow T$, $u_\alpha : \mathbb{K} \rightarrow U_\alpha$ and $u_{-\alpha} : \mathbb{K} \rightarrow U_{-\alpha}$ of these groups given by $\hat{\alpha}(z) = \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & z \end{pmatrix}$, $u_\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $u_{-\alpha}(y) = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$. We get a valuation of the root group datum given by $\varphi_{\pm\alpha}(u_{\pm\alpha}(x)) = \omega(x)$. For $x, y \in \mathbb{K}$ with $\omega(x) \geq 0$ and $\omega(y) > 0$, define $u = u_\alpha(x) \in U_{\alpha,0}$ and $v = u_{-\alpha}(y) \in U'_{-\alpha,0}$. Then

$$[u, v] = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 - xy + x^2y^2 & x^2y \\ xy^2 & 1 + xy \end{pmatrix}$$

We set $z = 1 + xy \in \mathbb{K}^*$ since $\omega(xy) > 0$. Denote by $t = \hat{\alpha}(z) \in T$, by $u' = u_\alpha\left(\frac{x^2y}{z}\right)$ and by $v' = u_{-\alpha}\left(-\frac{xy^2}{z}\right)$. One can easily check that $[u, v] = u'tv'$ so that $t = \hat{\alpha}(1 + xy) = t(u, v) \in T'_{\alpha,0}$ for every x, y .

Assume, for instance, that $\omega : \mathbb{K} \rightarrow \mathbb{Z}$ is a discrete valuation of rank 1. Thus $T'_{\alpha,0} = \{\hat{\alpha}(z), \omega(z-1) \geq 1\}$. By the same way, we get $T'_{-\alpha,0} = T'_{\alpha,0}$ and $T'_0 = T \cap \langle U'_{\alpha,0}, U'_{-\alpha,0} \rangle = T \cap \langle U_{\alpha,1}, U_{-\alpha,1} \rangle = \{\hat{\alpha}(z), \omega(z-1) \geq 2\}$. In this case, we have $T'_0 \subsetneq \langle T'_{\alpha,0}, T'_{-\alpha,0} \rangle = T'_{\alpha,0}$.

This example and the second inclusion of Lemma 3.42 suggests to us introduce the following subgroup of T (which is denoted by H_{f,f^*} in [BT72, 6.4]).

3.52 Notation. Let T_Ω^* be the subgroup of T_b generated by the $T'_{\alpha,\Omega}$ for $\alpha \in \Phi$.

3.53 Lemma. *We have $T_\Omega^* \subset T_\Omega \subset T_b$ and all these groups are normal in T_b . In particular, all those inclusions are inclusions of normal subgroups.*

3.54 Remark. The second assertion of this lemma becomes obvious when $T = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})(\mathbb{K})$ is the group of rational points of the centralizer of a maximal split torus \mathbf{S} of a quasi-split reductive group \mathbf{G} . Indeed, in this case $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a maximal torus of \mathbf{G} by definition and thus T is commutative. But note that $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})(\mathbb{K})$ may not be commutative if \mathbf{G} is not quasi-split.

Proof. Since T_b normalizes the $U_{\alpha,\lambda}$ and the $U'_{\alpha,\lambda}$ for any $\lambda \in R$ and any $\alpha \in \Phi$, it normalizes $U_{\alpha,\Omega}$ and $U'_{\alpha,\Omega}$ as intersection of such groups. Therefore T_b normalizes U_Ω , U'_Ω and $L'_{\alpha,\Omega}$ for any $\alpha \in \Phi$. Since T_b is a subgroup of T , it normalizes the intersections $T_\Omega = T \cap U_\Omega$, $T'_\Omega = T \cap U'_\Omega$ and $T'_{\alpha,\Omega} = L'_{\alpha,\Omega} \cap T$ for any $\alpha \in \Phi$. In particular, T_b normalizes T_Ω^* being generated by the $T'_{\alpha,\Omega}$ for $\alpha \in \Phi$. Thus it suffices to prove the inclusions.

For $\alpha \in \Phi$, we have $T'_{\alpha,\Omega} = L'_{\alpha,\Omega} \cap T \subset U_\Omega \cap T$. Hence a generating set of T_Ω^* is contained in T_Ω and therefore $T_\Omega^* \subset T_\Omega$.

If $x \in \Omega$, then $U_{\alpha,\Omega} \subset U_{\alpha,x}$ for any $\alpha \in \Phi$. Hence $U_\Omega \subset U_x$ and thus $T_\Omega = T \cap U_\Omega \subset T_{\alpha,x} = T_{\alpha,-\alpha(x)} \subset T_b$ by Proposition 3.27(5). \square

We have the analogous to [BT72, 6.4.27]:

3.55 Lemma. *For any $\alpha \in \Phi$, the following commutator subgroup satisfies*

$$[T_\Omega^*, U_{\alpha, \Omega}] \subset U'_{\alpha, \Omega}.$$

Proof. Since $T_\Omega^* \subset T_b$ normalizes $U'_{\alpha, \Omega}$ and, by Lemma 3.42, $U_{\alpha, \Omega}$ normalizes $U'_{\alpha, \Omega}$, it suffices to prove that for a generating set X of T_Ω^* , for every $u \in U_{\alpha, \Omega}$ and every $x \in X$, we have $[x, u] \in U'_{\alpha, \Omega}$. We consider $X = \bigcup_{\beta \in \Phi} T'_{\beta, \Omega}$. Thus, let $\beta \in \Phi$, $u \in U_{\alpha, \Omega}$ and $t \in T'_{\beta, \Omega}$. Consider any $x \in \Omega$.

We firstly assume that α is non-divisible. Then $x \in U_{\alpha, x} = U_{\alpha, -\alpha(x)}$ and $t \in T'_{\beta, x} = T'_{\beta, -\beta(x)}$. We proceed by cases.

Case 1: $\beta \notin \mathbb{R}\alpha$ Then, by Proposition 3.45, we get that $[t, u] \in U'_{\alpha, -\alpha(x)} = U'_{\alpha, x}$.

Case 2: $\beta \in \mathbb{R}_{>0}\alpha$ Then $\beta \in \{\alpha, 2\alpha\}$ since α is non-divisible and $t \in T'_{\alpha, x}$ since $T'_{2\alpha, x} \subset T'_{\alpha, x}$. Thus, by Corollary 3.34(11), we get that $[t, u] \in U'_{\alpha, -\alpha(x)} = U'_{\alpha, x}$.

Case 3: $\beta \in \mathbb{R}_{<0}\alpha$ Then $\beta \in \{-\alpha, -2\alpha\}$ since α is non-divisible and $t \in T'_{-\alpha, x}$ since $T'_{-2\alpha, x} \subset T'_{-\alpha, x}$. Thus, by Corollary 3.34(12) applied to $-\alpha$, we get that $[t, u] \in U'_{\alpha, -\alpha(x)} = U'_{\alpha, x}$.

Finally, if α is divisible, we have that $u \in U_{\frac{\alpha}{2}, x}$ and we have ever shown that $[t, u] \in U'_{\frac{\alpha}{2}, x}$. But since T normalizes U_α , we have $[t, u] \in U_\alpha \cap U'_{\frac{\alpha}{2}, x} = U'_{\alpha, x}$.

Since the inclusion $[t, u] \in U'_{\alpha, x}$ holds for every $x \in \Omega$, we get $[t, u] \in \bigcap_{x \in \Omega} U'_{\alpha, x} = U'_{\alpha, \Omega}$. \square

3.56 Notation. For $\alpha \in \Phi$, we denote by

$$U_{\alpha, \Omega}^* = \begin{cases} U_{\alpha, \Omega} & \text{if } \alpha \notin \Phi_\Omega^* \\ U'_{\alpha, \Omega} & \text{if } \alpha \in \Phi_\Omega^* \end{cases}.$$

We denote by U_Ω^* the subgroup generated by T_Ω^* and the $U_{\alpha, \Omega}^*$ for $\alpha \in \Phi$.

3.57 Remark. If $\alpha \in \Phi_\Omega^*$, then $\mathbb{R}\alpha \cap \Phi \subset \Phi_\Omega^*$.

Indeed, let $\beta \in \mathbb{R}\alpha$ and write it $\beta = r\alpha$ with $2r \in \{\pm 1, \pm 2, \pm 4\}$. By definition, there is $\lambda_\alpha \in R$ such that $\forall x \in \Omega$, $-\alpha(x) = \lambda_\alpha$. Thus, with $2\lambda_\beta = 2r\lambda_\alpha$ (that belongs to R since it is a \mathbb{Z} -module), we have $\forall x \in \Omega$, $-2\beta(x) = -2r\alpha(x) = 2\lambda_\beta$ so that $\beta \in \Phi_\Omega^*$ since R is \mathbb{Z} -torsion free and $-\beta(x) \in R$ for any $x \in \mathbb{A}_R$.

The difference between Φ_Ω and Φ_Ω^* is that Φ_Ω takes the group structure into account whereas the set Φ_Ω^* only considers the structure of Ω . For instance, any point $x \in \mathbb{A}_R$ satisfies $\Phi_x^* = \Phi$ whereas we will have $W(\Phi_x) = W(\Phi)$ if, and only if, x is a special vertex.

We recall that, in Bruhat-Tits theory, there exist situations in which we never have $\Phi_x = \Phi$ (for instance a dense valuation with $\Gamma'_\alpha = \emptyset$ for a multipliable root $\alpha \in \Phi$, a discrete valuation with a totally ramified extension $\tilde{\mathbb{K}}/\mathbb{K}$ splitting \mathbf{G}).

The following Proposition corresponds to the beginning of the statement [BT72, 6.4.23] with a different proof since we do not define quasi-concave maps.

3.58 Proposition. *The family of groups $((U_{\alpha, \Omega}^*)_{\alpha \in \Phi}, T_\Omega^*)$ is quasi-concave and the group U_Ω^* is a normal subgroup of U_Ω .*

Proof. Since $U_{\alpha, \Omega}^* \subset U_\Omega$ for any $\alpha \in \Phi$ and, by Lemma 3.53, $T_\Omega^* \subset T_\Omega$, we know that U_Ω^* is a subgroup of U_Ω . Moreover, T_Ω^* normalizes the $U_{\alpha, \Omega}$ for $\alpha \in \Phi$ since T_Ω^* is a subgroup of T_b by Lemma 3.53, which gives (QC3). Let $\alpha \in \Phi$ be a non-divisible root. By Lemma 3.55, we have $[U_{\alpha, \Omega}, T_\Omega^*] \subset U'_{\alpha, \Omega} \subset U_{\alpha, \Omega}^*$. In particular, T_Ω^* normalizes $U_{\alpha, \Omega}^*$. By Example 3.44, since the families $(U_{\alpha, \Omega})_{\alpha \in \Phi}$ and $(U'_{\alpha, \Omega})_{\alpha \in \Phi}$ are quasi-concave and since $\alpha \in \Phi_\Omega^* \Leftrightarrow -\alpha \in \Phi_\Omega^*$, we deduce that the family $((U_{\alpha, \Omega}^*)_{\alpha \in \Phi}, T_\Omega^*)$ satisfies axiom (QC1).

Now, consider any $\beta \in \Phi$.

Case $\beta \in \Phi_\Omega^*$: If $\beta \in -\mathbb{R}_{>0}\alpha$, by Lemma 3.42, we get that $[U_{\alpha,\Omega}, U_{\beta,\Omega}^*] = [U_{\alpha,\Omega}, U'_{\beta,\Omega}] \subset U'_{\alpha,\Omega} U'_{-\alpha,\Omega} T'_{\alpha,\Omega} \subset U_\Omega^*$ (since $[U_{\alpha,\Omega}, U'_{-2\alpha,\Omega}] \subset [U_{\alpha,\Omega}, U'_{-\alpha,\Omega}]$ and α is non-divisible). Otherwise, by Lemma 3.43(QC2'), we get that $[U_{\alpha,\Omega}, U_{\beta,\Omega}^*] = [U_{\alpha,\Omega}, U'_{\beta,\Omega}] \subset \prod_{\gamma \in (\alpha,\beta)} U'_{\gamma,\Omega} \subset \prod_{\gamma \in (\alpha,\beta)} U_{\gamma,\Omega}^*$.

Case $\alpha, \beta \in \Phi \setminus \Phi_\Omega^*$: Then $[U_{\alpha,\Omega}, U_{\beta,\Omega}^*] = [U_{\alpha,\Omega}^*, U_{\beta,\Omega}^*] \subset U_\Omega^*$ by definition. Moreover, if $\beta \notin -\mathbb{R}_{>0}\alpha$, we have $[U_{\alpha,\Omega}^*, U_{\beta,\Omega}^*] \subset \prod_{\gamma \in (\alpha,\beta)} U_{\gamma,\Omega}^*$ since $(U_{\alpha,\Omega})_{\alpha \in \Phi}$ is quasi-concave.

Case $\beta \in \Phi \setminus \Phi_\Omega^*$ and $\alpha \in \Phi_\Omega^*$: By Remark 3.57, we know that $\beta \notin \mathbb{R}\alpha$. Since $U_{\beta,\Omega}^* = U_{\beta,\Omega}$, we have inclusion $[U_{\alpha,\Omega}, U_{\beta,\Omega}^*] \subset U_{(\alpha,\beta),\Omega}$. Let $\gamma = r\alpha + s\beta \in (\alpha,\beta)$ with $r, s \in \mathbb{Z}_{>0}$. By definition, the linear form α is constant on Ω but β is not. Therefore γ is non-constant so that $\gamma \notin \Phi_\Omega^*$. Thus for every $\gamma \in (\alpha,\beta)$, we have $U_{\gamma,\Omega} = U_{\gamma,\Omega}^*$.

Thus from the three cases, we deduce that axiom (QC2) is satisfied by the family $(U_{\alpha,\Omega}^*)_{\alpha \in \Phi}$ and that $U_{\alpha,\Omega}$ normalizes U_Ω^* since it is generated by T_Ω^* and the $U_{\beta,\Omega}^*$ for $\beta \in \Phi$. Also does $U_{2\alpha,\Omega}$ since it is a subgroup of $U_{\alpha,\Omega}$. Since $U_{\alpha,\Omega}$ normalizes U_Ω^* for every $\alpha \in \Phi$, the group U_Ω generated by the $U_{\alpha,\Omega}$ also normalizes U_Ω^* . \square

3.59 Notation. We denote by $\overline{G_\Omega}$ the quotient group U_Ω/U_Ω^* .

For $\alpha \in \Phi$, we denote by $\overline{U_{\alpha,\Omega}}$ the canonical image of $U_{\alpha,\Omega}$ in $\overline{G_\Omega}$.

We denote by $\overline{T_\Omega}$ the canonical image of T_Ω in $\overline{G_\Omega}$.

For $\alpha \in \Phi_\Omega^*$, we denote by $\overline{M_{\alpha,\Omega}}$ the canonical image of $M_{\alpha,\lambda_\alpha}$ with $\lambda_\alpha = -\alpha(x)$ for any $x \in \Omega$ (it does not depend on the choice of $x \in \Omega$ by definition of Φ_Ω^*).

The following Lemma corresponds to part of [BT72, 6.4.23] with a completely different proof.

3.60 Lemma. For any $\alpha \in \Phi$, we have $\overline{U_{\alpha,\Omega}} \subset \overline{U_{2\alpha,\Omega}} \iff \alpha \notin \Phi_\Omega$.

Proof. Assume that $\overline{U_{\alpha,\Omega}} \not\subset \overline{U_{2\alpha,\Omega}}$. Suppose, by contradiction that $U_{\alpha,\Omega} \subset U_{2\alpha} U'_{\alpha,\Omega}$. Let $u \in U_{\alpha,\Omega}$ and write it $u = vw$ with $v \in U'_{\alpha,\Omega}$, $w \in U_{2\alpha}$. Then $w = v^{-1}u \in U_{\alpha,\Omega} \cap U_{2\alpha} = U_{2\alpha,\Omega}$. Since $v \in U'_{\alpha,\Omega} \subset U_\Omega^*$, it contradicts the assumption. Hence there exists $u \in U_{\alpha,\Omega} \setminus U_{2\alpha} U'_{\alpha,\Omega}$. Let $\lambda_\alpha = \varphi_\alpha(u) \in \Gamma_\alpha$. We prove that $\lambda_\alpha \in \Gamma'_\alpha$. Indeed, let $v \in U_{2\alpha}$. If, by contradiction, we have $\varphi_\alpha(uv) > \varphi_\alpha(u)$, then $uv \in U'_{\alpha,\Omega}$ and therefore $u \in U'_{\alpha,\Omega} U_{2\alpha}$ which contradicts the definition of u . Thus $\varphi_\alpha(uv) \leq \varphi_\alpha(u)$ for every $v \in U_{2\alpha}$ and therefore $U_{\alpha,\varphi_\alpha(u)} = \bigcap_{v \in U_{2\alpha}} U_{\alpha,\varphi_\alpha(uv)}$ which means $\varphi_\alpha(u) = \lambda_\alpha \in \Gamma'_\alpha$. Now, for any $x \in \Omega$, we have $u \in U_{\alpha,x}$ and therefore $-\alpha(x) \leq \varphi_\alpha(u)$. But if $-\alpha(x) < \varphi_\alpha(u)$ for every $x \in \Omega$, then $u \in \bigcap_{x \in \Omega} U'_{\alpha,x} = U'_{\alpha,\Omega}$ which contradicts the definition of u . Thus, there exists an element $y \in \Omega$ such that $-\alpha(y) = \varphi_\alpha(u) = \lambda_\alpha \in \Gamma'_\alpha$. Finally, $\alpha \in \Phi_\Omega^*$ since $\overline{U_{\alpha,\Omega}}$ is not trivial. Thus for every $x \in \Omega$, we have $-\alpha(x) = -\alpha(y) = \lambda_\alpha \in \Gamma'_\alpha$ which proves $\alpha \in \Phi_\Omega$.

Conversely, let $\alpha \in \Phi_\Omega \subset \Phi_\Omega^*$. Then $U_{\alpha,\Omega}^* = U'_{\alpha,\Omega}$ by definition. Let $u \in U_\alpha$ such that $\forall x \in \Omega$, $\varphi_\alpha(u) = -\alpha(x)$ and $\forall v \in U_{2\alpha}$, $\varphi_\alpha(uv) \leq \varphi_\alpha(u)$ given by Lemma 3.47(ii). By contradiction, suppose that $u \in U_{\alpha,\Omega}^* U_{2\alpha,\Omega}$. Then we would have some $v \in U_{2\alpha,\Omega}$ such that $uv \in U_{\alpha,\Omega}^* = U'_{\alpha,\Omega} = U'_{\alpha,x}$ for any $x \in \Omega$. Thus $\varphi_\alpha(uv) > -\alpha(x) = \varphi_\alpha(u)$ which contradicts the assumption on u . Hence $U_{\alpha,\Omega} \not\subset U_{\alpha,\Omega}^* U_{2\alpha,\Omega}$. This gives $\overline{U_{\alpha,\Omega}} \not\subset \overline{U_{2\alpha,\Omega}}$ because if we had $\overline{U_{\alpha,\Omega}} \subset \overline{U_{2\alpha,\Omega}}$, then we would have $U_{\alpha,\Omega} \subset U_{2\alpha,\Omega} U_\Omega^* \cap U_\alpha$. But $U_\Omega^* \cap U_\alpha = U_{\alpha,\Omega}^*$ according to Proposition 3.4(1) since the family $((U_{\alpha,\Omega}^*)_{\alpha \in \Phi}, T_\Omega^*)$ is quasi-concave. \square

The following theorem summarizes the work of this section and of the previous one. It corresponds to the last statement of [BT72, 6.4.23], applied to $X = T_\Omega$ and $X^* = T_\Omega^*$.

3.61 Theorem. The system $\left(\overline{T_\Omega}, (\overline{U_{\alpha,\Omega}}, \overline{M_{\alpha,\Omega}})_{\alpha \in \Phi_\Omega} \right)$ is a generating root group datum of type Φ_Ω in $\overline{G_\Omega}$.

Proof. Axiom (RGD1) is satisfied because if $\overline{U_{\alpha,\Omega}} = 1$, then $U_{\alpha,\Omega} \subset U_{\Omega}^* \cap U_{\alpha} = U'_{\alpha,\Omega}$ would contradict the assumption $\lambda_{\alpha} \in \Gamma'_{\alpha}$ in the definition of $\alpha \in \Phi_{\Omega}$.

Axiom (RGD2) is satisfied because the family $(U_{\alpha,\Omega})_{\alpha \in \Phi}$ is quasi-concave by Example 3.44.

Axiom (RGD3) is a consequence of Lemma 3.60.

Axiom (RGD4) is a consequence of Proposition 3.12.

Axiom (RGD5) is a consequence of Proposition 3.50.

Axiom (RGD6) is a consequence of Proposition 3.4.

It is a generating root group datum since the $U_{\alpha,\Omega}$ generates U_{Ω} by definition. \square

4 Parahoric subgroups and Bruhat decomposition

In this chapter, we work under data and notations of 3.22. We assume that the root system is non-empty. Any statement can be generalized for an empty root system by the trivial way ($G = N = T$ and $\mathbb{A}_R = \{o\}$ when Φ is empty).

4.1 Parahoric subgroups

In this section, we will consider various subsets Ω of \mathbb{A}_R .

4.1 Notation. For a non-empty subset $\Omega \subset \mathbb{A}_R$, we denote by N_{Ω} the subgroup of G generated by \tilde{N}_{Ω} and T_b (as defined in [BT72, 7.1.3]) and by P_{Ω} (resp. P'_{Ω}) the subgroup of G generated by T_b and U_{Ω} (resp. T_b and U'_{Ω}).

4.2 Notation. For any basis Δ of Φ , we denote by U_{Δ}^+ (resp. U_{Δ}^-) the subgroup of G generated by the U_{α} (resp. $U_{-\alpha}$) for $\alpha \in \Phi_{\Delta}^+$.

4.3 Lemma. *For any non-empty subset $\Omega \subset \mathbb{A}_R$, any basis Δ of Φ , we have:*

- $P_{\Omega} \cap N = N_{\Omega} = T_b \tilde{N}_{\Omega} = \tilde{N}_{\Omega} T_b$;
- $P_{\Omega} \cap U_{\Delta}^+ = U_{\Omega} \cap U_{\Delta}^+$;
- $P_{\Omega} \cap U_{\Delta}^- = U_{\Omega} \cap U_{\Delta}^-$;
- $P_{\Omega} = (P_{\Omega} \cap U_{\Delta}^+)(P_{\Omega} \cap U_{\Delta}^-)(P_{\Omega} \cap N) = T_b U_{\Omega}$.

Proof. Since $T_b \subset N$ normalizes U_{Ω} , it normalizes $\tilde{N}_{\Omega} = N \cap U_{\Omega}$ so that $N_{\Omega} = T_b \tilde{N}_{\Omega} = \tilde{N}_{\Omega} T_b$ and $P_{\Omega} = U_{\Omega} T_b = (U_{\Omega} \cap U_{\Delta}^+)(U_{\Omega} \cap U_{\Delta}^-) \tilde{N}_{\Omega} T_b$ according to example 3.44(3).

Let $p \in P_{\Omega}$ and write it as $p = uvn$ with $n \in \tilde{N}_{\Omega} T_b \subset N$, $u \in U_{\Omega} \cap U_{\Delta}^+$ and $v \in U_{\Omega} \cap U_{\Delta}^-$. If $p \in N$, then $pn^{-1} = uv$ gives $p = n$ by spherical Bruhat decomposition [BT72, 6.1.15(c)]. Thus $N \cap P_{\Omega} = N_{\Omega}$. If $p \in U_{\Delta}^+$, then $n = v^{-1}(u^{-1}p) \in U_{\Delta}^- U_{\Delta}^+$. Thus $n = 1$ by [BT72, 6.1.15(c)] again and $u^{-1}p = v \in U_{\Delta}^- \cap U_{\Delta}^+ = \{1\}$ by (RGD6). Thus $p = u$ and therefore $P_{\Omega} \cap U_{\Delta}^+ = U_{\Omega} \cap U_{\Delta}^+$. By an analogous method, we get that $P_{\Omega} \cap U_{\Delta}^- = U_{\Omega} \cap U_{\Delta}^-$. \square

4.4 Notation. We denote by:

$$\hat{N}_{\Omega} = \{n \in N, \forall x \in \Omega, \nu(n)(x) = x\}$$

the pointwise stabilizer of Ω in N and by \hat{P}_{Ω} (resp. \hat{P}'_{Ω}) the subgroup of G generated by U_{Ω} (resp. U'_{Ω}) and \hat{N}_{Ω} . The group \hat{P}_{Ω} is called the parahoric subgroup of Ω in G . It is constructed to be the pointwise stabilizer of Ω in G (see Lemma 5.5).

4.5 Fact. *If $\Omega' \subset \Omega \subset \mathbb{A}_R$ are two non-empty subsets, then U_{Ω} is a subgroup of $U_{\Omega'}$.*

Proof. For $\alpha \in \Phi$, we have $U_{\alpha,\Omega} = \bigcap_{x \in \Omega} U_{\alpha,x} \subset \bigcap_{x \in \Omega'} U_{\alpha,x} = U_{\alpha,\Omega'}$. Thus $U_{\Omega} \subset U_{\Omega'}$ since these groups are respectively generated by the $U_{\alpha,\Omega}$, $U_{\alpha,\Omega'}$ for $\alpha \in \Phi$. \square

4.6 Remark. As a consequence, the same inclusions hold for groups $\tilde{N}_{\Omega} = N \cap U_{\Omega}$, $T_{\Omega} = T \cap U_{\Omega}$, $P_{\Omega} = T_b U_{\Omega}$, $N_{\Omega} = N \cap P_{\Omega}$, \hat{N}_{Ω} being the pointwise stabilizer of Ω in N , and finally $\hat{P}_{\Omega} = \hat{N}_{\Omega} U_{\Omega}$.

4.2 Action of N on parahoric subgroups

The action of N on \mathbb{A}_R can be compared with the action of N on V^* as follows:

4.7 Lemma. *For any $\alpha \in \Phi$, any $n \in N$ and any $x \in \Omega$, we have:*

$${}^v\nu(n)(\alpha)(x - o) = \alpha(\nu(n^{-1})(x) - \nu(n^{-1})(o)).$$

Proof. We know that $\nu(N)$ is contained in the subgroup of $\text{Aff}_{\mathbb{Z}}(\mathbb{A}_R)$ generated by the $r_{\beta,\mu}$ for $\beta \in \Phi$ and $\mu \in R$ and that ${}^v\nu(M_{\beta}) = \{r_{\beta}\}$ by (CA1) and [BT72, 6.1.2(10)]. Moreover, N is generated by the M_{β} for $\beta \in \Phi$. Thus, it suffices to prove that for any $x \in \mathbb{A}_R$, any $\beta \in \Phi$ and any $\mu \in R$, we have:

$$r_{\beta}(\alpha)(x - o) = \alpha(r_{\beta,\mu}^{-1}(x) - r_{\beta,\mu}^{-1}(o)).$$

Note that $r_{\beta,\mu}^{-1} = r_{\beta,\mu}$. For any $x \in \mathbb{A}_R$, we have:

$$\begin{aligned} r_{\beta,\mu}(x) - r_{\beta,\mu}(o) &= (x - (\beta(x - o) + \mu)\beta^{\vee}) - (o - (\beta(o - o) + \mu)\beta^{\vee}) \\ &= x - o - \beta(x - o)\beta^{\vee} \end{aligned}$$

Hence

$$\alpha(r_{\beta,\mu}(x) - r_{\beta,\mu}(o)) = \alpha(x - o) - \alpha(\beta^{\vee})\beta(x - o) = r_{\beta}(\alpha)(x - o). \quad \square$$

4.8 Lemma. *For any $n \in N$ and any $\alpha \in \Phi$, we have $nU_{\alpha,\Omega}n^{-1} = U_{{}^v\nu(n)(\alpha),{}^v\nu(n)(\Omega)}$ and $nU'_{\alpha,\Omega}n^{-1} = U'_{{}^v\nu(n)(\alpha),{}^v\nu(n)(\Omega)}$.*

Proof. Let $x \in \Omega$. We have $nU_{\alpha,x}n^{-1} = U_{{}^v\nu(\alpha), -\alpha(x-o) + \alpha(\nu(n^{-1})(o)-o)}$ according to Corollary 3.21. But $-\alpha(x - o) + \alpha(\nu(n^{-1})(o) - o) = \alpha(\nu(n^{-1})(o) - \nu(n^{-1}) \circ \nu(n)(x)) = -{}^v\nu(n)(\alpha)(\nu(n)(x) - o)$ according to Lemma 4.7. Thus $nU_{\alpha,x}n^{-1} = U_{{}^v\nu(n)(\alpha),{}^v\nu(n)(x)}$ by definition. Hence $nU_{\alpha,\Omega}n^{-1} = \bigcap_{x \in \Omega} nU_{\alpha,x}n^{-1} = \bigcap_{x \in \Omega} U_{{}^v\nu(n)(\alpha),{}^v\nu(n)(x)} = U_{{}^v\nu(n)(\alpha),{}^v\nu(n)(\Omega)}$. We proceed in the same way for $U'_{\alpha,\Omega} = \bigcap_{x \in \Omega} \bigcup_{\varepsilon > 0} U_{\alpha, -\alpha(x) + \varepsilon}$. \square

4.9 Proposition. *For any $n \in N$, we have $nU_{\Omega}n^{-1} = U_{{}^v\nu(n)(\Omega)}$ and $nU'_{\Omega}n^{-1} = U'_{{}^v\nu(n)(\Omega)}$.*

Proof. By Lemma 4.8, we have $nU_{\alpha,\Omega}n^{-1} = U_{{}^v\nu(n)(\alpha),{}^v\nu(n)(\Omega)}$. Since $U_{{}^v\nu(n)(\Omega)}$ is generated by the $U_{\beta,{}^v\nu(n)(\Omega)}$ for $\beta \in \Phi$ and ${}^v\nu(\Phi) = \Phi$ by definition of root systems, we are done. We proceed in the same way for U'_{Ω} . \square

4.10 Corollary. *The group \hat{N}_{Ω} normalizes U_{Ω} and U'_{Ω} .*

Proof. Since ${}^v\nu(n)(\Omega) = \Omega$ for any $n \in \hat{N}_{\Omega}$, we are done by Proposition 4.9. \square

4.11 Corollary. *For any $n \in N$, we have $n\hat{P}_{\Omega}n^{-1} = \hat{P}_{{}^v\nu(n)(\Omega)}$ and $n\hat{P}'_{\Omega}n^{-1} = \hat{P}'_{{}^v\nu(n)(\Omega)}$.*

Proof. By definition, $n\hat{N}_{\Omega}n^{-1} = \hat{N}_{{}^v\nu(n)(\Omega)}$ being a pointwise stabilizer in N . We have $\hat{P}_{\Omega} = \hat{N}_{\Omega}U_{\Omega}$ since \hat{N}_{Ω} normalizes U_{Ω} by Corollary 4.10. Thus, by applying Proposition 4.9, we get that $n\hat{P}_{\Omega}n^{-1} = \hat{N}_{{}^v\nu(n)(\Omega)}U_{{}^v\nu(n)(\Omega)}$. The same holds for \hat{P}'_{Ω} . \square

4.12 Corollary. For any non-empty subset $\Omega \subset \mathbb{A}_R$, any basis Δ of Φ , we have:

- $\widehat{P}_\Omega \cap N = \widehat{N}_\Omega$;
- $\widehat{P}_\Omega \cap U_\Delta^+ = U_\Omega \cap U_\Delta^+$;
- $\widehat{P}_\Omega \cap U_\Delta^- = U_\Omega \cap U_\Delta^-$;
- $\widehat{P}_\Omega = (\widehat{P}_\Omega \cap U_\Delta^+)(\widehat{P}_\Omega \cap U_\Delta^-)(\widehat{P}_\Omega \cap N) = U_\Omega \widehat{N}_\Omega$.

Proof. By Corollary 4.10, we have $\widehat{P}_\Omega = U_\Omega \widehat{N}_\Omega$. Thus, one can conclude as in Lemma 4.3 using [BT72, 6.1.15(c)]. \square

4.13 Corollary. We have $\widetilde{N}_\Omega \subset N_\Omega \subset \widehat{N}_\Omega$. In particular, \widetilde{N}_Ω and N_Ω fix Ω .

Proof. By definition, we have $U_\Omega \subset P_\Omega \subset \widehat{P}_\Omega$. Thus, by intersection with N , by Lemma 4.3 and Corollary 4.12, we deduce $\widetilde{N}_\Omega \subset N_\Omega \subset \widehat{N}_\Omega$. \square

4.14 Lemma. The group \widehat{N}_Ω normalizes T_Ω^* .

Proof. For any $\alpha \in \Phi$ and any $n \in \widehat{N}_\Omega$, we get that $nU_{\alpha,\Omega}n^{-1} = U_{\nu(n)(\alpha),\Omega}$ and $nU'_{-\alpha,\Omega}n^{-1} = U'_{-\nu(n)(\alpha),\Omega}$ by Lemma 4.8 since $\nu(n)(\Omega) = \Omega$. Thus $nL'_{\alpha,\Omega}n^{-1} = L'_{\nu(n)(\alpha),\Omega}$. Since N normalizes T , by intersection with T , we get $nT'_{\alpha,\Omega}n^{-1} = T'_{\nu(n)(\alpha),\Omega}$. Thus n normalizes T_Ω^* since it is generated by the $T'_{\alpha,\Omega}$ for $\alpha \in \Phi$ and $\nu(n)(\Phi) = \Phi$. Hence \widehat{N}_Ω normalizes T_Ω^* . \square

4.15 Proposition. The subset $U'_\Omega T_\Omega^*$ is a normal subgroup of \widehat{P}_Ω .

Proof. Let us recall that $T_\Omega^* \subset T_\Omega \subset T_b$ by Lemma 3.53. Thus T_Ω^* normalizes U'_Ω so that $U'_\Omega T_\Omega^*$ is a subgroup of $U_\Omega = U_\Omega T_\Omega$, therefore of \widehat{P}_Ω .

Let $\alpha \in \Phi$ and $u \in U_{\alpha,\Omega}$. By Lemma 3.55, we have $[U_{\alpha,\Omega}, T_\Omega^*] \subset U'_{\alpha,\Omega}$. Thus $uT_\Omega^*u^{-1} \subset U'_\Omega T_\Omega^*$. By Lemmas 3.42 and 3.43(QC2'), we get that $[U_{\alpha,\Omega}, U'_{\beta,\Omega}] \subset U'_\Omega T_\Omega^*$. Thus $uU'_{\beta,\Omega}u^{-1} \subset U'_\Omega T_\Omega^*$. Hence $U_{\alpha,\Omega}$ normalizes $U'_\Omega T_\Omega^*$ for any $\alpha \in \Phi$. Therefore U_Ω normalizes $U'_\Omega T_\Omega^*$.

Moreover, according to Lemma 4.14 and Corollary 4.10, we deduce that \widehat{N}_Ω and thus \widehat{P}_Ω normalizes $U'_\Omega T_\Omega^*$. \square

4.16 Proposition. The group \widehat{N}_Ω normalizes U_Ω^*

Proof. We firstly prove that the action of \widehat{N}_Ω on Φ via ν stabilizes Φ_Ω^* . Let $\alpha \in \Phi_\Omega^*$ and $n \in \widehat{N}_\Omega$. By definition, there is a constant $\lambda_\alpha \in R$ such that $\forall x \in \Omega$, $-\alpha(x) = \lambda_\alpha$. For any $x \in \mathbb{A}_R$, we have $\nu(n)(\alpha)(x - o) = \alpha(\nu(n^{-1})(x) - \nu(n^{-1})(o))$ by Lemma 4.7. If $x \in \Omega$, then $n\nu(n^{-1})(x) = x$ so that $-\nu(n)(\alpha)(x - o) - \alpha(x - \nu(n^{-1})(o)) = -\alpha(x - o) + \alpha(\nu(n^{-1})(o) - o) = \lambda_\alpha + \alpha(\nu(n^{-1})(o) - o)$. Hence, $\nu(n)(\alpha) \in \Phi_\Omega^*$ and therefore \widehat{N}_Ω stabilizes Φ_Ω^* .

Let $n \in \widehat{N}_\Omega$ and $\alpha \in \Phi$. If $\alpha \in \Phi_\Omega^*$, we have $U_{\alpha,\Omega}^* = U'_{\alpha,\Omega}$ by definition. Thus, by Lemma 4.8, we have $nU_{\alpha,\Omega}^*n^{-1} = U'_{\nu(n)(\alpha),\Omega} = U_{\nu(n)(\alpha),\Omega}^*$ since $\nu(n)(\alpha) \in \Phi_\Omega^*$ and $\nu(n)(\Omega) = \Omega$. If $\alpha \notin \Phi_\Omega^*$, then we have $U_{\alpha,\Omega}^* = U_{\alpha,\Omega}$ by definition. Thus, by Lemma 4.8, we have $nU_{\alpha,\Omega}^*n^{-1} = U_{\nu(n)(\alpha),\Omega} = U_{\nu(n)(\alpha),\Omega}^*$ since $\nu(n)(\alpha) \notin \Phi_\Omega^*$ and $\nu(n)(\Omega) = \Omega$.

Since U_Ω^* is generated by the $U_{\alpha,\Omega}^*$ for $\alpha \in \Phi$, we got that n normalizes U_Ω^* for any $n \in \widehat{N}_\Omega$. \square

4.17 Corollary. The group U_Ω^* is a normal subgroup of \widehat{P}_Ω .

Proof. It is a immediate consequence of Proposition 4.16 and Proposition 3.58 since \widehat{P}_Ω is generated by \widehat{N}_Ω and U_Ω . \square

4.3 Parahoric subgroups as intersections over their fixed points

We get propositions analogous to that of [BT72, 7.1.4] as follows:

4.18 Lemma. *Let Δ be a basis of Φ with order Φ_Δ^+ in Φ . Let $C_{R,\Delta}^v$ be the vector chamber over Δ . Then, for any $x \in \mathbb{A}_R$, we have $U_{x+C_{R,\Delta}^v} \subset U_\Delta^+$. In particular, $\tilde{N}_{x+C_{R,\Delta}^v} = \{1\}$ and $N_{x+C_{R,\Delta}^v} = T_b$.*

Proof. Let $\Omega = x + C_{R,\Delta}^v$. Let $v \in C_{\mathbb{Z},\Delta}^v$ so that $x + \delta_v \subset x + C_{R,\Delta}^v$ according to Lemma 2.5. Let $\alpha \in \Phi_\Delta^+$. Thus $\alpha(v) \in \mathbb{Z}_{>0}$ by definition of v .

Let $\varepsilon \in R_{>0}$. Then $y \in x + \varepsilon v \in x + \delta_v \subset \Omega$. Moreover, we have $\alpha(y) = \alpha(x) + \alpha(\varepsilon v) = \alpha(x) + \alpha(v)\varepsilon$. Hence $U_{-\alpha,\Omega} \subset U_{-\alpha,y} = \varphi_{-\alpha}^{-1}([\alpha(x) + \alpha(v)\varepsilon, \infty])$. Since this is true for any $\varepsilon > 0$ and $\bigcap_{\varepsilon>0} [\alpha(x) + \alpha(v)\varepsilon, \infty] = \bigcap_{\varepsilon>0} [\alpha(x) + \varepsilon, \infty] = \{\infty\}$ (because $\alpha(v) \in \mathbb{Z}_{>0}$ and R is \mathbb{Z} -torsion free), we get that $U_{-\alpha,\Omega} = U_{-\alpha,\infty} = \{1\}$. Thus, U_Ω is generated by $U_{\alpha,\Omega}$ for $\alpha \in \Phi_\Delta^+$, therefore is a subgroup of U_Δ^+ . We conclude by applying Lemma 4.3. \square

4.19 Lemma. *For any basis Δ of Φ , any positive root $\alpha \in \Phi_\Delta^+$ and any non-empty subset $\Omega \subset \mathbb{A}_R$, we have*

$$U_{\alpha,\Omega+C_{R,\Delta}^v} = U_{\alpha,\Omega+\overline{C}_{R,\Delta}^v} = U_{\alpha,\Omega}.$$

Proof. Let $v \in C_{\mathbb{Z},\Delta}^v$ so that $\alpha(v) \in \mathbb{Z}_{>0}$. Since $\Omega + \overline{C}_{R,\Delta}^v \supset \Omega + C_{R,\Delta}^v \supset \Omega + \delta_v$, we have $U_{\alpha,\Omega+\overline{C}_{R,\Delta}^v} \subset U_{\alpha,\Omega+C_{R,\Delta}^v} \subset U_{\alpha,\Omega+\delta_v}$. Thus, it suffices to prove that $U_{\alpha,\Omega+\delta_v} \subset U_{\alpha,\Omega} \subset U_{\alpha,\Omega+\overline{C}_{R,\Delta}^v}$.

Let $u \in U_{\alpha,\Omega+\delta_v}$. Then for any $y \in \Omega$, any $\varepsilon \in R_{>0}$, if $\lambda = \frac{1}{\alpha(v)}\varepsilon$, then we have by definition:

$$\varphi_\alpha(u) \geq -\alpha(y + \lambda v) = -\alpha(y) - \varepsilon$$

since $y + \lambda v \in \Omega + \delta_v$. Since this inequality holds for every $\varepsilon > 0$, we get that $\varphi_\alpha(u) \geq -\alpha(y)$. Thus $u \in U_{\alpha,\Omega}$ by definition. For any $x \in \Omega + \overline{C}_{R,\Delta}^v$, write $x = y + z$ with $y \in \Omega$ and $z \in \overline{C}_{R,\Delta}^v$. Then $-\alpha(z) \leq 0$ by definition so that $-\alpha(x) \leq -\alpha(y) \leq \varphi_\alpha(u)$. Thus, $U_{\alpha,\Omega} \subset U_{\alpha,\Omega+\overline{C}_{R,\Delta}^v}$. \square

4.20 Proposition. *For any non-empty subset $\Omega \subset \mathbb{A}_R$, and any basis Δ of Φ , we have*

$$P_\Omega \cap U_\Delta^+ = U_{\Omega+C_{R,\Delta}^v} = U_{\Omega+\overline{C}_{R,\Delta}^v} \quad P_\Omega \cap U_\Delta^- = U_{\Omega-\overline{C}_{R,\Delta}^v} = U_{\Omega-C_{R,\Delta}^v}$$

Proof. On the one hand, we have $\Omega + C_{R,\Delta}^v \subset \Omega + \overline{C}_{R,\Delta}^v$ so that $U_{\Omega+\overline{C}_{R,\Delta}^v} \subset U_{\Omega+C_{R,\Delta}^v} \subset U_\Delta^+$ by Lemma 4.18. Hence, $U_{\Omega+\overline{C}_{R,\Delta}^v}$ and $U_{\Omega+C_{R,\Delta}^v}$ are generated by the same subgroups $U_{\alpha,\Omega+\overline{C}_{R,\Delta}^v} = U_{\alpha,\Omega+C_{R,\Delta}^v} = U_{\alpha,\Omega}$ for $\alpha \in (\Phi_\Delta^+)_{\text{nd}}$ according to Example 3.44 and Lemma 4.19. Thus $U_{\Omega+\overline{C}_{R,\Delta}^v} = U_{\Omega+C_{R,\Delta}^v} \subset P_\Omega \cap U_\Delta^+$.

On the other hand, we have $P_\Omega \cap U_\Delta^+ = U_\Omega \cap U_\Delta^+$ by Lemma 4.3 and this group is generated by the $U_{\alpha,\Omega}$ for $\alpha \in (\Phi_\Delta^+)_{\text{nd}}$ according to Example 3.44. Hence we get the first equalities.

Since $U_\Delta^- = U_{-\Delta}^+$ and $-\overline{C}_{R,\Delta}^v = \overline{C}_{R,-\Delta}^v$, we get the second equalities by applying the first ones with $-\Delta$ instead of Δ . \square

4.21 Corollary. *For any non-empty subset $\Omega \subset \mathbb{A}_R$ and any basis Δ of Φ , we have $P_\Omega = \tilde{N}_\Omega U_{\Omega+\overline{C}_{R,\Delta}^v} U_{\Omega-\overline{C}_{R,\Delta}^v} = \tilde{N}_\Omega U_{\Omega+C_{R,\Delta}^v} U_{\Omega-C_{R,\Delta}^v}$.*

Proof. This is a combination of Proposition 4.20 with Lemma 4.3. \square

Thus we get, as in [BT72, 7.1.8]:

4.22 Corollary. For any non-empty subset $\Omega \subset \mathbb{A}_R$ and any basis Δ of Φ , we have $\widehat{P}_\Omega = \widehat{N}_\Omega U_{\Omega + \overline{C}_{R,\Delta}^v} U_{\Omega - \overline{C}_{R,\Delta}^v} = \widehat{N}_\Omega U_{\Omega + C_{R,\Delta}^v} U_{\Omega - C_{R,\Delta}^v}$. Moreover

$$\widehat{P}_\Omega \cap N = \widehat{N}_\Omega \quad \widehat{P}_\Omega \cap U_\Delta^+ = U_{\Omega + \overline{C}_{R,\Delta}^v} = U_{\Omega + C_{R,\Delta}^v} \quad \widehat{P}_\Omega \cap U_\Delta^- = U_{\Omega - \overline{C}_{R,\Delta}^v} = U_{\Omega - C_{R,\Delta}^v}$$

Moreover, P_Ω and U_Ω are normal subgroups of \widehat{P}_Ω .

Proof. This is a combination of Proposition 4.20, Corollary 4.10 and spherical Bruhat decomposition [BT72, 7.1.15(c)]. \square

The previous Proposition 4.20 and Corollary 4.22 enables us to state the following proposition (see [BT72, 7.1.5]).

4.23 Proposition. Let Ω and Ω' be two subsets of \mathbb{A}_R . Let Δ be any basis of Φ .

(1) If $\Omega' \subset \Omega + \overline{C}_{R,\Delta}^v$, then $P_\Omega P_{\Omega'} \subset \widetilde{N}_\Omega U_\Delta^- U_{\Omega' + \overline{C}_{R,\Delta}^v} \widetilde{N}_{\Omega'}$.

(2) If $\Omega' \subset \Omega + \overline{C}_{R,\Delta}^v$ and $\Omega \subset \Omega' - \overline{C}_{R,\Delta}^v$, then $P_\Omega P_{\Omega'} = \widetilde{N}_\Omega U_{\Omega - \overline{C}_{R,\Delta}^v} U_{\Omega' + \overline{C}_{R,\Delta}^v} \widetilde{N}_{\Omega'}$.

Proof. According to Corollary 4.21, one can write

$$P_\Omega P_{\Omega'} = \widetilde{N}_\Omega U_{\Omega - \overline{C}_{R,\Delta}^v} U_{\Omega + \overline{C}_{R,\Delta}^v} U_{\Omega' + \overline{C}_{R,\Delta}^v} U_{\Omega' - \overline{C}_{R,\Delta}^v} \widetilde{N}_{\Omega'}$$

Assume that $\Omega' \subset \Omega + \overline{C}_{R,\Delta}^v$. Thus $\Omega' + \overline{C}_{R,\Delta}^v \subset \Omega + \overline{C}_{R,\Delta}^v$ so that $U_{\Omega + \overline{C}_{R,\Delta}^v} \subset U_{\Omega' + \overline{C}_{R,\Delta}^v}$. Hence $P_\Omega P_{\Omega'} = \widetilde{N}_\Omega U_{\Omega - \overline{C}_{R,\Delta}^v} P_{\Omega'}$. Now, write $P_{\Omega'} = U_{\Omega' - \overline{C}_{R,\Delta}^v} U_{\Omega' + \overline{C}_{R,\Delta}^v} \widetilde{N}_{\Omega'}$. Then

$$P_\Omega P_{\Omega'} = \widetilde{N}_\Omega U_{\Omega - \overline{C}_{R,\Delta}^v} U_{\Omega' - \overline{C}_{R,\Delta}^v} U_{\Omega' + \overline{C}_{R,\Delta}^v} \widetilde{N}_{\Omega'}$$

and, since $U_{\Omega - \overline{C}_{R,\Delta}^v} U_{\Omega' - \overline{C}_{R,\Delta}^v} \subset U_\Delta^-$ according to Proposition 4.20, we get (1).

Assume, furthermore, that $\Omega \subset \Omega' - \overline{C}_{R,\Delta}^v$. Then $\Omega - \overline{C}_{R,\Delta}^v \subset \Omega' - \overline{C}_{R,\Delta}^v$ so that $U_{\Omega - \overline{C}_{R,\Delta}^v} \subset U_{\Omega' - \overline{C}_{R,\Delta}^v}$. Thus

$$P_\Omega P_{\Omega'} = \widetilde{N}_\Omega U_{\Omega - \overline{C}_{R,\Delta}^v} U_{\Omega' + \overline{C}_{R,\Delta}^v} \widetilde{N}_{\Omega'}$$

\square

Thus, we deduce from Proposition 2.7, as in [BT72, 7.1.6 & 7.1.7]:

4.24 Corollary. For any non-empty subset $\Omega \subset \mathbb{A}_R$ and any point $x \in \mathbb{A}_R$, there is a basis Δ of Φ such that

$$P_\Omega P_x \subset \widetilde{N}_\Omega U_\Delta^- U_{x + \overline{C}_{R,\Delta}^v} \widetilde{N}_x$$

Proof. Let $y \in \Omega$ be any point. By Proposition 2.7, there exists a basis Δ of Φ such that $x - y \in \overline{C}_{R,\Delta}^v$. Thus $x \in y + \overline{C}_{R,\Delta}^v \subset \Omega + \overline{C}_{R,\Delta}^v$. Thus, by Proposition 4.23(1) applied to $\Omega' = \{x\}$, we have the desired inclusion. \square

4.25 Corollary. Let $x, y \in \mathbb{A}_R$. There exists a basis Δ of Φ such that

$$P_y P_x = \widetilde{N}_y U_{y - \overline{C}_{R,\Delta}^v} U_{x + \overline{C}_{R,\Delta}^v} \widetilde{N}_x$$

Proof. By Proposition 2.7, there exists a basis Δ of Φ such that $x - y \in \overline{C}_{R,\Delta}^v$. Then $x \in y + \overline{C}_{R,\Delta}^v \subset \Omega + \overline{C}_{R,\Delta}^v$ and Then $y \in x - \overline{C}_{R,\Delta}^v \subset \Omega + \overline{C}_{R,\Delta}^v$. Thus, by Proposition 4.23(2) applied to $\Omega' = \{x\}$ and $\Omega = \{y\}$, we have the desired equality. \square

Finally, we deduce that \widehat{P}_Ω can be written as an intersection, as in [BT72, 7.1.11]:

4.26 Proposition. *Let $\Omega \subset \mathbb{A}$, $\Omega \neq \emptyset$. Then $\widehat{P}_\Omega = \bigcap_{x \in \Omega} \widehat{P}_x$.*

Proof. The inclusion $\bigcap_{x \in \Omega} \widehat{P}_x \supset \widehat{P}_\Omega$ is a consequence of the fact $\Omega \mapsto \widehat{P}_\Omega$ is decreasing. Let us prove that $\bigcap_{x \in \Omega} \widehat{P}_x \subset \widehat{P}_\Omega$.

Let $\Omega_0 \subset \mathbb{A}_R$. We begin by proving that if $x \in \mathbb{A}_R$, one has $\widehat{P}_{\Omega_0} \cap \widehat{P}_x = \widehat{P}_{\Omega_0 \cup \{x\}}$. Let $y \in \Omega_0$ and C^v be a vectorial chamber such that $y - x \in \overline{C^v}$. Then $U_{\Omega_0 + C^v} \subset U_{y + C^v} \subset U_{x + C^v}$. Let $g \in \widehat{P}_{\Omega_0} \cap \widehat{P}_x$. Write $g = nvu$, with $n \in \widehat{N}_{\Omega_0}$, $v \in U_{\Omega_0 - C^v}$ and $u \in U_{\Omega_0 + C^v}$, which is possible by Corollary 4.22. By Lemma 4.32 and Lemma 4.27, one has $U_{x + C^v} \subset \widehat{P}_x$. Therefore $u \in \widehat{P}_x$ and $g^{-1}u \in \widehat{P}_x$. Thus $nv = n'u'v'$, with $n' \in \widehat{N}_x$, $u' \in U_{x + C^v}$ and $v' \in U_{x - C^v}$. Therefore, $n'^{-1}n = u'(v'v^{-1}) \in U_{\Delta_{C^v}}^+ \cdot U_{\Delta_{C^v}}^-$ (where Δ_{C^v} denotes the basis of Φ associated to C^v). By [BT72, 6.1.15 c)], $n' = n$ and by axiom (RGD6), $v = v'$. Therefore, $n \in \widehat{N}_x \cap \widehat{N}_{\Omega_0} \subset \widehat{N}_{\Omega_0 \cup \{x\}}$, $v \in U_{\Omega_0 - C^v} \cap U_{x - C^v}$ and $u \in U_{\Omega_0 + C^v} \cap \widehat{P}_x$. Moreover, by Corollary 4.22 and by Proposition 3.4, one has:

$$U_{\Omega_0 - C^v} \cap \widehat{P}_x \subset U_{\Omega_0 - C^v} \cap (U_{\Delta_{C^v}}^- \cap \widehat{P}_x) = U_{\Omega_0 - C^v} \cap U_{x - C^v} \subset U_{(\Omega_0 \cup \{x\}) - C^v}$$

and symmetrically, $U_{\Omega_0 + C^v} \cap \widehat{P}_x \subset U_{(\Omega_0 \cup \{x\}) + C^v}$. Therefore $\widehat{P}_{\Omega_0} \cap \widehat{P}_x = \widehat{P}_{\Omega_0 \cup \{x\}}$ (by Corollary 4.22).

By induction, we deduce that for each finite subset Ω' of Ω , one has:

$$\widehat{P}_{\Omega'} = \bigcap_{x \in \Omega'} \widehat{P}_x.$$

Let $x_0 \in \Omega$. Let $\text{Fin}(\Omega, x_0)$ be the set of finite subsets of Ω containing x_0 . Let $g \in \bigcap_{x \in \Omega} \widehat{P}_x$. Then for all $\Omega' \in \text{Fin}(\Omega, x_0)$, $g \in \bigcap_{x \in \Omega'} \widehat{P}_x = \widehat{P}_{\Omega'}$ and thus $g \in \bigcap_{\Omega' \in \text{Fin}(\Omega, x_0)} \widehat{P}_{\Omega'}$. Let us prove that $\bigcap_{\Omega' \in \text{Fin}(\Omega, x_0)} \widehat{P}_{\Omega'} \subset \widehat{P}_\Omega$.

Let $g \in \bigcap_{\Omega' \in \text{Fin}(\Omega, x_0)} \widehat{P}_{\Omega'}$. Since \widehat{N}_{x_0} is, by definition the stabilizer of x_0 and T_b is the kernel of the action $\nu : N \rightarrow \text{Aff}(\mathbb{A}_R)$, the quotient group \widehat{N}_{x_0}/T_b can be identified with a subgroup of W^v which is finite. We write the cosets n_1T_b, \dots, n_kT_b , with $k \in \mathbb{Z}_{\geq 0}$ and $n_i \in \widehat{N}_{x_0}$, for all $i \in \llbracket 1, k \rrbracket$. Choose a vectorial chamber C^v (for example, $C^v = C_f^v$). For $\Omega' \in \text{Fin}(\Omega, x_0)$, one can write $g = n_{\Omega'}u_{\Omega'}v_{\Omega'}$, with $n_{\Omega'} \in \widehat{N}_{\Omega'}$, $u_{\Omega'} \in U_{\Omega' + C^v}$ and $v_{\Omega'} \in U_{\Omega' - C^v}$. Let

$$J = \{j \in \llbracket 1, k \rrbracket \mid \exists \Omega' \in \text{Fin}(\Omega, x_0) \mid \forall \tilde{\Omega} \in \text{Fin}(\Omega, x_0) \mid \tilde{\Omega} \supset \Omega', n_{\tilde{\Omega}} \notin n_jT_b\}.$$

For $j \in J$, we pick $\Omega_j \in \text{Fin}(\Omega, x_0)$ such that for every $\Omega' \in \text{Fin}(\Omega, x_0)$ such that $\Omega' \supset \Omega_j$, $n_{\Omega'} \notin n_jT_b$. Let $\tilde{\Omega} = \bigcup_{j \in J} \Omega_j$ and $\ell \in \llbracket 1, k \rrbracket$ be such that $n_{\tilde{\Omega}} \in n_\ell T_b$. Then $\ell \in \llbracket 1, k \rrbracket \setminus J$.

Let

$$\mathcal{F} = \{\Omega' \in \text{Fin}(\Omega, x_0) \mid n_{\Omega'} \in n_\ell T_b\}.$$

Then for every $\Omega' \in \text{Fin}(\Omega, x_0)$, there exists $\Omega'' \in \mathcal{F}$ such that $\Omega' \subset \Omega''$ and in particular, $\Omega = \bigcup_{\Omega' \in \mathcal{F}} \Omega'$.

Let $\Omega' \in \text{Fin}(\Omega, x_0)$. Let $\Omega'' \in \text{Fin}(\Omega, x_0)$ be such that $\Omega'' \in \mathcal{F}$. As $T_b \subset \widehat{N}_{\Omega''}$, we deduce that $n_\ell \in \widehat{N}_{\Omega''} \subset \widehat{N}_{\Omega'}$. Consequently, $n_\ell \in \bigcap_{\Omega' \in \text{Fin}(\Omega, x_0)} \widehat{N}_{\Omega'} \subset \widehat{N}_\Omega$.

For $\Omega' \in \mathcal{F}$, write $n_{\Omega'} = n_\ell h_{\Omega'}$, with $h_{\Omega'} \in H$. Let $\Omega_1, \Omega_2 \in \mathcal{F}$. Then $h_{\Omega_1}u_{\Omega_1}v_{\Omega_1} = h_{\Omega_2}u_{\Omega_2}v_{\Omega_2}$ and by axiom (RGD6), we deduce that $h_{\Omega_1} = h_{\Omega_2} := h$, $u_{\Omega_1} = u_{\Omega_2} := u$ and $v_{\Omega_1} = v_{\Omega_2} := v$. Then $g = n_\ell h u v$. Moreover $u \in \bigcap_{\Omega' \in \mathcal{F}} U_{\Omega' + C^v} \subset U_{\Omega + C^v}$ and $v \in \bigcap_{\Omega' \in \mathcal{F}} U_{\Omega' - C^v} \subset U_{\Omega - C^v}$. Therefore, $g = n_\ell h u v \in \widehat{N}_\Omega U_{\Omega + C^v} U_{\Omega - C^v} = \widehat{P}_\Omega$ (by Corollary 4.21) and hence $g \in \widehat{P}_\Omega$. Consequently,

$$\bigcap_{x \in \Omega} \widehat{P}_x \subset \bigcap_{\Omega' \in \text{Fin}(\Omega, x_0)} \widehat{P}_{\Omega'} \subset \widehat{P}_\Omega \subset \bigcap_{x \in \Omega} \widehat{P}_x,$$

which proves the proposition. \square

4.4 Subgroups associated to a filter

If \mathcal{E} is a set, we denote by $\mathcal{P}(\mathcal{E})$ the set of subsets of \mathcal{E} . Let $X : \mathcal{P}(\mathbb{A}_R) \rightarrow \mathcal{P}(G)$, $\Omega \rightarrow X_\Omega$ be a decreasing map (for example, $X = U, N, \widehat{P}, \dots$). If \mathcal{V} is a filter on \mathbb{A}_R , we set $X_{\mathcal{V}} = \bigcup_{\Omega \in \mathcal{V}} X_\Omega$. If X_Ω is a subgroup of G for every subset Ω of \mathbb{A}_R , then $X_{\mathcal{V}}$ is a subgroup of G .

4.27 Lemma. *Let \mathcal{V} be a filter on \mathbb{A}_R . Then $U_{\mathcal{V}} = U_{\text{cl}(\mathcal{V})}$, $\widetilde{N}_{\mathcal{V}} = \widetilde{N}_{\text{cl}(\mathcal{V})}$ and $T_{\mathcal{V}} = T_{\text{cl}(\mathcal{V})}$. Moreover $N_{\mathcal{V}} = N_{\text{cl}(\mathcal{V})}$, $P_{\mathcal{V}} = P_{\text{cl}(\mathcal{V})}$ and $U_{\alpha, \mathcal{V}} = U_{\alpha, \text{cl}(\mathcal{V})}$ for every $\alpha \in \Phi$.*

Proof. As $\mathcal{V} \in \text{cl}(\mathcal{V})$, one has $U_{\text{cl}(\mathcal{V})} \subset U_{\mathcal{V}}$. Let us prove the reverse inclusion. We first assume that $\mathcal{V} = \Omega$ is a set. Let $u \in U_\Omega$. Then by definition of U_Ω , there exists $k \in \mathbb{Z}_{\geq 0}$, roots $\alpha_1, \dots, \alpha_k \in \Phi$ and elements $u_i \in U_{\alpha_i, \Omega}$ such that $u = \prod_{i=1}^k u_i$. For $\alpha \in \Phi$, set $\lambda_\alpha = \min\{\varphi_{\alpha_i}(u_i), i \in \llbracket 1, k \rrbracket \text{ and } \alpha_i = \alpha\}$ (one may have $\lambda_\alpha = \infty$). Set $\Omega' = \bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_\alpha}$. For $i \in \llbracket 1, k \rrbracket$ and $x \in D_{\alpha_i, \lambda_{\alpha_i}}$, we have $\varphi_{\alpha_i}(u_i) \geq \lambda_{\alpha_i} \geq -\alpha_i(x)$ so that $u_i \in U_{\alpha_i, D_{\alpha_i, \lambda_{\alpha_i}}} \subset U_{\alpha_i, \Omega'}$. Hence $u \in U_{\Omega'}$ and $\Omega' \in \text{cl}(\Omega)$. Thus $u \in U_{\text{cl}(\Omega)}$ and therefore, $U_\Omega = U_{\text{cl}(\Omega)}$.

We no longer assume that \mathcal{V} is a set. Let $u \in U_{\mathcal{V}}$. Then there exists $\Omega \in \mathcal{V}$ such that $u \in U_\Omega$. Therefore, $u \in U_{\text{cl}(\Omega)}$ and thus there exists $\Omega' \in \text{cl}(\Omega)$ such that $u \in U_{\Omega'}$. As $\Omega' \in \text{cl}(\Omega)$, $\Omega' \in \text{cl}(\mathcal{V})$ and thus $u \in U_{\text{cl}(\mathcal{V})}$. Hence $U_{\mathcal{V}} \subset U_{\text{cl}(\mathcal{V})}$, which proves that $U_{\mathcal{V}} = U_{\text{cl}(\mathcal{V})}$.

For any filter \mathcal{V} , by definition, we have $\widetilde{N}_{\mathcal{V}} = \bigcup_{\Omega \in \mathcal{V}} \widetilde{N}_\Omega = \bigcup_{\Omega \in \mathcal{V}} N \cap U_\Omega = N \cap U_{\mathcal{V}}$. Thus $\widetilde{N}_{\mathcal{V}} = N \cap U_{\mathcal{V}} = N \cap U_{\text{cl}(\mathcal{V})} = \widetilde{N}_{\text{cl}(\mathcal{V})}$. By the same way, we have $T_{\mathcal{V}} = T_{\text{cl}(\mathcal{V})}$ and $U_{\alpha, \mathcal{V}} = U_{\alpha, \text{cl}(\mathcal{V})}$ for any $\alpha \in \Phi$, since $U_\Omega \cap U_\alpha = U_{\alpha, \Omega}$ according to Example 3.44.

For any filter \mathcal{V} , by definition and Lemma 4.3, we have $P_{\mathcal{V}} = \bigcup_{\Omega \in \mathcal{V}} T_b U_\Omega = T_b U_{\mathcal{V}}$. Thus $P_{\mathcal{V}} = P_{\text{cl}(\mathcal{V})}$ and, by intersection with N , we have $N_{\mathcal{V}} = N_{\text{cl}(\mathcal{V})}$. \square

4.28 Remark. As $\mathcal{V} \in \text{cl}(\mathcal{V})$, one has $\widehat{N}_{\text{cl}(\mathcal{V})} \subset \widehat{N}_{\mathcal{V}}$ but this inclusion is strict in general. For instance, if $\mathbf{G} = \text{PGL}(2)$ and \mathbb{K} is a local field, if x is the center of an edge of the Bruhat-Tits tree, there is an element of $N \subset G = \mathbf{G}(\mathbb{K})$ permuting the two vertices of the edge and, therefore, fixing x . The enclosure $\text{cl}(\mathcal{V})$ of $\{x\}$ is the edge and the pointwise stabilizer of $\text{cl}(\mathcal{V})$ cannot exchange the vertices of the edge so that $\widehat{N}_{\text{cl}(\mathcal{V})} \neq \widehat{N}_{\mathcal{V}}$. Thus $\widehat{P}_{\text{cl}(\mathcal{V})} \neq \widehat{P}_{\mathcal{V}}$ in general.

4.29 Remark. Using Example 3.44(3) and Lemma 4.3 we deduce the following decompositions, when \mathcal{V} is a filter on \mathbb{A}_R :

$$U_{\mathcal{V}} = (U_{\mathcal{V}} \cap U_{\Delta}^+) (U_{\mathcal{V}} \cap U_{\Delta}^-) (U_{\mathcal{V}} \cap N) \quad \text{with } U_{\mathcal{V}} \cap N = \widetilde{N}_{\mathcal{V}}$$

$$P_{\mathcal{V}} = (P_{\mathcal{V}} \cap U_{\Delta}^+) (P_{\mathcal{V}} \cap U_{\Delta}^-) (P_{\mathcal{V}} \cap N) \quad \text{with } P_{\mathcal{V}} \cap N = N_{\mathcal{V}}$$

Indeed, let $u \in U_{\mathcal{V}}$. Let $\Omega \in \mathcal{V}$ be such that $u \in U_\Omega$. Then one can write $u = u^- u^+ n$, with $u^+ \in U_\Omega \cap U_{\Delta}^+$, $u^- \in U_\Omega \cap U_{\Delta}^-$ and $n \in U_\Omega \cap N$ and thus $u \in (U_{\mathcal{V}} \cap U_{\Delta}^+) (U_{\mathcal{V}} \cap U_{\Delta}^-) (U_{\mathcal{V}} \cap N)$. Thus $U_{\mathcal{V}} \subset (U_{\mathcal{V}} \cap U_{\Delta}^+) (U_{\mathcal{V}} \cap U_{\Delta}^-) (U_{\mathcal{V}} \cap N) \subset U_{\mathcal{V}}$ and we get similarly the statement with P .

Recall that given a vector facet F^v and a point $x \in \mathbb{A}_R$, we denote by \mathcal{F}_{x, F^v} the germ of $x + F^v$ at x (see definition in section 2.1.7).

4.30 Proposition. *Let C^v be a vector chamber of V_R and $x \in \mathbb{A}_R$. Denote by $\mathcal{F} = \mathcal{F}_{x, C^v}$ the germ of $x + C^v$ at x . Then, for any basis Δ of Φ , we have:*

- $\widehat{P}_{\mathcal{F}} \cap U_\alpha = U_{\alpha, \mathcal{F}}$ for any root $\alpha \in \Phi$;
- $\widehat{P}_{\mathcal{F}} \cap N = \widehat{N}_{\mathcal{F}} = T_b$;
- $\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^+ = U_{\mathcal{F}} \cap U_{\Delta}^+$;

- $\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{-} = U_{\mathcal{F}} \cap U_{\Delta}^{-}$;
- $\widehat{P}_{\mathcal{F}} = (\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{+})(\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{-})(\widehat{P}_{\mathcal{F}} \cap N)$.

Proof. Let $\alpha \in \Phi_{\Delta}^{+}$ and $\Omega \in \mathcal{F}$. According to Corollary 4.12 and Example 3.44, we have $\widehat{P}_{\Omega} \cap U_{\alpha} = U_{\Omega} \cap U_{\alpha} = U_{\alpha, \Omega}$ since U_{α} is a subgroup of U_{Δ}^{+} by definition. Thus

$$\widehat{P}_{\mathcal{F}} \cap U_{\alpha} = \left(\bigcup_{\Omega \in \mathcal{F}} \widehat{P}_{\Omega} \right) \cap U_{\alpha} = \bigcup_{\Omega \in \mathcal{F}} \widehat{P}_{\Omega} \cap U_{\alpha} = \bigcup_{\Omega \in \mathcal{F}} U_{\alpha, \Omega} = U_{\alpha, \mathcal{F}}$$

and we proceed in the same way for $\alpha \in \Phi_{\Delta}^{-}$.

According to Corollary 4.12, we have

$$\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{+} = \bigcup_{\Omega \in \mathcal{F}} \widehat{P}_{\Omega} \cap U_{\Delta}^{+} = \bigcup_{\Omega \in \mathcal{F}} U_{\Omega} \cap U_{\Delta}^{+} = U_{\mathcal{F}} \cap U_{\Delta}^{+}$$

and the same is for U_{Δ}^{-} . Let $\Omega \in \mathcal{F}$. There exists $\varepsilon > 0$ such that $\Omega \supset B(0, \varepsilon) \cap (x + C^v)$. Since \widehat{N}_{Ω} is the pointwise stabilizer of Ω and, for any $\alpha \in \Phi$, the affine map α is non-constant on Ω , there exists no $\lambda \in R$ such that $r_{\alpha, \lambda}$ fixes Ω pointwise. Thus $\nu(\widehat{N}_{\Omega})$ is trivial and, therefore, $\widehat{N}_{\Omega} = T_b$. Hence

$$\widehat{P}_{\mathcal{F}} \cap N = \bigcup_{\Omega \in \mathcal{F}} \widehat{P}_{\Omega} \cap N = \bigcup_{\Omega \in \mathcal{F}} \widehat{N}_{\Omega} = T_b = \widehat{N}_{\mathcal{F}}.$$

Finally, if $p \in \widehat{P}_{\mathcal{F}}$, there exists $\Omega \in \mathcal{F}$ such that $p \in \widehat{P}_{\Omega}$. By Corollary 4.12,

$$p \in \left(\widehat{P}_{\Omega} \cap U_{\Delta}^{+} \right) \left(\widehat{P}_{\Omega} \cap U_{\Delta}^{-} \right) \left(\widehat{P}_{\Omega} \cap N \right) \subset \left(\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{+} \right) \left(\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{-} \right) \left(\widehat{P}_{\mathcal{F}} \cap N \right).$$

Hence $\widehat{P}_{\mathcal{F}} = \left(\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{+} \right) \left(\widehat{P}_{\mathcal{F}} \cap U_{\Delta}^{-} \right) \left(\widehat{P}_{\mathcal{F}} \cap N \right)$. \square

4.31 Corollary. *Let $x \in \mathbb{A}_R$ and C^v be a vector chamber of V_R . Consider the sector $Q = x + C^v$ and the filter $\mathcal{F} = \mathcal{F}_{x, C}$ be the germ of Q at x . Then $\widehat{P}_Q = P_Q$ and $\widehat{P}_{\mathcal{F}} = P_{\mathcal{F}}$.*

The following three statements are intended to prove an analogous to [BT72, 7.2.6]

4.32 Lemma. *Suppose that $R = R_Q$. Let $x \in \mathbb{A}_R$ and $F^v \subset V_R$ be a vector facet. Then $\{x\} \in \text{cl}(\mathcal{F}_{x, F^v})$.*

More generally, write $F^v = F_R^v(\Delta, \Delta_P)$, where Δ is a basis of Φ and $\Delta_P \subset \Delta$. Let $\tilde{\Delta}$ be a set such that $\Delta \supset \tilde{\Delta}_P \supset \Delta_P$. Set $\tilde{F}^v = F^v(\Delta, \tilde{\Delta}_P)$. Then $\text{cl}(\mathcal{F}_{x, F^v}) \ni \mathcal{F}_{x, \tilde{F}^v}$.

Proof. Let $\Omega \in \text{cl}(\mathcal{F}_{x, F^v})$. By definition, there exists $\varepsilon > 0$ and values $\lambda_{\alpha} \in \Gamma_{\alpha} \cup \{\infty\}$ such that $\Omega \supset \bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_{\alpha}} \supset B(x, \varepsilon) \cap (x + F^v)$. By definition, there exists a basis Δ of Φ and a subset Δ_P of Δ such that $F^v = F_R^v(\Delta, \Delta_P)$. If $\Delta_P = \Delta$, then $F^v = 0$ and we have $x \in \Omega$.

We now assume that $\Delta_P \neq \Delta$. Let $v \in F_{\mathbb{Z}}^v(\Delta, \Delta_P)$. Then $|v| := \sum_{w \in W(\Phi)} |w \cdot v|_1 \in \mathbb{Z}_{>0}$ since $0 \notin F_{\mathbb{Z}}^v(\Delta, \Delta_P)$ by assumption on Δ_P .

Let $\lambda \in \mathfrak{R}_{>0}$. Then $\lambda v \in \delta_v$ and thus $\lambda v \neq 0$. Suppose moreover that $0 < |v| \lambda < \varepsilon$. Then $y := x + \lambda v \in B(x, \varepsilon) \cap (x + \delta_v) \subset B(x, \varepsilon) \cap (x + F^v)$ by Lemma 2.5. Let $\alpha \in \Phi$. Then $-\alpha(y) = -\alpha(x) - \lambda \alpha(v) \leq \lambda_{\alpha}$ since $y \in D_{\alpha, \lambda_{\alpha}}$. If $\alpha(v) \in \mathbb{Z}_{\leq 0}$, then $\lambda_{\alpha} + \alpha(x) \geq -\alpha(v) \lambda \geq 0$. Otherwise, $\alpha(v) \in \mathbb{Z}_{>0}$ and therefore

$$\lambda_{\alpha} + \alpha(x) \in \bigcap_{\lambda \in]0, \frac{\varepsilon}{|v|}[} [-\lambda \alpha(v), \infty[= \bigcap_{\lambda \in]0, \frac{\varepsilon}{|v| \alpha(v)}[} [-\lambda, \infty[.$$

Let $\mu \in \mathfrak{R}_{<0}$. If $\mu < -\varepsilon$, then $\mu \notin [-\frac{\varepsilon}{|v|\alpha(v)}, \infty[$. If $\mu \geq -\varepsilon$, then $\mu \notin [-\frac{-\mu}{2|v|\alpha(v)}, \infty[$. Consequently $\bigcap_{\lambda \in]0, \frac{\varepsilon}{|v|\alpha(v)}[} [-\lambda, \infty[\subset \mathfrak{R}_{\geq 0}$ and hence $\lambda_\alpha + \alpha(x) \geq 0$. As this is true for every $\alpha \in \Phi$, we deduce that $x \in \Omega$, which proves that $\{x\} \in \text{cl}(\mathcal{F}_{x,F^v})$.

Let now $\tilde{\Delta}$ be a set such that $\Delta \supset \tilde{\Delta}_P \supset \Delta_P$ and $\tilde{F}^v = F_R^v(\Delta, \tilde{\Delta}_P)$. Let $\Psi = \Phi \cap \bigoplus_{\alpha \in \Delta_P} \mathbb{R}\alpha$, $\tilde{\Psi} = \Phi \cap \bigoplus_{\alpha \in \tilde{\Delta}_P} \mathbb{R}\alpha$ and $\mathcal{X} = \bigcap_{\alpha \in \Phi_- \setminus \Psi} \mathring{D}_{\alpha, \lambda_\alpha}$. Let $\lambda \in]0, \frac{1}{|v|}\varepsilon[$ and $y = x + \lambda v$. Let $\alpha \in \Phi_- \setminus \Psi$. Then $y \in \Omega$ and $\alpha(x) > \alpha(y) > -\lambda_\alpha$. Therefore $x \in \mathcal{X}$. Let $z \in \mathcal{X} \cap B(x, \varepsilon) \cap (x + \tilde{F}^v)$. Let $\alpha \in \Phi_+$. Then $\alpha(z) \geq \alpha(x) \geq -\lambda_\alpha$. Let $\alpha \in \Phi_-$. Then if $\alpha \in \tilde{\Psi}$, $\alpha(z) = \alpha(x) \geq -\lambda_\alpha$. If $\alpha \in \Phi_- \setminus \tilde{\Psi}$, then $\alpha \in \Phi_- \setminus \Psi$ and thus $\alpha(z) > -\lambda_\alpha$. Therefore $z \in \Omega$ and hence $\Omega \supset \mathcal{X} \cap B(x, \varepsilon) \cap (x + \tilde{F}^v)$. Consequently $\mathcal{F}_{x, \tilde{F}^v} \in \text{cl}(\mathcal{F}_{x, F^v})$, which proves the lemma. \square

4.33 Proposition. *Suppose that $R = R_{\mathbb{Q}}$. Let $\alpha \in \Phi$, $x \in \mathbb{A}_R$ and F^v be a vector facet in V_R . Then*

$$U_{\alpha, \mathcal{F}_{x, F^v}} \subseteq \begin{cases} U_{\alpha, x} & \text{if } \alpha \in \Phi_{F^v}^+ \sqcup \Phi_{F^v}^0 \\ U'_{\alpha, x} & \text{if } \alpha \in \Phi_{F^v}^- \end{cases}$$

Proof. By Lemma 4.32, $\{x\} \in \text{cl}(\mathcal{F}_{x, F^v})$, and thus $U_{\alpha, \text{cl}(\mathcal{F}_{x, F^v})} = U_{\alpha, \mathcal{F}_{x, F^v}} \subset U_{\alpha, x}$ according to Lemma 4.27.

By definition, there exists a basis Δ of Φ and a subset Δ_P of Δ such that $F^v = F_R^v(\Delta, \Delta_P)$. If $\Delta_P = \Delta$, then $F^v = 0$ and we have $\Phi_{F^v}^- = \emptyset$: there is nothing to prove. Suppose now that $\Delta_P \neq \Delta$ and consider $\alpha \in \Phi_{F^v}^+$. Let $u \in U_{-\alpha, \mathcal{F}_{x, F^v}}$. Let $\Omega \in \mathcal{F}_{x, F^v}$ be such that $u \in U_{-\alpha, \Omega}$. Then there exists $\varepsilon \in R_{>0}$ such that $\Omega \supset B(x, \varepsilon) \cap (x + F^v)$. Let $v \in F_{\mathbb{Z}}^v(\Delta, \Delta_P)$. Then $|v| := \sum_{w \in W(\Phi)} |w \cdot v|_1 \in \mathbb{Z}_{>0}$ since $0 \notin F_{\mathbb{Z}}^v(\Delta, \Delta_P)$ by assumption on Δ_P .

Let $\lambda = \frac{\varepsilon}{2|v|}$ and $y = x + \lambda v$. Then $y \in B(x, \varepsilon) \cap (x + \delta_v) \subset B(x, \varepsilon) \cap (x + F^v) \subset \Omega$ by Lemma 2.5 and thus $\varphi_{-\alpha}(u) \geq -(-\alpha)(y)$. Moreover $\alpha(\lambda v) > 0$, thus $\varphi_{-\alpha}(u) > -(-\alpha)(x)$ and thus $u \in U'_{-\alpha, x}$, which proves the Lemma since $\Phi_{F^v}^- = -\Phi_{F^v}^+$. \square

4.34 Corollary. *Suppose that $R = R_{\mathbb{Q}}$. Let $\alpha \in \Phi$, $x \in \mathbb{A}_R$ and C^v be a vector chamber in V_R . Then*

$$U_{\alpha, \mathcal{F}_{x, C^v}} = \begin{cases} U_{\alpha, x} & \text{if } \alpha \in \Phi_{C^v}^+ \\ U'_{\alpha, x} & \text{if } \alpha \in \Phi_{C^v}^- \end{cases}$$

Proof. Let $\alpha \in \Phi$. If $\alpha \in \Phi_{C^v}^+$, then we have $U_{\alpha, x} = U_{\alpha, x+C^v}$ according to Lemma 4.19. By definition $U_{\alpha, x+C^v} \subseteq U_{\alpha, \mathcal{F}_{x, C^v}}$ and $U_{\alpha, \mathcal{F}_{x, C^v}} \subseteq U_{\alpha, x}$ by Proposition 4.33. Thus $U_{\alpha, x} = U_{\alpha, x+C^v} = U_{\alpha, \mathcal{F}_{x, C^v}}$.

If $\alpha \in \Phi_{C^v}^-$, let $u \in U'_{\alpha, x}$. For any $y \in x + C^v$, we have $\alpha(y - x) < 0$ by definition. Thus $\varphi_\alpha(u) > -\alpha(x)$. Let $\varepsilon = \varphi_\alpha(u) + \alpha(x) > 0$. Consider

$$\Omega = x + \left(C^v \cap \alpha^{-1}([-\varepsilon, 0]) \right) = x + \left(C^v \cap \mathring{D}_{\alpha, \varepsilon} \right) = x + \left(\bigcap_{\beta \in \Phi_{C^v}^+} \mathring{D}_{\beta, 0} \right) \cap \mathring{D}_{\alpha, \varepsilon}.$$

Then $\Omega \in \mathcal{F}_{x, C^v}$ as non-empty intersection of open neighbourhoods of x in $x + C^v$. Moreover, for any $y \in \Omega$, we have $-\varepsilon < \alpha(y - x) < 0$. Thus $-\alpha(y) < -\alpha(x) + \varepsilon = \varphi_\alpha(u)$. Hence $u \in U_{\alpha, \Omega} \subset U_{\alpha, \mathcal{F}_{x, C^v}}$. Therefore $U'_{\alpha, x} \subset U_{\alpha, \mathcal{F}_{x, C^v}}$. Hence Proposition 4.33, we have $U_{\alpha, \mathcal{F}_{x, C^v}} = U'_{\alpha, x}$. \square

4.35 Corollary. *Suppose that $R = R_{\mathbb{Q}}$. Let $x \in \mathbb{A}_R$ and C be a vector chamber in V_R . Let $\Delta = \Delta_C$. For any ordering of $(\Phi_{\Delta}^+)_{\text{nd}}$ and of $(\Phi_{\Delta}^-)_{\text{nd}}$, we have*

$$\hat{P}_{\mathcal{F}_{x, C}} = \left(\prod_{\alpha \in (\Phi_{\Delta}^+)_{\text{nd}}} U_{\alpha, x} \right) \left(\prod_{\alpha \in (\Phi_{\Delta}^-)_{\text{nd}}} U'_{\alpha, x} \right) T_b = \left(\prod_{\alpha \in (\Phi_{\Delta}^-)_{\text{nd}}} U'_{\alpha, x} \right) \left(\prod_{\alpha \in (\Phi_{\Delta}^+)_{\text{nd}}} U_{\alpha, x} \right) T_b$$

Proof. It is an immediate consequence of Proposition 4.30 and Corollary 4.34. \square

4.5 Iwasawa decomposition

Thus, we obtain the Iwasawa decomposition whose proof can be conducted in the same way as in [BT72, 7.3.1]:

4.36 Theorem (Iwasawa decomposition). *Suppose that the totally ordered commutative pseudo-ring R is equipped with a \mathbb{Q} -module structure. Let C, C' be two vector chambers of V_R and $x \in \mathbb{A}_R$. Let $\mathcal{F} = \mathcal{F}_{x,C}$ be the germ of C at x . Then*

$$G = U_{C'} N \widehat{P}_{\mathcal{F}}$$

and there is a natural one-to-one correspondence

$$W^{\text{aff}} = N/T_b \rightarrow U_{C'} \backslash G / \widehat{P}_{\mathcal{F}}.$$

Proof. In this proof, we denote by $U^+ = U_{C'}$, by $B = \widehat{P}_{\mathcal{F}}$ and by $Z = U^+NB$. Let $\Delta = \Delta_C$ and $\Delta' = \Delta_{C'}$ and $\Phi_{\Delta}^+, \Phi_{\Delta'}^+$, the associated subsets of positive roots.

First step: rank-one Levi subgroups are contained in Z . Let $\alpha \in \Phi$ be any root. Let $L_{\alpha} = \langle U_{-\alpha}, U_{\alpha}, T \rangle$ and $m_{\alpha} = m(u)$ for some $u \in U_{\alpha} \setminus \{1\}$. By [BT72, 6.1.2(5)], we recall that $M_{\alpha} = T\{1, m_{\alpha}\}$. By [BT72, 6.1.2(7)], we know that

$$L_{\alpha} = U_{\alpha}T \cup U_{\alpha}Tm_{\alpha}U_{\alpha}.$$

Let $B_{\alpha} = \langle U_{\alpha, \mathcal{F}}, U_{-\alpha, \mathcal{F}}, T_b \rangle$. Consider the set $Z_{\alpha} = U_{\alpha}M_{\alpha}B_{\alpha}$. We want to prove that $L_{\alpha} \subset Z_{\alpha}$. The inclusion $U_{\alpha}T \subset Z_{\alpha}$ is obvious. Hence, it suffices to prove that $m_{\alpha}u \in Z_{\alpha}$ for any $u \in U_{\alpha}$ since $U_{\alpha}T$ is a group (indeed, the subgroup T normalizes U_{α} by definition of U_{α}).

Suppose that $\alpha \in \Phi_{\Delta}^+$ (resp. $\alpha \in \Phi_{\Delta}^-$). If $\varphi_{\alpha}(u) \geq -\alpha(x)$ (resp. $\varphi_{\alpha}(u) > -\alpha(x)$), then $u \in U_{\alpha, -\alpha(x)} = U_{\alpha, \mathcal{F}}$ (resp. $u \in U'_{\alpha, -\alpha(x)} = U_{\alpha, \mathcal{F}}$) by Corollary 4.34. Thus $u \in B_{\alpha} \subset Z_{\alpha}$. Otherwise, $\varphi_{\alpha}(u) < -\alpha(x)$ (resp. $\varphi_{\alpha}(u) \leq -\alpha(x)$). Write $u = v'mv''$ for some $m \in M_{\alpha}$ and $v', v'' \in U_{-\alpha}$. Write $m = tm_{\alpha}$ for $t \in T$. By axiom (V5bis), we know that $\varphi_{-\alpha}(v'') = -\varphi_{\alpha}(u)$ so that $\varphi_{-\alpha}(v'') > -(-\alpha(x))$ (resp. $\varphi_{-\alpha}(v'') \geq -(-\alpha(x))$). Thus $v'' \in U'_{-\alpha, x} = U_{-\alpha, \mathcal{F}}$ (resp. $v'' \in U_{-\alpha, x} = U_{-\alpha, \mathcal{F}}$) by Corollary 4.34 since $-\alpha \in -\Phi_{\Delta}^+ = \Phi_{\Delta}^-$ (resp. $-\alpha \in \Phi_{\Delta}^+ = -\Phi_{\Delta}^-$). Hence $m_{\alpha}u = m_{\alpha}v'tm_{\alpha}^{-1}v'' = \underbrace{(m_{\alpha}v'm_{\alpha}^{-1})}_{\in U_{\alpha}} \underbrace{(m_{\alpha}tm_{\alpha}^{-1})}_{\in T} v'' \in Z_{\alpha}$.

Second step: Z is stable by left multiplication by root groups of simple roots with respect to C' . Let $\alpha \in \Delta'$ and $V_{\alpha} = \langle U_{\beta} \mid \beta \in \Phi_{\Delta'}^+ \setminus \{\alpha\} \rangle$. We recall that V_{α} is normalized by U_{α} and $U_{-\alpha}$ by axiom (RGD2) and that $U_{\Delta'}^+ = V_{\alpha}U_{\alpha} = U_{\alpha}V_{\alpha}$ by [BT72, 6.1.6]. Hence

$$U_{-\alpha}Z = U_{-\alpha}V_{\alpha}U_{-\alpha}NB = V_{\alpha}U_{-\alpha}U_{\alpha}NB \subset V_{\alpha}L_{\alpha}NB.$$

But since $L_{\alpha} \subset Z_{\alpha}$, we have

$$U_{-\alpha}Z \subset V_{\alpha}U_{\alpha}T\{1, m_{\alpha}\}B_{\alpha}NB.$$

On the one hand, since $B_{\alpha} \subset U_{\alpha}U_{-\alpha}N$ by Remark 4.29, we get

$$\begin{aligned} V_{\alpha}U_{\alpha}T\{1\}B_{\alpha}NB &\subset U^+TU_{\alpha}U_{-\alpha}NB \\ &= U^+TU_{-\alpha}NB \end{aligned}$$

On the other hand, since $B_{\alpha} \subset U_{-\alpha}U_{\alpha}N$ by Remark 4.29, we get

$$\begin{aligned} V_{\alpha}U_{\alpha}T\{m_{\alpha}\}B_{\alpha}NB &\subset U^+Tm_{\alpha}U_{-\alpha}U_{\alpha}NB \\ &= U^+Tm_{\alpha}U_{-\alpha}m_{\alpha}^{-1}m_{\alpha}U_{\alpha}m_{\alpha}^{-1}NB \\ &= U^+TU_{\alpha}U_{-\alpha}NB \\ &= U^+TU_{-\alpha}NB \end{aligned}$$

Hence $U_{-\alpha}Z \subset U^+TU_{-\alpha}NB$.

To conclude, we want to show that $U_{-\alpha}N \subset Z$. Consider any $u \in U_{-\alpha}$ and any $n \in N$. Let $v = n^{-1}un$. Then $v \in U_{\beta}$ with $\beta = {}^v\nu(n^{-1})(-\alpha)$. From the first step applied to β , we get that since $v \in L_{-\beta}$, we have

$$\begin{aligned} un &= nv \in nU_{-\beta}T\{1, m_{\beta}\}B_{-\beta} \\ &\subset nU_{-\beta}n^{-1}nNB \\ &= U_{\alpha}NB \\ &\subset Z \end{aligned}$$

Hence $U_{-\alpha}Z \subset U^+TZB = U^+TU^+NBB = U^+NB = Z$.

Third step: Z contains a generating subset of G . Z is stable by left multiplication by $U_{-\alpha}$ for $\alpha \in \Delta_{C'}$, by U_{α} for $\alpha \in \Phi_{\Delta'}^+$, and by T .

Since these groups generate G , we deduce that $G = Z$. Indeed, let H the group generated by the U_{α} for $\alpha \in \Phi$, by $U_{-\alpha}$ for $\alpha \in \Delta$ and by T . For any $\alpha \in \Delta$, the element m_{α} belongs to L_{α} and since W is a Coxeter group generated by the m_{α} for $\alpha \in \Delta$, the group N is contained in H . Thus for any $\beta \in \Phi^-$, there exist a root $\alpha \in \Phi$ and an element $n \in N$ such that the root group $U_{\beta} = nU_{\alpha}n^{-1}$ is contained in H . Since the root group datum is generating, we deduce that $H = G$.

Fourth step: determination of double cosets From the equality $G = Z = U_{\Delta'}^+N\widehat{P}_{\mathcal{F}}$, we deduce a natural surjective map:

$$\begin{aligned} N &\rightarrow U_{\Delta'}^+ \backslash G / \widehat{P}_{\mathcal{F}} \\ n &\mapsto U_{\Delta'}^+ n \widehat{P}_{\mathcal{F}} \end{aligned}$$

Let $n, n' \in N$ such that $n' \in U_{\Delta'}^+ n \widehat{P}_{\mathcal{F}}$. Denote $C'' = n^{-1} \cdot C'$ another vector chamber. Then

$$\begin{aligned} (n')^{-1}n &\in \widehat{P}_{\mathcal{F}} n^{-1} U_{\Delta'}^+ n \\ &= \widehat{P}_{\mathcal{F}} U_{n^{-1} \cdot C'}^+ && \text{by [BT72, 6.1.2(10)]} \\ &= \widehat{P}_{\mathcal{F}} U_{\Delta''}^+ \\ &= T_b \left(U_{\Delta''}^- \cap \widehat{P}_{\mathcal{F}} \right) U_{\Delta''}^+ && \text{by Proposition 4.30} \\ &= \left(U_{\Delta''}^- \cap \widehat{P}_{\mathcal{F}} \right) T_b U_{\Delta''}^+ \end{aligned}$$

Hence, by [BT72, 6.1.15(c)], the element $(n')^{-1}n$ is sent onto the double cosets of $U_{\Delta''}^- \backslash G / U_{\Delta''}^+$ arising from some element in T_b and therefore $(n')^{-1}n \in T_b$. Hence, it induces a bijection $N/T_b \rightarrow U_{\Delta'}^+ \backslash G / \widehat{P}_{\mathcal{F}}$. \square

4.6 Bruhat decomposition

4.37 Notation. Given a point $x \in \mathbb{A}_R$ and a vector chamber C of V_R , we denote by $\mathcal{F}_{x,C}$ the germ at x of the sector $x + C$.

In the following theorem, we generalize the affine Bruhat decomposition [BT72, 7.3.4]. In the particular case where $G = \mathbf{G}(\mathbb{K})$ for a split reductive group \mathbf{G} over a 2-local field \mathbb{K} , this is a particular case of [Kap01, Proposition (1.2.3)].

4.38 Theorem. *Suppose that the totally ordered commutative pseudo-ring R is equipped with a \mathbb{Q} -module structure. Let C, C' be two vector chambers of V_R and $x, x' \in \mathbb{A}_R$ be two points. Then*

$$G = \widehat{P}_{\mathcal{F}_{x,C}} N \widehat{P}_{\mathcal{F}_{x',C'}}$$

and there is a natural one-to-one correspondence

$$N/T_b \rightarrow \widehat{P}_{\mathcal{F}_{x,C}} \backslash G / \widehat{P}_{\mathcal{F}_{x',C'}}$$

4.39 Notation. Let C, C' be two vector chambers of V_R and $x, x' \in \mathbb{A}_R$ be two points. We denote by $Q = x + C$ and $Q' = x' + C'$ the corresponding sectors in \mathbb{A}_R . Denote by $\Delta = \Delta_C$ and $\Delta' = \Delta_{C'}$ respectively the bases of Φ defining $C = C_{R,\Delta}^v$ and $C' = C_{R,\Delta'}^v$.

According to Proposition 2.7, there is a unique $w \in W(\Phi)$ such that $x' - x \in \overline{C}_{w(\Delta)}^v$ and $C_{w(\Delta)}^v \cap (x' - x + C') \neq \emptyset$. We denote it by $w(x, C, x', C') = w$. We also denote by $\ell(x, C, x', C')$ the length of w in the Coxeter system $(W(\Phi), (r_\alpha)_{\alpha \in \Delta})$.

We denote by $\Delta'' = w(\Delta)$, by $C'' = C_{w(\Delta)}^v$ and by $Q'' = x + C''$ so that $x' \in \overline{Q''}$ and $Q' \cap Q'' \neq \emptyset$.

Note that the uniqueness of $w \in W(\Phi)$ and the simple transitivity of the action of $W(\Phi)$ imply that Δ'' , and therefore C'' , is uniquely determined by x, x', C' . We denote it by $C(x, x', C') = C''$.

4.40 Lemma. *Let $x \in \mathbb{A}_R$ and C a vector chamber of V_R . Then for any $n \in N$, we have $n\widehat{P}_{\mathcal{F}_{x,C}}n^{-1} = \widehat{P}_{\mathcal{F}_{\nu(n)(x), \nu(n)(C)}}$.*

Proof. We have $n\widehat{P}_{\mathcal{F}_{x,C}}n^{-1} = \bigcup_{\Omega \in \mathcal{F}_{x,C}} n\widehat{P}_\Omega n^{-1}$. By 4.11, we have $n\widehat{P}_\Omega n^{-1} = \widehat{P}_{\nu(n)(\Omega)}$. Moreover, we have $\{\nu(n)(\Omega), \Omega \in \mathcal{F}_{x,C}\} = \mathcal{F}_{\nu(n)(x), \nu(n)(C)}$ since $\nu(n)(x + C) = \nu(n)(x) + \nu(n)(C)$ by definitions. Thus $n\widehat{P}_{\mathcal{F}_{x,C}}n^{-1} = \bigcup_{\Omega \in \mathcal{F}_{\nu(n)(x), \nu(n)(C)}} \widehat{P}_\Omega = \widehat{P}_{\mathcal{F}_{\nu(n)(x), \nu(n)(C)}}$. \square

We follow the proof given by Bruhat and Tits. It relies on the following technical Lemma from [BT72, 7.3.6]. In our context, the notion of half-line is not obvious but it has been given in Definition 2.3 so that for a well-chosen $v \in V_{\mathbb{Z}}$, given by Lemma 2.4 one can follow word by word the proof given by Bruhat and Tits, since all the intermediate results have been generalized.

4.41 Lemma. *Suppose that R is equipped with a \mathbb{Q} -module structure. Let C, C' be two vector chambers of V_R and $x, x' \in \mathbb{A}_R$ be two points. Let $g \in G$ and $n \in N$ be such that $g \in \widehat{P}_{\mathcal{F}_{x,C}} n \widehat{P}_{\mathcal{F}_{x',C'}}$. Let $\Delta = \Delta_C$ the basis of the vector chamber C . Let $v \in C_{\mathbb{Z}, \Delta}^v$.*

(1) *Then we have either:*

- (a) $g \in \widehat{P}_{\mathcal{F}_{z,C}} n \widehat{P}_{\mathcal{F}_{x',C'}}$ for any $z \in x + \delta_v$ or
- (b) *there exist a value $\lambda \in R_{>0}$ and an element $n' \in N$ such that:*
 - i. $g \in \widehat{P}_{\mathcal{F}_{z,C}} n' \widehat{P}_{\mathcal{F}_{x',C'}}$ for any $z = x + \mu v$ with $\mu \in]0, \lambda[$,
 - ii. $g \in \widehat{P}_{\mathcal{F}_{y,C}} n' \widehat{P}_{\mathcal{F}_{x',C'}}$ with $y = x + \lambda v$,
 - iii. $\ell(y, C, n' \cdot x', n' \cdot C') > \ell(x, C, n \cdot x', n \cdot C')$.

(2) *Moreover, we have $g \in \widehat{P}_{\mathcal{F}_{z,C}} N \widehat{P}_{\mathcal{F}_{x',C'}}$ for any $z \in x + \delta_v$.*

Proof of (1). We denote by $\Delta = \Delta_C$ and by $\Delta' = \Delta_{C'}$.

Reduction step: If $g \in \widehat{P}_{\mathcal{F}_{x,C}} n \widehat{P}_{\mathcal{F}_{x',C'}}$, then $gn^{-1} \in \widehat{P}_{\mathcal{F}_{x,C}} \widehat{P}_{n \cdot x', n \cdot C'}$ by Lemma 4.40. Thus, up to replace g by gn^{-1} and x', C' by $n \cdot x', n \cdot C'$, we can assume that $n = 1$.

Let $C'' = C(x, x', C')$, $\Delta'' = \Delta_{C''}$ and Q, Q', Q'' as in Notation 4.39. Denote $\Psi^- = (\Phi_{\Delta}^-)_{\text{nd}} \cap \Phi_{\Delta''}^-$ and $\Psi^+ = (\Phi_{\Delta}^-)_{\text{nd}} \cap \Phi_{\Delta''}^+$. Let $b \in \widehat{P}_{\mathcal{F}_{x,C}}$. By Corollary 4.35, we can write b as a product:

$$b = \left(\prod_{\alpha \in (\Phi_{\Delta}^+)_{\text{nd}}} u_{\alpha} \right) \left(\prod_{\alpha \in \Psi^-} u_{\alpha} \right) \left(\prod_{\alpha \in \Psi^+} u_{\alpha} \right) t$$

where $u_{\alpha} \in U_{\alpha,x}$ for every $\alpha \in (\Phi_{\Delta}^+)_{\text{nd}}$ and $u_{\alpha} \in U'_{\alpha,x}$ for every $\alpha \in (\Phi_{\Delta}^-)_{\text{nd}}$ and $t \in T_b$.

For $\alpha \in \Phi_{\Delta}^+$ and $z \in x + \delta_v$, write $z = x + \lambda v$ with $\lambda \in R_{>0}$. Then, we have $\alpha(z) = \alpha(x + \lambda v) = \alpha(x) + \lambda \alpha(v) > \alpha(x)$ with last inequality given by assumption on v . Thus

$$\begin{aligned} u_{\alpha} &\in U_{\alpha, \mathcal{F}_{x,C}} = U_{\alpha,x} && \text{by Corollary 4.34 since } \alpha \in \Phi_{\Delta}^+ \\ &= \varphi_{\alpha}^{-1}([- \alpha(x), +\infty]) \subset \varphi_{\alpha}^{-1}([- \alpha(z), +\infty]) && \text{since } -\alpha(z) < -\alpha(x) \\ &= U_{\alpha,z} = U_{\alpha, \mathcal{F}_{z,C}} && \text{by Corollary 4.34 since } \alpha \in \Phi_{\Delta}^+ \\ &\subset \widehat{P}_{\mathcal{F}_{z,C}} \end{aligned}$$

In the other side, we have $u_{\alpha} \in U_{\alpha, \mathcal{F}_{x,C}} \subset U_{\alpha,x}$ since $U_{\alpha, \mathcal{F}_{x,C}}$ is either equal to $U_{\alpha,x}$ or $U'_{\alpha,x}$ by Corollary 4.34 and $U'_{\alpha,x} \subset U_{\alpha,x}$. By definition of C'' , we know that $x' \in x + \overline{C''} = \overline{Q''}$ and there is some element $y \in Q' \cap Q''$. Write $y = x' + v' \in x + C''$ with $v' \in C'$. For any $\alpha \in \Phi_{\Delta''}^+$, we have $\alpha(x') - \alpha(x) \geq 0$ since $x' - x \in \overline{D''}$ and $\alpha(y - x) = \alpha(x') + \alpha(v') - \alpha(x) > 0$ since $(x' - x) + v' \in D''$. Hence, if $\alpha \in \Phi_{\Delta'}^+$, then

$$u_{\alpha} \in U_{\alpha,x} = \varphi_{\alpha}^{-1}([- \alpha(x), +\infty]) \subset \varphi_{\alpha}^{-1}([- \alpha(x'), +\infty]) = U_{\alpha,x'} = U_{\alpha, \mathcal{F}_{x',C'}}$$

because $U_{\alpha, \mathcal{F}_{x',C'}} = U_{\alpha,x'}$ for $\alpha \in \Phi_{\Delta'}^+$ by Corollary 4.34. Otherwise, $\alpha \in \Phi_{\Delta'}^-$ and $\alpha(x') - \alpha(x) > -\alpha(v') > 0$. Hence

$$u_{\alpha} \in U_{\alpha,x} = \varphi_{\alpha}^{-1}([- \alpha(x), +\infty]) \subset \varphi_{\alpha}^{-1}([- \alpha(x'), +\infty]) = U'_{\alpha,x'} = U_{\alpha, \mathcal{F}_{x',C'}}$$

because $U_{\alpha, \mathcal{F}_{x',C'}} = U'_{\alpha,x'}$ for $\alpha \in \Phi_{\Delta'}^-$ by Corollary 4.34. Thus $u_{\alpha} \in \widehat{P}_{\mathcal{F}_{x',C'}}$ for every $\alpha \in \Phi_{\Delta''}^+$ and, in particular, for every $\alpha \in \Psi^+$. Therefore, we have

$$u_+ = \left(\prod_{\alpha \in (\Phi_{\Delta}^+)_{\text{nd}}} u_{\alpha} \right) \in \bigcap_{z \in x + \delta_v} \widehat{P}_{\mathcal{F}_{z,C}} \quad \text{and} \quad \left(\prod_{\alpha \in \Psi^+} u_{\alpha} \right) t \in \widehat{P}_{\mathcal{F}_{x',C'}}.$$

Hence, up to replace g by $u_+^{-1}g$, we can assume that $g \in u \widehat{P}_{\mathcal{F}_{x',C'}}$ for some $u = \prod_{\alpha \in \Psi^-} u_{\alpha}$ with $u_{\alpha} \in U_{\alpha, \mathcal{F}_{x,C}} = U'_{\alpha,x}$ for $\alpha \in \Psi^-$.

Decomposition step: If $u = 1$, then the condition (a) is satisfied. In particular, if $C'' = -C$, then $\ell(x, C, x', C')$ is maximal by definition of the length in W with respect to Δ_D and, in this case, $u = 1$ since $\Psi^- = (\Phi_{\Delta}^-)_{\text{nd}} \cap \Phi_{\Delta''}^- = (\Phi_{\Delta}^-)_{\text{nd}} \cap \Phi_{\Delta}^+ = \emptyset$.

Suppose that $u \neq 1$ (and therefore $C'' \neq -C$). By assumption on v and R , for $\alpha \in \Psi^-$, we can define $\lambda_{\alpha} = \frac{-\varphi_{\alpha}(u_{\alpha}) - \alpha(x)}{\alpha(v)}$ since $\alpha(v) \in \mathbb{Z}_{<0}$ (since $-\alpha \in \Phi_{\Delta}^+$). Because $u_{\alpha} \in U'_{\alpha,x} = \varphi_{\alpha}^{-1}([- \alpha(x), +\infty])$, we have that $-\varphi_{\alpha}(u_{\alpha}) - \alpha(x) < 0$. Hence $\lambda_{\alpha} \in R$ is positive. By finiteness of Ψ^- , there is a maximal element $\lambda = \lambda_{\beta} \in R_{>0}$ for some $\beta \in \Psi^-$. If we set $y = x + \lambda v \in x + \delta_v$, we have that for any $\alpha \in \Psi^-$:

$$\alpha(y) = \alpha(x) + \lambda \alpha(v) \geq \alpha(x) + \lambda_{\alpha} \alpha(v) = -\varphi_{\alpha}(u_{\alpha})$$

with equality at least for $\alpha = \beta$.

We firstly prove (1)((b))ii. We have shown that $u_\beta \in U_{\beta,y} \setminus U'_{\beta,y}$ and that $u_\alpha \in U_{\alpha,y}$ for any $\alpha \in \Psi^-$. Thus $u \in U_y$. Since $U'_y \cap U_\beta = U'_{\beta,y}$ according to Example 3.44, we have $u \in U_y \setminus U'_y$. Since $U_{\beta,y} \neq U'_{\beta,y}$, we know by Lemma 3.60 that either $\beta \in \Phi_y$ or $2\beta \in \Phi_y$ so that, in particular, the image $\overline{u_\beta}$ of u_β in \overline{G}_y is non-trivial. Let $C'''' = C(y, x', C')$ and $\Delta'''' = \Delta_{C''''}$. Thus, by Theorem 3.61 and spherical Bruhat decomposition [BT72, 6.1.15], we know that there are $\overline{u'} \in \prod_{\alpha \in \Delta} \overline{U_\alpha}$, $\overline{u''} \in \prod_{\alpha \in \Delta''''} \overline{U_\alpha}$ and $\overline{n'} \in \overline{N}_y$ such that $\overline{u} = \overline{u'n'u''}$. According to Proposition 4.20, one can lift those elements in elements $u' \in U_{y+C} = \prod_{\alpha \in \Delta} U_{\alpha,y}$, $u'' \in U_{y+C''''} = \prod_{\alpha \in \Delta''''} U_{\alpha,y}$ and $n' \in N_y$ such that there exists an element $v \in U_y^* = U'_y$ (indeed, y is a point so that $\Phi_y^* = \Phi$ by definition) so that $u = vu'n'u''$. Since $v \in U'_y \subset \widehat{P}_{\mathcal{F}_{y,C}}$, $u' \in U_{y+C} \subset \widehat{P}_{\mathcal{F}_{y,C}}$ and $u'' \in U_{y+C''''} \subset \widehat{P}_{\mathcal{F}_{x',C'}}$ since $\mathcal{F}_{x',C'} \cap (y+C''') \neq \emptyset$ by construction of $C'''' = C(y, x', C')$, we get that $u \in \widehat{P}_{\mathcal{F}_{y,C}} n' \widehat{P}_{\mathcal{F}_{x',C'}}$.

We secondly prove (1)((b))i. Consider any value $\mu \in]0, \lambda[$ and denote by $z = x + \mu v$. For any $\alpha \in \Psi^-$, since $\alpha(v) \in \mathbb{Z}_{<0}$, we have that $\alpha(z) = \alpha(x) + \mu\alpha(v) > \alpha(x) + \lambda\alpha(v) \geq -\varphi_\alpha(u_\alpha)$. Hence $u_\alpha \in U'_{\alpha,z} \subset U_{\mathcal{F}_{z,C}}$. Hence $g \in \widehat{P}_{\mathcal{F}_{z,C}} \widehat{P}_{\mathcal{F}_{x',C'}}$.

It remains to prove (1)((b))iii.

We firstly prove that $\ell(y, C, n' \cdot x', n' \cdot C') \geq \ell(x, C, x', C')$. Let \mathcal{H} be the set of hyperplanes in V_R that are kernels of elements in Φ . We know that the length $\ell(w)$ of an element $w \in W$ is equal to the number of hyperplanes in \mathcal{H} separating $C_{\mathbb{R},\Delta}^v$ and $C_{\mathbb{R},w(\Delta)}^v$ [Bou81, chap. VI § 1, no. 6] and that two vector chambers of $V_{\mathbb{R}}$ intersect if and only if the corresponding vector chamber in V_R intersect, according to Lemma 2.2. Hence, for $z \in \mathbb{A}_R$, the number $\ell(z, C, x', C')$ is the cardinality of \mathcal{H}_z which is the set of elements $H \in \mathcal{H}$ such that $z+H$ separates $z+C$ and a neighbourhood of x in $x+\overline{C'}$. But, for $H \in \mathcal{H}$, the set of elements $z \in \mathbb{A}_R$ such that $H \in \mathcal{H}_z$ is either the open half-space $x' + H + C$ or the closed half-space $x' + H + \overline{C}$ and its intersection with the open half-line $x + \delta_v$ is an (open or closed) half-line. Hence $\mathcal{H}_y \supset \mathcal{H}_x$ so that $\ell(y, C, n' \cdot x', n' \cdot C') \geq \ell(x, C, x', C')$. More precisely, $w' = w(y, C, x', C')$ is upper than $w = w(x, C, x', C')$ in the sense that there exists $w'' \in W$ such that $w' = ww''$ with $\ell(w') = \ell(w) + \ell(w'')$.

We prove that $\ell(y, C, n' \cdot x', n' \cdot C') > \ell(x, C, x', C')$ whenever $n' \in T_b$. Indeed, suppose by contradiction that it is an equality. Then $w' = w$ and therefore $C'''' = C''$. But since the image $\overline{u} \in \overline{G}_y$ of u is in the subgroup $\overline{U_\Delta^-} \cap \overline{U_{\Delta''}^-}$ generated by the root groups $\overline{U_\alpha}$ for $\alpha \in \Phi_y \cap \Phi_\Delta^- \cap \Phi_{\Delta''}^-$, whereas the image $\overline{u'} \in \overline{G}_y$ (resp. $\overline{u''} \in \overline{G}_y$) of u' (resp. u'') is contained in $\overline{U_\Delta^+}$ (resp. $\overline{U_{\Delta''}^+}$) generated by the root groups $\overline{U_\alpha}$ for $\alpha \in \Phi_y \cap \Phi_\Delta^+$ (resp. $\Phi_y \cap \Phi_{\Delta''}^+$). If $n' \in T_b$ and $\Delta'' = \Delta'''$, then we have $\overline{u} \in \overline{U_\Delta^-} \cap \overline{U_{\Delta''}^-} \cap \overline{U_\Delta^+} \cdot \overline{T} \cdot \overline{U_{\Delta''}^+}$. In particular, $\overline{u} = 1$ because of axiom (RGD6) of spherical root groups data and [BT72, 6.1.6]. In particular $u \in U'_y$ which is a contradiction. As a consequence, when $n' \in T_b$, we have $\ell(y, C, n' \cdot x', n' \cdot C') > \ell(x, C, x', C')$.

We prove that $\ell(y, C, n' \cdot x', n' \cdot C') \geq \ell(y, C, x', C')$ with equality if, and only if $n' \in T_b$. Since $n' \in N_y$, we get that $w(y, D, n' \cdot x, n' \cdot D') = \nu(n')w(y, C, x', C')$. Denote $t = \nu(n')$ and $w' = w(y, C, x', C')$. Hence, we want to prove that $\ell(tw') \geq \ell(w')$ with equality if, and only if, $t = 1$. Let $W_y = W(\Phi_y)$ the Weyl group of the root system Φ_y and identify it with a subgroup of $W = W(\Phi)$. Any vector chamber of Φ_y , which is a simplicial cone in some quotient of V_R , can be identified with its inverse image in V_R . Let \tilde{C} be the vector chamber of Φ_y containing C . Let R (resp. R_y) the generating system of W (resp. W_y) of reflections with respect to the walls of C (resp. \tilde{C}). Let w_0 (resp. w'_0) the longest element with respect to R (resp. R_y) of W (resp. of W_y). We have $w_0(C) = -C$ and $w'_0(\tilde{C}) = -\tilde{C}$ so that $w'_0(\tilde{C}) \supset w_0(C)$.

In the quotient group \overline{G}_y , consider the minimal parabolic subgroup \overline{B} associated to the vector chamber \tilde{C} . Write $\overline{B} = \overline{T} \cdot \prod_{\alpha \in \Phi_y \cap \Phi_\Delta^+} \overline{U_\alpha}$. Since $w'(C) = C''''$, we have $\overline{U_{y+C''''}} \subset w' \overline{B} w'^{-1}$. Since $u \in U_{y-C}$, we have $\overline{u} \in w'_0 \overline{B} w_0^{-1}$. Since $u \in U'_y U_{y+C} n' U_{y+C''''}$, we have $\overline{u} \in \overline{B} t w' \overline{B} w'^{-1}$. Let (r_k, \dots, r_1) be a reduced decomposition of $w'^{-1} w_0 \in W$ with respect to R .

Then $w_0 = w'r_k \cdots r_1$ and we know that $\ell(w') = \ell(w_0) - k$ [Bou81, chap.VI,§1,no.6cor.3 of prop.17]. For $1 \leq i \leq k-1$, denote $w_i = w'r_k \cdots r_{i+1}$ and, $w_k = w'$. Then $w_i = w_{i+1}r_i$ so that the vector chambers $w_i(C)$ and $w_{i+1}(C)$ have a wall H_i in common and the reflection s_i with respect to H_i is $s_i = w_{i+1}w_i^{-1} = w_{i+1}r_iw_{i+1}^{-1} = w_i r_i w_i^{-1}$.

On the other hand, let w'_i be the unique element in W_y such that the chamber $w'_i(\tilde{C})$ of Φ_y contains the chamber $w_i(C)$ of Φ . Note that for $i=0$, we have w'_0 equals to the w'_0 previously defined. We have $w'_k = w_k = w'$ since $w' \in W_y$. Finally, the two chambers $w'_i(\tilde{C})$ and $w'_{i+1}(\tilde{C})$ are adjoining or equal according to the fact that s_i belongs or not to W_y . When $s_i \in W_y$, we also have that $s_i = w'_{i+1}w_i'^{-1}$. By [BT72, 6.1.15(a)], for any $0 \leq i \leq k$, there exists a unique $\overline{w'_i} \in W_y$ such that $\overline{B\overline{w'_i}B} = \overline{Bw'_iB}$. Since $\overline{w'_i} \in w'_0\overline{B}w_0'^{-1}$, we have $\overline{w'_0} = w'_0$. Since $\overline{w'_i} \in \overline{Btw'_iB}w_i'^{-1}$ and $w'_k = w'$, we have $\overline{w'_k}tw'$.

Let $I = \llbracket 0, k-1 \rrbracket$ and let $I_1 = \{i \in I, w'_{i+1} = w'_i\}$. For $i \in I_1$, we have $w'_{i+1} = w'_i$ and $\overline{w'_{i+1}} = \overline{w'_i}$. Moreover, we have ever seen that if $i \notin I_1$, then we have $w'_{i+1}w_i'^{-1} = w_{i+1}w_i^{-1} = s_i$. If $i \notin I_1$, we observe that $r'_i = w_{i+1}'^{-1}w'_i = w_i'^{-1}s_iw'_i$ is a reflection with respect to a wall of \tilde{C} : in other words, we have $r'_i \in R_y$. From [BT72, 6.1.15] and axiom (T3) of Tits systems, we therefore deduce that for $i \notin I_1$, we have

$$\overline{Bw'_{i+1}B} = \overline{B\overline{w'_i}B} = \overline{B\overline{w'_i}r'_iB} \subset \overline{Bw'_iBR'_iB} \subset \overline{Bw'_iB} \cup \overline{Bw'_i r'_i B}.$$

Hence, we get a partition of $I \setminus I_1$ in two (possibly empty) subsets:

$$I_2 = \{i \in I \setminus I_1, \overline{w'_{i+1}} = \overline{w'_i}\} \quad \text{and} \quad I_3 = \{i \in I \setminus I_1, \overline{w'_{i+1}} = \overline{w'_i r'_i}\}$$

so that $I = I_1 \sqcup I_2 \sqcup I_3$. Finally, we denote $\overline{w'_i} = \overline{w'_i}w_i'^{-1}w_i$ for $0 \leq i \leq k$. Note that we have $\overline{w'_0} = w_0$ and $\overline{w'_k} = tw'w'^{-1}w' = tw'$.

For $i \in I$, define $d_i = \overline{w'_{i+1}}^{-1}\overline{w'_i} = w_{i+1}'^{-1}w_{i+1}'\overline{w'_{i+1}}^{-1}\overline{w'_i}w_i'^{-1}w_i$.

If $i \in I_1$, since $w'_{i+1} = w'_i$ and $\overline{w'_{i+1}} = \overline{w'_i}$, we have $d_i = w_{i+1}'^{-1}w_{i+1}'\overline{w'_{i+1}}^{-1}\overline{w'_i}w_i'^{-1}w_i = w_{i+1}'^{-1}w_i = r_i$.

If $i \in I_2$, since $w'_{i+1} = s_iw'_i$ and $w_{i+1} = s_iw_i$, we have $d_i = w_{i+1}'^{-1}w_{i+1}'\overline{w'_{i+1}}^{-1}\overline{w'_i}w_i'^{-1}w_i = w_i'^{-1}s_i s_i w_i' w_i'^{-1} w_i = 1$.

If $i \in I_3$, then $d_i = w_{i+1}'^{-1}w_{i+1}'\overline{w'_{i+1}}^{-1}\overline{w'_i}w_i'^{-1}w_i = w_{i+1}'^{-1}w_{i+1}'r'_i w_i'^{-1}w_i = w_{i+1}'^{-1}w_i = r_i$.

As a consequence, we have $w_0 = \overline{w'_0} = \overline{w'_k}(\overline{w_k'}^{-1}\overline{w_{k-1}'}^{-1}) \cdots (\overline{w_1'}^{-1}\overline{w_0'}) = \overline{w'_k}d_{k-1} \cdots d_0 = tw' \prod_{i \in I_1 \sqcup I_3} r_i$. Hence $\ell(tw') \geq \ell(w_0) - \text{Card}(I_1 \sqcup I_3) \geq \ell(w_0) - k = \ell(w')$ and, if this is an equality, then we necessarily have that $I_2 = \emptyset$. In that case, $w_0 = tw_0$ and therefore $t = 1$.

To conclude, if we have $\ell(y, C, n' \cdot x', n' \cdot C') > \ell(y, C, x', C')$, then $\ell(y, C, n' \cdot x', n' \cdot C') > \ell(x, C, x', C')$ since $\ell(y, C, x', C') \geq \ell(x, C, x', C')$. Otherwise, we have $\ell(y, C, n' \cdot x', n' \cdot C') = \ell(y, C, x', C')$ and $n' \in T_b$. We have seen that in this case $\ell(y, C, x', C') > \ell(x, C, x', C')$ and therefore we also have that $\ell(y, C, n' \cdot x', n' \cdot C') > \ell(x, C, x', C')$. \square

Proof of (2). If g satisfies condition (a), there is nothing to prove. Otherwise, we define a strictly increasing sequence $(\lambda_i)_i$ of values in R and a sequence of elements $(n_i)_i$ in N by $\lambda_0 = 0$, $n_0 = n$ and for $i \geq 1$, while g is in the case (b) when we apply step (1) to $(x + \lambda_{i-1}v, C)$, $x' + C'$ and $n_{i-1} \in N$, we set $\lambda_i = \lambda_{i-1} + \lambda$ and $n_i = n'$ where $\lambda > 0$ and $n' \in N$ are both given by (b). At some point $k \in \mathbb{N}$, the element g will be in case (a) for $y_k = x + \lambda_k v$ and n_k since the length in the spherical (hence finite) Weyl group is bounded so that this process stops. Thus $g \in \widehat{P}_{\mathcal{F}_z, C} n_k \widehat{P}_{\mathcal{F}_{x'}, C'} \subset \widehat{P}_{\mathcal{F}_z, C} N(K) \widehat{P}_{\mathcal{F}_{x'}, C'}$ for any $z \in y_k + \delta_v$ and for any $i \in \llbracket 0, k-1 \rrbracket$ we have $g \in \widehat{P}_{\mathcal{F}_z, C} n_i \widehat{P}_{\mathcal{F}_{x'}, C'} \subset \widehat{P}_{\mathcal{F}_z, C} N \widehat{P}_{\mathcal{F}_{x'}, C'}$ for any $z = x + \mu v$ with $\mu \in]\lambda_i, \lambda_{i+1}[$ and $g \in \widehat{P}_{\mathcal{F}_{y_{i+1}}, C} n_{i+1} \widehat{P}_{\mathcal{F}_{x'}, C'} \subset \widehat{P}_{\mathcal{F}_{y_{i+1}}, C} N \widehat{P}_{\mathcal{F}_{x'}, C'}$ for $y_{i+1} = x + \lambda_{i+1}v$. We get the result from the decomposition $R_{>0} = \left(\bigsqcup_{i=0}^k]\lambda_i, \lambda_{i+1}[\sqcup \{\lambda_{i+1}\} \right) \sqcup]\lambda_k, \infty[$. \square

Since the topology of \mathbb{A}_R is less usual than the topology of $\mathbb{A}_{\mathbb{R}}$, we detail how to generalize the proof of the affine Bruhat decomposition.

Proof of Theorem 4.38. Let $g \in G$. By Iwasawa Decomposition 4.36 applied to $-C$ and $\mathcal{F}_{x',C'}$, there exist $u \in U_{\Delta}^-$, $n \in N$ and $u' \in \widehat{P}_{\mathcal{F}_{x',C'}}$ such that $g = unu'$. Write $u = \prod_{\beta \in (\Phi_{\Delta}^+)_{\text{nd}}} u_{\beta}$ with $u_{\beta} \in U_{-\beta}$ for $\beta \in (\Phi_{\Delta}^+)_{\text{nd}}$.

Let $v \in C_{\mathbb{Z},\Delta}^v$ so that $\beta(v) \in \mathbb{Z}_{>0}$ for any $\beta \in \Phi_{\Delta}^+$. For $\beta \in \Phi_{\Delta}^+$, define $\lambda_{\beta} = 0$ if $\varphi_{-\beta}(u_{\beta}) = \infty$ and $\lambda_{\beta} = \frac{1}{\beta(v)} \left(\beta(x) - \varphi_{-\beta}(u_{\beta}) \right)$ otherwise. Let $\lambda \in R_{>0}$ be such that $\lambda > \max\{\lambda_{\beta}, \beta \in \Phi_{\Delta}^+\}$ and consider $y = x - \lambda v \in x - \delta_v$. For any $\beta \in \Phi_{\Delta}^+$ such that $u_{\beta} \neq 1$, we have $\beta(v)\lambda > \beta(v)\lambda_{\beta} = \beta(x) - \varphi_{-\beta}(u_{\beta})$. Thus $-\beta(x - \lambda v) + \varphi_{-\beta}(u_{\beta}) > 0$. Hence we get that $u_{\beta} \in U'_{-\beta,y}$ for any $\beta \in \Phi_{\Delta}^+$.

By Proposition 4.30 and Corollary 4.34, we know that $U'_{\beta,y} \subset \widehat{P}_{\mathcal{F}_{y,C}}$ for any $\beta \in \Phi$. Therefore $u \in \widehat{P}_{\mathcal{F}_{y,C}}$. Hence $g = unu' \in \widehat{P}_{\mathcal{F}_{y,C}} N \widehat{P}_{\mathcal{F}_{x',C'}}$. Since $x = y + \lambda v \in y + \delta_v \subset y + C$ according to Lemma 2.5, Lemma 4.41(2) gives that $g \in \widehat{P}_{\mathcal{F}_{x,C}} N \widehat{P}_{\mathcal{F}_{x',C'}}$. Hence, $G = \widehat{P}_{\mathcal{F}_{x,C}} N \widehat{P}_{\mathcal{F}_{x',C'}}$.

Now, let $n, n' \in N$ be such that $n' \in \widehat{P}_{\mathcal{F}_{x,C}} n \widehat{P}_{\mathcal{F}_{x',C'}}$. Let $x'' = \nu(n')(x')$ and $C'' = \nu(n')(C')$ so that $n' \widehat{P}_{\mathcal{F}_{x',C'}} (n')^{-1} = \widehat{P}_{\mathcal{F}_{x'',C''}}$ according to Lemma 4.40. Let $n'' = n(n')^{-1}$. Then $1 \in \widehat{P}_{\mathcal{F}_{x,C}} n'' \widehat{P}_{\mathcal{F}_{x'',C''}}$ which gives $n'' \in \widehat{P}_{\mathcal{F}_{x,C}} \widehat{P}_{\mathcal{F}_{x'',C''}}$. By Proposition 4.30 and Remark 4.29, we have $\widehat{P}_{\mathcal{F}_{x,C}} = P_{\mathcal{F}_{x,C}}$ and $\widehat{P}_{\mathcal{F}_{x'',C''}} = P_{\mathcal{F}_{x'',C''}}$. By Lemma 4.27, we have $P_{\mathcal{F}_{x,C}} = P_{\text{cl}(\mathcal{F}_{x,C})}$ and $P_{\mathcal{F}_{x'',C''}} = P_{\text{cl}(\mathcal{F}_{x'',C''})}$.

Let $C^{(3)} = C(x, x'', C'')$ (see Notation 4.39) so that $(x + C^{(3)}) \cap (x'' + C'') \neq \emptyset$ and $x'' \in x + \overline{C^{(3)}}$. Let $\mathcal{X} = \{\Omega'' \in \mathcal{F}_{x'',C''} \mid \Omega'' \subset x + C^{(3)}\}$. Then $\mathcal{X} \subset \mathcal{F}_{x'',C''}$ (as sets of subsets of $\mathbb{A}_{\mathbb{R}}$) and thus $P_{\mathcal{F}_{x'',C''}} \supset \bigcup_{\Omega'' \in \mathcal{X}} P_{\Omega''}$. Let $y \in (x + C^{(3)}) \cap (x'' + C'')$. Then by Lemma 2.17, $\text{cl}(\{x'', y\}) \ni \mathcal{F}_{x'',C''}$. Moreover $x + \overline{C^{(3)}} \in \text{cl}(\{x'', y\})$ and thus $\mathcal{F}_{x'',C''} \in x + \overline{C^{(3)}}$. In other words, $x + \overline{C^{(3)}} \in \mathcal{F}_{x'',C''}$. Therefore, for every $\tilde{\Omega}'' \in \mathcal{F}_{x'',C''}$, there exists $\Omega'' \in \mathcal{X}$ such that $\tilde{\Omega}'' \supset \Omega''$ (one can take $\Omega'' = \Omega \cap (x + \overline{C^{(3)}})$ for example). Consequently,

$$P_{\mathcal{F}_{x'',C''}} = \bigcup_{\Omega'' \in \mathcal{X}} P_{\Omega''}.$$

Let $\Omega \in \text{cl}(\mathcal{F}_{x,C})$ and $\Omega'' \in \mathcal{X}$. Set $\Omega' = \Omega \cap (\Omega'' - \overline{C^{(3)}})$. By Lemma 4.32, $x \in \Omega$, $x'' \in \Omega''$ and thus $x \in \Omega'$ (since $x'' \in x + \overline{C^{(3)}}$). Therefore

$$\Omega' \subset \Omega'' - C^{(3)} \text{ and } \Omega'' \subset x + C^{(3)} \subset \Omega' + C^{(3)}.$$

Using Proposition 4.23(2), we get that $P_{\Omega'} P_{\Omega''} \subset N_{\Omega'} U_{\Omega' - \overline{C^{(3)}}} U_{\Omega'' + \overline{C^{(3)}}} N_{\Omega''}$. By Proposition 4.30, we have $N_{\Omega'} = N_{\Omega''} = T_b$. By Proposition 4.20, we have $U_{\Omega' - \overline{C^{(3)}}} \subset U_{\Delta^{(3)}}^-$ and $U_{\Omega'' + \overline{C^{(3)}}} \subset U_{\Delta^{(3)}}^+$, where $\Delta^{(3)}$ is the base associated to $C^{(3)}$. Thus $P_{\Omega'} P_{\Omega''} \subset P_{\Omega'} P_{\Omega''} \subset U_{\Delta^{(3)}}^- T_b U_{\Delta^{(3)}}^+$ for every $\Omega \in \mathcal{F}_{x,C}$ and every $\Omega'' \in \mathcal{X}$. Hence $n'' \in \widehat{P}_{\mathcal{F}_{x,C}} \widehat{P}_{\mathcal{F}_{x'',C''}} = P_{\text{cl}(\mathcal{F}_{x,C})} \bigcup_{\Omega'' \in \mathcal{X}} P_{\Omega''} \subset U_{\Delta^{(3)}}^- T_b U_{\Delta^{(3)}}^+$. By [BT72, 6.1.15], we get that $n'' \in T_b$. Hence $n' \in n T_b$. This provides the correspondence between the quotient group N/T_b and the double cosets $\widehat{P}_{\mathcal{F}_{x,C}} \backslash G / \widehat{P}_{\mathcal{F}_{x',C'}}$. \square

5 Building associated to a valued root group datum

Let $V_{\mathbb{R}}$ be a finite dimensional \mathbb{R} -vector space and let $\Phi \subseteq V_{\mathbb{R}}^*$ be a root system that such $V_{\mathbb{R}}^* = \langle \Phi \rangle_{\mathbb{R}}$. Let $V_{\mathbb{Z}}$ be the free \mathbb{Z} -submodule of $V_{\mathbb{R}}$ spanned by Φ^{\vee} . Let R be a nonzero totally ordered commutative pseudo-ring such that $R_{\mathbb{Q}} = R$. Let \mathbb{A} be an R -affine space

with some origin o and underlying R -module $V_R := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. Let G be a group and let $(T, (U_{\alpha})_{\alpha \in \Phi}, (M_{\alpha})_{\alpha \in \Phi}, (\varphi_{\alpha})_{\alpha \in \Phi})$ be a generating R -valued root group datum of G of type Φ . Denote by N the subgroup of G that is generated by the M_{α} for $\alpha \in \Phi$, and assume that we are given a compatible action $\nu : N \rightarrow \text{Aff}_{\mathbb{Z}}(\mathbb{A})$ of N on \mathbb{A} . We adopt all the notations that have been introduced in sections 3 and 4 and that are associated to the data we are given.

Under these assumptions, we can define the following relation on $G \times \mathbb{A}$:

$$(g, x) \sim (h, y) \Leftrightarrow \exists n \in N, \begin{cases} y = \nu(n)(x), \\ g^{-1}hn \in U_x. \end{cases}$$

This relation is reflexive and symmetric. Moreover, for any $(g_1, g_2, g_3) \in G$ and $(x_1, x_2, x_3) \in \mathbb{A}$, if we can find $n_{12}, n_{23} \in N$ such that

$$\begin{cases} x_2 = \nu(n_{12})(x_1), \\ x_3 = \nu(n_{23})(x_2), \\ g_1^{-1}g_2n_{12} \in U_{x_1}, \\ g_2^{-1}g_3n_{23} \in U_{x_2}, \end{cases}$$

then:

$$\begin{cases} x_3 = \nu(n_{23}n_{12})(x_1), \\ g_1^{-1}g_3n_{23}n_{12} \in g_1^{-1}g_2U_{x_2}n_{12} \subseteq U_{x_1}n_{12}^{-1}U_{x_2}n_{12} = U_{x_1} \end{cases}$$

where the last equality is given by Proposition 4.9. Hence \sim is an equivalence relation on $G \times \mathbb{A}$.

5.1 Definition. The R -building associated to the datum:

$$(G, T, (U_{\alpha})_{\alpha \in \Phi}, (M_{\alpha})_{\alpha \in \Phi}, (\varphi_{\alpha})_{\alpha \in \Phi}, \nu)$$

is the quotient:

$$\mathcal{I}(G, T, (U_{\alpha})_{\alpha \in \Phi}, (M_{\alpha})_{\alpha \in \Phi}, (\varphi_{\alpha})_{\alpha \in \Phi}, \nu) := (G \times \mathbb{A}) / \sim.$$

To simplify notations, we will denote it $\mathcal{I}(G)$ in the rest of this paragraph.

We denote by $[g, x]$ the class in $\mathcal{I}(G)$ of $(g, x) \in G \times \mathbb{A}$.

The group G then acts on $\mathcal{I}(G)$ by:

$$g \cdot [h, x] := [gh, x].$$

5.2 Lemma. *The map:*

$$\begin{aligned} i : \mathbb{A} &\rightarrow \mathcal{I}(G) \\ x &\mapsto [1, x] \end{aligned}$$

is injective.

Proof. Let $x, y \in \mathbb{A}$ such that $i(x) = i(y)$. We can then find $n \in N \cap U_x = \tilde{N}_x$ such that $y = \nu(n)(x)$. By Corollary 4.13, we deduce that $x = y$. \square

5.3 Fact. *For any $n \in N$ and any $x \in \mathbb{A}$, we have $n \cdot [1, x] = [n, x] = [1, \nu(n)(x)]$. In particular, the subgroup N stabilizes $i(\mathbb{A})$.*

In particular, we identify \mathbb{A} with the subset $i(\mathbb{A})$ of $\mathcal{I}(G)$. More generally, we identify any subset Ω of \mathbb{A} with the subset $i(\Omega)$ of $\mathcal{I}(G)$.

5.4 Definition. An **apartment** of $\mathcal{I}(G)$ is a subset of $\mathcal{I}(G)$ of the form:

$$A = g \cdot \mathbb{A} = \{[g, x], x \in \mathbb{A}\}$$

for some $g \in G$, endowed with the set $\text{Isom}(\mathbb{A}, A)$ of bijections $\iota : \mathbb{A} \rightarrow A$ given by $\iota : x \mapsto [g, \nu(n)(x)]$ for some $n \in N$.

A **local face** (resp. **local chamber**) of $\mathcal{I}(G)$ is a filter on $\mathcal{I}(G)$ of the form:

$$\mathcal{F} = g \cdot \text{germ}_x(x + F^v) = \{g \cdot \Omega, \Omega \in \text{germ}_x(x + F^v)\}$$

for some $g \in G$, some $x \in \mathbb{A}$ and some vector face (resp. vector chamber) F^v in V_R .

5.5 Lemma. [see [BT72, 7.4.4]] Let Ω be a non-empty subset of \mathbb{A} . The group \widehat{P}_Ω is the pointwise stabilizer of Ω in G .

In particular, for any filter \mathcal{V} on \mathbb{A} , the group $\widehat{P}_\mathcal{V}$ fixes \mathcal{V} .

Proof. If $\Omega = \{x\}$ is a single point and $g \in G$, then

$$[1, x] = g \cdot [1, x] \iff \exists n \in N, \begin{cases} x = \nu(n^{-1})(x) \\ gn^{-1} \in U_x \end{cases} \iff \exists n \in \widehat{N}_x, g \in U_x n$$

since \widehat{N}_x is, by definition, the stabilizer of x in N . Thus, the stabilizer of x in G is $U_x \widehat{N}_x$, which is \widehat{P}_x by Corollary 4.12. Hence, Proposition 4.26 gives that \widehat{P}_Ω is the pointwise stabilizer of Ω . \square

5.6 Proposition. (see [BT72, 7.4.8]) The set \mathcal{I} satisfies (A2). More precisely, let $g \in G$. Then:

1. there exists $n \in N$ such that $g^{-1} \cdot x = n \cdot x$ for all $x \in \mathbb{A} \cap g \cdot \mathbb{A}$.
2. $\mathbb{A} \cap g \cdot \mathbb{A}$ is enclosed.

Proof. We may assume that $\Omega := \mathbb{A} \cap g \cdot \mathbb{A}$ is nonempty. Let \mathcal{X} be the set of subsets $\widetilde{\Omega} \subset \Omega$ such that $g \cdot N \cap \widehat{P}_{\widetilde{\Omega}} \neq \emptyset$. By definition of $\mathcal{I}(G)$, \mathcal{X} contains $\{x\}$, for all $x \in \Omega$. Let $\Omega_1 \in \mathcal{X}$, $x_2 \in \Omega$ and $n_1, n_2 \in N$ be such that $gn_1 \in \widehat{P}_{\Omega_1}$ and $gn_2 \in \widehat{P}_{\{x_2\}}$. Let us prove that $\Omega_1 \cup \{x_2\} \in \mathcal{X}$.

By Corollary 4.24, there exists a vector chamber $C^v \subset V_R$ such that $n_1^{-1}n_2 \in \widehat{P}_{\Omega_1} \cdot \widehat{P}_{x_2} \subset \widehat{N}_{\Omega_1} \cdot U_{C^v}^- \cdot U_{C^v}^+ \cdot \widehat{N}_{x_2}$ (we used the relations $\widehat{P}_{\Omega_1} = \widehat{N}_{\Omega_1} P_{\Omega_1}$ and $\widehat{P}_{x_2} = P_{x_2} \widehat{N}_{x_2}$ from Lemma 4.3 and Corollary 4.12). Therefore, there exists $n'_1 \in \widehat{N}_{\Omega_1}$ and $n'_2 \in \widehat{N}_{x_2}$ such that $n_1^{-1}n_2 = n'_1 n'_2$ and $n'_1 \in N \cap U_{C^v}^- U_{C^v}^+ = \{1\}$ by [BT72, 6.1.15(c)]. Set $n = n_1 n'_1 = n_2 n'_2$. Then $gn \in \widehat{P}_{\Omega_1} \cap \widehat{P}_{x_2} = \widehat{P}_{\Omega_1 \cup \{x_2\}}$ (by Proposition 4.26). Consequently, $\Omega_1 \cup \{x_2\} \in \mathcal{X}$ and by induction, every nonempty finite subset of Ω is in \mathcal{X} .

The group N/T_b is finite. Indeed, \widehat{N}_{x_0} is by definition the stabilizer of x_0 and T_b is the kernel of the action $\nu : N \rightarrow \text{Aff}_{\mathbb{Z}}(\mathbb{A})$. Thus the quotient group \widehat{N}_{x_0}/T_b can be identified with a subgroup of W^v which is finite. Write $N/T_b = \{n_1 T_b, \dots, n_k T_b\}$, with $k \in \mathbb{Z}_{\geq 0}$ and $n_1, \dots, n_k \in N$. Choose $x_0 \in \Omega$. Let $\text{Fin}(\Omega, x_0)$ be the set of finite subsets $\widetilde{\Omega}$ of Ω such that $x_0 \in \widetilde{\Omega}$. Let J be the set of elements of $j \in \llbracket 1, k \rrbracket$ such that there exists $\Omega_j \in \text{Fin}(\Omega, x_0)$ such that $gn_j \notin \widehat{P}_{\Omega_j}$. Let $\widetilde{\Omega} = \bigcup_{j \in J} \Omega_j$. Then $\widetilde{\Omega} \in \text{Fin}(\Omega, x_0)$. Moreover, if $j \in J$, $gn_j \notin \widehat{P}_{\Omega_j} \supset \widehat{P}_{\widetilde{\Omega}}$. Let $i \in \llbracket 1, k \rrbracket$ be such that $gn_i \in \widehat{P}_{\widetilde{\Omega}}$. Then $i \notin J$ and thus for all $\Omega' \in \text{Fin}(\Omega, x_0)$, $gn_i \in \widehat{P}_{\Omega'}$. In particular, for all $x \in \Omega$, $gn_i \in \widehat{P}_x$. Consequently, $gn_i \in \bigcap_{x \in \Omega} \widehat{P}_x = \widehat{P}_\Omega$ (by Proposition 4.26) and thus for all $x \in \Omega$, $g^{-1} \cdot x = n_i \cdot x$.

It remains to prove that Ω is enclosed. Let $\tilde{g} = gn_i$. Then $\tilde{g} \cdot \mathbb{A} \cap \mathbb{A} = g \cdot \mathbb{A} \cap \mathbb{A} = \Omega$. Moreover, $\tilde{g} \in \widehat{P}_\Omega$ and thus there exists $\tilde{n} \in \widehat{N}_\Omega$ such that $p := \tilde{n} \tilde{g} \in P_\Omega$. By Lemma 4.27, $P_\Omega = P_{\text{cl}(\Omega)}$: there exists $\Omega' \in \text{cl}(\Omega)$ such that $p \in P_{\Omega'}$. Then for $x \in \Omega'$, one has

$\tilde{g}.x = \tilde{n}^{-1}p.x = \tilde{n}^{-1}.x \in \mathbb{A}$ and thus $\tilde{g}.x \in \Omega$ for all $x \in \Omega'$. Let $y \in \Omega'$. Then $\tilde{g}.y \in \Omega$ and thus $\tilde{g}\tilde{g}.y = \tilde{g}.y$. Consequently, $\tilde{g}.y = y$ and thus $y \in \Omega$. Therefore, $\Omega' \subset \Omega$ and thus $\Omega = \Omega' \in \text{cl}(\Omega)$: Ω is enclosed, which proves the proposition. \square

5.7 Remark. Let $x \in \mathcal{I}(G)$ and Q_∞ be a sector-germ at infinity. Then by Lemma 5.10, there exists an apartment A containing x and Q_∞ . Let $Q \subset A$ be a sector whose germ at infinity is Q_∞ . Then we denote by $x + Q_\infty$ the translate of Q at x . This does not depend on the choice of A by (A2) (Proposition 5.6).

5.8 Corollary. [see [Lan96, 9.7(i)]] *Let Ω be a non-empty subset of \mathbb{A} . The group U_Ω acts transitively on the set of apartments of $\mathcal{I}(G)$ containing Ω .*

Proof. Let A' an apartment containing Ω and $g \in G$ such that $A' = g \cdot A$. Let $n \in N$ such that $g^{-1} \cdot x = n \cdot x$ for all $x \in \mathbb{A} \cap g \cdot \mathbb{A} \supset \Omega$ as in Proposition 5.6. Hence $gn \in \widehat{P}_\Omega$ by Lemma 5.5. By Corollary 4.12, there exist $u \in U_\Omega$ and $n' \in \widehat{N}_\Omega \subset N$ such that $gn = un'$. Hence $A' = g \cdot A = gn \cdot A = un' \cdot A = u \cdot A$. This proves the transitivity. \square

5.9 Corollary (see [BT72, 7.4.10]). *The group N is the stabilizer of $i(\mathbb{A})$ in G .*

The group T_b is the pointwise stabilizer of $i(\mathbb{A})$ in G .

Proof. We firstly prove that $U_{\mathbb{A}} = \{1\}$. Let $\alpha \in \Phi$. Then $U_{\alpha, \mathbb{A}} = \bigcap_{x \in \mathbb{A}} U_{\alpha, -\alpha(x-o)}$. Considering the elements $x = o + \lambda\alpha^\vee \in \mathbb{A}$ for $\lambda \in R$, we get that

$$U_{\alpha, \mathbb{A}} \subset \bigcap_{\lambda \in R} U_{\alpha, -\alpha(\lambda\alpha^\vee)} = \varphi_\alpha^{-1} \left(\bigcap_{\lambda \in R} [-2\lambda, \infty] \right).$$

But the last intersection is reduced to ∞ since for any $\varepsilon \in R_{>0}$ and any $\mu \in R$, we have $\mu \notin [-2(-\mu - \varepsilon), \infty]$. Thus $U_{\alpha, \mathbb{A}} = \varphi_\alpha^{-1}(\{\infty\}) = \{1\}$ for any $\alpha \in \Phi$. Hence $U_{\mathbb{A}} = \{1\}$.

Let $g \in G$ be such that $g \cdot \mathbb{A} = \mathbb{A}$. By Proposition 5.6, there is $n \in N$ such that $\forall x \in \mathbb{A}$, $g^{-1} \cdot x = n \cdot x$. Hence $gn \in \widehat{P}_{\mathbb{A}}$ by Lemma 5.5. Since $\widehat{P}_{\mathbb{A}} = \widehat{N}_{\mathbb{A}}$ by Corollary 4.12, we have that $gn \in N$. Thus $g \in N$. Moreover, since the action of N on \mathbb{A} is induced by that of ν via $n \cdot [1, x] = [1, \nu(n)(x)]$, we deduce the result. \square

5.10 Lemma. (1) *Any two local faces are contained in a single apartment of $\mathcal{I}(G)$. In particular, $\mathcal{I}(G)$ satisfies axioms (A3) and (GG).*

(2) *Any two sector-germs are contained in a single apartment of $\mathcal{I}(G)$. In other words, $\mathcal{I}(G)$ satisfies axiom (A4).*

(3) *If \mathcal{F} is a local face and Q_∞ is the germ at infinity of a sector, there exists an apartment containing \mathcal{F} and Q_∞ .*

Proof. (1) Consider an element $g \in G$ and a local sector-germ $\text{germ}_{x''}(Q'') \in \mathbb{A}$ such that $\text{germ}_{x'}(Q') = g \cdot \text{germ}_{x''}(Q'')$. By the Bruhat decomposition 4.38, we can write $g = g_x n g_{x''}$ with $g_x \in \widehat{P}_{\text{germ}_x(Q)}$, $g_{x''} \in \widehat{P}_{\text{germ}_{x''}(Q')}$ and $n \in N$. By Lemma 5.5, we then have $\text{germ}_x(Q) \in g_x n \cdot \mathbb{A}$ and:

$$\text{germ}_{x'}(Q') = g \cdot \text{germ}_{x''}(Q'') = g_x n \cdot \text{germ}_{x''}(Q'') \in g_x n \cdot \mathbb{A}.$$

(2) Let Q_∞ and Q'_∞ be two sector-germs at infinity in $\mathcal{I}(G)$. Since G acts transitively on the set of apartments in $\mathcal{I}(G)$, we may assume that $Q_\infty \in \mathbb{A}$. Consider an element $g \in G$ such that $Q'_\infty = g \cdot Q_\infty$. Let $\Phi_{Q_\infty}^+$ be the set of roots in Φ that are positive on Q_∞ . By [BT72, 6.1.15(a)], we can write $g = u_+ n v_+$ with $u_+, v_+ \in U_{\Phi_{Q_\infty}^+}$ and $n \in N$. Since u_+ and v_+ fix Q_∞ , we then have $Q_\infty \in u_+ n \cdot \mathbb{A}$ and:

$$Q'_\infty = g \cdot Q_\infty = u_+ n \cdot Q_\infty \in u_+ n \cdot \mathbb{A}.$$

(3) We obtain it similarly as (i), by replacing the Bruhat decomposition by the Iwasawa decomposition (4.36). \square

6 Valuation for quasi-split reductive groups

6.1 Notations and recalls for quasi-split reductive groups

Let \mathbb{K} be any field and \mathbf{G} be any reductive \mathbb{K} -group. Recall that \mathbf{G} splits over a finite Galois extension of \mathbb{K} and denote by $\tilde{\mathbb{K}}/\mathbb{K}$ the minimal one [BT84, 4.1.2]. Denote by $\tilde{\mathbf{G}} = \mathbf{G}_{\tilde{\mathbb{K}}}$ the $\tilde{\mathbb{K}}$ -group obtained by a base change from \mathbb{K} to $\tilde{\mathbb{K}}$.

When \mathbb{K} is algebraically closed, the theory of structure of reductive groups enables us to consider Borel subgroups. In general, over an arbitrary field, a reductive group does not admit any Borel subgroup defined over the ground field and we therefore need to consider the minimal parabolic subgroups. The intermediate situation is that of quasi-split reductive groups:

6.1 Proposition-Definition. [BT84, 4.1.1] One says that a reductive \mathbb{K} -group G is *quasi-split* if it satisfies the following equivalent conditions:

- (i) \mathbf{G} contains a Borel subgroup defined over \mathbb{K} ;
- (ii) \mathbf{G} contains a maximal \mathbb{K} -split torus \mathbf{S} such that its centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a torus;
- (iii) for any maximal \mathbb{K} -split torus \mathbf{S} of \mathbf{G} , its centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a torus.

We will now assume that \mathbf{G} is a quasi-split reductive \mathbb{K} -group. We provide a choice of a maximal \mathbb{K} -split torus \mathbf{S} and a Borel subgroup \mathbf{B} such that $\mathbf{T} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is a maximal torus of \mathbf{G} contained in \mathbf{B} . This is always possible [Bor91, 20.5, 20.6 (iii)]. Thus $\tilde{\mathbf{T}} = \mathbf{T}_{\tilde{\mathbb{K}}}$ is a maximal $\tilde{\mathbb{K}}$ -torus of $\tilde{\mathbf{G}}$ containing $\tilde{\mathbf{S}} = \mathbf{S}_{\tilde{\mathbb{K}}}$.

We denote by $\Phi = \Phi(\mathbf{G}, \mathbf{S})$ the root system of \mathbf{G} with respect to \mathbf{S} and we call it the **relative root system**. We denote by $\tilde{\Phi} = \Phi(\tilde{\mathbf{G}}, \tilde{\mathbf{T}})$ the root system of the split group $\tilde{\mathbf{G}}$ with respect to $\tilde{\mathbf{T}}$ and we call it the **absolute root system**.

6.2 Recalls on root groups and their parametrizations

6.2.1 Definition of root groups

Given a basis Δ of Φ (resp. $\tilde{\Delta}$ of $\tilde{\Phi}$), we denote $\text{Dyn}(\Delta)$ (resp. $\text{Dyn}(\tilde{\Delta})$) its Dynkin diagram. The edges represent orthogonal defects of the basis which will translate defects of commutativity between root groups. Multiple edges appear between two simple roots of different lengths and are oriented from the long root to the short root.

Given a reductive \mathbb{K} -group \mathbf{G} and a maximal \mathbb{K} -split torus \mathbf{S} , the choice of a minimal \mathbb{K} -parabolic subgroup of \mathbf{G} containing $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ is equivalent to the choice of a basis Δ of the relative root system [BT65, 4.15]. In particular, if \mathbf{G} is quasi-split, the choice of $\mathbf{S} \subset \mathbf{T} \subset \mathbf{B}$ as before naturally determines a basis $\tilde{\Delta} = \Delta(\tilde{\mathbf{G}}, \tilde{\mathbf{T}}, \tilde{\mathbf{B}})$ of $\tilde{\Phi} = \Phi(\tilde{\mathbf{G}}, \tilde{\mathbf{T}})$ and a basis $\Delta = \Delta(\mathbf{G}, \mathbf{S}, \mathbf{B})$ of $\Phi = \Phi(\mathbf{G}, \mathbf{S})$.

Recall that root groups of \mathbf{G} over \mathbb{K} are defined by the following proposition:

6.2 Proposition-Definition ([Bor91, 14.5 & 21.9]). For any root $\alpha \in \Phi$, there exists a unique \mathbb{K} -subgroup of \mathbf{G} , denoted by \mathbf{U}_{α} , which is closed, connected, unipotent, normalized by $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ and whose Lie algebra is $\mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$. It is called the *root group* of \mathbf{G} with respect to α .

If Ψ is a positively closed subset of Φ , then there exists a unique \mathbb{K} -subgroup of \mathbf{G} , denoted by \mathbf{U}_{Ψ} , which is closed, connected, unipotent, normalized by $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ and whose Lie algebra is $\sum_{\alpha \in \Psi} \mathfrak{g}_{\alpha}$.

Note that the definition depends on Φ and, therefore, on the choice of the maximal split torus \mathbf{S} defining $\mathbf{T} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$.

Moreover, these root groups satisfy the following proposition:

6.3 Proposition ([Bor91, 21.9]). *For any ordering on a positively closed subset Ψ of Φ , the product map $\prod_{\alpha \in \Psi_{nd}} \mathbf{U}_{\alpha} \rightarrow \mathbf{U}_{\Psi}$ is an isomorphism of \mathbb{K} -varieties.*

For any pair of non-collinear roots $a, b \in \Phi$, the subset $(\alpha, \beta) = \{r\alpha + s\beta \in \Phi, r, s \in \mathbb{Z}_{>0}\}$ is positively closed and $[\mathbf{U}_{\alpha}, \mathbf{U}_{\beta}] \subset \mathbf{U}_{(\alpha, \beta)}$.

We denote by $\tilde{\mathbf{U}}_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \tilde{\Phi}$ the root groups of $\tilde{\mathbf{G}}$ with respect to $\tilde{\mathbf{T}}$.

6.2.2 The Galois action on the absolute root system

Even if we can define a $*$ -action on the Dynkin diagram for an arbitrary reductive \mathbb{K} -group, we assume for simplicity that \mathbf{G} is a quasi-split reductive \mathbb{K} -group.

We consider the canonical action of the absolute Galois group $\Sigma = \text{Gal}(\mathbb{K}_s/\mathbb{K})$ on the abstract group $\mathbf{G}(\mathbb{K}_s)$. Since \mathbf{G} is quasi-split, we can choose a maximal \mathbb{K} -split torus \mathbf{S} and we get a maximal torus $\mathbf{T} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ of \mathbf{G} defined over \mathbb{K} . Thus, we define an action of Σ on $X^*(\mathbf{T}_{\mathbb{K}_s})$ by:

$$\forall \sigma \in \Sigma, \forall \chi \in X^*(\mathbf{T}_{\mathbb{K}_s}), \sigma \cdot \chi = t \mapsto \sigma(\chi(\sigma^{-1}(t)))$$

6.4 Notation (The Galois action on the absolute root system). This is a summary of [BT65, §6] for a quasi-split reductive \mathbb{K} -group \mathbf{G} . Denote by $\tilde{\Delta}$ a set of absolute simple roots and by $\text{Dyn}(\tilde{\Delta})$ its associated Dynkin diagram. There exists an action of the Galois group $\Sigma = \text{Gal}(\mathbb{K}/\mathbb{K})$ on $\text{Dyn}(\tilde{\Delta})$ which preserves the diagram structure. This action can be extended, by linearity, to an action of Σ on $\tilde{V}^* = X^*(\mathbf{T}_{\tilde{\mathbb{K}}}) \otimes_{\mathbb{Z}} \mathbb{R}$, and on $\tilde{\Phi}$. The restriction morphism $j = \iota^* : X^*(\mathbf{T}) \rightarrow X^*(\mathbf{S})$, where $\iota : \mathbf{S} \subset \mathbf{T}$ is the inclusion morphism, can be extended to an endomorphism $\rho : \tilde{V}^* \rightarrow \tilde{V}^*$ of the Euclidean space \tilde{V}^* . This morphism ρ is the orthogonal projection onto the subspace V^* of fixed points by the action of Σ on \tilde{V}^* . The inclusion of $\tilde{\Phi}$ in the Euclidean space \tilde{V}^* provides a geometric realization of the absolute roots from which we deduce a geometric realization of $\Phi = \rho(\tilde{\Phi})$ in V^* . The orbits of the action of Σ on $\tilde{\Phi}$ are the fibers of the map $\rho : \tilde{\Phi} \rightarrow \Phi$.

6.5 Definition. Let $\tilde{\alpha} \in \tilde{\Phi}$ be an absolute root. Denote by $\Sigma_{\tilde{\alpha}}$ be the stabilizer of $\tilde{\alpha}$ for the canonical Galois action. The **field of definition** of the root $\tilde{\alpha}$ is the subfield of $\tilde{\mathbb{K}}$ fixed by $\Sigma_{\tilde{\alpha}}$, denoted by $\mathbb{L}_{\tilde{\alpha}} = \tilde{\mathbb{K}}^{\Sigma_{\tilde{\alpha}}}$.

This is determined, up to isomorphism by the relative root $\alpha = \tilde{\alpha}|_{\mathbf{S}}$. Indeed, α is an orbit of absolute roots, which means that if $\tilde{\beta}|_{\mathbf{S}} = \tilde{\alpha}|_{\mathbf{S}} = \alpha$, then $\tilde{\beta} = \sigma \cdot \tilde{\alpha}$ and $\mathbb{L}_{\tilde{\beta}} = \sigma(\mathbb{L}_{\tilde{\alpha}})$. For a root $\alpha \in \Phi$, we denote by \mathbb{L}_{α} the class of $\mathbb{L}_{\tilde{\alpha}}$ for $\tilde{\alpha}|_{\mathbf{S}} = \alpha$. We call it the **splitting field** of α .

6.6 Remark. If $\alpha \in \Phi$ is a multipliable root, then there exists $\tilde{\alpha}, \tilde{\alpha}' \in \alpha$ such that $\tilde{\alpha} + \tilde{\alpha}' \in \tilde{\Phi}$ [BT84, 4.1.4 Cas II]. Because α is an orbit, we can write $\tilde{\alpha}' = \sigma(\tilde{\alpha})$ where $\sigma \in \Sigma$ is of order 2. As a consequence, the extension of fields $\mathbb{L}_{\tilde{\alpha}}/\mathbb{L}_{\tilde{\alpha}+\tilde{\alpha}'}$ is quadratic. By abuse of notation, we denote this extension, determined up to isomorphism, by $\mathbb{L}_{\alpha}/\mathbb{L}_{2\alpha}$.

6.2.3 Parametrization of root groups

In order to valuate the root groups thanks to the Λ -valuation of the field, we have to define a parametrization of each root group. Moreover, these valuations have to be compatible. That is why we furthermore have to get relations between the parametrizations.

A Chevalley-Steinberg system of $(\mathbf{G}, \widetilde{\mathbb{K}}, \mathbb{K})$ is the datum of morphisms: $\tilde{x}_{\tilde{\alpha}} : \mathbb{G}_{a, \widetilde{\mathbb{K}}} \rightarrow \widetilde{\mathbf{U}}_{\tilde{\alpha}}$ parametrizing the various root groups of $\widetilde{\mathbf{G}}$, and satisfying some axioms of compatibility, given in [BT84, 4.1.3], taking into account the commutation relations of absolute root groups and the $\text{Gal}(\widetilde{\mathbb{K}}/\mathbb{K})$ -action on root groups. Note that despite the morphisms parametrize root groups of $\widetilde{\mathbf{G}}$, a Chevalley-Steinberg system also depends on the quasi-split group \mathbf{G} because of the relations between the $\tilde{x}_{\tilde{\alpha}}$ where $\tilde{\alpha} \in \widetilde{\Phi}$. According to [BT84, 4.1.3], a quasi-split reductive \mathbb{K} -group always admits a Chevalley-Steinberg system $(\tilde{x}_{\tilde{\alpha}})_{\tilde{\alpha} \in \widetilde{\Phi}}$.

6.7 Notation. Let us recall that there are elements in $\mathcal{N}_{\mathbf{G}}(\mathbf{S})(\widetilde{\mathbb{K}})$ defined by:

$$m_{\tilde{\alpha}} = \tilde{x}_{\tilde{\alpha}}(1)\tilde{x}_{-\tilde{\alpha}}(1)\tilde{x}_{\tilde{\alpha}}(1)$$

for $\tilde{\alpha} \in \widetilde{\Phi}$ such that for any $\tilde{\beta} \in \widetilde{\Phi}$ and any $u \in \widetilde{\mathbb{K}}$, we have:

$$m_{\tilde{\alpha}}\tilde{x}_{\tilde{\beta}}(u)m_{-\tilde{\alpha}} \in \{\tilde{x}_{r_{\tilde{\alpha}}(\tilde{\beta})}(\pm u)\}$$

according to second axiom defining Chevalley systems. Moreover, one can observe that $m_{-\tilde{\alpha}} = m_{\tilde{\alpha}}$ from the matrix realization in SL_2 .

Let $\alpha \in \Phi$ be a relative root. Let $\pi : \mathbf{G}^{\alpha} \rightarrow \langle \mathbf{U}_{-\alpha}, \mathbf{U}_{\alpha} \rangle$ be the universal covering of the quasi-split semi-simple \mathbb{K} -subgroup of relative rank 1 generated by \mathbf{U}_{α} and $\mathbf{U}_{-\alpha}$. The group \mathbf{G}^{α} splits over \mathbb{L}_{α} (this explains the terminology of splitting field of a root). A parametrization of the simply-connected group \mathbf{G}^{α} is given by [BT84, 4.1.1 to 4.1.9]. We now recall it to fix the notation.

The non-multipliable case Let $\alpha \in \Phi_{\text{nd}}$ be a relative root such that $2\alpha \notin \Phi$ and choose $\tilde{\alpha} \in \alpha$. By [BT84, 4.1.4], the rank-1 group \mathbf{G}^{α} is isomorphic to $R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}})$. Inside the classical group $\text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}}$, a maximal $\mathbb{L}_{\tilde{\alpha}}$ -split torus of $\text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}}$ can be parametrized by the following homomorphism:

$$\begin{aligned} z : \mathbb{G}_{m, \mathbb{L}_{\tilde{\alpha}}} &\rightarrow \text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}} \\ t &\mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \end{aligned}$$

The corresponding root groups can be parametrized by the following homomorphisms:

$$\begin{aligned} y_- : \mathbb{G}_{a, \mathbb{L}_{\tilde{\alpha}}} &\rightarrow \text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}} & y_+ : \mathbb{G}_{a, \mathbb{L}_{\tilde{\alpha}}} &\rightarrow \text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}} \\ v &\mapsto \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} & u &\mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \end{aligned}$$

According to [BT84, 4.1.5], there exists a unique $\mathbb{L}_{\tilde{\alpha}}$ -group isomorphism $\xi_{\tilde{\alpha}} : \text{SL}_{2, \mathbb{L}_{\tilde{\alpha}}} \rightarrow \widetilde{\mathbf{G}}^{\tilde{\alpha}}$ satisfying $\tilde{x}_{\pm\tilde{\alpha}} = \pi \circ \xi_{\tilde{\alpha}} \circ y_{\pm}$, where $\widetilde{\mathbf{G}}^{\tilde{\alpha}}$ is the simple factor of $\mathbf{G}_{\mathbb{K}}^{\alpha}$ of index $\tilde{\alpha}$.

6.8 Notation. Thus, we define \mathbb{K} -homomorphisms

$$x_{\alpha} = \pi \circ R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\xi_{\tilde{\alpha}} \circ y_+) \quad x_{-\alpha} = \pi \circ R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\xi_{\tilde{\alpha}} \circ y_-)$$

which are \mathbb{K} -group isomorphisms between $R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\mathbb{G}_{a, \mathbb{L}_{\tilde{\alpha}}})$ and respectively \mathbf{U}_{α} and $\mathbf{U}_{-\alpha}$.

We also define the following \mathbb{K} -group homomorphism:

$$\hat{\alpha} = \pi \circ R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\xi_{\tilde{\alpha}} \circ z) : R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\mathbb{G}_{m, \mathbb{L}_{\tilde{\alpha}}}) \rightarrow \mathbf{T}^{\alpha}$$

where $\mathbf{T}^{\alpha} = \mathbf{T} \cap \langle \mathbf{U}_{-\alpha}, \mathbf{U}_{\alpha} \rangle$.

6.9 Fact. For any $\tilde{\alpha} \in \tilde{\Phi}$, any $t \in \mathbf{T}(\mathbb{L}_{\tilde{\alpha}})$ and any $u \in \mathbb{L}_{\tilde{\alpha}}$, we have $t\tilde{x}_{\tilde{\alpha}}(u)t^{-1} = \tilde{x}_{\tilde{\alpha}}(\tilde{\alpha}(t)u)$ by definition of root groups. Thus, for any $\tilde{\alpha} \in \alpha$, we have that:

$$tx_{\alpha}(u)t^{-1} = x_{\alpha}(\tilde{\alpha}(t)u)$$

by definition of the Weil restriction. Thus, since a matrix calculation gives $\hat{\alpha}(z)x_{\alpha}(u)\hat{\alpha}(z^{-1}) = x_{\alpha}(z^2u)$, we get that for any $\tilde{\alpha} \in \alpha$, any $z \in \mathbb{L}_{\tilde{\alpha}}^*$, we have:

$$\tilde{\alpha}(\hat{\alpha})(z) = z^2.$$

6.10 Notation. We define maps $m_{\alpha} : \mathbb{G}_{a,\mathbb{L}_{\alpha}} \rightarrow \mathcal{N}_{\mathbf{G}}(\mathbf{S})$ and $m_{-\alpha} : \mathbb{G}_{a,\mathbb{L}_{\alpha}} \rightarrow \mathcal{N}_{\mathbf{G}}(\mathbf{S})$ [BT84, 4.1.5] given by:

$$m_{\alpha}(u) = x_{\alpha}(u)x_{-\alpha}(u^{-1})x_{\alpha}(u) \quad m_{-\alpha}(u) = x_{-\alpha}(u)x_{\alpha}(u^{-1})x_{-\alpha}(u)$$

whose matrix realizations in $\mathrm{SL}_{2,\mathbb{L}_{\alpha}}$ are respectively $\begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix}$.

The unique elements $m(x_{\alpha}(u))$ and $m(x_{-\alpha}(v))$ defined in [BT72, 6.1.2(2)] are then:

$$m(x_{\alpha}(u)) = m_{-\alpha}(u^{-1}) \quad m(x_{-\alpha}(v)) = m_{\alpha}(v^{-1})$$

We define an element $m_{\alpha} = m_{\alpha}(1) = m_{-\alpha}(1) = m_{-\alpha} \in \mathcal{N}_{\mathbf{G}}(\mathbf{S})(\mathbb{K})$.

6.11 Fact. We observe that $m_{\alpha} = m_{\tilde{\alpha}}$ for any $\tilde{\alpha} \in \alpha$ by definition of the x_{α} as Weil restriction.

From matrix realization in SL_2 we can easily check that:

- $\forall u \in \mathbb{L}_{\alpha}$, $x_{-\alpha}(u) = m_{\alpha}x_{\alpha}(u)m_{\alpha}^{-1}$;
- $\forall u \in \mathbb{L}_{\alpha}^*$, $m_{\alpha}(u) = \hat{\alpha}(u)m_{\alpha} = m_{\alpha}\hat{\alpha}(u^{-1})$;
- $m_{\alpha}^4 = \mathrm{id}$.

The multipliable case: Let $\alpha \in \Phi_{\mathrm{nd}}$ be a relative root such that $2\alpha \in \Phi$. Let $\tilde{\alpha} \in \alpha$ be an absolute root from which α arises, and let $\tau \in \Sigma$ be an element of the Galois group such that $\tilde{\alpha} + \tau(\tilde{\alpha})$ is again an absolute root. To simplify notations, we let (up to compatible isomorphisms in Σ) $\mathbb{L} = \mathbb{L}_{\tilde{\alpha}}$ and $\mathbb{L}_2 = \mathbb{L}_{\tilde{\alpha} + \tau(\tilde{\alpha})}$ in this paragraph. For any $x \in \mathbb{L}$, we denote ${}^{\tau}x$ instead of $\tau(x)$. By [BT84, 4.1.4], the \mathbb{K} -group \mathbf{G}^{α} is isomorphic to $R_{\mathbb{L}_2/\mathbb{K}}(\mathrm{SU}(h))$, where h denotes the hermitian form on $\mathbb{L} \times \mathbb{L} \times \mathbb{L}$ given by the formula:

$$h : (x_{-1}, x_0, x_1) \mapsto \sum_{i=-1}^1 x_i {}^{\tau}x_{-i}.$$

The group $\mathbf{G}_{\mathbb{L}_2}^{\alpha}$ can be written as $\mathbf{G}_{\mathbb{L}_2}^{\alpha} = \prod_{\sigma \in \mathrm{Gal}(\mathbb{L}_2/\mathbb{K})} \tilde{\mathbf{G}}^{\sigma(\tilde{\alpha}), \sigma(\tau(\tilde{\alpha}))}$ where each $\tilde{\mathbf{G}}^{\sigma(\tilde{\alpha}), \sigma(\tau(\tilde{\alpha}))}$

denotes a simple factor isomorphic to $\mathrm{SU}(h)$, so that $\mathrm{SU}(h)_{\mathbb{L}} \simeq \mathrm{SL}_{3,\mathbb{L}}$.

We define a connected unipotent \mathbb{L}_2 -group scheme by providing the \mathbb{L}_2 -subvariety of $R_{\mathbb{L}/\mathbb{L}_2}(\mathbb{A}_{2,\mathbb{L}})$:

$$H_0(\mathbb{L}, \mathbb{L}_2) = \{(u, v), u^{\tau}u = v + {}^{\tau}v\}$$

with the following group law:

$$(u, v), (u', v') \mapsto (u + u', v + v' + u^{\tau}u').$$

Then, we let $H(\mathbb{L}, \mathbb{L}_2) = R_{\mathbb{L}_2/\mathbb{K}}(H_0(\mathbb{L}, \mathbb{L}_2))$. For the rational points, we get

$$H(\mathbb{L}, \mathbb{L}_2)(\mathbb{K}) = \{(u, v) \in \mathbb{L} \times \mathbb{L}, u^{\tau}u = v + {}^{\tau}v\}.$$

We parametrize a maximal torus of $SU(h)$ by the isomorphism

$$z : R_{\mathbb{L}/\mathbb{L}_2}(\mathbb{G}_{m,\mathbb{L}}) \rightarrow SU(h)$$

$$t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1}\tau t & 0 \\ 0 & 0 & \tau t^{-1} \end{pmatrix}$$

We parametrize the corresponding root groups of $SU(h)$ by the homomorphisms:

$$y_- : H_0(\mathbb{L}, \mathbb{L}_2) \rightarrow SU(h) \quad y_+ : H_0(\mathbb{L}, \mathbb{L}_2) \rightarrow SU(h)$$

$$(u, v) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -v & -\tau u & 1 \end{pmatrix} \quad (u, v) \mapsto \begin{pmatrix} 1 & -\tau u & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}$$

By [BT84, 4.1.9], there exists a unique \mathbb{L}_2 -group isomorphism $\xi_{\tilde{\alpha}} : SU(h) \rightarrow \tilde{\mathbf{G}}^{\tilde{\alpha}, \tau(\tilde{\alpha})}$ satisfying:

$$y_+(u, v) = \tilde{x}_{\tilde{\alpha}}(u)\tilde{x}_{\tilde{\alpha}+\tau\tilde{\alpha}}(-v)\tilde{x}_{\tau\tilde{\alpha}}(\tau u) \quad y_-(u, v) = \tilde{x}_{-\tau\tilde{\alpha}}(u)\tilde{x}_{-\tilde{\alpha}-\tau\tilde{\alpha}}(v)\tilde{x}_{-\tilde{\alpha}}(\tau u)$$

6.12 Notation. From this, we define \mathbb{K} -homomorphisms

$$x_\alpha = \pi \circ R_{\mathbb{L}_2/\mathbb{K}}(\xi_{\tilde{\alpha}} \circ y_+) \quad x_{-\alpha} = \pi \circ R_{\mathbb{L}_2/\mathbb{K}}(\xi_{\tilde{\alpha}} \circ y_-)$$

which are \mathbb{K} -group isomorphism between the \mathbb{K} -group $H(\mathbb{L}, \mathbb{L}_2)$ and the root groups \mathbf{U}_α and $\mathbf{U}_{-\alpha}$ respectively and the group law is given by $x_{\pm\alpha}(u, v)x_{\pm\alpha}(u', v') = x_{\pm\alpha}(u + u', v + v' + u^\tau u')$.

We also define the following \mathbb{K} -group homomorphism:

$$\hat{\alpha} = \pi \circ R_{\mathbb{L}_2/\mathbb{K}}(\xi_{\tilde{\alpha}} \circ z) : R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\mathbb{G}_{m,\mathbb{L}_{\tilde{\alpha}}}) \rightarrow \mathbf{T}^\alpha$$

where $\mathbf{T}^\alpha = \mathbf{T} \cap \langle \mathbf{U}_{-\alpha}, \mathbf{U}_\alpha \rangle$.

6.13 Fact. Let $\tilde{\alpha}$ and τ be as before. For any $t \in \mathbf{T}(\mathbb{L}_{\tilde{\alpha}})$ and any $(u, v) \in H(\mathbb{L}, \mathbb{L}_2)$, we have

$$t\tilde{x}_{\tilde{\alpha}}(u)\tilde{x}_{\tilde{\alpha}+\tau\tilde{\alpha}}(-v)\tilde{x}_{\tau\tilde{\alpha}}(\tau u)t^{-1} = \tilde{x}_{\tilde{\alpha}}(\tilde{\alpha}(t)u)\tilde{x}_{\tilde{\alpha}+\tau\tilde{\alpha}}(-(\tilde{\alpha} + \tau\tilde{\alpha})(t)v)\tilde{x}_{\tau\tilde{\alpha}}(\tau\tilde{\alpha}(t)\tau u)$$

by definition of root groups. Thus, we have that:

$$tx_\alpha(u, v)t^{-1} = x_\alpha(\tilde{\alpha}(t)u, \tilde{\alpha}(t)\tau\tilde{\alpha}(t)v)$$

by definition of the Weil restriction. Thus, since a matrix calculation gives $\hat{\alpha}(z)x_\alpha(u, v)\hat{\alpha}(z^{-1}) = x_\alpha(\tau z^2 z^{-1}u, z^\tau z v)$, we get that for any $z \in \mathbb{L}_{\tilde{\alpha}}^*$, we have:

$$\tilde{\alpha}(\hat{\alpha}(z)) = (\tau z)^2 z^{-1}.$$

6.14 Notation. We define maps $m_\alpha : H(\mathbb{L}, \mathbb{L}_2) \rightarrow \mathcal{N}_{\mathbf{G}}(\mathbf{S})$ and $m_{-\alpha} : H(\mathbb{L}, \mathbb{L}_2) \rightarrow \mathcal{N}_{\mathbf{G}}(\mathbf{S})$ given by [BT84, 4.1.11]:

$$m_\alpha(u, v) = x_\alpha(uv^{-1}, (\tau v)^{-1})x_{-\alpha}(u, v)x_\alpha(u(\tau v)^{-1}, (\tau v)^{-1})$$

$$m_{-\alpha}(u, v) = x_{-\alpha}(uv^{-1}, (\tau v)^{-1})x_\alpha(u, v)x_{-\alpha}(u(\tau v)^{-1}, (\tau v)^{-1})$$

whose matrix realizations in $SU(h)$ are respectively

$$\begin{pmatrix} 0 & 0 & -(\tau v)^{-1} \\ 0 & -(\tau v)v^{-1} & 0 \\ -v & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -v \\ 0 & -(\tau v)v^{-1} & 0 \\ -(\tau v)^{-1} & 0 & 0 \end{pmatrix}$$

The unique elements $m(x_\alpha(u, v))$ and $m(x_{-\alpha}(u, v))$ defined in [BT72, 6.1.2(2)] are then:

$$m(x_\alpha(u, v)) = m_{-\alpha}(u, v) \qquad m(x_{-\alpha}(u, v)) = m_\alpha(u, v)$$

Even if $(0, 1) \notin H(\mathbb{L}, \mathbb{L}_2)(\mathbb{K})$ in general, one can define an element $m_\alpha = m_\alpha(0, 1) = m_{-\alpha}(0, 1) = m_{-\alpha}$, that in fact belongs to $\mathcal{N}_{\mathbf{G}}(\mathbf{S})(\mathbb{K})$.

By convention, we set $m_{2\alpha} = m_\alpha$.

6.15 Fact. *We observe that $\tilde{x}_{\tilde{\alpha}}(1)\tilde{x}_{-\tilde{\alpha}}(1)\tilde{x}_{\tilde{\alpha}}(1) = m_{\tilde{\alpha}}$ (resp. $m_{\tau_{\tilde{\alpha}}}$) has matrix realization:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{resp.} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that we have:

$$m_\alpha = m_{\tilde{\alpha}}m_{\tau_{\tilde{\alpha}}}^{-1}m_{\tilde{\alpha}} = m_{\tau_{\tilde{\alpha}}}m_{\tilde{\alpha}}^{-1}m_{\tau_{\tilde{\alpha}}}.$$

Moreover

- $\forall (u, v) \in H(\mathbb{L}, \mathbb{L}_2)$, $x_{-\alpha}(u, v) = m_\alpha x_\alpha(u, v) m_\alpha^{-1}$;
- $\forall (u, v) \in H(\mathbb{L}, \mathbb{L}_2) \setminus \{(0, 0)\}$, $m_\alpha(u, v) = \hat{\alpha}(\tau_v^{-1})m_\alpha = m_\alpha \hat{\alpha}(v)$;
- $m_\alpha^2 = \text{id}$.

6.3 \mathfrak{R}^S -valuation of a root groups datum

As before, let \mathbf{G} be a quasi-split reductive \mathbb{K} -group with a choice of a maximal split torus \mathbf{S} contained in the maximal \mathbb{K} -torus $\mathbf{T} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ contained in a Borel subgroup \mathbf{B} , together with a parametrization of root groups $(x_\alpha)_{\alpha \in \Phi}$ deduced from a Chevalley-Steinberg system, defined in Notations 6.8 and 6.12.

Denote $G = \mathbf{G}(\mathbb{K})$, $T = \mathbf{T}(\mathbb{K})$ and $N = \mathbf{N}(\mathbb{K})$. For any relative root $\alpha \in \Phi$, denote $U_\alpha = \mathbf{U}_\alpha(\mathbb{K})$ and $M_\alpha = Tm_\alpha$ where the element $m_\alpha \in G$ is defined as in Notations 6.10 and 6.14. Then, by [BT84, 4.1.19(ii)], we know that $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ is a generating root group datum of G of type Φ .

From now on, we assume that the extension $\tilde{\mathbb{K}}/\mathbb{K}$ is **univalent** (generalizing definition of [BT84, 1.6.1]). This means that the Λ -valued ground field \mathbb{K} satisfies the following:

6.16 Assumption. *There is a unique valuation $\omega' : \tilde{\mathbb{K}}^\times \rightarrow \Lambda'$ such that:*

- Λ' is a totally ordered abelian group;
- there is a strictly increasing map $\Lambda \rightarrow \Lambda'$ that identifies Λ with a finite index subgroup of Λ' ;
- for all $x \in \mathbb{K}^\times$, we have $\omega'(x) = \omega(x)$.

Thus Λ' identifies with a subgroup of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathfrak{R}^S$. Note that for any sub-extension $\tilde{\mathbb{K}}/\mathbb{L}/\mathbb{K}$ and any $\sigma \in \text{Aut}_{\mathbb{K}}(\mathbb{L})$, we have $\omega' \circ \sigma = \omega'$. We still denote, by abuse of notation, the valuation $\omega : \mathbb{L}^\times \rightarrow \mathfrak{R}^S$ for any sub-extension $\tilde{\mathbb{K}}/\mathbb{L}/\mathbb{K}$.

6.17 Example. According to corollary 3.2.3 and section 4.1 of [EP05], the assumptions 6.16 are all satisfied if \mathbb{K} is Henselian.

6.18 Notation. For each root $\alpha \in \Phi$, we use the parametrization x_α of the root group \mathbf{U}_α , given by the choice of a Chevalley-Steinberg system and the choice of an absolute root $\tilde{\alpha}$ in the orbit α , to define a map $\varphi_\alpha : U_\alpha \rightarrow \mathfrak{R}^S \cup \{\infty\}$ as follows:

- $\varphi_\alpha(x_\alpha(y)) = \omega(y)$ if α is a non-multipliable and non-divisible root, and if $y \in \mathbb{L}_\alpha$;
- $\varphi_\alpha(x_\alpha(y, y')) = \frac{1}{2}\omega(y')$ if α is a multipliable root and if $(y, y') \in H(\mathbb{L}_\alpha, \mathbb{L}_{2\alpha})$;
- $\varphi_{2\alpha}(x_\alpha(0, y')) = \omega(y')$ if α is a multipliable root and if $y' \in \mathbb{L}_\alpha^0$.

Note that, by convention, we set $\omega(0) = \infty = \frac{1}{2}\omega(0)$.

6.19 Remark. Despite the parametrization x_α depends on the choice of $\tilde{\alpha} \in \tilde{\Phi}$ such that $\tilde{\alpha}|_{\mathbf{S}} = \alpha$, the value $\varphi_\alpha(u) \in \mathfrak{R}^S \cup \{\infty\}$ for $u \in U_\alpha$ does not depend on this choice. Indeed, assume for instance that $\alpha \in \Phi_{\text{nd}}$ is non-multipliable. For any $\sigma \in \text{Gal}(\tilde{\mathbb{K}}/\mathbb{K})$, the isomorphism $\sigma^{-1} : \mathbb{L}_{\sigma(\tilde{\alpha})} \rightarrow \mathbb{L}_\alpha$ induces a \mathbb{K} -isomorphism of the Weil restrictions: $j : R_{\mathbb{L}_{\sigma(\tilde{\alpha})}/\mathbb{K}}(\mathbb{G}_{a, \mathbb{L}_{\sigma(\tilde{\alpha})}}) \rightarrow R_{\mathbb{L}_{\tilde{\alpha}}/\mathbb{K}}(\mathbb{G}_{a, \mathbb{L}_{\tilde{\alpha}}})$. Thus the parametrization of U_α defined by $\tilde{\alpha}$ instead of α would be precisely $x_\alpha \circ j : R_{\mathbb{L}_{\sigma(\tilde{\alpha})}/\mathbb{K}}(\mathbb{G}_{a, \mathbb{L}_{\sigma(\tilde{\alpha})}}) \rightarrow U_\alpha$. Since $\omega = \omega \circ j$, we deduce that the value of φ_α does not depend on σ . If α is a multipliable root, we can make a similar observation.

6.20 Proposition. *The datum $(\varphi_\alpha)_{\alpha \in \Phi}$ is an \mathfrak{R}^S -valuation of the root group datum $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$, i.e. it satisfies axioms (V0) to (V5).*

According to Bruhat-Tits [BT84, 4.1.11], it is easy to check it. Such a verification is carried out by Landvogt [Lan96, 7.4].

Proof. Axiom (V0) is immediate since $\varphi_\alpha(\mathbf{U}_\alpha(\mathbb{K}))$ contains $\Lambda \cup \{\infty\}$ and the totally ordered group Λ is not trivial by assumption. Axiom (V4) is immediate by definition.

Axioms (V1), (V2) and (V5) for a non-multipliable: Let $\lambda \in \mathfrak{R}^S$ and let $g_1 = x_\alpha(u_1)$, $g_2 = x_\alpha(u_2)$ be elements in $U_{\alpha, \lambda}$ for some parameters $u_1, u_2 \in \mathbb{L}_\alpha$, then $g_1 g_2^{-1} = x_\alpha(u_1) x_\alpha(u_2)^{-1} = x_\alpha(u_1) x_\alpha(-u_2) = x_\alpha(u_1 - u_2)$. Thus $\varphi_\alpha(g_1 g_2^{-1}) = \omega(u_1 - u_2) \geq \min(\omega(u_1), \omega(u_2)) \geq \lambda$. Moreover, $x_\alpha(0)$ is the only element with valuation ∞ which gives (V1).

Let $x_\alpha(u) \in U_\alpha$ with $u \in \mathbb{L}_\alpha^*$ and $m = m_\alpha t \in M_\alpha = M_{-\alpha}$ with $t \in T$. By formulas in 6.9, we have $m x_\alpha(u) m^{-1} = m_\alpha t x_\alpha(u) t^{-1} m_\alpha^{-1} = m_\alpha x_\alpha(\alpha(t)u) m_\alpha^{-1} = x_{-\alpha}(\alpha(t)u)$. Hence $\varphi_\alpha(x_\alpha(u)) - \varphi_\alpha(m x_\alpha(u) m^{-1}) = \omega(u) - \omega(\alpha(t)u) = -\omega(\alpha(t))$ which does not depend on u . This proves (V2).

Let $x_\alpha(u) \in U_\alpha$ and $x_{-\alpha}(u'), x_{-\alpha}(u'') \in U_{-\alpha}$ such that $x_{-\alpha}(u') x_\alpha(u) x_{-\alpha}(u'') \in M_\alpha$. By uniqueness in [BT72, 6.1.2(2)] and formula defining $m_{-\alpha}(u^{-1})$ in 6.10, we get $u' = u'' = u^{-1}$. Thus $\varphi_{-\alpha}(x_{-\alpha}(u')) = \omega(u^{-1}) = -\varphi_\alpha(x_\alpha(u))$ which gives (V5).

Axioms (V1), (V2) and (V5) for α multipliable: Let $\lambda \in \mathfrak{R}^S$ and let $g_1 = x_\alpha(u_1, v_1)$, $g_2 = x_\alpha(u_2, v_2)$ are in $U_{\alpha, \lambda}$ for some parameters $(u_1, v_1), (u_2, v_2) \in H(\mathbb{L}_\alpha, \mathbb{L}_{2\alpha})$, then $u_i \tau u_i = v_i + \tau v_i$ gives $\omega(u_i) \geq \frac{1}{2}\omega(v_i) \geq \lambda$. Thus $g_1 g_2^{-1} = x_\alpha(u_1, v_1) x_\alpha(u_2, v_2)^{-1} = x_\alpha(u_1, v_1) x_\alpha(-u_2, \tau v_2) = x_\alpha(u_1 - u_2, \tau u_1 u_2 - v_1 - \tau v_2)$. Hence $\varphi_\alpha(g_1 g_2^{-1}) = \frac{1}{2}\omega(\tau u_1 u_2 - v_1 - \tau v_2) \geq \frac{1}{2} \min(\omega(u_1) + \omega(u_2), \omega(v_1), \omega(v_2)) \geq \lambda$. Since, for $(u, v) \in H(\mathbb{L}_\alpha, \mathbb{L}_{2\alpha})$, we have $u = 0$ whenever $v = 0$, we get that $x_\alpha(0, 0)$ is the only element with valuation ∞ . This gives (V1).

Let $x_\alpha(u, v) \in U_\alpha$ with $(u, v) \in H(\mathbb{L}_\alpha, \mathbb{L}_{2\alpha}) \setminus \{(0, 0)\}$ and $m = m_\alpha t \in M_\alpha = M_{-\alpha}$ with $t \in T$. By formulas in 6.13, we have $m x_\alpha(u, v) m^{-1} = m_\alpha t x_\alpha(u, v) t^{-1} m_\alpha^{-1} = m_\alpha x_\alpha(\tilde{\alpha}(t)u, \tilde{\alpha}(t)\tau \tilde{\alpha}(t)v) m_\alpha^{-1} = x_{-\alpha}(-\tilde{\alpha}(t)u, \tilde{\alpha}(t)\tau \tilde{\alpha}(t)v)$. Hence $\varphi_\alpha(x_\alpha(u, v)) - \varphi_{-\alpha}(m x_\alpha(u, v) m^{-1}) = \frac{1}{2}\omega(v) - \frac{1}{2}\omega(\tilde{\alpha}(t)\tau \tilde{\alpha}(t)v) = -\omega(\alpha(t))$ which does not depend on (u, v) . This proves (V2).

Let $x_\alpha(u, v) \in U_\alpha$ and $x_{-\alpha}(u', v'), x_{-\alpha}(u'', v'') \in U_{-\alpha}$ such that $x_{-\alpha}(u', v') x_\alpha(u, v) x_{-\alpha}(u'', v'') \in M_\alpha$. By uniqueness in [BT72, 6.1.2(2)] and formula defining $m_{-\alpha}(u, v)$ in 6.14, we get $(u', v') = (u v^{-1}, \tau v^{-1})$. Thus $\varphi_{-\alpha}(x_{-\alpha}(u', v')) = \frac{1}{2}\omega(\tau v^{-1}) = -\varphi_\alpha(x_\alpha(u, v))$ which gives (V5).

Axiom (V3) of commutation: It is a consequence of [BT84, Annexe A].

□

6.4 Action of N on an affine space

In the sequel, we adopt the following notations:

- \mathbf{G} : a quasi-split reductive group over \mathbb{K} ,
- $\tilde{\mathbb{K}}$: a finite Galois extension of \mathbb{K} on which \mathbf{G} splits,
- \mathbf{S} : a maximal \mathbb{K} -split torus of \mathbf{G} ,
- \mathbf{T} : the centralizer of \mathbf{S} in \mathbf{G} ,
- \mathbf{N} : the normalizer of \mathbf{S} in \mathbf{G} ,

and for any algebraic \mathbb{K} -group \mathbf{H} :

- $\tilde{\mathbf{H}}$: the scalar extension $\mathbf{H}_{\tilde{\mathbb{K}}}$ of \mathbf{H} to $\tilde{\mathbb{K}}$,
- $X_*(\mathbf{H})$: the group of cocharacters of \mathbf{H} ,
- $X^*(\mathbf{H})$: the group of characters of \mathbf{H} ,
- $X_{\mathbb{K}}^*(\tilde{\mathbf{H}})$: the group of rational characters of $\tilde{\mathbf{H}}$ over \mathbb{K} .

The natural pairing of abelian groups:

$$X_*(\mathbf{S}) \otimes X^*(\mathbf{S}) \rightarrow \mathbb{Z}$$

is perfect and therefore induces an isomorphism:

$$X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}}(X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R}).$$

By tensorization by the pseudo-ring \mathfrak{A}^S , we obtain an isomorphism of \mathfrak{A}^S -modules:

$$V_1 := X_*(\mathbf{S}) \otimes \mathfrak{A}^S \cong \mathrm{Hom}_{\mathbb{R}}(X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R}) \otimes \mathfrak{A}^S \cong \mathrm{Hom}_{\mathbb{R}}(X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}, \mathfrak{A}^S).$$

The Weyl group $W := \mathbf{N}(\mathbb{K})/\mathbf{T}(\mathbb{K})$ acts \mathbb{R} -linearly on $X_*(\mathbf{S}) \otimes \mathbb{R}$ and hence \mathfrak{A}^S -linearly on V_1 . Since $X_{\mathbb{K}}^*(\mathbf{T})$ is a finite index subgroup of $X^*(\mathbf{S})$, we have:

$$X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} = X_{\mathbb{K}}^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Moreover, for each $t \in \mathbf{T}(\mathbb{K})$, the map:

$$\begin{aligned} \rho(t) : X_{\mathbb{K}}^*(\mathbf{T}) \otimes \mathbb{R} &\rightarrow \mathfrak{A}^S \\ \chi \otimes \lambda &\mapsto -\lambda\omega(\chi(t)) \end{aligned}$$

is well-defined and is \mathbb{R} -linear.

We can therefore see $\rho(t)$ as an element in V_1 and we get a group homomorphism:

$$\begin{aligned} \rho : \mathbf{T}(\mathbb{K}) &\rightarrow V_1 \\ t &\mapsto \rho(t). \end{aligned}$$

Let $\mathbf{T}_b(\mathbb{K})$ be the kernel of ρ and let V_0 be the subspace of V_1 given by vectors v such that $\alpha(v) = 0$ for every root $\alpha \in \Phi$. The quotient $V := V_1/V_0$ is then an \mathfrak{A}^S -module that is endowed with the following structures:

- a morphism:

$$\bar{\rho} : \mathbf{T}(\mathbb{K})/\mathbf{T}_b(\mathbb{K}) \rightarrow V.$$

induced by ρ ;

- a morphism:

$$j : W \rightarrow GL(V)$$

induced by the action of W on V_1 .

Let W' be the push-out of the morphism $\bar{\rho} : \mathbf{T}(\mathbb{K})/\mathbf{T}_b(\mathbb{K}) \rightarrow V$ and the inclusion $\mathbf{T}(\mathbb{K})/\mathbf{T}_b(\mathbb{K}) \subseteq \mathbf{N}(\mathbb{K})/\mathbf{T}_b(\mathbb{K})$. The group W' is then an extension of W by V :

$$1 \rightarrow V \rightarrow W' \rightarrow W \rightarrow 1. \quad (13)$$

If $\kappa : W \rightarrow \text{Aut}_{\text{group}}(V)$ is the induced action by conjugation of W on V , we can see the previous exact sequence as a class in the cohomology group $H^2(W, V)$. But this group is trivial since W is finite and V is uniquely divisible. Hence exact sequence (13) splits and $W' = V \rtimes_{\kappa} W$. The action κ is computed as follows:

$$\forall v \in V, \forall w \in W, \kappa(w)(v) = j(w)(v),$$

Hence j induces a morphism:

$$j' : W' = V \rtimes_{\kappa} W \rightarrow V \rtimes GL(V) = \text{Aff}(V).$$

By composing the projection $\mathbf{N}(\mathbb{K}) \rightarrow \mathbf{N}(\mathbb{K})/\mathbf{T}_b(\mathbb{K})$, the natural morphism $\mathbf{N}(\mathbb{K})/\mathbf{T}_b(\mathbb{K}) \rightarrow W'$ and j' , we get a morphism:

$$\nu : \mathbf{N}(\mathbb{K}) \rightarrow \text{Aff}(V)$$

with kernel $\mathbf{T}_b(\mathbb{K})$ such that the following diagram commutes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{T}(\mathbb{K}) & \longrightarrow & \mathbf{N}(\mathbb{K}) & \longrightarrow & W \longrightarrow 1 \\ & & \downarrow \rho & & \downarrow \nu & & \downarrow j \\ 1 & \longrightarrow & V & \longrightarrow & \text{Aff}(V) & \longrightarrow & GL(V) \longrightarrow 1. \end{array} \quad (14)$$

6.21 Lemma (see [BT84, 4.2.5, 4.2.6 and 4.2.7]). *For any relative root $\alpha \in \Phi$, any absolute root $\tilde{\alpha} \in \tilde{\Phi}$ such that $\tilde{\alpha}|_{\mathbf{S}} = \alpha$ and any $t \in \mathbf{T}(\mathbb{K})$, we have:*

$$\alpha(\nu(t)) = -\omega(\tilde{\alpha}(t)).$$

Proof. For $\chi \in X_{\mathbb{K}}^*(\mathbf{T})$, we have by definition of the action that

$$\chi(\nu(t)) = \chi(\rho(t)) = -\omega(\chi(t)). \quad (15)$$

Inside the \mathbb{R} -module $X^*(\mathbf{S}) \otimes \mathbb{R} = X_{\mathbb{K}}^*(\mathbf{T}) \otimes \mathbb{R}$, we have the identification

$$\alpha \otimes 1 = \sum_{\sigma \in \text{Gal}(\tilde{\mathbb{K}}/\mathbb{K})} \sigma(\tilde{\alpha}) \otimes \frac{1}{[\tilde{\mathbb{K}} : \mathbb{K}]}.$$

Thus, applying the formula (15) to $\chi = \sigma(\tilde{\alpha})$ for $\sigma \in \text{Gal}(\tilde{\mathbb{K}}/\mathbb{K})$, we get

$$\alpha(\nu(t)) = -\frac{1}{[\tilde{\mathbb{K}} : \mathbb{K}]} \sum_{\sigma \in \text{Gal}(\tilde{\mathbb{K}}/\mathbb{K})} \omega(\sigma(\tilde{\alpha})(t)) = -\omega(\tilde{\alpha}(t)).$$

□

6.22 Lemma. *The subgroup of $\mathbf{N}(\widetilde{\mathbb{K}})$ generated by the $m_{\widetilde{\alpha}}$ for $\widetilde{\alpha} \in \widetilde{\Phi}$ is finite.*

Proof. As a split group is, in particular, a quasi-split group with $\widetilde{\mathbb{K}} = \mathbb{K}$, we keep notations $x_{\widetilde{\alpha}}, \widetilde{\alpha}, m_{\widetilde{\alpha}}$ of section 6.2.3 for $\widetilde{\alpha} \in \widetilde{\Phi} = \Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$. Denote by $\widetilde{T} = \widetilde{\mathbf{T}}(\widetilde{\mathbb{K}})$ and by $\widetilde{N} = \widetilde{\mathbf{N}}(\widetilde{\mathbb{K}})$. Let \widetilde{N}_1 be the subgroup of \widetilde{N} generated by the $m_{\widetilde{\alpha}}$ for $\widetilde{\alpha} \in \widetilde{\Phi}$. Let \widetilde{T}_1 be the subgroup of \widetilde{T} generated by the $\widetilde{\alpha}(-1) \in \widetilde{T}$ for $\widetilde{\alpha} \in \widetilde{\Phi}$. The group \widetilde{T}_1 is finite since it is a subgroup of a commutative group \widetilde{T} generated by finitely many elements of order 2. Moreover, for any $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\Phi}$, we have

$$\begin{aligned} \widetilde{\alpha}(-1)m_{\widetilde{\beta}}\widetilde{\alpha}(-1)^{-1} &= \widetilde{\alpha}(-1)x_{\widetilde{\beta}}(1)x_{-\widetilde{\beta}}(1)x_{\widetilde{\beta}}(1)\widetilde{\alpha}(-1)^{-1} \\ &= x_{\widetilde{\beta}}\left(\widetilde{\beta}\left(\widetilde{\alpha}(-1)\right)\right)x_{-\widetilde{\beta}}\left(\left(-\widetilde{\beta}\right)\left(\widetilde{\alpha}(-1)\right)\right)x_{\widetilde{\beta}}\left(\widetilde{\beta}\left(\widetilde{\alpha}(-1)\right)\right) \\ &= m_{\widetilde{\beta}}\widetilde{\beta}\left((-1)^{\langle \widetilde{\beta}, \widetilde{\alpha}^\vee \rangle}\right) \end{aligned}$$

Thus \widetilde{T}_1 normalizes \widetilde{N}_1 . Moreover for every $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\Phi}$, we have

$$\begin{aligned} m_{\widetilde{\alpha}}m_{\widetilde{\beta}}m_{\widetilde{\alpha}} &= m_{\widetilde{\alpha}}x_{\widetilde{\beta}}(1)x_{-\widetilde{\beta}}(1)x_{\widetilde{\beta}}(1)m_{\widetilde{\alpha}}^{-1} \\ &= x_{r_{\widetilde{\alpha}}(\widetilde{\beta})}(\varepsilon_1)x_{r_{\widetilde{\alpha}}(-\widetilde{\beta})}(\varepsilon_2)x_{r_{\widetilde{\alpha}}(\widetilde{\beta})}(\varepsilon_1) \end{aligned}$$

for two signs $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, according to axioms of Chevalley systems. Since this belongs to \widetilde{N} , we necessarily have that $\varepsilon_1 = \varepsilon_2$. Thus $m_{\widetilde{\alpha}}m_{\widetilde{\beta}}m_{\widetilde{\alpha}} = m_{r_{\widetilde{\alpha}}(\widetilde{\beta})}r_{\widetilde{\alpha}}(\widetilde{\beta})(\varepsilon_1)$. Thus $\widetilde{N}_1\widetilde{T}_1/\widetilde{T}_1$ identifies to a subgroup of the Weyl group $W(\widetilde{\Phi})$ and therefore \widetilde{N}_1 is finite. \square

6.23 Lemma (see [BT84, 4.2.9]). *Let m_α for $\alpha \in \Phi$ be defined as in Notations 6.10 and 6.14. There is a point $o \in V$ such that $\nu(m_\alpha)(o) = o$ for every $\alpha \in \Phi$.*

Proof. According to Facts 6.11 and 6.15, the subgroup N_1 of $\mathbf{N}(\mathbb{K})$ generated by the m_α for $\alpha \in \Phi$ is contained in the subgroup of $\mathbf{N}(\mathbb{K})$ generated by the $m_{\widetilde{\alpha}}$ for $\widetilde{\alpha} \in \widetilde{\Phi}$. Thus, according to Lemma 6.22, the group N_1 is a finite subgroup of $N = \mathbf{N}(\mathbb{K})$. Since V is, in particular, an \mathbb{R} -vector space on which N acts by affine transformation, the group N_1 has to fix a point $o \in V$. In particular, it is a fixed point by the m_α for $\alpha \in \Phi$. \square

6.24 Notation. We denote by \mathbb{A} the \mathfrak{R}^S affine space with underlying \mathfrak{R}^S module V and origin o chosen as in Lemma 6.23.

Thus, we have by definition that $\nu(m_\alpha) = r_{\alpha,0} = \text{id}_{\mathbb{A}} - \alpha(\cdot - o)\alpha^\vee$ for any $\alpha \in \Phi$.

6.25 Proposition. *The action $\nu : \mathbf{N}(\mathbb{K}) \rightarrow \text{Aff}(\mathbb{A})$ satisfies (CA1) and (CA2).*

Proof. The condition (CA1) is a consequence of the commutation of the diagram (14) since it is well-known in reductive groups that $W \simeq N/T$ naturally identifies with $W(\Phi)$.

We use notations of sections 6.2.3 and 2.1.3. Let $\alpha \in \Phi$ and $u \in U_\alpha \setminus \{1\}$. Consider the unique element $m(u) \in M_\alpha$ given by [BT72, 6.1.2(2)] and consider $t = m_\alpha m(u) \in T$. Let $\lambda \in \mathfrak{R}^S$ such that $\nu(m(u)) = r_{\alpha,\lambda}$. On the one hand, $\nu(t) = \nu(m_\alpha m(u)) = r_{\alpha,0} \circ r_{\alpha,\lambda} = (x \mapsto x + \lambda\alpha^\vee)$ so that $\alpha(\nu(m_\alpha m(u))(o) - o) = 2\lambda$. On the other hand, according to Lemma 6.21, we have $\alpha(\nu(t)) = -\omega(\widetilde{\alpha}(t))$ for any $\widetilde{\alpha} \in \widetilde{\Phi}$ such that $\widetilde{\alpha}|_S = \alpha$. Thus $2\lambda = -\omega(\widetilde{\alpha}(t))$. Moreover $\nu(m(u))(o) - o = r_{\alpha,\lambda}(o) - o = -\lambda\alpha^\vee$. Thus $\alpha(\nu(m(u))(o) - o) = -2\lambda = \omega(\widetilde{\alpha}(t))$. if we show that $\omega(\widetilde{\alpha}(t)) = -2\varphi_\alpha(u)$, then condition (CA2) will be proven.

If α is non-multipliable and non-divisible, then one can write $u = x_\alpha(z)$ with $z \in \mathbb{L}_\alpha^*$. Then $m(u) = m_{-\alpha}(z^{-1}) = m_{-\alpha}\widehat{-\alpha}(z)$ according to Notation 6.10 and Fact 6.11. Thus $m_\alpha m(u) = -\widehat{-\alpha}(z)$ and therefore $\widetilde{\alpha}(t) = (-\widetilde{\alpha})(t)^{-1} = (-\widetilde{\alpha})(-\widehat{-\alpha}(z))^{-1} = -z^{-2}$. Thus $\omega(\widetilde{\alpha}(t)) = -2\omega(z) = -2\varphi_\alpha(u)$.

If α is multipliable, then one can write $u = x_\alpha(y, z)$ with $(y, z) \in H(\mathbb{L}_\alpha, \mathbb{L}_{2\alpha}) \setminus \{(0, 0)\}$. Then $m(u) = m_{-\alpha}(y, z) = m_{-\alpha}\widehat{-\alpha}(z)$ according to Notation 6.14. Thus $m_\alpha m(u) = \widehat{-\alpha}(z)$

and therefore $\tilde{\alpha}(t) = (-\tilde{\alpha})(t)^{-1} = (-\tilde{\alpha})(\widehat{-\alpha}(z))^{-1} = z^\tau z^{-2}$. Thus $\omega(\tilde{\alpha}(t)) = -\omega(z) = -2\varphi_\alpha(u)$.

If α is divisible, then there are $\beta \in \Phi$, $\tilde{\beta} \in \tilde{\Phi}$ and $\tau \in \text{Gal}(\mathbb{L}_\beta/\mathbb{L}_{2\beta})$ such that $\alpha = 2\beta$, $\tilde{\beta}|_{\mathbf{S}} = \beta$ and $\tilde{\alpha} = \tilde{\beta} + \tau(\tilde{\beta})$. By definition $m_\beta = m_\alpha$. In particular, $u \in U_\beta$ and $t = m_\beta m(u)$. Thus, we have shown that $\omega(\tilde{\beta}(t)) = -2\varphi_\beta(u)$. Hence, using (V4), we get $\omega(\tilde{\alpha}(t)) = \omega(\tilde{\beta}(t)) + \omega(\tau(\tilde{\beta})(t)) = -4\varphi_\beta(u) = -2\varphi_\alpha(u)$. \square

6.26 Remark. By definition, the groups $\mathbf{T}_b(\mathbb{K})$ and $\mathbf{T}(\mathbb{K})_b$ are respectively kernels of ρ and ν . Thus, by commutativity of the diagram (14), we have that $\mathbf{T}_b(\mathbb{K}) = \mathbf{T}(\mathbb{K})_b$.

6.5 The \mathfrak{R}^S -building of a quasi-split reductive group

In this section, we have defined a generating root group datum $(T, (U_\alpha, M_\alpha)_{\alpha \in \Phi})$ (in the sense of Definition 3.1) together with an \mathfrak{R}^S valuation $(\varphi_\alpha)_{\alpha \in \Phi}$ of this root group datum (in the sense of Definition 3.5, see Proposition 6.20). Moreover, in Notation 6.24 we defined an \mathfrak{R}^S -affine space \mathbb{A} together with an action ν of N by affine transformations given by commutative diagram (14). According to Proposition 6.25, this action is compatible with the valuation in the sense of Definition 3.14. Thus assumption 3.22 is satisfied and by construction $\mathfrak{R}^S = \mathfrak{R}_\mathbb{Q}^S$. Thus, we provided a datum as in assumptions of section 5 so that we can define, as in Definition 5.1, the following space:

6.27 Definition. The \mathfrak{R}^S -building associated to the quasi-split reductive \mathbb{K} -group \mathbf{G} is:

$$\mathcal{I}(\mathbf{G}) = \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) := \mathcal{I}(G, T, (U_\alpha)_{\alpha \in \Phi}, (M_\alpha)_{\alpha \in \Phi}, (\varphi_\alpha)_{\alpha \in \Phi}, \nu).$$

7 Projection maps

7.1 Construction and explicit description of the fibers

Let Λ_0 be a convex subgroup of Λ , and set $\Lambda_1 := \Lambda/\Lambda_0$. The group Λ_1 is then naturally endowed with a structure of totally ordered abelian group.

Denote by $\omega_1 : \mathbb{K} \rightarrow \Lambda_1 \cup \{\infty\}$ the composite of the valuation ω followed by the projection $\Lambda \rightarrow \Lambda_1$.

In the sequel, we set:

$$\begin{aligned} \mathbb{O} &:= \omega^{-1}(\Lambda_{\geq 0}), & \mathbb{M} &:= \omega^{-1}(\Lambda_{> 0}), \\ \mathcal{O} &:= \omega_1^{-1}((\Lambda_1)_{\geq 0}), & \mathcal{M} &:= \omega_1^{-1}((\Lambda_1)_{> 0}), \\ \mathcal{K}_1 &:= \mathcal{O}/\mathcal{M}, & \mathcal{O}_1 &:= \mathbb{O}/\mathcal{M}, & \mathcal{M}_1 &:= \mathbb{M}/\mathcal{M}, \\ \kappa &:= \mathbb{O}/\mathbb{M} = \mathcal{O}_1/\mathcal{M}_1. \end{aligned}$$

Observe that the rank $\text{rk}(\Lambda)$ is isomorphic to the set $\text{rk}(\Lambda_1) \amalg \text{rk}(\Lambda_0)$ endowed with the total order such that $s_1 < s_0$ for any $s_0 \in \text{rk}(\Lambda_0)$ and $s_1 \in \text{rk}(\Lambda_1)$. Hence $\mathcal{R}^{\text{rk}(\Lambda_0)}$ is an ideal of $\mathcal{R}^{\text{rk}(\Lambda)}$ such that:

$$\mathcal{R}^{\text{rk}(\Lambda)} / \mathcal{R}^{\text{rk}(\Lambda_0)} \cong \mathcal{R}^{\text{rk}(\Lambda_1)}$$

as ordered \mathbb{R} -algebras. Let $\pi : \mathcal{R}^{\text{rk}(\Lambda)} \rightarrow \mathcal{R}^{\text{rk}(\Lambda_1)}$ be the projection. By tensorization, it induces an $\mathcal{R}^{\text{rk}(\Lambda)}$ -affine linear epimorphism between apartments:

$$\pi_{\text{aff}} : \mathbb{A}(\mathbb{K}, \omega, G) \rightarrow \mathbb{A}(\mathbb{K}, \omega_1, G)$$

which is compatible with the action of N . By taking the product with G and by passing to the quotient, we get a surjective map:

$$\pi : \mathcal{I}(\mathbb{K}, \omega, G) \rightarrow \mathcal{I}(\mathbb{K}, \omega_1, G)$$

which is compatible with the G -action.

Take now $g_1 \in G$ and $x_1 \in \mathbb{A}(\mathbb{K}, \omega_1, G)$, and consider the point $X_1 := [(g_1, x_1)]$ in $\mathcal{I}(\mathbb{K}, \omega_1, G)$. By setting $U_{X_1} := g_1 U_{x_1} g_1^{-1}$, we can define the map:

$$\begin{aligned} \varphi_{X_1} : U_{X_1} \times \pi_{\text{aff}}^{-1}(\{x_1\}) &\rightarrow \mathcal{I}(\mathbb{K}, \omega, G) \\ (u, z) &\mapsto [(pg_1, z)]. \end{aligned}$$

7.1 Proposition. *The image of φ_{X_1} is the fiber $\pi^{-1}(\{X_1\})$.*

Proof. For any $(u, z) \in U_{X_1} \times \pi_{\text{aff}}^{-1}(\{x_1\})$, we have:

$$\pi(\varphi_{X_1}(u, z)) = [(ug_1, \pi(z))] = [(ug_1, x_1)] = u \cdot X_1 = X_1.$$

Hence $\text{Im}(\varphi_{X_1}) \subseteq \pi^{-1}(\{X_1\})$. Conversely, choose an element $X := [(g, x)]$ in the fiber $\pi^{-1}(\{X_1\})$. We then have $[(g, \pi(x))] = [(g_1, x_1)] \in \mathcal{I}(\mathbb{K}, \omega_1, G)$ and hence we can find $n \in N$ such that $\nu(n)(x_1) = \pi(x)$ and $g_1^{-1}gn \in U_{x_1} = g_1^{-1}U_{X_1}g_1$. Set $u := gng_1^{-1}$ and $z := \nu(n^{-1})(x)$. Then $u \in U_{X_1}$, $z \in \pi_{\text{aff}}^{-1}(\{x_1\})$, and :

$$\pi(\varphi_{X_1}(u, z)) = [(gn, \nu(n^{-1})(x))] = gnn^{-1} \cdot [(1, x)] = g \cdot [(1, x)] = [(g, x)] = X.$$

Hence $X \in \text{Im}(\varphi_{X_1})$, and $\text{Im}(\varphi_{X_1}) = \pi^{-1}(\{X_1\})$. \square

As a consequence, if $U_{\pi^{-1}(X_1)}$ stands for the pointwise stabilizer of $\pi^{-1}(X_1)$ in U_{X_1} , then for any $u_0 \in U_{\pi^{-1}(X_1)}$, $u \in U_{X_1}$ and $z \in \pi_{\text{aff}}^{-1}(\{x_1\})$, we have $[(ug_1, z)] \in \pi^{-1}(X_1)$, so that:

$$\varphi_{X_1}(u_0u, z) = [(u_0ug_1, z)] = u_0 \cdot [(ug_1, z)] = [(ug_1, z)] = \varphi_{X_1}(u, z).$$

Hence φ_{X_1} induces a surjective map:

$$\bar{\varphi}_{X_1} : (U_{X_1}/U_{\pi^{-1}(\{X_1\})}) \times \pi_{\text{aff}}^{-1}(\{x_1\}) \rightarrow \pi^{-1}(\{X_1\}).$$

Consider now two elements (u, z) and (u', z') in $U_{X_1} \times \pi_{\text{aff}}^{-1}(\{x_1\})$ such that $\varphi_{X_1}(u, z) = \varphi_{X_1}(u', z')$. Then we have $[(ug_1, z)] = [(u'g_1, z')]$, and hence we can find $n \in N$ such that

$$\begin{cases} z' = \nu(n)(z), \\ g_1^{-1}u^{-1}u'g_1n \in U_z \subseteq U_{x_1}. \end{cases} \quad (16)$$

By setting $m := g_1ng_1^{-1}$, we get $m \in U_{X_1} \cap g_1Ng_1^{-1}$ and :

$$\begin{cases} z' = \nu(g_1^{-1}mg_1)(z), \\ u^{-1}u'm \in g_1U_zg_1^{-1}. \end{cases}$$

In other words, if we introduce the groups:

$$\begin{aligned} N_{0, X_1} &:= U_{X_1} \cap g_1Ng_1^{-1}, \\ U_{0, z} &:= g_1U_zg_1^{-1}, \end{aligned}$$

and the group homomorphism:

$$\begin{aligned} \nu_{0, X_1} : N_{0, X_1} &\rightarrow \text{Aff}(\pi_{\text{aff}}^{-1}(\{x_1\})) \\ \eta &\mapsto \nu(g_1^{-1}\eta g_1)|_{\pi_{\text{aff}}^{-1}(\{x_1\})}, \end{aligned}$$

then $m \in N_{0,X_1}$ and:

$$\begin{cases} z' = \nu_{0,X_1}(m)(z), \\ u^{-1}u'm \in U_{0,z}. \end{cases} \quad (17)$$

Conversely, if we can find $m \in N_{0,X_1}$ satisfying (17), then we can write $u^{-1}u'm = g_1vg_1^{-1}$ with $v \in U_z$, and hence:

$$\begin{aligned} \varphi_{X_1}(u', z') &= [(u'g_1, z')] = u'g_1[(1, z')] \\ &= ug_1vg_1^{-1}m^{-1}g_1[1, z'] = ug_1v([1, \nu_{0,X_1}(m^{-1})z']) \\ &= ug_1v([1, z]) = ug_1[1, z] = u[g_1, z] = \varphi_{X_1}(u, z). \end{aligned}$$

We have thus proved the following proposition:

7.2 Proposition. *Consider the group $\overline{N}_{0,X_1} := N_{0,X_1} / (U_{\pi^{-1}(X_1)} \cap g_1Ng_1^{-1})$ and the group homomorphism:*

$$\overline{\nu}_{0,X_1} : \overline{N}_{0,X_1} \rightarrow \text{Aff}(\pi_{\text{aff}}^{-1}(\{x_1\}))$$

induced by ν_{0,X_1} . For each $z \in \pi_{\text{aff}}^{-1}(\{x_1\})$, set $\overline{U}_{0,z} := U_{0,z} / (U_{\pi^{-1}(X_1)} \cap g_1U_zg_1^{-1})$. Endow the set $(U_{X_1}/U_{\pi^{-1}(\{X_1\})}) \times \pi_{\text{aff}}^{-1}(\{x_1\})$ with the equivalence relation defined in the following way: $(p, z) \sim (p', z')$ if, and only if, there exists $n \in \overline{N}_{0,x_1}$ satisfying equations:

$$\begin{cases} z' = \overline{\nu}_{0,X_1}(n)(z), \\ p^{-1}p'n \in \overline{U}_{0,z}. \end{cases} \quad (18)$$

Then the map $\overline{\varphi}_{X_1}$ induces a bijection:

$$\frac{(U_{X_1}/U_{\pi^{-1}(\{X_1\})}) \times \pi_{\text{aff}}^{-1}(\{x_1\})}{\sim} \rightarrow \pi^{-1}(\{X_1\}).$$

which is compatible with the action of $U_{X_1}/U_{\pi^{-1}(\{X_1\})}$ and which will still be denoted $\overline{\varphi}_{X_1}$.

7.2 The root group data axioms for the fibers

For $\alpha \in \Phi_{x_1}$, set:

$$\begin{aligned} U_{0,\alpha} &:= U_{X_1} \cap g_1U_\alpha g_1^{-1}, \\ \overline{U}_{0,\alpha} &:= U_{0,\alpha} / (U_{\pi^{-1}(\{X_1\})} \cap g_1U_\alpha g_1^{-1}), \\ S_{0,X_1} &:= U_{X_1} \cap g_1Sg_1^{-1}, \\ \overline{S}_{0,X_1} &:= S_{0,X_1} / (U_{\pi^{-1}(\{X_1\})} \cap g_1Sg_1^{-1}), \\ T_{0,X_1} &:= U_{X_1} \cap g_1Tg_1^{-1}, \\ \overline{T}_{0,X_1} &:= T_{0,X_1} / (U_{\pi^{-1}(\{X_1\})} \cap g_1Tg_1^{-1}), \\ M_{0,\alpha} &:= U_{X_1} \cap g_1M_\alpha g_1^{-1}, \\ \overline{M}_{0,\alpha} &:= \text{Im}(M_{0,\alpha} \hookrightarrow N_{0,X_1} \twoheadrightarrow \overline{N}_{0,X_1}). \end{aligned}$$

7.3 Lemma. *Let L/K be any finite extension of K and consider the extension of the valuations v_1 and v to L . For any $x \in L$, if $v_1(x) > 0$ then $v(x) > 0$.*

Proof. We have $v(L \setminus \{0\}) \subseteq \mathfrak{R}^{\text{rk}(\Lambda)}$ and $v_1 = \pi \circ v$ where $\pi : \mathfrak{R}^{\text{rk}(\Lambda)} \rightarrow \mathfrak{R}^{\Lambda_1}$ is the natural projection. The result then follows from the fact that π is non-decreasing. \square

7.4 Proposition. *Let Ω_1 be a subset of $\mathbb{A}(\mathbb{K}, \omega_1, \mathbf{G})$. Let $T_{\Omega_1,1}$ be the subgroup of T defined as in notation 3.52 and associated to Ω_1 . We then have $T_{\Omega_1,1}^* \subset T_b$.*

7.5 Remark. Note that, in the previous proposition, the group $T_{\Omega_1,1}^*$ is defined thanks to the apartment $\mathbb{A}(\mathbb{K}, \omega_1, \mathbf{G})$ while the group T_b is defined thanks to the apartment $\mathbb{A}(\mathbb{K}, \omega, \mathbf{G})$

Proof. Let $x_1 \in \Omega_1$. Since $T_{\Omega_1,1}^* \subset T_{x_1,1}^*$, it suffices to prove it for $\Omega_1 = \{x_1\}$. Let $a \in \Phi$. Consider any $u^+ \in U_{a,x_1}$ and any $u^- \in U'_{-a,x_1}$ so that $\varphi_a^1(u^+) + \varphi_{-a}^1(u^-) > 0$. Let $t(u^+, u^-)$ be the unique element in T given by Lemma 3.26(1) so that $u^-(u^+)^{-1} \in U_a T U_{-a}$.

We firstly prove that $\varphi_a(u^+) + \varphi_{-a}(u^-) > 0$.

Case a non-multiplicable root: Write $u^+ = x_a(x)$ and $u^- = x_{-a}(y)$ with $x, y \in L_a$. By definition $\varphi_a^1(u^+) = v_1(x)$ and $\varphi_{-a}^1(u^-) = v_1(y)$. We have $\varphi_a^1(u^+) + \varphi_{-a}^1(u^-) = v_1(x) + v_1(y) = v_1(xy) > 0$. Hence $\varphi_a(u^+) + \varphi_{-a}(u^-) = v(x) + v(y) = v(xy) > 0$ by Lemma 7.3.

Case a multiplicable root: Write $u^+ = x_a(u, x)$ and $u^- = x_{-a}(v, y)$ with $(u, x), (v, y) \in H(L_a, L_{2a})$. By definition $\varphi_a^1(u^+) = \frac{1}{2}v_1(x)$ and $\varphi_{-a}^1(u^-) = \frac{1}{2}v_1(y)$. We have $\varphi_a^1(u^+) + \varphi_{-a}^1(u^-) = \frac{1}{2}v_1(x) + \frac{1}{2}v_1(y) = \frac{1}{2}v_1(xy) > 0$. Hence $\varphi_a(u^+) + \varphi_{-a}(u^-) = \frac{1}{2}v(x) + \frac{1}{2}v(y) = \frac{1}{2}v(xy) > 0$ by Lemma 7.3.

Thus, in both cases, we get that $\varphi_a(u^+) + \varphi_{-a}(u^-) > 0$ which implies that $t(u^+, u^-) \in T_b$ according to Lemma 3.26(2).

According to definition of T'_{a,x_1} and Proposition 3.33, we know that the group $T'_{a,x_1,1}$ is generated by the $t(u^+, u^-)$ $u^+ \in U_{a,x_1}$ and $u^- \in U_{-a,x_1}$. But we have shown that the $t(u^+, u^-)$ all are contained in the group T_b , hence we get that $T'_{a,x_1,1} \subset T_b$. Since, by definition, the group $T_{x_1,1}^*$ is generated by the $T'_{a,x_1,1}$ for $a \in \Phi$, then it is contained in T_b . \square

7.6 Corollary. For any non-empty subset $\Omega_1 \subset \mathbb{A}_1$, any $p \in \widehat{P_{\Omega_1}}$ and any $u \in U'_{\Omega_1}$, we have $pup^{-1} \in U'_{\Omega_1} T_b$.

Proof. It is an immediate consequence of Proposition 4.15 and Proposition 7.4. \square

7.7 Corollary. For $\alpha \in \Phi$, we have:

$$\begin{aligned} U_{X_1} \cap g_1 U_{\alpha} g_1^{-1} &= g_1 U_{\alpha, x_1} g_1^{-1}, \\ U_{\pi^{-1}(X_1)} \cap g_1 U_{\alpha} g_1^{-1} &= g_1 U'_{\alpha, x_1} g_1^{-1}. \end{aligned}$$

Proof. The first equality immediately follows from:

$$U_{X_1} \cap g_1 U_{\alpha} g_1^{-1} = g_1 (U_{x_1} \cap U_{\alpha}) g_1^{-1} = g_1 U_{\alpha, x_1} g_1^{-1}.$$

The second equality is a bit more delicate. If we choose $x \in \mathbb{A}(\mathbb{K}, v, G)$ such that $\pi(x) = x_1$ and we set $X := [(g_1, x)] \in \pi^{-1}(X_1)$, then we have:

$$\begin{aligned} U_{\pi^{-1}(X_1)} \cap g_1 U_{\alpha} g_1^{-1} &\subseteq \hat{P}_X \cap g_1 U_{\alpha} g_1^{-1} \\ &\subseteq g_1 \left(\hat{P}_x \cap U_{\alpha} \right) g_1^{-1} \\ &\subseteq g_1 U_{\alpha, x} g_1^{-1}. \end{aligned}$$

Hence:

$$U_{\pi^{-1}(X_1)} \cap g_1 U_{\alpha} g_1^{-1} \subseteq g_1 \left(\bigcap_{\substack{x \in \mathbb{A}(\mathbb{K}, v, G) \\ \pi(x) = x_1}} U_{\alpha, x} \right) g_1^{-1} = g_1 U'_{\alpha, x_1} g_1^{-1}.$$

Conversely, let's take $u \in U'_{\alpha, x_1}$ and let's prove that $g_1 u g_1^{-1} \in U_{\pi^{-1}(X_1)}$. In other words, we have to check that $g_1 u g_1^{-1}$ fixes $\pi^{-1}(X_1)$. To do so, take $X := [(v g_1, x)] \in \pi^{-1}(X_1)$ with $v \in U_{X_1}$ and $x \in \pi_{\text{aff}}^{-1}(x_1)$. By corollary 7.6, we have $(g_1^{-1} v g_1) u (g_1^{-1} v g_1)^{-1} \in U'_{x_1} T_b$. But the groups U'_{x_1} and T_b both fix x . Hence $v (g_1 u g_1^{-1}) v^{-1}$ fixes $[(g_1, x)]$, so that $g_1 u g_1^{-1}$ fixes $v \cdot [(g_1, x)] = X$, as wished. \square

7.8 Lemma. For $\alpha \in \Phi_{x_1}$, the subset $\overline{M}_{0,\alpha}$ of \overline{N}_{0,X_1} is a right coset of \overline{T}_{0,X_1} in \overline{N}_{0,X_1} .

Proof. For any $m, m' \in M_{0,\alpha}$ and $t \in T_{0,X_1}$, we have $m^{-1}m' \in T_{0,X_1}$ and $mt \in M_{0,\alpha}$. It is therefore enough to check that $M_{0,\alpha}$ is not empty. To do so, we distinguish two cases.

Assume first that α is non-multipliable. Since $\alpha \in \Phi_{x_1}$, we can find $y \in L_\alpha$ such that $\omega_1(y) = -\alpha(x_1)$. We then have:

$$\begin{cases} x_\alpha(y) \in U_{\alpha, -\alpha(x_1)} = U_\alpha \cap U_{x_1} \\ x_{-\alpha}(y^{-1}) \in U_{-\alpha, \alpha(x_1)} = U_{-\alpha} \cap U_{x_1}, \end{cases}$$

so that:

$$\begin{cases} g_1 x_\alpha(y) g_1^{-1} \in U_{0,\alpha} \\ g_1 x_{-\alpha}(y^{-1}) g_1^{-1} \in U_{0,-\alpha}. \end{cases}$$

Hence:

$$m_{0,\alpha}(y) := g_1 m_\alpha(y) g_1^{-1} = g_1 x_\alpha(y) x_{-\alpha}(y^{-1}) x_\alpha(y) g_1^{-1} \in M_{0,\alpha}.$$

Assume now that α is multipliable. Since $\alpha \in \Phi_{x_1}$, we can find $(y, y') \in H(L_\alpha, L_{2\alpha})$ such that $\omega_1(y') = 2\alpha(x_1)$. We then have:

$$\begin{cases} x_\alpha(yy'^{-1}, (\tau y')^{-1}) \in U_{\alpha, -\alpha(x_1)} = U_\alpha \cap U_{x_1} \\ x_\alpha(y(\tau y')^{-1}, (\tau y')^{-1}) \in U_{\alpha, -\alpha(x_1)} = U_\alpha \cap U_{x_1} \\ x_{-\alpha}(y, y') \in U_{-\alpha, \alpha(x_1)} = U_{-\alpha} \cap U_{x_1}, \end{cases}$$

so that:

$$\begin{cases} g_1 x_\alpha(yy'^{-1}, (\tau y')^{-1}) g_1^{-1} \in U_{0,\alpha} \\ g_1 x_\alpha(y(\tau y')^{-1}, (\tau y')^{-1}) g_1^{-1} \in U_{0,\alpha} \\ g_1 x_{-\alpha}(y, y') g_1^{-1} \in U_{0,-\alpha}. \end{cases}$$

Hence:

$$m_{0,\alpha}(y, y') := g_1 m_\alpha(y, y') g_1^{-1} = g_1 x_\alpha(yy'^{-1}, (\tau y')^{-1}) x_{-\alpha}(y, y') x_\alpha(y(\tau y')^{-1}, (\tau y')^{-1}) g_1^{-1} \in M_{0,\alpha}.$$

□

In the sequel, we will keep the notations $m_{0,\alpha}(y)$ and $m_{0,\alpha}(y, y')$ that have been used in the previous proof.

7.9 Proposition. The system $(\overline{T}_{0,X_1}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}})$ is a generating root group datum in $U_{X_1}/U_{\pi^{-1}(X_1)}$.

Proof.

Axiom (RGD1). Since $\alpha \in \Phi_{x_1}$, we can find $u \in U_\alpha$ such that $\pi(\varphi_\alpha(u)) = -\alpha(x_1)$. By lemma 7.7, we have:

$$g_1 u g_1^{-1} \in g_1 U_\alpha g_1^{-1} \cap (U_{X_1} \setminus U_{\pi^{-1}(X_1)}).$$

Hence $\overline{U}_{0,\alpha}$ is not trivial.

Axiom (RGD2). By proposition 6.20, given two roots $\alpha, \beta \in \Phi_{x_1}$ such that $\alpha \notin -\mathbb{R}_+\beta$, the group $[U_{\alpha, -\alpha(x_1)}, U_{\beta, -\beta(x_1)}]$ is contained in the group spanned by the groups $U_{\gamma, -\gamma(x_1)}$ with $\gamma \in \Phi \cap (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta)$. Hence the group $[U_{0,\alpha}, U_{0,\beta}]$ is contained in the group spanned by the groups $U_{0,\gamma}$ with $\gamma \in \Phi \cap (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta)$.

Fix now $\gamma \in \Phi \setminus \Phi_{x_1}$ and $u \in U_{0,\gamma} \setminus (U_{\pi^{-1}(X_1)} \cap g_1 U_\gamma g_1^{-1}) = g_1(U_{\gamma,x_1} \setminus U'_{\gamma,x_1})g_1^{-1}$. We then have $\pi(\varphi_\gamma(g_1^{-1}ug_1)) = -\gamma(x_1)$. But $\gamma \notin \Phi_{x_1}$, hence:

$$U_{\gamma,\pi(\varphi_\gamma(g_1^{-1}ug_1))} \neq \bigcap_{u' \in U_{2\gamma}} U_{\gamma,\pi(\varphi_\gamma(g_1^{-1}ug_1u'))}.$$

In other words, there exists $u' \in U_{2\gamma}$ such that:

$$-\gamma(x_1) = \pi(\varphi_\gamma(g_1^{-1}ug_1)) < \pi(\varphi_\gamma(g_1^{-1}ug_1u')),$$

and we necessarily have $\pi(\varphi_{2\gamma}(u')) = 2\pi(\varphi_\gamma(u')) = 2\pi(\varphi_\gamma(g_1^{-1}ug_1)) = -2\gamma(x_1)$. We deduce that $2\gamma \in \Phi_{x_1}$ and that:

$$g_1^{-1}ug_1 = (g_1^{-1}ug_1u') \cdot u'^{-1}$$

with:

$$\begin{aligned} ug_1u'g_1^{-1} &\in g_1U'_{\gamma,x_1}g_1^{-1} = U_{\pi^{-1}(X_1)} \cap g_1U_\gamma g_1^{-1}, \\ g_1^{-1}u'^{-1}g_1 &\in g_1U_{2\gamma,x_1}g_1^{-1} = U_{0,2\gamma}. \end{aligned}$$

The last equalities show that the image of u in $\overline{U}_{0,\gamma}$ belongs to $\overline{U}_{0,2\gamma}$, and hence the group $[\overline{U}_{0,\alpha}, \overline{U}_{0,\beta}]$ is necessarily contained in the group spanned by the groups $\overline{U}_{0,\gamma}$ with $\gamma \in \Phi_{x_1} \cap (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta)$.

Axiom (RGD3). If α and 2α belong to Φ_{x_1} , then $U_{2\alpha} \subseteq U_\alpha$, and hence $\overline{U}_{0,2\alpha} \subseteq \overline{U}_{0,\alpha}$. Let's check that this inclusion is strict. The condition that $\alpha \in \Phi_{x_1}$ implies that we can find $(y, y') \in H_0(L_\alpha, L_{2\alpha})$ such that $\text{Tr}_{L_\alpha/L_{2\alpha}}(y') = N_{L_\alpha/L_{2\alpha}}(y)$ and $\omega_1(y') = -2\alpha(x_1)$.

If $y \neq 0$, then:

$$g_1x_\alpha(y, y')g_1^{-1} \in (U_{X_1} \cap g_1U_\alpha g_1^{-1}) \setminus (U_{\pi^{-1}(X_1)} \cap g_1U_\alpha g_1^{-1})$$

and:

$$g_1x_\alpha(y, y')g_1^{-1} \notin (U_{X_1} \cap g_1U_{2\alpha}g_1^{-1}).$$

Hence the class of $g_1x_\alpha(y, y')g_1^{-1}$ in $\overline{U}_{0,\alpha}$ is not in $\overline{U}_{0,2\alpha}$.

Now assume that $y = 0$. Let (z, z') be any element of $H_0(L_\alpha, L_{2\alpha})$ with $z \neq 0$. Let $\lambda \in L_\alpha^\times$ such that $\omega_1(\lambda^2 z') > \omega_1(y')$. We then have:

$$\begin{aligned} (\lambda z, \lambda^2 z' + y') &\in H_0(L_\alpha, L_{2\alpha}), \\ \lambda z &\neq 0, \\ \omega_1(\lambda^2 z' + y') &= -2\alpha(x_1). \end{aligned}$$

Hence the class of $g_1x_\alpha(\lambda z, \lambda^2 z' + y')$ in $\overline{U}_{0,\alpha}$ is not in $\overline{U}_{0,2\alpha}$.

Axiom (RGD4). Take any element $\bar{u} \in \overline{U}_{0,-\alpha} \setminus \{1\}$ and fix a lifting

$$u \in (U_{X_1} \cap g_1U_{-\alpha}g_1^{-1}) \setminus (U_{\pi^{-1}(X_1)} \cap g_1U_{-\alpha}g_1^{-1})$$

of \bar{u} .

Assume first that α is non-multipliable. We can then find $y \in L_{-\alpha}^\times$ such that:

$$u = g_1x_{-\alpha}(y^{-1})g_1^{-1}.$$

By lemma 7.7, we have $\omega_1(y) = -\alpha(x_1)$. Hence:

$$u = (g_1x_\alpha(y)^{-1}g_1^{-1})m_\alpha(y)(g_1x_\alpha(y)^{-1}g_1^{-1}) \in U_{0,\alpha}m_\alpha(y)U_{0,\alpha} \subseteq U_{0,\alpha}M_{0,\alpha}U_{0,\alpha}.$$

Now assume that α is multipliable. We can then find $(y, y') \in H_0(L_\alpha, L_{2\alpha})$ such that:

$$u = g_1 x_{-\alpha}(y, y') g_1^{-1}.$$

By lemma 7.7, we have $\omega_1(y') = 2\alpha(x_1)$. Hence:

$$\begin{aligned} u &= (g_1 x_\alpha(y y'^{-1}, (\tau y')^{-1})^{-1} g_1^{-1}) \cdot m_\alpha(y, y') \cdot (g_1 x_\alpha(y (\tau y')^{-1}, (\tau y')^{-1})^{-1} g_1^{-1}) \\ &\in U_{0,\alpha} m_\alpha(y, y') U_{0,\alpha} \subseteq U_{0,\alpha} M_{0,\alpha} U_{0,\alpha}. \end{aligned}$$

Axiom (RGD5). Take $\alpha, \beta \in \Phi_{x_1}$ and $\bar{m} \in \bar{M}_{0,\alpha}$. Let $m \in M_{0,\alpha}$ be a lifting of \bar{m} . Since $g_1^{-1} m g_1 \in M_\alpha$, proposition 6.20 implies that:

$$(g_1^{-1} m g_1) U_\beta (g_1^{-1} m g_1)^{-1} = U_{r_\alpha(\beta)}.$$

Since $m \in U_{X_1}$, we deduce that $m U_{0,\beta} m^{-1} = U_{0,r_\alpha(\beta)}$, and hence:

$$\bar{m} \bar{U}_{0,\beta} \bar{m}^{-1} = \bar{U}_{0,r_\alpha(\beta)}.$$

Axiom (RGD6). Let \bar{U}_{0,X_1}^\pm be the subgroup of $U_{X_1}/U_{\pi^{-1}(X_1)}$ generated by the $\bar{U}_{0,\alpha}$ for $\alpha \in \Phi_{x_1}^\pm$. Take $\bar{u}_+ \in \bar{U}_{0,X_1}^+$, $\bar{u}_- \in \bar{U}_{0,X_1}^-$ and $\bar{t} \in \bar{T}_{0,X_1}$ such that $\bar{t} \bar{u}_+ = \bar{u}_-$. By setting $U_{0,X_1}^\pm := \langle U_{0,\alpha}, \alpha \in \Phi_{x_1}^\pm \rangle$, we can find a lifting u_+ of \bar{u}_+ in U_{0,X_1}^+ , a lifting u_- of \bar{u}_- in U_{0,X_1}^- , a lifting t of \bar{t} in T_{0,X_1} and an element $u \in U_{\pi^{-1}(X_1)}$ such that $tu_+ = uu_-$. Since $\hat{P}_{g_1 \mathbb{A} \cap \pi^{-1}(X_1)} = g_1 \hat{N}_{\mathbb{A} \cap \pi^{-1}(X_1)} U_{\mathbb{A} \cap \pi^{-1}(X_1)}^+ U_{\mathbb{A} \cap \pi^{-1}(X_1)}^- g_1^{-1}$, we can find $u'_+ \in U_{\mathbb{A} \cap \pi^{-1}(X_1)}^+$, $u'_- \in U_{\mathbb{A} \cap \pi^{-1}(X_1)}^-$ and $n' \in \hat{N}_{\mathbb{A} \cap \pi^{-1}(X_1)}$ such that $u = g_1 n' u'_+ u'_- g_1^{-1}$. We therefore have:

$$g_1^{-1} t u_+ g_1 = n' u'_+ u'_- (g_1^{-1} u_- g_1),$$

and hence:

$$U^+ g_1^{-1} t u_+ g_1 U^- = U^+ n' u'_+ U^-.$$

But $g_1^{-1} t g_1 \in T$, $n' \in N_{\mathbb{A} \cap \pi^{-1}(X_1)} \subseteq T$ and T normalizes U^+ . Hence:

$$U^+ (g_1^{-1} t g_1) U^- = U^+ n' U^-.$$

We deduce that:

$$\begin{aligned} g_1^{-1} t g_1 &= n', \\ u'_+{}^{-1} (g_1^{-1} u_+ g_1) &= u'_- (g_1^{-1} u_- g_1) \in U^+ \cap U^-. \end{aligned}$$

But $U^+ \cap U^- = \{1\}$. Hence, by using corollary 7.7, we get:

$$\begin{aligned} u_+ &= g_1 u'_+ g_1^{-1} \in g_1 U_{\mathbb{A} \cap \pi^{-1}(X_1)} g_1^{-1} \subseteq U_{\pi^{-1}(X_1)}, \\ u_- &= g_1 u'_-{}^{-1} g_1^{-1} \in g_1 U_{\mathbb{A} \cap \pi^{-1}(X_1)} g_1^{-1} \subseteq U_{\pi^{-1}(X_1)}, \\ t &= g_1 n' g_1^{-1} = u u'_-{}^{-1} u'_+ \in U_{\pi^{-1}(X_1)}, \end{aligned}$$

so that $\bar{u}_+ = \bar{u}_- = \bar{t} = 1$.

The root group datum is generating. Indeed, U_{X_1} is spanned by the $g_1 U_{\alpha, x_1} g_1^{-1}$ for $\alpha \in \Phi$. So it suffices to check that:

$$g_1 U_{\alpha, x_1} g_1^{-1} \subseteq \langle U_{\pi^{-1}(X_1)}, T_{0, X_1}, U_{0,\beta} \mid \beta \in \Phi_{x_1} \rangle$$

for each $\alpha \in \Phi$. To do so, fix a root $\alpha \in \Phi$ and an element $u \in g_1 U_{\alpha, x_1} g_1^{-1}$. If $u \in g_1 U'_{\alpha, x_1} g_1^{-1}$, then $u \in U_{\pi^{-1}(X_1)}$ by corollary 7.7. Otherwise, two cases arise:

- if $\alpha \in \Phi_{x_1}$, then corollary 7.7 implies that $U_{0,\alpha} = g_1 U_{\alpha, x_1} g_1^{-1}$, and hence $u \in U_{0,\alpha}$.
- if $\alpha \notin \Phi_{x_1}$, then, by proceeding as in the proof of axiom (RGD2), we have $2\alpha \in \Phi_{x_1}$ and $u \in U_{\pi^{-1}(X_1)} U_{0,2\alpha}$.

□

7.3 The valuation axioms for the fibers

Fix $\alpha \in \Phi_{x_1}$. Observe that, according to corollary 7.7, given $u \in (U_{X_1} \setminus U_{\pi^{-1}(X_1)}) \cap g_1 U_\alpha g_1^{-1}$ and $u' \in U_{\pi^{-1}(X_1)} \cap g_1 U_\alpha g_1^{-1}$, we have $\pi(\varphi_\alpha(u)) = -\alpha(x_1)$ and $\pi(\varphi_\alpha(u')) > -\alpha(x_1)$. Hence:

$$\varphi_\alpha(u) < \varphi_\alpha(u'),$$

so that:

$$\varphi_\alpha(uu') = \varphi_\alpha(u).$$

By choosing an element $\tilde{x}_1 \in \pi_{\text{aff}}(\{x_1\})$, we can therefore define the map:

$$\bar{\varphi}_\alpha : \bar{U}_{0,\alpha} \rightarrow \mathcal{R}^{\text{rk}(\Lambda_0)} \cup \{\infty\}$$

that sends the class in $\bar{U}_{0,\alpha}$ of an element $u \in U_{X_1} \cap g_1 U_\alpha g_1^{-1}$ to $\varphi_\alpha(g_1^{-1} u g_1) + \alpha(\tilde{x}_1)$ if $u \notin U_{\pi^{-1}(X_1)}$ and to ∞ otherwise.

7.10 Proposition. *The system $(\bar{T}_{0,X_1}, (\bar{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\bar{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\bar{\varphi}_\alpha)_{\alpha \in \Phi_{x_1}})$ is a valued generating root group datum in $U_{X_1}/U_{\pi^{-1}(X_1)}$.*

Proof. Observe that axiom (V4) is obviously satisfied. We will therefore use it freely to prove the other axioms.

Axiom (V0). By corollary 7.7, the image of $\bar{\varphi}_\alpha$ contains ∞ as well as $\{\lambda \in \Gamma_\alpha + \alpha(\tilde{x}_1) | \pi(\lambda) = 0\}$. Since $\alpha \in \Phi_{x_1}$, we can find $\gamma \in \Gamma_\alpha$ such that $\alpha(x_1) = \pi(\alpha(\tilde{x}_1)) = \pi(\gamma)$, and hence:

$$\{\lambda \in \Gamma_\alpha + \alpha(\tilde{x}_1) | \pi(\lambda) = 0\} = \{\mu - \gamma + \alpha(\tilde{x}_1) | \Gamma_\alpha \cap \mathcal{R}^{\text{rk}(\Lambda_0)}\}.$$

If α is non-multipliable, then the set $\Gamma_\alpha \cap \mathcal{R}^{\text{rk}(\Lambda_0)}$ contains Λ_0 and is hence infinite. If α is multipliable, then, by axiom (V4), the set $\text{Im}(\bar{\varphi}_\alpha)$ contains $\frac{1}{2}\text{Im}(\bar{\varphi}_{2\alpha})$, and is hence also infinite.

Axiom (V1). Fix $\lambda \in \mathcal{R}^{\text{rk}(\Lambda_0)} \subseteq \mathcal{R}^{\text{rk}(\Lambda)}$, and take $u, v \in U_{X_1} \cap g_1 U_\alpha g_1^{-1}$ such that:

$$\begin{aligned} \varphi_\alpha(g_1^{-1} u g_1) + \alpha(\tilde{x}_1) &\geq \lambda, \\ \varphi_\alpha(g_1^{-1} v g_1) + \alpha(\tilde{x}_1) &\geq \lambda. \end{aligned}$$

Then:

$$\begin{aligned} \varphi_\alpha(g_1^{-1} u v g_1) + \alpha(\tilde{x}_1) &\geq \min\{\varphi_\alpha(g_1^{-1} u g_1), \varphi_\alpha(g_1^{-1} v g_1)\} + \alpha(\tilde{x}_1) \geq \lambda, \\ \varphi_\alpha(g_1^{-1} u^{-1} g_1) + \alpha(\tilde{x}_1) &= \varphi_\alpha(g_1^{-1} u g_1) + \alpha(\tilde{x}_1) \geq \lambda. \end{aligned}$$

Hence $\bar{U}_{0,\alpha,\lambda} := \bar{\varphi}_\alpha^{-1}([\lambda, +\infty])$ is a subgroup of $\bar{U}_{0,\alpha}$. Moreover, $\bar{\varphi}_\alpha^{-1}(\{\infty\})$ is the trivial subgroup of $\bar{U}_{0,\alpha}$ by definition.

Axiom (V2). Let $\alpha \in \Phi_{x_1}$, $\bar{n} \in \bar{M}_{0,\alpha}$ and $\bar{u} \in \bar{U}_{0,-\alpha} \setminus \{1\}$. Denote by n (resp. u) a lifting of \bar{n} to $M_{0,\alpha}$ (resp. of \bar{u} to $U_{0,-\alpha}$). Since $\bar{u} \neq 1$, we have $u \notin U_{\pi^{-1}(\{X_1\})}$ and $nun^{-1} \notin U_{\pi^{-1}(\{X_1\})}$. Hence:

$$\bar{\varphi}_{-\alpha}(\bar{u}) - \bar{\varphi}_\alpha(\overline{nun^{-1}}) = \varphi_{-\alpha}(g_1^{-1} u g_1) - \varphi_\alpha(g_1^{-1} nun^{-1} g_1).$$

But the function:

$$u \in U_{0,\alpha} \mapsto \varphi_{-\alpha}(g_1^{-1} u g_1) - \varphi_\alpha(g_1^{-1} nun^{-1} g_1)$$

is constant by proposition 6.20.

Axiom (V3). Fix $\alpha, \beta \in \Phi_{x_1}$ and $\lambda, \mu \in \mathcal{R}^{\Lambda_0}$ such that $\beta \notin -\mathbb{R}_+\alpha$. By setting $U_{0,\alpha,\lambda} := g_1 U_{\alpha,\lambda-\alpha(\tilde{x}_1)} g_1^{-1}$ and $\overline{U}_{0,\alpha,\lambda} := \overline{\varphi}_\alpha^{-1}([\lambda, \infty])$, corollary 7.7 implies that:

$$\overline{U}_{0,\alpha,\lambda} = (U_{0,\alpha,\lambda}) / (g_1 U_\alpha g_1^{-1} \cap U_{\pi^{-1}(X_1)}) = U_{0,\alpha,\lambda} / (g_1 U'_{\alpha,x_1} g_1^{-1}).$$

By proposition 6.20, the group $[U_{0,\alpha,\lambda}, U_{0,\beta,\mu}]$ is contained in:

$$\langle U_{0,p\alpha+q\beta,p\lambda+q\mu} \mid p, q \in \mathbb{Z}_{>0}, p\alpha + q\beta \in \Phi \rangle.$$

Now take $p, q \in \mathbb{Z}_{>0}$ such that $\gamma := p\alpha + q\beta \in \Phi \setminus \Phi_{x_1}$ and fix an element:

$$u \in U_{0,\gamma,p\lambda+q\mu} \setminus (g_1 U'_{\gamma,x_1} g_1^{-1}) \subseteq g_1 (U_{\gamma,x_1} \setminus U'_{\gamma,x_1}) g_1^{-1}.$$

Since $\pi(\varphi_\gamma(g_1^{-1} u g_1)) = -\gamma(x_1)$ and $\gamma \notin \Phi_{x_1}$, we have:

$$U_{\gamma,x_1} \neq \bigcap_{u' \in U_{2\gamma}} U_{\gamma,\pi(\varphi_\gamma(g_1^{-1} u g_1 u'))}.$$

In other words, there exists $u' \in U_{2\gamma}$ such that:

$$-\gamma(x_1) = \pi(\varphi_\gamma(g_1^{-1} u g_1)) < \pi(\varphi_\gamma(g_1^{-1} u g_1 u')),$$

and we necessarily have:

$$\begin{aligned} \varphi_{2\gamma}(u') &= 2\varphi_\gamma(u') = 2\varphi_\gamma(g_1^{-1} u g_1) \geq 2(p\lambda + q\mu - \gamma(\tilde{x}_1)), \\ \pi(\varphi_{2\gamma}(u')) &= 2\pi(\varphi_\gamma(g_1^{-1} u g_1)) = -2\gamma(x_1). \end{aligned}$$

We deduce that $2\gamma \in \Phi_{x_1}$ and that:

$$g_1^{-1} u g_1 = (g_1^{-1} u g_1 u') \cdot u'^{-1}$$

with:

$$\begin{aligned} u g_1 u' g_1^{-1} &\in g_1 U'_{\gamma,x_1} g_1^{-1} = P_{\pi^{-1}(X_1)} \cap g_1 U_\gamma g_1^{-1}, \\ g_1 u'^{-1} g_1^{-1} &\in U_{0,2\gamma,2p\lambda+2q\mu}. \end{aligned}$$

The last equalities show that the image of u in $\overline{U}_{0,\gamma,p\lambda+q\mu}$ belongs to $\overline{U}_{0,2\gamma,2p\lambda+2q\mu}$, and hence the group $[\overline{U}_{0,\alpha,\lambda}, \overline{U}_{0,\beta,\mu}]$ is necessarily contained in:

$$\langle \overline{U}_{0,p\alpha+q\beta,p\lambda+q\mu} \mid p, q \in \mathbb{Z}_{>0}, p\alpha + q\beta \in \Phi_{x_1} \rangle.$$

Axiom (V5). Let $\alpha \in \Phi_{x_1}$, $\overline{u} \in \overline{U}_{0,\alpha}$ and $\overline{u}', \overline{u}'' \in \overline{U}_{0,-\alpha}$ such that $\overline{u}' \overline{u} \overline{u}'' \in \overline{M}_{0,\alpha}$. Let u be a lifting of \overline{u} in $U_{0,-\alpha}$. Note that $\overline{u} \neq 1$.

Assume first that α is non-multipliable. By proceeding as in the proof of the axiom (RGD4), there exists $y \in L_\alpha^\times$ such that:

$$\begin{aligned} \omega_1(y) &= -\alpha(x_1), \\ (g_1 x_\alpha(y) g_1^{-1}) u (g_1 x_\alpha(y) g_1^{-1}) &= m_{0,\alpha}(y). \end{aligned}$$

Since $g_1 x_\alpha(y) g_1^{-1} \in U_{0,\alpha}$ and $m_{0,\alpha}(y) \in M_{0,\alpha}$, uniqueness in paragraph 6.1.2.(2) of [BT84] implies that $g_1 x_\alpha(y) g_1^{-1}$ is a lifting of both \overline{u}' and \overline{u}'' . Hence:

$$\overline{\varphi}_{-\alpha}(\overline{u}) = \varphi_{-\alpha}(g_1^{-1} u g_1) - \alpha(\tilde{x}_1) = -\varphi_\alpha(x_\alpha(y)) - \alpha(\tilde{x}_1) = -\overline{\varphi}_\alpha(\overline{u}').$$

Assume now that α is multipliable. By proceeding as in the proof of the axiom (RGD4), there exists $(y, y') \in H_0(L_\alpha, L_{2\alpha})$ such that:

$$\begin{aligned} \omega_1(y') &= -2\alpha(x_1), \\ (g_1 x_\alpha(y y'^{-1}, (\tau y')^{-1}) g_1^{-1}) \cdot u \cdot (g_1 x_\alpha(y (\tau y')^{-1}, (\tau y')^{-1}) g_1^{-1}) &= m_{0,\alpha}(y, y'). \end{aligned}$$

Since $g_1 x_\alpha(y y'^{-1}, (\tau y')^{-1}) g_1^{-1}$ and $g_1 x_\alpha(y (\tau y')^{-1}, (\tau y')^{-1}) g_1^{-1}$ are both in $U_{0,\alpha}$ and $m_{0,\alpha}(y, y') \in M_{0,\alpha}$, uniqueness in paragraph 6.1.2.(2) of [BT84] implies that $g_1 x_\alpha(y y'^{-1}, (\tau y')^{-1}) g_1^{-1}$ is a lifting of \overline{u}' . Hence:

$$\overline{\varphi}_{-\alpha}(\overline{u}) = \varphi_{-\alpha}(g_1^{-1} u g_1) - \alpha(\tilde{x}_1) = -\varphi_\alpha(x_\alpha(y y'^{-1}, (\tau y')^{-1})) - \alpha(\tilde{x}_1) = -\overline{\varphi}_\alpha(\overline{u}').$$

□

7.4 Compatibility axioms for the N -action

7.11 Lemma. *The group \overline{N}_{0,X_1} is the subgroup of $U_{X_1}/U_{\pi^{-1}(X_1)}$ spanned by the $\overline{M}_{0,\alpha}$ for $\alpha \in \Phi_{x_1}$.*

Proof. Example 3.44 shows that the group N_{0,X_1} is spanned by the $g_1(N \cap L_{\alpha,x_1})g_1^{-1}$ for $\alpha \in \Phi$. We distinguish 3 cases:

- if $-\alpha(x_1) \notin \Gamma_\alpha$, then $g_1 L_{\alpha,x_1} g_1^{-1} \subseteq U_{\pi^{-1}(X_1)}$. Hence $g_1(N \cap L_{\alpha,x_1})g_1^{-1} \subseteq U_{\pi^{-1}(X_1)} \cap g_1 N g_1^{-1}$.
- if $-\alpha(x_1) \in \Gamma_\alpha \setminus \Gamma'_\alpha$, then, by proceeding as in the proof of the axiom (RGD2), one can see that $2\alpha \in \Phi_{x_1}$ and:

$$\begin{aligned} g_1(N \cap L_{\alpha,x_1})g_1^{-1} &\subseteq g_1(N \cap L_\alpha \cap U_{x_1})g_1^{-1} \subseteq g_1(T \cup M_\alpha)g_1^{-1} \cap U_{X_1} \\ &= g_1(T \cup M_{2\alpha})g_1^{-1} \cap U_{X_1} \subseteq T_{0,X_1} \cup M_{0,2\alpha}. \end{aligned}$$

- if $-\alpha(x_1) \in \Gamma'_\alpha$, then $\alpha \in \Phi_{x_1}$ and:

$$g_1(N \cap L_{\alpha,x_1})g_1^{-1} \subseteq g_1(N \cap L_\alpha \cap U_{x_1})g_1^{-1} \subseteq g_1(T \cup M_\alpha)g_1^{-1} \cap U_{X_1} = T_{0,X_1} \cup M_{0,\alpha}.$$

We deduce that N_{0,X_1} is the subgroup of U_{X_1} spanned by $U_{\pi^{-1}(X_1)} \cap g_1 N g_1^{-1}$ and the $M_{0,\alpha}$ for $\alpha \in \Phi_{x_1}$. Hence \overline{N}_{0,X_1} is the subgroup of $U_{X_1}/U_{\pi^{-1}(X_1)}$ spanned by the $\overline{M}_{0,\alpha}$ for $\alpha \in \Phi_{x_1}$. \square

Consider the $\mathfrak{R}^{\text{rk}(\Lambda_0)}$ -module $V_{x_1} = \ker(\pi) / (\langle \Phi_{x_1} \rangle^\perp \cap \ker(\pi))$, the affine space:

$$\mathbb{A}_{x_1} := \pi_{\text{aff}}^{-1}(x_1) / (\langle \Phi_{x_1} \rangle^\perp \cap \ker(\pi)),$$

and the projection:

$$\text{pr} : \pi_{\text{aff}}^{-1}(x_1) \rightarrow \mathbb{A}_{x_1}.$$

By the previous lemma, the image of $\overline{\nu}_{0,X_1}(\overline{N}_{0,X_1})$ in $GL(V)$ is $\overline{N}_{0,X_1}/\overline{T}_{0,X_1} \cong W(\Phi_{x_1}) \subseteq \text{Fix}(\langle \Phi_{x_1} \rangle^\perp)$. Hence $\overline{\nu}_{0,X_1}(\overline{N}_{0,X_1})$ induces a morphism:

$$\overline{\nu}_{X_1} : \overline{N}_{0,X_1} \rightarrow \text{Aff}(\mathbb{A}_{x_1}).$$

7.12 Proposition. *The action $\overline{\nu}_{0,X_1}$ is compatible with the valuation $(\overline{\varphi}_\alpha)_{\alpha \in \Phi_{x_1}}$.*

Proof. Axiom (CA1) is obvious. Let's prove axiom (CA2). To do so, let α be a root in Φ_{x_1} and take $\overline{u} \in \overline{U}_{0,\alpha} \setminus \{1\}$. We can find $y \in L_\alpha$ such that $u := g_1 x_\alpha(y) g_1^{-1}$ is a lifting of \overline{u} in $U_{0,\alpha}$ and $v_1(y) = -\alpha(x_1)$. We then have:

$$\begin{aligned} \alpha(\nu_{0,X_1}(m_{0,\alpha}(y))(\tilde{x}_1) - \tilde{x}_1) &= \alpha(\nu_{0,X_1}(m_{0,\alpha}(y))(o) - o) + \alpha(\nu_{0,X_1}^v(m_{0,\alpha}(y))(\tilde{x}_1 - o) - (\tilde{x}_1 - o)) \\ &= -2\varphi_\alpha(x_\alpha(y)) - 2\alpha(\tilde{x}_1) \\ &= -2\overline{\varphi}_\alpha(\overline{u}). \end{aligned}$$

The multipliable case is analogous. \square

7.5 Conclusion

7.13 Theorem. *The fiber $\pi^{-1}(X_1)$ is isomorphic to:*

$$\mathcal{I}(U_{X_1}/U_{\pi^{-1}(X_1)}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{\varphi}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, \overline{\nu}_{X_1}) \times \langle \Phi_{x_1} \rangle^\perp.$$

Proof. All the previous considerations show that the 5-tuple:

$$(U_{X_1}/U_{\pi^{-1}(X_1)}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{\varphi}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, \overline{\nu}_{X_1})$$

is a generating and valued root group datum together with a compatible action. We can therefore consider the building:

$$\mathcal{I}(U_{X_1}/U_{\pi^{-1}(X_1)}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{\varphi}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, \overline{\nu}_{X_1})$$

and it can be described as:

$$(U_{X_1}/U_{\pi^{-1}(X_1)} \times \mathbb{A}_{x_1})^\perp / \sim,$$

where:

$$(u, x) \sim (v, y) \Leftrightarrow \exists n \in \overline{N}_{0,X_1}, \begin{cases} y = \overline{\nu}_{X_1}(n)(x), \\ u^{-1}vn \in U_x. \end{cases}$$

Consider now the surjective map:

$$\text{pr} : \pi^{-1}(X_1) \rightarrow \mathcal{I}(U_{X_1}/U_{\pi^{-1}(X_1)}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{\varphi}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, \overline{\nu}_{X_1})$$

induced by the projection $\text{pr} : \pi_{\text{aff}}^{-1}(x_1) \rightarrow \mathbb{A}_{x_1}$. It is compatible with the U_{X_1} -action. Take a point:

$$X_0 := [(u_0, x_0)] \in \mathcal{I}(U_{X_1}/U_{\pi^{-1}(X_1)}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{\varphi}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, \overline{\nu}_{X_1}),$$

and fix a lifting \tilde{x}_0 of x_0 in $\pi_{\text{aff}}^{-1}(x_1)$. Observe that, if $X := [(u, x)] \in \text{pr}^{-1}(X_0)$, then there exists $n \in \overline{N}_{0,X_1}$ such that:

$$\begin{cases} \text{pr}(x) = \overline{\nu}_{X_1}(n)(x_0) \\ u_0^{-1}un \in \overline{U}_{0,\tilde{x}_0}. \end{cases}$$

Hence $x - \overline{\nu}_{0,X_1}(n)(\tilde{x}_0) \in \langle \Phi_{x_1} \rangle^\perp$, and we can find $y \in \langle \Phi_{x_1} \rangle^\perp$ such that $x - \overline{\nu}_{0,X_1}(n)(\tilde{x}_0) = \overline{\nu} \cdot y$ where $\overline{\nu}$ is the image of n in $W(\Phi_{x_1})$. Since $\alpha(\tilde{x}_0 + y) = \alpha(\tilde{x}_0) = \alpha(x_0)$ for each $\alpha \in \Phi_{x_1}$, we deduce that:

$$\begin{cases} x = \overline{\nu}_{0,X_1}(n)(\tilde{x}_0 + y) \\ u_0^{-1}un \in \overline{U}_{0,\tilde{x}_0} = \langle \overline{U}_{\alpha,\tilde{x}_0} \mid \alpha \in \Phi_{x_1} \rangle = \langle \overline{U}_{\alpha,\tilde{x}_0+y} \mid \alpha \in \Phi_{x_1} \rangle = \overline{U}_{\tilde{x}_0+y} \end{cases}$$

Hence $X = [(u_0, \tilde{x}_0 + y)]$, so that the map:

$$\begin{aligned} \psi_{X_0} : (\text{pr})^{-1}(x_0) &\rightarrow \text{pr}^{-1}(X_0) \\ x &\mapsto [(u_0, x)] \end{aligned}$$

is surjective. Now take $x, x' \in (\text{pr})^{-1}(x_0)$ such that $\psi_{X_0}(x) = \psi_{X_0}(x')$. We can then find $\overline{N}_{0,X_1} \cap \overline{U}_x$ such that $x' = \overline{\nu}_{0,X_1}(n)(x)$. Hence $x = x'$, and ψ_{X_0} is a bijection. \square

7.14 Remark. (i) A subset of $\pi^{-1}(X_1)$ is the intersection of an apartment of $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ with $\pi^{-1}(X_1)$ if, and only if, it is of the form $A_{x_1} \times \langle \Phi_{x_1} \rangle^\perp$ with A_{x_1} an apartment of $\mathcal{I}(U_{X_1}/U_{\pi^{-1}(X_1)}, (\overline{U}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{M}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, (\overline{\varphi}_{0,\alpha})_{\alpha \in \Phi_{x_1}}, \overline{\nu}_{X_1})$.

(ii) By considering all the intersections of an apartment of $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ with $\pi^{-1}(X_1)$, one endows the fiber $\pi^{-1}(X_1)$ with a system of apartments of type:

$$(\pi_{\text{aff}}^{-1}(X_1), \Phi, (\tilde{\Gamma}_\alpha)_{\alpha \in \Phi})$$

where $\tilde{\Gamma}_\alpha = \Gamma_\alpha$ if $\alpha \in \Phi_{x_1}$ and $\tilde{\Gamma}_\alpha = \emptyset$ otherwise. The fiber $\pi^{-1}(X_1)$ then automatically satisfies axioms (A2), (A3), (A4) and (GG).

7.15 Example. Let Λ_0 be a totally ordered abelian group and let k be a field endowed with a valuation $\omega_0 : k \rightarrow \Lambda_0 \cup \{\infty\}$. Set $\mathbb{K} := k((t))$, let ω_1 be the t -adic valuation on \mathbb{K} , and let $\omega : \mathbb{K} \rightarrow \mathbb{Z} \times \Lambda_0$ be the valuation defined by $\omega(x) = (\omega_1(x), \omega_0(xt^{-\omega_1(x)}))$.

Consider a reductive k -group \mathbf{G}_k and set $\mathbf{G} := \mathbf{G}_k \times_k \mathbb{K}$. One can find a finite extension \mathbb{K}' of \mathbb{K} over which ω_1 does not ramify and \mathbf{G} splits. Thus $0 \in \Gamma'_{k,\alpha}$ for every $\alpha \in \Phi$, and hence the point $X_1 := [(1, 0)] \in \mathcal{I}(\mathbb{K}, \omega_1, \mathbf{G})$ is a hyperspecial point.

Now take $u \in U_{X_1}$ and $x \in \pi_{\text{aff}}^{-1}(\{0\})$ such that $[(u, x)] \in \pi^{-1}(X_1)$. Since $u \in U_{X_1}$, we can find $\alpha_1, \dots, \alpha_n \in \Phi$ so that u can be written as $u_1 \dots u_n$ with $u_i \in U_{\alpha_i, 0}$ for each $i \in \{1, \dots, n\}$. Each u_i can then be written as $v_i u'_i$ with $v_i \in U_{k, \alpha_i}$ and $u'_i \in U'_{\alpha_i, 0}$. As a consequence, the product $u_1 \dots u_n$ can be written as $u' v_1 \dots v_n$ for some $u' \in U_{X_1}$ that fixes $\pi^{-1}(X_1)$. Hence:

$$[(u, x)] = [(u' v_1 \dots v_n, x)] = u' \cdot [(v_1 \dots v_n, x)] = [(v_1 \dots v_n, x)],$$

and the map:

$$\begin{aligned} \psi_{X_1} : G_k \times \pi_{\text{aff}}^{-1}(\{0\}) &\rightarrow \pi^{-1}(X_1) \\ (g, x) &\mapsto [(g, x)] \end{aligned}$$

is surjective. Now, if $\psi_{X_1}(g, x) = \psi_{X_1}(g', x')$ for some g, g', x, x' , then we can find $n \in N$ such that:

$$\begin{cases} x' = \nu(n)(x) \\ g' g^{-1} n \in U_x. \end{cases}$$

In particular:

$$n \in g g'^{-1} U_x \cap N \subseteq \mathbf{G}(k[[t]]) \cap N = N_k.$$

Hence, ψ_{X_1} induces a bijection:

$$\bar{\psi}_{X_1} : \mathcal{I}(\mathbf{G}_k, k, \omega_0) \rightarrow \pi^{-1}(X_1).$$

7.6 Further notations related to the projection maps

In this section, we fix some notations that will be used throughout the paper. Let's fix some $s_0 \in \text{rk}(\Lambda)$. Given a positive element λ_{s_0} in the archimedean class of Λ corresponding to s_0 , we define:

$$\begin{aligned} \Lambda_{\geq s_0} &:= \{\lambda \in \Lambda \mid \exists n > 0, \lambda < n\lambda_{s_0}\}, \\ \Lambda_{> s_0} &:= \{\lambda \in \Lambda \mid \forall n > 0, n\lambda < \lambda_{s_0}\}. \end{aligned}$$

The sets $\Lambda_{\geq s_0}$ and $\Lambda_{> s_0}$ are both convex subgroups of Λ and they do not depend on the choice of λ_{s_0} . We may therefore introduce the quotients:

$$\begin{aligned} \Lambda_{< s_0} &:= \Lambda / \Lambda_{\geq s_0}, \\ \Lambda_{\leq s_0} &:= \Lambda / \Lambda_{> s_0}. \end{aligned}$$

Denote by $\omega_{< s_0} : \mathbb{K} \rightarrow \Lambda_{< s_0} \cup \{\infty\}$ (resp. $\omega_{\leq s_0} : \mathbb{K} \rightarrow \Lambda_{\leq s_0} \cup \{\infty\}$) the composite of the valuation ω followed by the projection $\Lambda \rightarrow \Lambda_{< s_0}$ (resp. $\Lambda \rightarrow \Lambda_{\leq s_0}$). According to the previous section, we have three projection maps:

$$\begin{aligned} \pi_{\leq s_0} &: \mathcal{I}(\mathbb{K}, \omega, G) \rightarrow \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, G), \\ \pi_{< s_0}^{\leq s_0} &: \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, G) \rightarrow \mathcal{I}(\mathbb{K}, \omega_{< s_0}, G), \\ \pi_{< s_0} &= \pi_{< s_0}^{\leq s_0} \circ \pi_{\leq s_0} : \mathcal{I}(\mathbb{K}, \omega, G) \rightarrow \mathcal{I}(\mathbb{K}, \omega_{< s_0}, G). \end{aligned}$$

The map $\pi_{\leq s_0} : \mathcal{I}(\mathbb{K}, \omega, G) \rightarrow \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, G)$ then induces a surjection:

$$\pi_{=s_0, X} : \pi_{< s_0}^{-1}(X) \rightarrow (\pi_{< s_0}^{\leq s_0})^{-1}(X)$$

for each $X \in \mathcal{I}(\mathbb{K}, \omega_{< s_0}, G)$.

8 The axiom (CO)

In this section, we prove that the building \mathcal{I} that we constructed satisfies (CO). Let us sketch the ideas of our proof. We proceed as follows.

1. (see Subsection 8.1) We begin by giving a sufficient condition for an \mathbb{R} -building to satisfy (CO).
2. (see Subsection 8.2) We prove that when S admits a minimum, if C and \tilde{C} are two sectors opposed at $x \in \mathcal{I}$, then the corresponding sectors of $\pi_{\leq s}(\mathcal{I})$ are opposed at $\pi_{\leq s}(x)$.
3. (see Subsection 8.3) We still assume that S admits a minimum and we prove (CO). Let $x \in \mathcal{I}$ and C, \tilde{C} be two sectors opposed at x . Let A be the apartment containing the germs at infinity $C_\infty, \tilde{C}_\infty$ of C, \tilde{C} . We want to prove that $x \in A$. We consider an apartment A_C containing C and an isomorphism $\phi : A_C \rightarrow A$ fixing $A_C \cap A$. We then prove that $\phi(x) = x$. For this, we act by contradiction and assume that $\phi(x) \neq x$. Then we prove that there exists a minimal $s \in S$ such that $\pi_{\leq s}(\phi(x)) \neq \pi_{\leq s}(x)$. By working in the \mathbb{R} -building $\pi_{=s}(\pi_{\leq s}^{-1}(\pi_{\leq s}(x)))$ and by using steps (1) and (2), we reach a contradiction.
4. (see Subsection 8.4) We then prove (CO) in the general case, by considering $\mathbf{G}(\mathbb{K}((t)))$.

8.1 A sufficient condition for (CO) for \mathbb{R} -buildings

In this subsection, we assume that S is reduced to a single element. Let $\mathcal{I}_{\mathbb{R}}$ be a set covered with apartments and satisfying (A1), (A2), (GG), (A4) and

(Iwa) : for all local face F , for all sector-germ at infinity C_∞ , there exists an apartment containing F and C_∞ .

8.1 Lemma. *Let $x, y \in \mathcal{I}_{\mathbb{R}}$ and C_∞ be a sector-germ at infinity of $\mathcal{I}_{\mathbb{R}}$. Then there exists $n \in \mathbb{Z}_{\geq 0}$, $x_1, \dots, x_n \in [x, y]$ such that $[x, y] = \bigcup_{i=1}^{n-1} [x_i, x_{i+1}]$ and for all $i \in \llbracket 1, n-1 \rrbracket$, there exists an apartment A_i containing $[x_i, x_{i+1}]$ and C_∞ .*

Proof. This is a standard result. Let A be an apartment containing $[x, y]$. For $a \in [x, y]$ (resp. $a \in (x, y)$) we choose a local chamber C_a^+ (resp. C_a^-) of A such that $C_a^+ \cap [x, y] \ni \text{germ}_a([a, y])$ (resp. $C_a^- \cap [x, a] \ni \text{germ}_a([a, x])$). For $a \in (x, y)$ (resp. $a \in [x, y]$) we choose an apartment A_a^+ (resp. A_a^-) containing C_a^+ and C_∞ (resp. C_a^- and C_∞), which is possible by (Iwa). We choose a neighborhood V_a^+ (resp. V_a^-) of a in $[a, y]$ (resp. in $[x, a]$) such that $V_a^+ \subset A_a^+$ (resp. $V_a^- \subset A_a^-$). Set $V_x = V_x^+, V_y = V_y^-$ and $V_a = V_a^+ \cup V_a^-$, for $a \in (x, y)$. Then by compactness of $[x, y]$, there exists a finite subset $\{a_1, \dots, a_k\}$ of $[x, y]$ such that $[x, y] = \bigcup_{i=1}^k V_{a_i}$, and the result follows. \square

8.2 Lemma. *Let $C_\infty, \tilde{C}_\infty$ be two sector-germs at infinity of $\mathcal{I}_{\mathbb{R}}$. Let $x \in \mathcal{I}_{\mathbb{R}}$. We assume that $\text{germ}_x(x + C_\infty)$ and $\text{germ}_x(x + \tilde{C}_\infty)$ are opposite. Then:*

1. C_∞ and \tilde{C}_∞ are opposite,
2. there exists a unique apartment $A_{C_\infty, \tilde{C}_\infty}$ containing C_∞ and \tilde{C}_∞ ,
3. the point x belongs to $A_{C_\infty, \tilde{C}_\infty}$.

Proof. We follow [Par00a, Proposition 1.12] and [Rou11, Proposition 5.4 2)]. Let $A_{C_\infty, \tilde{C}_\infty}$ be an apartment containing C_∞ and \tilde{C}_∞ . Let $C = x + C_\infty, \tilde{C} = x + \tilde{C}_\infty, C_x = \text{germ}_x(C)$ and $\tilde{C}_x = \text{germ}_x(\tilde{C})$. Let A_C and $A_{\tilde{C}}$ be apartments containing C and \tilde{C} . Let A_{C_x, \tilde{C}_x} be an apartment containing C_x, \tilde{C}_x . Let $y, \tilde{y} \in A_{C_x, \tilde{C}_x}$ be such that $y \in C, \tilde{y} \in \tilde{C}$ and

$x \in [\tilde{y}, y]$. Let δ (resp $\tilde{\delta}$) be the ray of A_C (resp. $A_{\tilde{C}}$) based at x and containing y (resp. \tilde{y}).

The ray δ meets $A_{C_\infty, \tilde{C}_\infty}$. We choose y' in $\delta \cap A_{C_\infty, \tilde{C}_\infty}$. Let $r : [0, \infty[\rightarrow \delta$ be the affine parametrization of δ such that $r(0) = x$ and $r(1) = y'$. Using Lemma 8.1, we choose $n \in \mathbb{Z}_{\geq 0}$, $t_0 = 0 < t_1 < \dots < t_n = 1$ such that for all $i \in \llbracket 0, n-1 \rrbracket$, $[r(t_i), r(t_{i+1})]$ and \tilde{C}_∞ are contained in an apartment A_i .

Let $r : (-\infty, 0] \rightarrow \tilde{\delta}$ be the affine parametrization of \tilde{R} such that $r(0) = x$ and $r(-1) = \tilde{y}$. We set $t_{-1} = -1$. Then $\tilde{y} = r(t_{-1}) \in r(t_0) + \tilde{C}_\infty = x + \tilde{C}_\infty$. Let $i \in \llbracket 0, n-1 \rrbracket$. We assume that $r(t_i) + \tilde{C}_\infty$ contains $r(t_{i-1})$. Let B_i be an apartment containing $[r(t_i - \epsilon), r(t_i)] \cup [r(t_i), r(t_i + \epsilon)]$, for $\epsilon > 0$ small enough (one may take $B_0 = A_{C_x, \tilde{C}_x}$ and $B_i = A_C$, for $i > 0$). By assumption, A_i contains $r(t_{i-1})$ and as $[r(t_i - \epsilon), r(t_i + \epsilon)]$ is a segment in B_i , $[r(t_{i-1}), r(t_{i+1})]$ is a segment in A_i . In the apartment A_i , $r(t_i) + \tilde{C}_\infty$ is a sector parallel to $r(t_{i+1}) + \tilde{C}_\infty$. As $r(t_i) + \tilde{C}_\infty$ contains $r(t_{i-1})$, we deduce that $r(t_{i+1}) + \tilde{C}_\infty$ contains $r(t_i)$. By induction, we deduce that $r(t_0) = x \in r(t_n) + \tilde{C}_\infty = y' + \tilde{C}_\infty$. Therefore $x \in A_{C_\infty, \tilde{C}_\infty}$. Consequently, $x + C_\infty$ and $x + \tilde{C}_\infty$ are two opposite sectors of $A_{C_\infty, \tilde{C}_\infty}$. Therefore, C_∞ and \tilde{C}_∞ are opposite and by (A2), there exists at most one apartment containing C_∞ and \tilde{C}_∞ , which proves the lemma. \square

8.2 Preservation of the opposition

The aim of this subsection is to prove that if S admits a minimum and if C and \tilde{C} are two sectors opposite at some point $x \in \mathcal{I}$, then for every $s \in S$, $C_{\leq s}$ and $\tilde{C}_{\leq s}$ are opposite at $\pi_{\leq s}(x)$. For this, our idea is to prove that if $C_{1,\infty}, \dots, C_{n,\infty}$ is a gallery of sector-germs at infinity such that $\text{germ}_{\pi_{\leq s}(x)}(\pi_{\leq s}(x) + C_{1,\infty, \leq s}), \dots, \text{germ}_{\pi_{\leq s}(x)}(\pi_{\leq s}(x) + C_{n,\infty, \leq s})$ is a gallery from $\text{germ}_{\pi_{\leq s}(x)}(C)$ to $\text{germ}_{\pi_{\leq s}(x)}(\tilde{C})$, then $\text{germ}_x(x + C_{1,\infty}), \dots, \text{germ}_x(x + C_{n,\infty})$ is a gallery from $\text{germ}_x(C)$ to $\text{germ}_x(\tilde{C})$.

8.2.1 Preliminaries on enclosed sets

Let C be a sector of \mathbb{A}_S and $s \in S$. Write $C = x + w.C_{f, \mathbb{A}_S}^v$, where $x \in \mathbb{A}_S$ and $w \in W^v$. One sets $C_{\leq s} = x + w.\{y_{\leq s} \in \mathbb{A}_{\leq s} \mid \alpha(y_{\leq s}) > 0, \forall \alpha \in \Delta_f\}$. This is the sector of $\mathbb{A}_{\leq s}$ corresponding to C . One has $C_{\leq s} \subset \pi_{\leq s}(\mathbb{A}_S)$ but this containment is strict, because $\pi_{\leq s}$ preserves large inequalities but not strict inequalities.

Let C be a sector of \mathcal{I} . Write $C = g.\tilde{C}$, with $g \in G$ and \tilde{C} a subsector of \mathbb{A}_S . We set $C_{\leq s} = g.\tilde{C}_{\leq s}$. This does not depend on the choice of g and \tilde{C} by the lemma below.

8.3 Lemma. *Let $g_1, g_2 \in G$, $C^{(1)}, C^{(2)}$ be sectors of \mathbb{A}_S and $s \in S$. Suppose that $g_1.C^{(1)} = g_2.C^{(2)}$. Then $g_1.C_{\leq s}^{(1)} = g_2.C_{\leq s}^{(2)}$.*

Proof. For $i \in \{1, 2\}$, set $A_i = g_i.\mathbb{A}_S$. Let $h \in G$ inducing an isomorphism $\phi : A_1 \rightarrow A_2$ fixing $A_1 \cap A_2$. Then maybe considering hg_1 instead of g_1 , we may assume that $A_1 = A_2$. Let $n = g_2^{-1}.g_1 \in N$. Then $n.C^{(1)} = C^{(2)}$. As the restriction of n to \mathbb{A}_S is induced by an element of W^{aff} , we deduce that $n.C_{\leq s}^{(1)} = C_{\leq s}^{(2)}$. Therefore $g_1.C_{\leq s}^{(1)} = g_2.C_{\leq s}^{(2)}$, which proves the lemma. \square

8.4 Lemma. *Let $(\lambda_\alpha) \in (\mathfrak{R}^S \cup \{\infty\})^\Phi$ and $\Omega = \bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_\alpha}$. We assume that there exists a vector chamber C_S^v of \mathbb{A}_S and $x_{\leq s} \in \mathbb{A}_{\leq s}$ such that $\text{germ}_{x_{\leq s}}(x_{\leq s} + C_{\leq s}^v) \in \bigcap_{\alpha \in \Phi} D_{\alpha, \pi_{\leq s}(\lambda_\alpha)}$. Then $\pi_{\leq s}(\Omega) = \bigcap_{\alpha \in \Phi} D_{\alpha, \pi_{\leq s}(\lambda_\alpha)}$.*

Proof. Let $\Omega' = \bigcap_{\alpha \in \Phi} D_{\alpha, \pi_{\leq s}(\lambda_\alpha)}$. The inclusion $\pi_{\leq s}(\Omega) \subset \Omega'$ is clear. Let us prove the reverse inclusion. Let $\Phi^+ = \Phi_{C_{\mathbb{R}}^v}^+$.

Let $z'_{\leq s} \in \Omega' \cap (x_{\leq s} + C_{\leq s}^v)$ and $z_{\leq s} \in (x_{\leq s} + C_{\leq s}^v) \cap (z'_{\leq s} - C_{\leq s}^v)$. Then for all $\alpha \in \Phi^+$, one has $-\pi_{\leq s}(\lambda_\alpha) \leq \alpha(x_{\leq s}) < \alpha(z_{\leq s}) < \alpha(z'_{\leq s})$ and for all $\alpha \in \Phi^-$, $-\pi_{\leq s}(\lambda_\alpha) \leq \alpha(z'_{\leq s}) < \alpha(z_{\leq s})$. In particular,

$$\alpha(z_{\leq s}) > -\pi_{\leq s}(\lambda_\alpha),$$

for every $\alpha \in \Phi$.

Let $y_{\leq s} \in \Omega'$. Set $\Phi(y_{\leq s}) = \{\alpha \in \Phi \mid \alpha(y_{\leq s}) = -\pi_{\leq s}(\lambda_\alpha)\}$. Then by definition,

$$\Omega \cap (\{y_{\leq s}\} \times \mathbb{A}_{>s}) = \{y_{\leq s}\} \times \bigcap_{\alpha \in \Phi(y_{\leq s})} D_{\alpha, \pi_{>s}(\lambda_\alpha)}.$$

If $\Phi(y_{\leq s})$ is empty, then $\{y_{\leq s}\} \times \mathbb{A}_{>s} \subset \Omega$ and thus $y_{\leq s} \in \pi_{\leq s}(\Omega)$. We now assume that $\Phi(y_{\leq s})$ is nonempty. By choice of $z_{\leq s}$, one has $\alpha(z_{\leq s} - y_{\leq s}) > 0$, for all $\alpha \in \Phi(y_{\leq s})$. Write $y_{\leq s} = (y_t)_{t \leq s}$ and $z_{\leq s} = (z_t)_{t \leq s}$. For $\alpha \in \Phi(y_{\leq s})$, set

$$s_\alpha = \min\{t \in \text{supp}(z_{\leq s} - y_{\leq s}) \mid \alpha(z_t) > \alpha(y_t)\}.$$

Let $k = |\Phi(y_{\leq s})|$. We write $\Phi(y_{\leq s}) = \{\alpha_1, \dots, \alpha_k\}$ in such a way that $(s_{\alpha_i})_{i \in \llbracket 1, k \rrbracket}$ is increasing. For $i \in \llbracket 1, k \rrbracket$, one sets $u_i = z_{s_{\alpha_i}} - y_{s_{\alpha_i}} \in \mathbb{A}_{\mathbb{R}}$. Then for all $j \in \llbracket 1, k \rrbracket$, one has $\alpha_j(u_i) > 0$ if $s_{\alpha_i} = s_{\alpha_j}$ and $\alpha_j(u_i) = 0$ if $s_{\alpha_j} > s_{\alpha_i}$. In particular, for all $i, j \in \llbracket 1, k \rrbracket$, one has:

$$\alpha_i(u_i) > 0 \text{ and } j \geq i \implies \alpha_j(u_i) \geq 0. \quad (19)$$

If $(y_{\leq s}, 0) \in \Omega$, then we are done. Suppose that $(y_{\leq s}, 0) \notin \Omega$. Let

$$\tilde{s} = \min\{t \in \text{supp}(\lambda_\alpha) \mid \exists \alpha \in \Phi(y_{\leq s}) \mid -\pi_{=t}(\lambda_\alpha) > 0\}.$$

One chooses $t_k \in \mathbb{R}$ such that $\alpha_k(t_k u_k) > -\pi_{=\tilde{s}}(\lambda_{\alpha_k})$. Let $i \in \llbracket 2, k \rrbracket$. Suppose we have constructed $(t_j)_{j \in \llbracket i, k \rrbracket}$ such that

$$\sum_{j=i}^k t_j \alpha_\ell(u_j) > -\pi_{=\tilde{s}}(\lambda_{\alpha_\ell})$$

for all $\ell \in \llbracket i, k \rrbracket$. Then one chooses $t_{i-1} \in \mathbb{R}_{\geq 0}$ such that $\sum_{j=i-1}^k t_j \alpha_{i-1}(u_j) > -\pi_{=\tilde{s}}(\lambda_{\alpha_{i-1}})$. Then by (19), one has $\sum_{j=i-1}^k \alpha_\ell(t_j u_j) > -\pi_{=\tilde{s}}(\lambda_{\alpha_\ell})$ for all $\ell \in \llbracket i-1, k \rrbracket$. By induction, we thus find $(t_1, \dots, t_k) \in \mathbb{R}$ such that $\sum_{j=1}^k \alpha_\ell(t_j u_j) > -\pi_{=\tilde{s}}(\lambda_{\alpha_\ell})$ for all $\ell \in \llbracket 1, k \rrbracket$.

Let $y_{<\tilde{s}} = (a_t y_t) \in \mathbb{A}_{<\tilde{s}}$, where $a_t = 1$ if $t \leq s$ and $a_t = 0$ if $t \geq s$. Let

$$y = (y_{<\tilde{s}}, \sum_{i=1}^k t_i u_i, 0) \in \mathbb{A}_S.$$

Let $\alpha \in \Phi(y_{\leq s})$. Then:

$$\alpha(y) = \left(-\pi_{<\tilde{s}}(\lambda_\alpha), \sum_{j=i}^k t_j \alpha(u_j), 0 \right) > -\lambda_{\alpha_\ell}.$$

Therefore, $y \in \Omega$. Moreover $\pi_{\leq s}(y) = y_{\leq s}$ and thus $\pi_{\leq s}(\Omega) = \Omega'$. □

8.5 Remark. The above lemma is not true when Ω is an arbitrary enclosed subset of \mathbb{A}_S . For example if $S = \{1, 2\}$, $\mathbb{A}_{\mathbb{R}} = \mathbb{R}$ and $\alpha = \text{Id}$, one has $D_{\alpha, (0, -1)} \cap D_{-\alpha, (0, 1)} = \emptyset$ but $D_{\alpha, 0} \cap D_{-\alpha, 0}$ is nonempty.

8.2.2 Projection of an intersection of apartments

8.6 Lemma. *Let $s \in S$. Then the map $A \mapsto \pi_{\leq s}(A)$ from the set of apartments of \mathcal{I} to the set of apartments of $\pi_{\leq s}(\mathcal{I})$ is a bijection.*

Proof. This map is surjective by definition. Let A and B be two apartments such that $\pi_{\leq s}(A) = \pi_{\leq s}(B)$. As the action of $G(\mathbb{K})$ commutes with $\pi_{\leq s}$, we may assume that $A = \mathbb{A}_S$. Let $g \in G(\mathbb{K})$ be such that $B = g.\mathbb{A}_S$. Then $\pi_{\leq s}(g.\mathbb{A}_S) = \pi_{\leq s}(B) = \pi_{\leq s}(\mathbb{A}_S) = g.\pi_{\leq s}(\mathbb{A}_S)$.

By Proposition 5.6, there exists $n \in N$ such that $g.x = n.x$ for all $x \in \mathbb{A}_{\leq s}$. Then by Lemma 5.5 $g^{-1}.n \in \widehat{P}_{\mathbb{A}_{\leq s}}$.

By Corollary 4.12, if Δ is a basis of Φ , then:

$$\widehat{P}_{\mathbb{A}_{\leq s}} = (U_{\mathbb{A}_{\leq s}} \cap U_{\Delta}^+)(U_{\mathbb{A}_{\leq s}} \cap U_{\Delta}^-)\widehat{N}_{\mathbb{A}_{\leq s}}.$$

By Example 3.44, $U_{\mathbb{A}_{\leq s}} \cap U_{\Delta}^+ = U_{\mathbb{A}_{\leq s}} \cap U_{\Delta}^- = \{1\}$, thus $g^{-1}.n \in N$, hence $g \in N$ and $B = \mathbb{A}_S$, which proves the lemma. \square

8.7 Lemma. *Let A, B be two apartments and $s \in S$. We assume that there exists a sector C based at some $x \in \mathcal{I}$ such that $\pi_{\leq s}(A) \cap \pi_{\leq s}(B) \ni \text{germ}_{\pi_{\leq s}(x)}(C_{\leq s})$. Then $\pi_{\leq s}(A \cap B) = \pi_{\leq s}(A) \cap \pi_{\leq s}(B)$.*

Proof. Using some $g \in G(\mathbb{K})$, we may assume that $A = \mathbb{A}_S$ and $C = x + \text{germ}_{\infty}(C_f^v)$. Let $\Omega_{\leq s} = \pi_{\leq s}(\mathbb{A}_S) \cap \pi_{\leq s}(B)$. One has $\Omega_{\leq s} \supset \pi_{\leq s}(\mathbb{A}_S \cap B)$. Let us prove the reverse inclusion.

By Corollary 5.8, there exists $u \in U_{\Omega_{\leq s}}$ such that $u.\pi_{\leq s}(\mathbb{A}_S) = \pi_{\leq s}(B)$. By Lemma 8.6, $u.\mathbb{A}_S = B$. By definition of U_{Ω} , there exist $k \in \mathbb{Z}_{\geq 0}$, $\alpha_1, \dots, \alpha_k \in \Phi$ and elements $u_i \in U_{\alpha_i, \Omega_{\leq s}}$ such that $u = \prod_{i=1}^k u_i$. For $\alpha \in \Phi$, set $\lambda_{\alpha} = \min\{\varphi_{\alpha_i}(u_i) | i \in \llbracket 1, k \rrbracket \text{ and } \alpha_i = \alpha\}$ (one may have $\lambda_{\alpha} = \infty$). Then $\Omega_{\leq s} \subset \bigcap_{\alpha \in \Phi} D_{\alpha, \pi_{\leq s}(\lambda_{\alpha})}$. As $\Omega_{\leq s}$ contains a local chamber, we can apply Lemma 8.4 and one has $\pi_{\leq s}(\bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_{\alpha}}) = \bigcap_{\alpha \in \Phi} D_{\alpha, \pi_{\leq s}(\lambda_{\alpha})}$. Moreover u fixes $\bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_{\alpha}}$ and thus $\mathbb{A}_S \cap B$ contains $\bigcap_{\alpha \in \Phi} D_{\alpha, \lambda_{\alpha}}$. Thus

$$\pi_{\leq s}(\mathbb{A}_S \cap B) \supset \bigcap_{\alpha \in \Phi} D_{\alpha, \pi_{\leq s}(\lambda_{\alpha})} \supset \Omega_{\leq s}.$$

Therefore $\Omega_{\leq s} = \pi_{\leq s}(\mathbb{A}_S \cap B)$, which proves the lemma. \square

8.8 Remark. If we already knew that \mathcal{I} is a building, we could use [SS12, Lemma 3.7 and 3.10], but their proof uses the fact that retractions are 1-Lipschitz continuous, which we want to prove.

8.2.3 The exchange condition

We now prove that \mathcal{I} satisfies the exchange condition (EC) (see Lemma 8.10 or [BS14, Section 2] for the definition of (EC)). We will use it to prove that \mathcal{I} satisfies a property called the sundial configuration (SC) in [BS14, Section 2] (see Lemma 8.13).

8.9 Lemma. *Let A be an apartment of \mathcal{I} . Let D be a half-space of \mathbb{A}_S . Suppose that $A \cap \mathbb{A}_S$ contains D . Then either $A = \mathbb{A}_S$ or there exists $\alpha \in \Phi$ and $\lambda \in \Gamma_{\alpha}$ such that $A \cap \mathbb{A}_S = D_{\alpha, \lambda}$.*

Proof. Using (A2) we write $A \cap \mathbb{A}_S = \bigcap_{\alpha \in \Phi} D_{\alpha, \ell_{\alpha}}$, where $(\ell_{\alpha}) \in \prod_{\alpha \in \Phi} (\Gamma_{\alpha} \cup \{\infty\})$. Let $\alpha \in \Phi$. If the hyperplane defining D is not parallel to $\alpha^{-1}(\{0\})$, then $\lambda_{\alpha} = \infty$. Lemma follows. \square

8.10 Lemma. *The set \mathcal{I} satisfies the exchange condition (EC): if A and B are apartments of \mathcal{I} such that $A \cap B$ is a half-apartment, then $(A \cup B) \setminus (A \cap B) \cup M$ is an apartment of \mathcal{I} , where M is the wall of $A \cap B$.*

Proof. Using isomorphisms of apartments, we may assume that $A = \mathbb{A}_S$. Then by Lemma 8.9, there exists $\alpha \in \Phi$ and $\lambda \in \Gamma_\alpha$ such that $D := \mathbb{A}_S \cap B = D_{\alpha,\lambda}$. By Corollary 5.8, one has $B = u \cdot \mathbb{A}_S$, where $u \in U_D$. By Example 3.44, we can write $u = u_D^+ u_D^- n_D$, where $u_D^+ \in U_D^+$, $u_D^- \in U_D^-$ and $n_D \in N$. Moreover, either $U_D^+ = \{1\}$ or $U_D^- = \{1\}$ (depending on whether $\alpha \in \Phi_+$ or $\alpha \in \Phi_-$). By symmetry, we may assume that $U_D^+ = \{1\}$. Then $U_D^- = U_{\alpha,D}$.

One has $\lambda = \varphi_\alpha(u_D^-)$. Using (RGD4), we write $u_D^- = v_+ m u_+^{-1}$, with $u_+, v_+ \in U^+$ and $m \in M_\alpha$. By (V5) and Lemma 3.7, we have $\lambda = -\varphi_{-\alpha}(u_+) = -\varphi_{-\alpha}(v_+) = \varphi_\alpha(u_D^-)$. Let $D_- = D_{\alpha,\lambda}$, $D_+ = D_{-\alpha,-\lambda}$ and $M = H_{\alpha,\lambda} = D_- \cap D_+$. Then one has:

$$u_D^- u_+ \cdot D_- = v_+ m \cdot D_- = v_+ \cdot D_+ = D_+ \text{ and } u_D^- u_+ \cdot D_+ = u_D^- \cdot D_+ = v_+ m \cdot D_+ = v_+ \cdot D_-.$$

Therefore:

$$(u_D^- \cdot \mathbb{A}_S \cup \mathbb{A}_S) \setminus (\mathbb{A}_S \cap u_D^- \cdot \mathbb{A}_S) \cup M = u_D^- \cdot D_+ \cup D_+ = v_+ \cdot D_+ \cup D_+ = v_+ \cdot \mathbb{A}_S,$$

which is an apartment of \mathbb{A}_S , which proves the lemma. \square

8.2.4 Germ of a gallery of local chambers and conclusion

8.11 Lemma. *Let $C_\infty, \tilde{C}_\infty$ be two sector-germs at infinity and $x \in \mathcal{I}$. Let $C = x + C_\infty$ and $\tilde{C} = x + \tilde{C}_\infty$. We assume that there exists $s \in S$ such that $\text{germ}_{\pi_{\leq s}(x)}(C_{\leq s}) = \text{germ}_{\pi_{\leq s}}(\tilde{C}_{\leq s})$. Then $\text{germ}_x(C) = \text{germ}_x(\tilde{C})$.*

Proof. Let A be an apartment containing C . Then $\pi_{\leq s}(A) \ni \text{germ}_{\pi_{\leq s}(x)}(C_{\leq s}) = \text{germ}_{\pi_{\leq s}(x)}(\tilde{C}_{\leq s})$. Let \tilde{A} be an apartment containing \tilde{C} . By Lemma 8.7, $\pi_{\leq s}(A \cap \tilde{A}) = \pi_{\leq s}(A) \cap \pi_{\leq s}(\tilde{A})$. Let $\tilde{y}_{\leq s} \in \tilde{C}_{\leq s} \cap \pi_{\leq s}(A)$. Then there exists $\tilde{y} \in A \cap \tilde{A}$ such that $\pi_{\leq s}(\tilde{y}) = \tilde{y}_{\leq s}$. As $\tilde{y} \in A$ we have $\tilde{y} \in C$ and as $\tilde{y} \in \tilde{A}$, we have $\tilde{y} \in \tilde{C}$. Using Lemma 2.17, we deduce that $C \cap \tilde{C} \ni \text{cl}(\{x, y\}) \ni \text{germ}_x(C), \text{germ}_x(\tilde{C})$ and thus $\text{germ}_x(C) = \text{germ}_x(\tilde{C})$. \square

Two sector-germs $Q_{1,\infty}$ and $Q_{2,\infty}$ are said to be **adjacent** if there exists an apartment A containing $Q_{1,\infty}$ and $Q_{2,\infty}$ and such that $x + Q_{1,\infty}$ and $x + Q_{2,\infty}$ are adjacent, for any $x \in A$. If such an A exists, then any apartment containing $Q_{1,\infty}$ and $Q_{2,\infty}$ satisfies this property.

We now assume that S admits a minimum. We denote it 0_S . We denote $\pi_{=0_S}$ instead of $\pi_{\leq 0_S}$ and if C is a sector of \mathcal{I} , we denote $C_{=0_S}$ instead of $C_{\leq 0_S}$.

8.12 Lemma. *Let C be a sector of \mathcal{I} . Then $\text{germ}_\infty(C_{=0_S}) = \pi_{=0_S}(\text{germ}_\infty(C))$. In particular the sector-germs at infinity of $\pi_{=0_S}(\mathcal{I})$ are exactly the $\pi_{=0_S}(C_\infty)$ such that C_∞ is a sector-germ at infinity of \mathcal{I} .*

Proof. As $\pi_{=0_S}$, germ_∞ and $C \mapsto C_{=0_S}$ commute with the action of G , it suffices to check it when C is a subsector of $C_{f,S}^v$, which is straightforward. \square

8.13 Lemma. *Let A be an apartment of \mathcal{I} and C_∞ be a sector-germ of A . Let \tilde{C}_∞ be a sector-germ of \mathcal{I} adjacent to C_∞ and different from it. Then one can write $\pi_{=0_S}(A) = D_{1,0_S} \cup D_{2,0_S}$, where for both $i \in \{1, 2\}$, $D_{i,0_S}$ is a half-apartment of $\pi_{=0_S}(A)$ and there exists an apartment $\pi_{0_S}(A_i)$ containing $D_{i,0_S}$ and $\pi_{=0_S}(\tilde{C}_\infty)$.*

Proof. By Lemma 5.10, Proposition 5.6, Lemma 8.2 and by [BS14, Theorem 3.3], $\pi_{=0_S}(\mathcal{I})$ is an \mathbb{R} -building and in particular, it satisfies the axiom (SC) of [BS14]. Moreover $\pi_{=0_S}(C_\infty)$ and $\pi_{=0_S}(\tilde{C}_\infty)$ are adjacent and thus the lemma is a consequence of the paragraph after (SC) in [BS14, page 385]. \square

Galleries of local chambers Let $x \in \mathcal{I}$. Two local chambers $C_{1,x}, C_{2,x}$ are called **adjacent** if there exists an apartment A containing $C_{1,x}, C_{2,x}$ and two adjacent sectors Q_1, Q_2 of A based at x such that $\text{germ}_x(Q_1) = C_{1,x}$ and $\text{germ}_x(Q_2) = C_{2,x}$. By (A2), this does not depend on the choice of apartment A . A **gallery of local chambers based at x** is a finite sequence $\Gamma_x = (C_{1,x}, \dots, C_{k,x})$ such that for all $i \in \llbracket 1, k-1 \rrbracket$, $C_{i,x}$ and $C_{i+1,x}$ are adjacent local chambers based at x . The **length** of Γ_x is then k . The gallery is called **minimal** if k is the minimal possible length for a gallery joining $C_{1,x}$ and $C_{k,x}$.

If C_x, \tilde{C}_x are two local chambers based at x , there exists a gallery Γ_x joining C_x to \tilde{C}_x . Indeed, by (GG), there exists an apartment A containing C_x and \tilde{C}_x . Let Q and \tilde{Q} be the sectors of A corresponding to C_x and \tilde{C}_x . Then if Γ is a gallery of sectors from Q to \tilde{Q} , then the germ Γ_x of Γ at x is a gallery joining C_x to \tilde{C}_x . The minimal length of a gallery joining C_x to \tilde{C}_x is called the **distance between C_x and \tilde{C}_x** and we denote it $d(C_x, \tilde{C}_x)$.

8.14 Lemma. *Let A be an apartment and C_∞ be a sector-germ at infinity of \mathcal{I} . We assume that $\pi_{=0_S}(A) \ni \pi_{=0_S}(C_\infty)$. Then $A \ni C_\infty$.*

Proof. Let $\Omega \in C_\infty$ be such that $\pi_{0_S}(A) \supset \pi_{=0_S}(\Omega)$. Let B be an apartment containing C_∞ . Then $\Omega \cap B \in C_\infty$. Moreover $\pi_{=0_S}(A) \cap \pi_{=0_S}(B) \supset \pi_{=0_S}(\Omega \cap B)$ and by Lemma 8.7, $\pi_{=0_S}(A \cap B) = \pi_{=0_S}(A) \cap \pi_{=0_S}(B)$. We identify B and \mathbb{A}_S and we assume that C_∞ is the germ of $C_{f,S}^v$. We choose $(y_{n,=0_S}) \in \pi_{=0_S}(A \cap \mathbb{A}_S)_{\geq 0}^{\mathbb{Z}}$ such that $\beta(y_{n,=0_S}) \rightarrow +\infty$ for all $\beta \in \Delta_f$. Let $(y_n) \in (A \cap \mathbb{A}_S)_{\geq 0}^{\mathbb{Z}}$ be such that $\pi_{=0_S}(y_n) = y_{n,=0_S}$ for all $n \in \mathbb{Z}_{\geq 0}$. Then $A \supset A \cap B \ni \text{cl}(\{y_n | n \in \mathbb{Z}_{\geq 0}\}) \ni C_\infty$, which proves the lemma. \square

8.15 Lemma. *1. Let C and \tilde{C} be two sectors of \mathcal{I} and $C_\infty, \tilde{C}_\infty$ be their germs. We assume that C_∞ and \tilde{C}_∞ are adjacent. Let A be an apartment containing C_∞ . Then one can write $A = D_1 \cup D_2$, where D_1, D_2 are half-apartments of A which have the same wall and such that for both $i \in \{1, 2\}$, there exists an apartment containing D_i and \tilde{C}_∞ .*

2. Let $y \in \mathcal{I}$. Then $\text{germ}_y(y + C_\infty)$ and $\text{germ}_y(y + \tilde{C}_\infty)$ are adjacent.

Proof. Let $\pi = \pi_{=0_S}$. By Lemma 8.13, one can write $\pi(A) = D_{1,\pi} \cup D_{2,\pi}$, where $D_{1,\pi}$ and $\pi(\tilde{C}_\infty)$ are contained in an apartment $\pi(A_1)$. Then $\pi(A) \cap \pi(A_1)$ is a half-apartment and thus by Lemma 8.7 and (A2), $A \cap A_1$ is a half-apartment. Let H be the wall of $A \cap A_1$. Then by (EC) (Lemma 8.10), $(A \cup A_1) \setminus (A \cap A_1) \sqcup H$ is an apartment A_2 of \mathcal{I} . By Lemma 8.14, $A_1 \ni \tilde{C}_\infty$. Let $D_1 = A \cap A_1$ and $D_2 = A_2 \cap A$. Then (A, D_1, D_2) satisfy the condition of (1).

Let now $y \in \mathcal{I}$. We want to prove that $\text{germ}_y(y + C_\infty)$ and $\text{germ}_y(y + \tilde{C}_\infty)$ are adjacent. As there exists an apartment containing y and C_∞ , there is no loss of generality in assuming that $y \in A$.

Maybe exchanging the roles of D_1 and D_2 , we may assume that $D_1 \ni C_\infty$. Suppose $y \in D_1$. Then A_1 contains $y + C_\infty$ and $y + \tilde{C}_\infty$ and thus $\text{germ}_y(y + C_\infty)$ and $\text{germ}_y(y + \tilde{C}_\infty)$ are adjacent (and distinct).

By construction, H is also the wall of $A \cap A_2$. Let $x \in H$, $C = x + C_\infty$ and $\tilde{C} = x + \tilde{C}_\infty$. Then $x + C_\infty \subset D_1$ and $x + \tilde{C}_\infty \not\subset A$. Therefore H separates $x + C_\infty$ and $x + \tilde{C}_\infty$ in A_1 .

Let $\phi : A \rightarrow A_2$ be the isomorphism of apartments fixing $A \cap A_2$. Then ϕ induces a map (still denoted ϕ) from the set of sector-germs at infinity of A to the set of sector-germs at infinity of A_2 . If $x \in H$, $x + C_\infty$ and $x + \tilde{C}_\infty$ share a panel in H . Let $C'_\infty \neq C_\infty$ be the sector-germ at infinity of A such that $x + C_\infty$ and $x + C'_\infty$ share a panel in H . Then $C'_\infty \in D_2$ and thus $\phi(C'_\infty) = C'_\infty$. Therefore $\phi(C_\infty) = \tilde{C}_\infty$ and thus for all $y \in A$, $\phi(y + C_\infty) = \phi(y) + \tilde{C}_\infty$.

Let now $y \in A \setminus D_1$. Let $\epsilon \in \mathfrak{R}^S$ be such that $B(y, \epsilon) \subset D_2$, which exists by Lemma 2.21 (2). Then $\phi(B(y, \epsilon) \cap (y + C_\infty)) = B(y, \epsilon) \cap (y + C_\infty) = B(y, \epsilon) \cap (y + \tilde{C}_\infty)$. Therefore $\text{germ}_y(y + C_\infty) = \text{germ}_y(y + \tilde{C}_\infty)$ and in particular $\text{germ}_y(y + C_\infty)$ and $\text{germ}_y(y + \tilde{C}_\infty)$ are adjacent. Lemma follows. \square

Using Lemma 8.15 (2) we deduce the following proposition.

8.16 Proposition. *Let $\Gamma = (C_{1,\infty}, \dots, C_{n,\infty})$ be a gallery of sector-germs at infinity. Then for all $x \in \mathcal{I}$, $\Gamma_x = (\text{germ}_x(x + C_{1,\infty}), \dots, \text{germ}_x(x + C_{n,\infty}))$ is a gallery.*

8.17 Lemma. *Let $x \in \mathcal{I}$. Let C_x, \tilde{C}_x be two local chambers based at x . Let w_0 be the longest element of W^v . Then:*

1. $d(C_x, \tilde{C}_x) \leq \ell(w_0)$ and
2. $d(C_x, \tilde{C}_x) = \ell(w_0)$ if and only if C_x and \tilde{C}_x are opposite.

Proof. Let A be an apartment such that $\pi_{\leq s}(A)$ contains C_x and \tilde{C}_x . Using an isomorphism of apartments, we identify A and \mathbb{A}_S . Let C_S^v and \tilde{C}_S^v be the vector chambers of \mathbb{A}_S such that $C_x = \text{germ}_x(x + C_S^v)$ and $\tilde{C}_x = \text{germ}_x(x + \tilde{C}_S^v)$. Write $\tilde{C}_S^v = w \cdot C_S^v$, with $w \in W^v$. By [Bro89, I Proposition 4], one has $\ell(w) = d(C_S^v, \tilde{C}_S^v) = d(C_x, \tilde{C}_x)$. By definition of w_0 we deduce that $d(C_x, \tilde{C}_x) \leq \ell(w_0)$. Point 2 is a consequence of the uniqueness of w_0 , see [BB05, Proposition 2.2.9] for example. \square

8.18 Lemma. *Let $x \in \mathcal{I}$ and C, \tilde{C} be two sectors opposite at x . Let $s \in S$. Then $C_{\leq s}$ and $\tilde{C}_{\leq s}$ are opposite at $\pi_{\leq s}(x)$.*

Proof. Let A be an apartment containing $\text{germ}_{\pi_{\leq s}(x)}(C_{\leq s})$ and $\text{germ}_{\pi_{\leq s}(x)}(\tilde{C}_{\leq s})$. Let Q and \tilde{Q} be the sectors of A based at $\pi_{\leq s}(x)$ and such that $Q_{\leq s}$ and $\tilde{Q}_{\leq s}$ contain $\text{germ}_{\pi_{\leq s}(x)}(C_{\leq s})$ and $\text{germ}_{\pi_{\leq s}(x)}(\tilde{C}_{\leq s})$. Let $\Gamma = (Q_{1,\infty}, Q_{2,\infty}, \dots, Q_{k,\infty})$ be a minimal gallery of sector-germs at infinity of A from $\text{germ}_\infty(Q)$ to $\text{germ}_\infty(\tilde{Q})$. Then by Proposition 8.16, $\Gamma_x = (\text{germ}_x(x + Q_{1,\infty}), \dots, \text{germ}_x(x + Q_{k,\infty}))$ is a gallery. By Lemma 8.11, $\text{germ}_x(x + Q_{1,\infty}) = \text{germ}_x(C)$ and $\text{germ}_x(x + Q_{k,\infty}) = \text{germ}_x(\tilde{C})$. As $\text{germ}_x(C)$ and $\text{germ}_x(\tilde{C})$ are opposite, we have $k \geq \ell(w_0)$. As Γ is minimal, one has $k \leq \ell(w_0)$. Consequently, $k = \ell(w_0)$ and thus by Lemma 8.17, $\text{germ}_{\pi_{\leq s}(x)}(C_{\leq s})$ and $\text{germ}_{\pi_{\leq s}(x)}(\tilde{C}_{\leq s})$ are opposite, which proves the lemma. \square

8.3 Proof of (CO) when S admits a minimum

8.19 Lemma. *Let $x \in \mathcal{I}$ and let C and \tilde{C} be two sectors opposite at x . Then there exists an apartment A containing C and \tilde{C} .*

Proof. Let A_C and $A_{\tilde{C}}$ be apartments containing C and \tilde{C} respectively. By (A4) (Lemma 5.10) we can find an apartment $A_{C', \tilde{C}'}$ containing subsectors C' and \tilde{C}' of C and \tilde{C} . Let us prove that $x \in A_{C', \tilde{C}'}$.

Let $\phi : A_C \rightarrow A_{C', \tilde{C}'}$ be an apartment isomorphism such that $\phi|_{A_C \cap A_{C', \tilde{C}'}} = \text{Id}|_{A_C \cap A_{C', \tilde{C}'}}$. Set $y := \phi(x)$. We want to prove that $x = y$. By contradiction, assume that $x \neq y$.

Let $A_{x,y}$ be an apartment containing both x and y and $\psi : A_{x,y} \rightarrow \mathbb{A}_S$. Let $\text{supp}(y - x) \subset S$ be the support of $\psi(y) - \psi(x) \in V_S$. Then $\text{supp}(y - x)$ depend neither on the choice of $A_{x,y}$ nor on the choice of ψ . By assumption, $\text{supp}(y - x)$ is nonempty. Let s_0 be the minimum of $\text{supp}(y - x)$. Let $g \in G(\mathbb{K})$ inducing the isomorphism ψ . Then we have:

$$g \cdot \pi_{< s_0}(x) = \pi_{< s_0}(g \cdot x) = \pi_{< s_0}(\psi(x)) = \pi_{< s_0}(\psi(y)) = \pi_{< s_0}(g \cdot y) = g \cdot \pi_{< s_0}(y)$$

and:

$$g \cdot \pi_{\leq s_0}(x) = \pi_{\leq s_0}(g \cdot x) = \pi_{\leq s_0}(\psi(x)) \neq \pi_{\leq s_0}(\psi(y)) = \pi_{\leq s_0}(g \cdot y) = g \cdot \pi_{\leq s_0}(y)$$

and hence $\pi_{\leq s_0}(x) \neq \pi_{\leq s_0}(y)$.

Set $X = \pi_{< s_0}(x) = \pi_{< s_0}(y)$ and $\mathcal{I}_X = \pi_{< s_0}^{-1}(X) \subset \mathcal{I}$. Set

$$\pi_{=s_0} = \pi_{=s_0, X} : \mathcal{I}_X \rightarrow (\pi_{\leq s_0}^{-1}(X)) \subset \mathcal{I}(\mathbb{K}, \omega_{\leq s_0}, G),$$

with the notation of 7.6. Then $\pi_{=s_0}(x) \neq \pi_{=s_0}(y)$ and $(\pi_{\leq s_0}^{-1}(X))$ is an \mathbb{R} -building.

Since $x \in A_C \cap \mathcal{I}_X$ and $y \in A_{C', \tilde{C}'} \cap \mathcal{I}_X$, the sets $A_C \cap \mathcal{I}_X$ and $A_{C', \tilde{C}'} \cap \mathcal{I}_X$ are nonempty. Moreover, let $h \in G(\mathbb{K})$ inducing ϕ . Then:

$$\begin{aligned} \phi(A_C \cap \mathcal{I}_X) &\subseteq A_{C', \tilde{C}'} \cap (h \cdot \mathcal{I}_X) = A_{C', \tilde{C}'} \cap \pi_{< s_0}^{-1}(h \cdot X) = A_{C', \tilde{C}'} \cap \pi_{< s_0}^{-1}(h \cdot X) \\ &= A_{C', \tilde{C}'} \cap \pi_{< s_0}^{-1}(\phi(x)) = A_{C', \tilde{C}'} \cap \pi_{< s_0}^{-1}(\pi_{< s_0}(y)) \\ &= A_{C', \tilde{C}'} \cap \mathcal{I}_X. \end{aligned}$$

Hence ϕ induces a map $\phi_{\geq s_0} : A_C \cap \mathcal{I}_X \rightarrow A_{C', \tilde{C}'} \cap \mathcal{I}_X$, which induces itself a map:

$$\phi_{=s_0} : A_{C, =s_0} \rightarrow A_{C', \tilde{C}', =s_0}$$

where $A_{C, =s_0} := \pi_{=s_0}(A_C \cap \mathcal{I}_X)$ and $A_{C', \tilde{C}', =s_0} = \pi_{=s_0}(A_{C', \tilde{C}'} \cap \mathcal{I}_X)$.

We know that $\phi_{=s_0}(\pi_{=s_0}(x)) = \pi_{=s_0}(y) \neq \pi_{=s_0}(x)$. But the sectors-germs $C_{=s_0} := \pi_{=s_0}(C_{\leq s_0} \cap \mathcal{I}_{\pi_{< s_0}(x)})$ and $\tilde{C}_{=s_0} := \pi_{=s_0}(\tilde{C}_{\leq s_0} \cap \mathcal{I}_{\pi_{< s_0}(x)})$ germ $_{\pi_{=s_0}(x)}(\pi_{=s_0}(C))$ are opposite in $\pi_{=s_0}(x)$ (in the \mathbb{R} -building $(\pi_{\leq s_0}^{-1}(X))$).

Hence, by Lemma 8.2, $\pi_{=s_0}(x)$ belongs to the unique apartment of $\pi_{=s_0}(\mathcal{I}_{\pi_{< s_0}(x)})$ containing germ $_{\infty}(\pi_{=s_0}(C))$ and germ $_{\infty}(\pi_{=s_0}(\tilde{C}))$. In other words, $\pi_{=s_0}(\mathcal{I}_{\pi_{< s_0}(x)}) \in A_{C, =s_0} \cap A_{C', \tilde{C}', =s_0}$. Since ϕ fixes $A_C \cap A_{C', \tilde{C}'}$, we deduce that $\phi_{=s_0}(\pi_{=s_0}(x)) = \pi_{=s_0}(y) \neq \pi_{=s_0}(x)$: contradiction! Hence $x = y$. \square

8.4 Proof of (CO) in the general case

Let $\hat{\mathbb{K}} = \mathbb{K}((t))$. Let $G = \mathbf{G}(\mathbb{K})$ and $\hat{G} = \mathbf{G}(\hat{\mathbb{K}})$. Let $\hat{\omega} : \hat{\mathbb{K}} \rightarrow \mathbb{Z} \times \Lambda$ be defined by $\hat{\omega}(\sum_{k=n}^{\infty} a_k t^k) = (n, \omega(a_n))$, if $a_n \neq 0$. Let $\mathcal{I} = \mathcal{I}(G, \omega)$ and $\hat{\mathcal{I}} = \mathcal{I}(\hat{G}, \hat{\omega})$. Note that the rank of the ordered abelian group $\mathbb{Z} \times \Lambda$ is $\hat{S} := \{0_{\hat{s}}\} \sqcup S$, where $0_{\hat{s}}$ is an element of \hat{S} such that $s > 0_{\hat{s}}$ for all $s \in S$. In particular, we have an isomorphism of ordered \mathbb{R} -algebras $\mathbb{R} \times \mathfrak{R}^S \cong \mathfrak{R}^{\hat{S}}$.

Let $x \in \mathcal{I}$ and consider C and \tilde{C} two sectors in \mathcal{I} opposite at x . According to example 7.15, if ω_t stands for the t -adic valuation on \mathbb{K} , we have a projection:

$$\pi : \hat{\mathcal{I}} \rightarrow \mathcal{I}(\hat{\mathbb{K}}, \omega_t, \mathbf{G})$$

such that the fiber of the point $X_1 := [(1, 0)]$ is \mathcal{I} . Let \hat{C} and $\hat{\tilde{C}}$ be sectors of $\hat{\mathcal{I}}$ such that $\hat{C} \cap \pi^{-1}(X_1) = C$ and $\hat{\tilde{C}} \cap \pi^{-1}(X_1) = \tilde{C}$. The sectors \hat{C} and $\hat{\tilde{C}}$ being opposite, we deduce that there is an apartment A of $\hat{\mathcal{I}}$ that contains both \hat{C} and $\hat{\tilde{C}}$. Hence $A \cap \pi^{-1}(X_1)$ is an apartment of $\mathcal{I} = \pi^{-1}(X_1)$ that contains both C and \tilde{C} .

8.5 Proof of (CO) for the fibers of the projection maps

Adopt the notations of section 7. In particular, we have a projection map:

$$\pi : \mathcal{I}(\mathbb{K}, \omega, \mathbf{G}) \rightarrow \mathcal{I}(\mathbb{K}, \omega_1, \mathbf{G}).$$

Fix a point $X_1 \in \mathcal{I}(\mathbb{K}, \omega_1, \mathbf{G})$, let $x \in \pi^{-1}(X_1) \subseteq \mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ and consider C and \tilde{C} two sectors of $\pi^{-1}(X_1)$ opposite at x . Let \hat{C} and $\hat{\tilde{C}}$ be sectors of $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ such that $\hat{C} \cap \pi^{-1}(X_1) = C$ and $\hat{\tilde{C}} \cap \pi^{-1}(X_1) = \tilde{C}$. The sectors \hat{C} and $\hat{\tilde{C}}$ being opposite, we deduce that there is an apartment A of $\mathcal{I}(\mathbb{K}, \omega, \mathbf{G})$ that contains both \hat{C} and $\hat{\tilde{C}}$. Hence $A \cap \pi^{-1}(X_1)$ is an apartment of $\pi^{-1}(X_1)$ that contains both C and \tilde{C} .

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