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GLOBAL WELL-POSEDNESS FOR THE HALL-MAGNETOHYDRODYNAMICS SYSTEM IN LARGER CRITICAL BESOV SPACES

LVQIAO LIU AND JIN TAN

ABSTRACT. We prove the global well-posedness of the Cauchy problem to the 3D incompressible Hall-magnetohydrodynamic system supplemented with initial data in critical Besov spaces, which generalize the result in [10]. Meanwhile, we analyze the long-time behavior of the solutions and get some decay estimates. Finally, a stability theorem for global solutions is established.

1. INTRODUCTION

This paper focuses on the following three dimensional incompressible resistive and viscous Hall-magnetohydrodynamics system (Hall-MHD) in \mathbb{R}^3 :

$$\partial_t u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b + \mu \Delta u, \quad (1.1)$$

$$\operatorname{div} u = \operatorname{div} b = 0, \quad (1.2)$$

$$\partial_t b - \nabla \times ((u - \varepsilon \nabla \times b) \times b) = \nu \Delta b, \quad (1.3)$$

with the initial data:

$$(u(0, x), b(0, x)) = (u_0(x), b_0(x)), \quad x \in \mathbb{R}^3. \quad (1.4)$$

Where $\pi = (P + \frac{|b|^2}{2})$, u , b and P stand for the velocity field, the magnetic field and the scalar pressure, respectively. The parameters μ and ν denote the fluid viscosity and the magnetic resistivity respectively, while the dimensionless number ε measures the magnitude of the Hall effect compared to the typical length scale of the fluid.

Hall-MHD is much different from the classical MHD equation, due to the appearance of the so-called *Hall-term* : $\varepsilon \nabla \times ((\nabla \times b) \times b)$. It does play an important role in magnetic reconnection, as observed in e.g. plasmas, star formation, solar flares, neutron stars or geo-dynamo. For more explanation on the physical background of Hall-MHD system, one can refer to [3, 6, 14, 20]. Meanwhile, it looks that the mathematical analysis of the Hall-MHD system is more complicated than that for the MHD system, since Hall-term makes Hall-MHD a quasi-linear PDEs.

Considering its physical significance and mathematical applications, Hall-MHD system has been considered by many researchers. The authors in [1] had derived the Hall-MHD equations from a two-fluid Euler-Maxwell system for electrons and ions by some scaling limit arguments, which also provided a kinetic formulation for the Hall-MHD. Then, in [7], Chae, Degond and Liu showed the global existence of weak solutions as well as the local well-posedness of classical solutions with initial data

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in sobolev spaces H^s with $s > 5/2$. Weak solutions have been further investigated by Dumas and Sueur in [12]. Moreover, Blow-up criteria for smooth solutions and the small data global existence of smooth solutions are obtained in [8, 21]. Later, in [23, 24], Weng studied the long-time behaviour and obtained optimal space-time decay rates of strong solutions. More recently, [4, 22] established the well-posedness of strong solutions with improved regularity conditions for initial data in sobolev or Besov spaces, and smooth data with arbitrarily large L^∞ norms giving rise to global unique solutions have been exhibited in [15]. Very recently, Danchin and the second author in [10, 11] establish well-posedness in critical spaces based on a new observation of the Hall-MHD system.

Our first goal here is to prove the global well-posedness of Hall-MHD system with initial data in *larger critical spaces* compared with [10]. Let us first recall the classical MHD system (corresponding to $\varepsilon = 0$):

$$(MHD) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b + \mu \Delta u, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ \partial_t b - \nabla \times (u \times b) = \nu \Delta b, \end{cases}$$

it is invariant for all $\lambda > 0$ by the rescaling

$$(u(t, x), b(t, x)) \rightsquigarrow \lambda(u(\lambda^2 t, \lambda x), b(\lambda^2 t, \lambda x)) \quad \text{and} \quad \pi(t, x) \rightsquigarrow \lambda^2 \pi(\lambda^2 t, \lambda x), \quad (1.5)$$

provided the initial data (u_0, b_0) is rescaled according to

$$(u_0(x), b_0(x)) \rightsquigarrow (\lambda u_0(\lambda x), \lambda b_0(\lambda x)). \quad (1.6)$$

One can refer [17] for the well-posedness of (MHD) in critical Besov spaces.

But we see the Hall-term in (1.3) breaks the above scaling and system (1.1)-(1.3) does not have any *scaling invariance*. It is pointed out in [8] that if we set the fluid velocity u to 0 in (1.3), then we get the following *Hall equation* for b :

$$(Hall) \quad \begin{cases} \partial_t b + \varepsilon \nabla \times ((\nabla \times b) \times b) = \nu \Delta b, \\ b|_{t=0} = b_0, \end{cases}$$

which is invariant by the rescaling

$$b(t, x) \rightsquigarrow b(\lambda^2 t, \lambda x), \quad (1.7)$$

provided the data b_0 is rescaled according to

$$b_0(x) \rightsquigarrow b_0(\lambda x). \quad (1.8)$$

In other words, ∇b has the same scaling invariance as the fluid velocity u in (MHD) .

On the another hand, Danchin and the second author [10] have transformed the Hall-MHD system into a system having some *scaling invariance*, they consider the current function $J := \nabla \times b$ as an additional unknown. Since b is divergence free, then thanks to the vector identity

$$\nabla \times (\nabla \times v) + \Delta v = \nabla \operatorname{div} v,$$

we have

$$b = \operatorname{curl}^{-1} J := (-\Delta)^{-1} \nabla \times J,$$

where the -1 -th order homogeneous Fourier multiplier curl^{-1} is defined on the Fourier side by

$$\mathcal{F}(\operatorname{curl}^{-1} J)(\xi) := \frac{i\xi \times \widehat{J}}{|\xi|^2}. \quad (1.9)$$

With that notation, the system (1.1)-(1.3) can be extended to the following *extended Hall-MHD system*:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \pi = b \cdot \nabla b, \\ \partial_t b - b \cdot \nabla (u - \varepsilon J) + (u - \varepsilon J) \cdot \nabla b - \nu \Delta b = 0, \\ \partial_t J - \nabla \times (\nabla \times ((u - \varepsilon J) \times \text{curl}^{-1} J)) - \nu \Delta J = 0, \\ \text{div } u = \text{div } b = \text{div } J = 0. \end{cases} \quad (1.10)$$

with initial data

$$(u(0, x), b(0, x), J(0, x)) = (u_0(x), b_0(x), \nabla \times b_0(x)), \quad x \in \mathbb{R}^3. \quad (1.11)$$

The reason of considering the extended system (1.10) rather than the initial system (1.1)-(1.3) is that it has a *scaling invariance*, which is actually the same as that of the incompressible MHD equations. Even better, the quadratic terms in the first two lines of (1.10) are keep the same type with the incompressible MHD equations. It is thus natural to study whether the above system goes beyond the theory of the generalized Navier-stokes equations as presented in e.g. [2].

Next, we focus on the large time behavior of the solution in critical Besov spaces by using time-weighted estimates.

Our third purpose is to prove the stability of an *a priori* global solution to Hall-MHD system. Compared with the classical incompressible Navier-Stokes equations which is semi-linear, the Hall-MHD system is quasi-linear, it may forces us to go beyond the theory of the generalized Navier-Stokes equations, since the differentiation is outside instead of being inside on the $\text{curl}^{-1} J$ in the last line of (1.10). However, in the case $\mu = \nu$, it is possible to take advantage of the cancellation property found in [10] combined with standard energy method to recover the stability results as Navier-Stokes equation (see [13]). To this, we have to take $p = q = 2$.

Throughout this paper, we use C to denote a general positive constant which may different from line to line. And we sometimes write $A \lesssim B$ instead of $A \leq CB$. Likewise, $A \sim B$ means that $C_1 B \leq A \leq C_2 B$ with absolute constants C_1, C_2 . For X a Banach space, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L_T^p(X)$ designates the set of measurable functions $f : [0, T] \rightarrow X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L_T^p(X)} := \|\|\cdot\|_X\|_{L^p(0, T)}$, and agree that $\mathcal{C}([0, T], X)$ denotes the set of continuous functions from $[0, T]$ to X . Slightly abusively, we will keep the same notations for multi-component functions.

2. MAIN RESULTS

After the work of [9], we know that the incompressible Navier-Stokes equations is locally well-posed in all homogeneous Besov spaces $\dot{B}_{p,r}^{\frac{3}{p}-1}$ with $1 \leq p < \infty$ and $1 \leq r \leq \infty$, for any initial data, and globally well-posed for small initial data (see [17] for similar results of MHD equations). Once the Hall-MHD system (1.1)-(1.3) has been recast into its extended version as in (1.10), compared to a recent work obtained in [10], they prove global well-posedness for initial data

$$(u_0, b_0, J_0) \in \dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{p,1}^{\frac{3}{p}-1} \quad \text{with } 1 \leq p < \infty.$$

It is interesting to consider well-posedness for initial data (u_0, b_0, J_0) in the general critical homogeneous Besov spaces

$$\dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{q,1}^{\frac{3}{q}-1} \times \dot{B}_{q,1}^{\frac{3}{q}-1}.$$

The solution (u, b) thus lies in the space $E_p \times E_q$, which defines as follow:

$$E_p(T) := \{v \in \mathcal{C}([0, T], \dot{B}_{p,1}^{\frac{3}{p}-1}), \nabla_x^2 v \in L^1(0, T; \dot{B}_{p,1}^{\frac{3}{p}-1}) \text{ and } \operatorname{div} v = 0\}$$

with

$$\|v\|_{E_p(T)} := \|v\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \|v\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})}.$$

or in its global version, denoted by E_p , for solutions defined on $\mathbb{R}_+ \times \mathbb{R}^3$.

Our first result states the global well-posedness of the Hall-MHD system (1.1)-(1.4) for all positive coefficients μ, ν, ε .

Theorem 2.1. *Let $1 \leq p \leq q < \infty$ be such that*

$$-\min\left\{\frac{1}{3}, \frac{1}{2p}\right\} \leq \frac{1}{q} - \frac{1}{p}. \quad (2.1)$$

Assume that $u_0 \in \dot{B}_{p,1}^{\frac{3}{p}-1}$ and $b_0, \nabla \times b_0 \in \dot{B}_{q,1}^{\frac{3}{q}-1}$. There exists a positive constant ε_0 depends on $\mu, \nu, \varepsilon, p, q$ such that if

$$\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|b_0\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} + \|\nabla \times b_0\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \leq \varepsilon_0, \quad (2.2)$$

then the Cauchy problem (1.1)-(1.4) admits a unique global-in-time solution

$$(u, b) \in E_p \times E_q, \quad \text{and} \quad \nabla \times b \in E_q, \quad (2.3)$$

with

$$\begin{aligned} & \|u\|_{L^\infty(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu \|u\|_{L^1(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{3}{p}+1})} \\ & + \|(b, J)\|_{L^\infty(\mathbb{R}_+; \dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|(b, J)\|_{L^1(\mathbb{R}_+; \dot{B}_{q,1}^{\frac{3}{q}+1})} \leq 2\varepsilon_0. \end{aligned} \quad (2.4)$$

If only $\nabla \times b_0$ fulfills (2.2) and in addition

$$-\frac{1}{3} < \frac{1}{q} - \frac{1}{p}, \quad (2.5)$$

there exists a time $T > 0$ such that Hall-MHD system admits a unique local-in-time solution

$$(u, b) \in E_p(T) \times E_q(T) \quad \text{with} \quad \nabla \times b \in E_q(T).$$

Next, we prove that the solution has the following decay estimates.

Theorem 2.2. *Let $1 \leq p \leq q < \infty$ satisfy (2.1) and (2.5). Let (u, b) be a solution of the Cauchy problem (1.1)-(1.4) supplemented with initial data (u_0, b_0) that satisfies (2.2). Then for any integer $m \geq 1$, we have*

$$\|D^m u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|D^m b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \leq C_0 \varepsilon_0 t^{-\frac{m}{2}},$$

for all $t > 0$, where

$$\|D^m u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} := \sup_{|\alpha|=m} \|D^\alpha u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}},$$

and the positive constant C_0 depends on $\mu, \nu, \varepsilon, p, q, m$.

The following Theorem states the global stability for possible large solutions of Hall-MHD system in critical spaces when $\mu = \nu$.

Theorem 2.3. *Assume that $(u_{0,i}, b_{0,i}) \in \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\operatorname{div} u_{0,i} = \operatorname{div} b_{0,i} = 0$ such that $v_{0,i} \in \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^3)$, where*

$$v_{0,i} := u_{0,i} - \varepsilon \nabla \times b_{0,i}, \quad i = 1, 2.$$

Suppose in addition that for $\mu = \nu$ the Cauchy problem (1.1)-(1.4) supplemented with initial data $(u_{0,1}, b_{0,1})$ admits a global solution (u_1, b_1) such that

$$(u_1, b_1, \nabla \times b_1) \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{5}{2}}(\mathbb{R}^3)).$$

There exist two positive constants η, C depend on μ, ε such that if

$$\|(u_{0,1} - u_{0,2}, b_{0,1} - b_{0,2}, v_{0,1} - v_{0,2})\|_{\dot{B}_{2,1}^{\frac{1}{2}}} \leq \eta, \quad (2.6)$$

then $(u_{0,2}, b_{0,2})$ generate a global solution $(u_2, b_2) \in E_2$, and

$$\|(u_1 - u_2, b_1 - b_2, \nabla \times b_1 - \nabla \times b_2)\|_{E_2} \leq \eta \exp(C\|(u_1, b_1, \nabla \times b_1)\|_{E_2}). \quad (2.7)$$

Remark 1. In this Theorem, we prove that the flow associated to the Hall-MHD system is Lipschitz in critical regularity setting: perturbing a global solution gives again a global solution, which moreover stays close to the given one. It improve the result in [4].

Remark 2. As proposed by Chae and Lee in [8], considering the $2\frac{1}{2}$ D flows for the Hall-MHD system, which reads:

$$\begin{aligned} \partial_t u + \tilde{u} \cdot \tilde{\nabla} u + \tilde{\nabla} \pi &= \tilde{b} \cdot \tilde{\nabla} b + \mu \tilde{\Delta} u && \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \widetilde{\operatorname{div}} \tilde{u} &= \widetilde{\operatorname{div}} b = 0 && \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t b - \tilde{\nabla} \times ((u - \varepsilon j) \times b) &= \nu \tilde{\Delta} b && \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, B)|_{t=0} &= (u_0, B_0) && \text{in } \mathbb{R}^2, \end{aligned}$$

where the unknowns u and b are functions from $\mathbb{R}_+ \times \mathbb{R}^2$ to \mathbb{R}^3 , $\tilde{u} := (u^1, u^2)$, $\tilde{b} := (b^1, b^2)$, $\tilde{\nabla} := (\partial_1, \partial_2)$, $\widetilde{\operatorname{div}} := \tilde{\nabla} \cdot$, $\tilde{\Delta} := \partial_1^2 + \partial_2^2$ and

$$j := \tilde{\nabla} \times b = \begin{pmatrix} \partial_2 b^3 \\ -\partial_1 b^3 \\ \partial_1 b^2 - \partial_2 b^1 \end{pmatrix}.$$

After a small diversification, our method may still works for this case. And we shall see that for any initial data $(u_0, 0) \in \dot{B}_{2,1}^0(\mathbb{R}^2)$, it will generate a global solution $(u, 0)$ for the above system due to the theory of $2\frac{1}{2}$ D Navier-Stokes equation in [16], thus one can conclude that if

$$\|(b_0, \nabla \times b_0)\|_{\dot{B}_{2,1}^0} < \eta$$

then supplemented with data (u_0, b_0) will also generate a global solution.

3. PRELIMINARIES

In this section, we first recall the Littlewood-Paley decomposition theory, the definition of homogeneous Besov space and some useful properties. More details and proofs may be found in e.g. [2].

Let $\varphi \in \mathcal{D}(\mathcal{C})$ be a smooth function supported in the annulus $\mathcal{C} = \{k \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}k) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

For $u \in \mathcal{S}'(\mathbb{R}^3)$, the frequency localization operator $\dot{\Delta}_j$ and \dot{S}_j are defined by

$$\forall j \in \mathbb{Z}, \quad \dot{\Delta}_j u := \varphi(2^{-j}D)u \text{ and } \dot{S}_j u := \sum_{\ell \leq j-1} \dot{\Delta}_\ell u.$$

Then we have the formal decomposition

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u, \quad \forall u \in \mathcal{S}'_h(\mathbb{R}^3) := \mathcal{S}'(\mathbb{R}^3) / \mathcal{P}[\mathbb{R}^3].$$

where $\mathcal{P}[\mathbb{R}^3]$ is the set of polynomials (see [19]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_j \dot{\Delta}_k u = 0, \quad \text{if } |j - k| \geq 2, \quad \dot{\Delta}_j (S_{k-1} u \dot{\Delta}_k u) = 0, \quad \text{if } |j - k| \geq 5.$$

We now recall the definition of homogeneous Besov spaces from [2].

Definition 3.1. Let s be a real number and (p, r) be in $[1, \infty]^2$, we set

$$\|u\|_{\dot{B}_{p,r}^s} := \begin{cases} \|2^{js} \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^d)}\|_{\ell^r(\mathbb{Z})} & \text{for } 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p} & \text{for } r = \infty. \end{cases}$$

The homogeneous Besov space $\dot{B}_{p,r}^s := \{u \in \mathcal{S}'_h(\mathbb{R}^3), \|u\|_{\dot{B}_{p,r}^s} < \infty\}$.

Next, we recall some basic facts on Littlewood-Paley theory and Besov spaces, one may check [2, 5] for more details.

Proposition 3.2. Fix some $0 < r < R$. A constant C exists such that for any nonnegative integer k , any couple (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$ and any function u of L^p with $\text{Supp } \hat{u} \subset \{\xi \in \mathbb{R}^d, |\xi| \leq \lambda R\}$, we have

$$\|D^k u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$

If u satisfies $\text{Supp } \hat{u} \subset \{\xi \in \mathbb{R}^d, r\lambda \leq |\xi| \leq R\lambda\}$, then we have

$$C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

Lemma 3.3. Let \mathcal{C} be a ring of \mathbb{R}^3 , if the support of \hat{u} is included in $\lambda\mathcal{C}$. Then, there exist two positive constants c and C such that for all $1 \leq p \leq \infty$,

$$\|e^{t\Delta} u\|_{L^p} \leq C e^{c\lambda^2 t} \|u\|_{L^p},$$

where $e^{t\Delta}$ denotes the heat semi-group operator.

Proposition 3.4. Let $1 \leq p \leq \infty$. Then there hold:

- for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we have

$$\|D^k u\|_{\dot{B}_{p,r}^s} \sim \|u\|_{\dot{B}_{p,r}^{s+k}}.$$

- for any $\theta \in (0, 1)$ and $s < \tilde{s}$, we have

$$\|u\|_{\dot{B}_{p,1}^{\theta s + (1-\theta)\tilde{s}}} \lesssim \|u\|_{\dot{B}_{p,1}^{\theta}} \|u\|_{\dot{B}_{p,1}^{\tilde{s}}}^{1-\theta}.$$

Lemma 3.5. *Let f be a smooth function on $\mathbb{R}^3 \setminus \{0\}$ which is homogeneous of degree m . Let $1 \leq p, r \leq \infty$. Assume that*

$$s - m < \frac{3}{p}, \quad \text{or} \quad s - m = \frac{3}{p} \quad \text{and} \quad r = 1.$$

Define $f(D)$ on $\mathcal{S}'_h(\mathbb{R}^3)$ by

$$\mathcal{F}(f(D)u)(\xi) := f(\xi)\mathcal{F}u(\xi),$$

and assume that $f(D)$ maps $\mathcal{S}'_h(\mathbb{R}^3)$ to itself. Then $f(D)$ is continuous from $\dot{B}_{p,r}^s(\mathbb{R}^3)$ to $\dot{B}_{p,r}^{s-m}(\mathbb{R}^3)$.

In the next, we shall need to use Bony's decomposition from [5] in the homogeneous context:

$$uv = T_u v + T_v u + R(u, v)$$

with

$$T_u v := \sum_q \dot{S}_{q-1} u \dot{\Delta}_q v, \quad \text{and} \quad R(u, v) := \sum_q \sum_{|q'-q| \leq 1} \dot{\Delta}_q u \dot{\Delta}_{q'} v.$$

The above operator T is called the "paraproduct" whereas R is called the "remainder".

Lemma 3.6. *Let $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ and $t < 0$, there exists a constant C such that*

$$\|T_u v\|_{\dot{B}_{p,r}^s} \leq C \|u\|_{L^\infty} \|v\|_{\dot{B}_{p,r}^s} \quad \text{and} \quad \|T_u v\|_{\dot{B}_{p,r}^{s+t}} \leq C \|u\|_{\dot{B}_{\infty,\infty}^t} \|v\|_{\dot{B}_{p,r}^s}.$$

For any (s_1, p_1, r_1) and (s_2, p_2, r_2) in $\mathbb{R} \times [1, \infty]^2$ there exists a constant C such that if $s_1 + s_2 > 0$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2} \leq 1$ then

$$\|R(u, v)\|_{\dot{B}_{p,r}^s} \leq C \|u\|_{\dot{B}_{p_1,r_1}^{s_1}} \|v\|_{\dot{B}_{p_2,r_2}^{s_2}},$$

with $s := s_1 + s_2 - 3(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p})$, provided that $s < \frac{3}{p}$ or $s = \frac{3}{p}$ and $r = 1$.

As an application of the above basic facts on Littlewood-Paley theory, the following product laws in Besov spaces will play a crucial role in the sequel.

Lemma 3.7. (see [18]) *Let $q \geq p \geq 1$, and $s_1 \leq \frac{3}{p}$, $s_2 \leq \frac{3}{q}$ with*

$$s_1 + s_2 \geq 3 \min\left\{0, \frac{1}{p} + \frac{1}{q} - 1\right\}.$$

Let $a \in \dot{B}_{p,1}^{s_1}(\mathbb{R}^3)$, $b \in \dot{B}_{q,1}^{s_2}(\mathbb{R}^3)$. Then $ab \in \dot{B}_{q,1}^{s_1+s_2-\frac{3}{p}}(\mathbb{R}^3)$, and

$$\|ab\|_{\dot{B}_{q,1}^{s_1+s_2-\frac{3}{p}}} \lesssim \|a\|_{\dot{B}_{p,1}^{s_1}} \|b\|_{\dot{B}_{q,1}^{s_2}}. \quad (3.1)$$

Lemma 3.8. *Let $q \geq p \geq 1$ and*

$$\frac{1}{q} - \frac{1}{p} \geq -\min\left\{\frac{1}{3}, \frac{1}{2p}\right\}.$$

Assume θ satisfies

$$\frac{3}{p} - \frac{3}{q} \leq \theta \leq 1.$$

Let $a, b \in \dot{B}_{q,1}^{\frac{3}{q}-\theta}(\mathbb{R}^3) \cap \dot{B}_{q,1}^{\frac{3}{q}+\theta}(\mathbb{R}^3)$. Then $ab \in \dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$, and

$$\|ab\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}-\theta}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}+\theta}} + \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}+\theta}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-\theta}}. \quad (3.2)$$

Let $a, b \in \dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}(\mathbb{R}^3) \cap \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$. Then $ab \in \dot{B}_{p,1}^{\frac{3}{q}}(\mathbb{R}^3)$, and

$$\|ab\|_{\dot{B}_{p,1}^{\frac{3}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} + \|a\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}}}. \quad (3.3)$$

Proof. The proof is standard, we follow the method of [18] and only give a proof of (3.2). In fact, we shall focus on the case $q > p$, since when $q = p$, it is obvious.

By Bony's decomposition, we can write

$$ab = T_a b + T_b a + R(a, b).$$

By Hölder's inequality and Proposition 3.2, under $p < q \leq 2p$ we have

$$\begin{aligned} 2^{\frac{3}{p}j} \|S_{j-1} a \dot{\Delta}_j b\|_{L^p} &\lesssim (2^{(\frac{3}{q}+\theta)j} \|\dot{\Delta}_j b\|_{L^q}) \sum_{k \leq j-2} 2^{(\frac{3}{p}-\frac{3}{q}-\theta)k} \|\dot{\Delta}_k a\|_{L^{\frac{pq}{q-p}}} 2^{(\frac{3}{p}-\frac{3}{q}-\theta)(j-k)} \\ &\lesssim (2^{(\frac{3}{q}+\theta)j} \|\dot{\Delta}_j b\|_{L^q}) \sum_{k \leq j-2} 2^{(\frac{3}{q}-\theta)k} \|\dot{\Delta}_k a\|_{L^q} 2^{(\frac{3}{p}-\frac{3}{q}-\theta)(j-k)}. \end{aligned}$$

Thus by Young's inequality with $\frac{3}{p} - \frac{3}{q} \leq \theta$

$$\|T_a b\|_{\dot{B}_{p,1}^{\frac{3}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}-\theta}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}+\theta}}.$$

Similarly,

$$\|T_b a\|_{\dot{B}_{p,1}^{\frac{3}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}+\theta}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-\theta}}.$$

Thanks to Lemma 3.6, we have

$$\begin{aligned} \|R(a, b)\|_{\dot{B}_{p,1}^{\frac{3}{q}}} &\lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}-\frac{3}{q}-\theta}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}+\theta}} \\ &\lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}-\theta}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}+\theta}}. \end{aligned}$$

□

Lemma 3.9. *Let $1 \leq q < \infty$. For any homogeneous function σ of degree -1 smooth outside of 0, there hold:*

- let $a \in \dot{B}_{q,1}^{\frac{3}{q}-1}$ and $b \in \dot{B}_{q,1}^{\frac{3}{q}+1}$, then

$$\|(\sigma(D)a) \cdot \nabla b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}}. \quad (3.4)$$

- let $a \in \dot{B}_{q,1}^{\frac{3}{q}}$ and $b \in \dot{B}_{q,1}^{\frac{3}{q}}$, then

$$\|a \cdot \nabla(\sigma(D)b)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}}}. \quad (3.5)$$

Proof. Thanks to Lemma 3.7, we know that when $1 \leq q < \infty$, $\dot{B}_{q,1}^{\frac{3}{q}}$ is an algebra, thus by Lemma 3.5, we have

$$\begin{aligned} \|(\sigma(D)a) \cdot \nabla b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} &\lesssim \|\sigma(D)a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\nabla b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\ &\lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}}. \end{aligned}$$

Similarly,

$$\|a \cdot \nabla(\sigma(D)b)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\nabla(\sigma(D)b)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}$$

$$\begin{aligned}
&\lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\sigma(D)\nabla b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\
&\lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}}}.
\end{aligned}$$

□

The basic heat equation reads:

$$\begin{cases} \partial_t u - \mu \Delta u = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^3. \end{cases} \quad (3.6)$$

Then, it is classical that for all $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}_+; \mathcal{S}'(\mathbb{R}^d))$, the heat equation (3.6) has a unique tempered distribution solution, which is given by the following Duhamel's formula:

$$u(t) = e^{\mu t \Delta} u_0 + \int_0^t e^{(t-s)\mu \Delta} f(\tau) ds, \quad t \geq 0. \quad (3.7)$$

The following fundamental results to heat semi-group has been first proved in [9].

Lemma 3.10. *Let $s > 0$, $1 \leq p < \infty$. Assume that $u_0 \in \dot{B}_{p,1}^s$, then for any $\rho \in [1, \infty)$,*

$$\lim_{T \rightarrow 0} \|e^{t\mu \Delta} u_0\|_{L_T^\rho(\dot{B}_{p,1}^{s+\frac{2}{\rho}})} = 0.$$

Lemma 3.11. *Let $T > 0$, $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. Assume that $u_0 \in \dot{B}_{p,1}^s$ and $f \in L_T^1(\dot{B}_{p,1}^s)$. Then (3.6) has a unique solution u in $\mathcal{C}([0, T]; \dot{B}_{p,1}^s) \cap L^1(0, T; \dot{B}_{p,1}^{s+2})$ and there exists a constant C such that*

$$\|u\|_{L_T^\infty L \dot{B}_{p,1}^s} + \mu \|u\|_{L_T^1(\dot{B}_{p,1}^{s+2})} \leq C \left(\|u_0\|_{\dot{B}_{p,1}^s} + \|f\|_{L_T^1(\dot{B}_{p,1}^s)} \right). \quad (3.8)$$

4. THE PROOF OF THEOREM 2.1

In this section, we shall give the proof of Theorem 2.1. We only look at the case $p < q$, since the case $p = q$ is been shown in [10].

By means of the **Leray projector** $\mathcal{P} := \text{Id} - \nabla(-\Delta)^{-1} \text{div}$ and the fact that u, b are divergence free vector fields, we can rewrite the system (1.10) as (see [10]):

$$\begin{cases} \partial_t u - \mu \Delta u = Q_a(b, b) - Q_a(u, u), \\ \partial_t b - \nu \Delta b = Q_b(u - \varepsilon J, b), \\ \partial_t J - \nu \Delta J = \nabla \times Q_b(u - \varepsilon J, \text{curl}^{-1} J), \end{cases} \quad (4.1)$$

and supplemented with divergence free initial data

$$(u(0, x), b(0, x), J(0, x)) = (u_0, b_0, J_0). \quad (4.2)$$

Where bi-linear forms

$$\begin{aligned}
Q_a(v, w) &:= \frac{1}{2} \mathcal{P}(\text{div}(v \otimes w) + \text{div}(w \otimes v)), \\
Q_b(v, w) &:= \text{div}(v \otimes w) - \text{div}(w \otimes v),
\end{aligned}$$

and

$$\left(\text{div}(v \otimes w) \right)^j := \sum_{k=1}^3 \partial_k (v^j w^k).$$

Define free solution

$$u_L := e^{\mu t \Delta} u_0, \quad b_L := e^{\nu t \Delta} b_0, \quad J_L := e^{\nu t \Delta} J_0.$$

Let $u_0 \in \dot{B}_{p,1}^{\frac{3}{p}-1}$ and $b_0, J_0 \in \dot{B}_{q,1}^{\frac{3}{q}-1}$, it is easy to find that by Lemma 3.11 that

$$(u_L, b_L, J_L) \in E_p \times E_q \times E_q$$

and there holds

$$\|u_L\|_{L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu \|u_L\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \leq C \|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}, \quad (4.3)$$

$$\|b_L\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|b_L\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \leq C \|b_0\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}, \quad (4.4)$$

$$\|J_L\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|J_L\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \leq C \|J_0\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}. \quad (4.5)$$

Define $(\bar{u}, \bar{b}, \bar{J}) := (u - u^L, b - b^L, J - J^L)$. Then (u, b, J) is a solution of (4.1) if and only if $(\bar{u}, \bar{b}, \bar{J})$ satisfies the following system:

$$\begin{cases} \partial_t \bar{u} - \mu \Delta \bar{u} = Q_a(\bar{b}, \bar{b}) + Q_a(b_L, \bar{b}) + Q_a(\bar{b}, b_L) + Q_a(b_L, b_L) \\ \quad - Q_a(\bar{u}, \bar{u}) - Q_a(u_L, \bar{u}) - Q_a(\bar{u}, u_L) - Q_a(u_L, u_L), \\ \partial_t \bar{b} - \nu \Delta \bar{b} = Q_b(\bar{u} - \varepsilon \bar{J}, \bar{b}) + Q_b(u_L - \varepsilon J_L, \bar{b}) + Q_b(\bar{u} - \varepsilon \bar{J}, b_L) \\ \quad + Q_b(u_L - \varepsilon J_L, b_L), \\ \partial_t \bar{J} - \nu \Delta \bar{J} = \nabla \times \left(Q_b(\bar{u} - \varepsilon \bar{J}, \text{curl}^{-1} \bar{J}) + Q_b(u_L - \varepsilon J_L, \text{curl}^{-1} \bar{J}) \right. \\ \quad \left. + Q_b(\bar{u} - \varepsilon \bar{J}, \text{curl}^{-1} J_L) + Q_b(u_L - \varepsilon J_L, \text{curl}^{-1} J_L) \right). \end{cases} \quad (4.6)$$

In what follows, we will employing the iterative method to prove that there exists an unique solution $(\bar{u}, \bar{b}, \bar{J})$ to the system (4.6). More precisely, the iterating approximate system is constructed as follows:

$$\begin{cases} \partial_t \bar{u}_n - \mu \Delta \bar{u}_n = Q_a(\bar{b}_{n-1}, \bar{b}_{n-1}) + Q_a(b_L, \bar{b}_{n-1}) + Q_a(\bar{b}_{n-1}, b_L) + Q_a(b_L, b_L) \\ \quad - Q_a(\bar{u}_{n-1}, \bar{u}_{n-1}) - Q_a(u_L, \bar{u}_{n-1}) - Q_a(\bar{u}_{n-1}, u_L) - Q_a(u_L, u_L), \\ \partial_t \bar{b}_n - \nu \Delta \bar{b}_n = Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, \bar{b}_{n-1}) + Q_b(u_L - \varepsilon J_L, \bar{b}_{n-1}) \\ \quad + Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, b_L) + Q_b(u_L - \varepsilon J_L, b_L), \\ \partial_t \bar{J}_n - \nu \Delta \bar{J}_n = Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, \text{curl}^{-1} \bar{J}_{n-1}) + Q_b(u_L - \varepsilon J_L, \text{curl}^{-1} \bar{J}_{n-1}) \\ \quad + Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, \text{curl}^{-1} J_L) + Q_b(u_L - \varepsilon J_L, \text{curl}^{-1} J_L). \end{cases} \quad (4.7)$$

We start the approximate system with

$$(\bar{u}_0, \bar{b}_0, \bar{J}_0)(t, x) = (0, 0, 0)$$

for all $t \geq 0$, and assume the initial data of the iterative approximate system (4.7) satisfied for all $n \in \mathbb{N}$

$$(\bar{u}_n, \bar{b}_n, \bar{J}_n)(0, x) = (0, 0, 0). \quad (4.8)$$

In the arguments proving the convergence ($n \rightarrow \infty$) of the iterative approximate solutions of (4.7)-(4.8), it is essential to obtain uniform estimates for it.

4.1. **Uniform boundedness of $(\bar{u}_n, \bar{b}_n, \bar{J}_n)$.** We claim that there exists a positive constant M such that for all $n \in \mathbb{N}$,

$$\begin{aligned} & \left(\|\bar{u}_n\|_{L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu \|\bar{u}_n\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \right) \\ & + \left(\|\bar{b}_n\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\bar{b}_n\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \right) \\ & + \left(\|\bar{J}_n\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\bar{J}_n\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \right) \leq M. \end{aligned} \quad (4.9)$$

Obviously, (4.9) is satisfied when $n = 0$. Assume the claim (4.9) holds true for $n - 1$, i.e.,

$$\begin{aligned} & \|\bar{u}_{n-1}\|_{L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu \|\bar{u}_{n-1}\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} + \|\bar{b}_{n-1}\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} \\ & + \nu \|\bar{b}_{n-1}\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} + \|\bar{J}_{n-1}\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\bar{J}_{n-1}\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \leq M. \end{aligned}$$

With smallness condition (2.2), we now devote to the proof of (4.9) through finding some suitable M . Firstly, we need to state the following product laws for quadratic terms Q_a, Q_b , it will play a significant role in the later parts.

If (2.1) is assumed, by using Lemma 3.7, Lemma 3.9, direct calculation tells us that:

$$\begin{aligned} \|Q_a(v, w)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} & \lesssim \|v \otimes w\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \\ & \lesssim \|v\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|w\|_{\dot{B}_{p,1}^{\frac{3}{p}}}, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \|Q_b(v, w)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} & \lesssim \|v \otimes w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\ & \lesssim \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}}, \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} \|\nabla \times Q_b(v, \text{curl}^{-1}w)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} & \lesssim \|Q_b(v, \text{curl}^{-1}w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\ & \lesssim \|v \cdot \nabla(\text{curl}^{-1}w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(\text{curl}^{-1}w) \cdot \nabla v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\ & \lesssim \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|v\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}} \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}, \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \|\nabla \times Q_b(v, \text{curl}^{-1}w)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} & \lesssim \|Q_b(v, \text{curl}^{-1}w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\ & \lesssim \|v \cdot \nabla(\text{curl}^{-1}w)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(\text{curl}^{-1}w) \cdot \nabla v\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\ & \lesssim \|v\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|v\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} \|w\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}. \end{aligned} \quad (4.13)$$

Thanks to (3.2) in Lemma 3.8 with $\theta = \frac{3}{p} - \frac{3}{q} \leq 1$, we have

$$\begin{aligned} \|Q_a(v, w)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} &\lesssim \|v \otimes w\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \\ &\lesssim \|v\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|w\|_{\dot{B}_{q,1}^{\frac{3}{p}}} + \|w\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|v\|_{\dot{B}_{q,1}^{\frac{3}{p}}}. \end{aligned} \quad (4.14)$$

Notice that from Lemma 3.11, there exists a constant C such that

$$\|u_L\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} + \|u_L\|_{L^2(\dot{B}_{p,1}^{\frac{3}{p}})} \leq C\varepsilon_0(1 + \frac{1}{\mu}), \quad (4.15)$$

$$\begin{aligned} &\|(b_L, J_L)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} + \|(b_L, J_L)\|_{L^2(\dot{B}_{q,1}^{\frac{3}{q}})} \\ &+ \|b_L\|_{L^{\frac{2pq}{pq-3q+3p}}(\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}})} + \|b_L\|_{L^{\frac{2pq}{pq+3q-3p}}(\dot{B}_{q,1}^{\frac{3}{q}})} \leq C\varepsilon_0(1 + \frac{1}{\nu}). \end{aligned} \quad (4.16)$$

Combining (4.10), (4.14), take use of interpolation inequality in Proposition 3.4 and Hölder inequality, remember the "norm" of free solution (u_L, b_L, J_L) is small thanks to (4.15) and (4.16), it follows from Lemma 3.11 that

$$\begin{aligned} &\|\bar{u}_n\|_{L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu\|\bar{u}_n\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \\ &\leq C\|Q_a(\bar{b}_{n-1}, \bar{b}_{n-1}) + Q_a(b_L, \bar{b}_{n-1}) + Q_a(\bar{b}_{n-1}, b_L) + Q_a(b_L, b_L)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \\ &\quad + C\|Q_a(\bar{u}_{n-1}, \bar{u}_{n-1}) + Q_a(u_L, \bar{u}_{n-1}) + Q_a(\bar{u}_{n-1}, u_L) + Q_a(u_L, u_L)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \\ &\leq C \int_0^\infty \left(\|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|b_L\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \right. \\ &\quad + \|b_L\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + 2\|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \\ &\quad \left. + \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \right) (t) dt \\ &\leq C \left(M^2 + \varepsilon_0 M + \varepsilon_0^2 \right) \left(1 + \frac{1}{\nu} + \frac{1}{\mu} \right)^2, \end{aligned}$$

where we have used the facts

$$\begin{aligned} &\int_0^\infty \|b_L\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} (t) dt \\ &\leq C \|b_L\|_{L^{\frac{2pq}{pq-3q+3p}}(\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}})} \|\bar{b}_{n-1}\|_{L^{\frac{2pq}{pq+3q-3p}}(\dot{B}_{q,1}^{\frac{3}{q}})}, \end{aligned}$$

and

$$\begin{aligned} &\int_0^\infty \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} (t) dt \\ &\leq C \|\bar{b}_{n-1}\|_{L^{\frac{2pq}{pq-3q+3p}}(\dot{B}_{q,1}^{\frac{3}{q}+1})} \|b_L\|_{L^{\frac{2pq}{pq+3q-3p}}(\dot{B}_{q,1}^{\frac{3}{q}})}. \end{aligned}$$

Again, taking advantage of (4.10), (4.11), interpolation and Hölder inequality, it follows from Lemma 3.11 that

$$\begin{aligned}
& \|\bar{b}_n\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\bar{b}_n\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \\
& \leq C \|Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, \bar{b}_{n-1}) + Q_b(u_L - \varepsilon J_L, \bar{b}_{n-1})\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} \\
& \quad + C \|Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, b_L) + Q_b(u_L - \varepsilon J_L, b_L)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} \\
& \leq C \int_0^\infty \left(\|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \right. \\
& \quad + \|\bar{b}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\
& \quad \left. + \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \right) (t) dt \\
& \leq C \left(M^2 + \varepsilon_0 M + \varepsilon_0^2 \right) \left(1 + \frac{1}{\nu} + \frac{1}{\mu} \right)^2,
\end{aligned}$$

and

$$\begin{aligned}
& \|\bar{J}_n\|_{L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \nu \|\bar{J}_n\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \\
& \leq C \|Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, \text{curl}^{-1} \bar{J}_{n-1}) + Q_b(u_L - \varepsilon J_L, \text{curl}^{-1} \bar{J}_{n-1})\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}})} \\
& \quad + C \|Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}_{n-1}, \text{curl}^{-1} J_L) + Q_b(u_L - \varepsilon J_L, \text{curl}^{-1} J_L)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}})} \\
& \leq C \int_0^\infty \left(\|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}} \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \right. \\
& \quad + \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} + \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\
& \quad + \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} + \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}} + \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\
& \quad + \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\bar{J}_{n-1}\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} + \|\bar{u}_{n-1}\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \\
& \quad + \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}} \\
& \quad \left. + \|u_L\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} \|J_L\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \right) (t) dt \\
& \leq C \left(M^2 + \varepsilon_0 M + \varepsilon_0^2 \right) \left(1 + \frac{1}{\nu} + \frac{1}{\mu} \right)^2.
\end{aligned}$$

By choosing ε_0, M sufficiently small such that

$$M \leq \frac{1}{9C \left(1 + \frac{1}{\mu} + \frac{1}{\nu} \right)^2}, \quad \varepsilon_0 \leq \frac{1}{9C \left(1 + \frac{1}{\mu} + \frac{1}{\nu} \right)^2}, \quad (4.17)$$

then one find that

$$\begin{aligned}
& \|\bar{u}_n\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu \|\bar{u}_n\|_{L_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \\
& \quad + \|\bar{b}_n\|_{L_T^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\bar{b}_n\|_{L_T^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \\
& \quad + \|\bar{J}_n\|_{L_T^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\bar{J}_n\|_{L_T^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \leq M.
\end{aligned}$$

Arguing by induction, we conclude that (4.9) holds true for all $n \in \mathbb{N}$.

Once the uniform bounds is established for $(\bar{u}_n, \bar{b}_n, \bar{J}_n)$, we shall use compactness arguments to prove convergence.

4.2. Convergence of $(\bar{u}_n, \bar{b}_n, \bar{J}_n)$. We claim that $(\bar{u}_n, \bar{b}_n, \bar{J}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in

$$\left(L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1}) \cap L^1(\dot{B}_{p,1}^{\frac{3}{p}+1}) \right) \times \left(L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}) \right) \times \left(L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}) \right).$$

For all $n \in \mathbb{N}$, let us consider the difference

$$(\delta \bar{u}^n, \delta \bar{b}^n, \delta \bar{J}^n) := (\bar{u}_{n+1}, \bar{b}_{n+1}, \bar{J}_{n+1}) - (\bar{u}_n, \bar{b}_n, \bar{J}_n),$$

and define

$$\begin{aligned} \delta^n &:= \|\delta \bar{u}^n\|_{L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \mu \|\delta \bar{u}^n\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \\ &\quad + \|\delta \bar{b}^n\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\delta \bar{u}^n\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \\ &\quad + \|\delta \bar{J}^n\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \nu \|\delta \bar{J}^n\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})}. \end{aligned}$$

Then thanks to Lemma 3.11, we only need to estimate the following terms

$$\begin{aligned} I_1^n &:= \|Q_a(\delta \bar{b}^{n-1}, \bar{b}_n)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \|Q_a(\bar{b}_{n-1}, \delta \bar{b}^{n-1})\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \\ &\quad + \|Q_a(b_L, \delta \bar{b}^{n-1})\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \|Q_a(\delta \bar{b}^{n-1}, b_L)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \\ &\quad + \|Q_a(\delta \bar{u}^{n-1}, \bar{u}_n)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \|Q_a(\bar{u}_{n-1}, \delta \bar{u}^{n-1})\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \\ &\quad + \|Q_a(u_L, \delta \bar{u}^{n-1})\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \|Q_a(\delta \bar{u}^{n-1}, u_L)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})}, \end{aligned}$$

$$\begin{aligned} I_2^n &:= \|Q_b(\delta \bar{u}^{n-1} - \varepsilon \delta \bar{J}^{n-1}, \bar{b}_n)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \|Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}^{n-1}, \delta \bar{b}^{n-1})\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} \\ &\quad + \|Q_a(u_L - \varepsilon J_L, \delta \bar{b}^{n-1})\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \|Q_a(\delta \bar{u}^{n-1} - \varepsilon \delta \bar{J}^{n-1}, b_L)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})}, \end{aligned}$$

$$\begin{aligned} I_3^n &:= \|Q_b(\delta \bar{u}^{n-1} - \varepsilon \delta \bar{J}^{n-1}, \text{curl}^{-1} \bar{J}_n) + Q_b(\bar{u}_{n-1} - \varepsilon \bar{J}^{n-1}, \text{curl}^{-1} \delta \bar{J}^{n-1})\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}})} \\ &\quad + \|Q_a(u_L - \varepsilon J_L, \text{curl}^{-1} \delta \bar{J}^{n-1}) + Q_a(\delta \bar{u}^{n-1} - \varepsilon \delta \bar{J}^{n-1}, \text{curl}^{-1} J_L)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}})}. \end{aligned}$$

Along extremely similar calculations as previous subsections, thanks to (4.17) and uniform bounds (4.9), one can get

$$\begin{aligned} \delta^n &\leq C(I_1^n + I_2^n + I_3^n) \\ &\leq C\delta^{n-1} \left(1 + \frac{1}{\mu} + \frac{1}{\nu} \right)^2 (M + \varepsilon_0) \\ &\leq \frac{1}{2} \delta^{n-1}. \end{aligned}$$

Thus, we know that $(\bar{u}_n, \bar{b}_n, \bar{J}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the space

$$\left(L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1}) \cap L^1(\dot{B}_{p,1}^{\frac{3}{p}+1}) \right) \times \left(L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}) \right) \times \left(L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}) \right)$$

and there exists a triplet $(\bar{u}, \bar{b}, \bar{J})$ such that

$$\begin{aligned}\bar{u}_n &\rightarrow \bar{u} \quad \text{in } L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1}) \cap L^1(\dot{B}_{p,1}^{\frac{3}{p}+1}), \\ \bar{b}_n &\rightarrow \bar{b} \quad \text{in } L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}), \\ \bar{J}_n &\rightarrow \bar{J} \quad \text{in } L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}).\end{aligned}$$

By the product laws that we used frequently before, it is not difficult to prove the convergence of non-linear terms in (4.7), by an example, we show that

$$\begin{aligned}& \|Q_b(\bar{u}_n - \varepsilon \bar{J}_n, \text{curl}^{-1} \bar{J}_n) - Q_b(\bar{u} - \varepsilon \bar{J}, \text{curl}^{-1} \bar{J})\|_{L_T^1(\dot{B}_{q,1}^{\frac{3}{q}})} \\ & \leq \|Q_b(\bar{u}_n - \bar{u} + \varepsilon \bar{J}_n - \varepsilon \bar{J}, \text{curl}^{-1} \bar{J}_n)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}})} \\ & \quad + \|Q_b(\bar{u} - \varepsilon \bar{J}, \text{curl}^{-1} \bar{J} - \text{curl}^{-1} \bar{J}_n)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}})} \\ & \lesssim M \left(\|\bar{u}_n - \bar{u}\|_{L_T^2(\dot{B}_{p,1}^{\frac{3}{p}}) \cap L^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} + \|\varepsilon \bar{J}_n - \varepsilon \bar{J}\|_{L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1})} \right).\end{aligned}$$

Hence, we conclude that $(\bar{u}, \bar{b}, \bar{J})$ is indeed a solution of (4.1). This implies that $(u, b, J) = (u_L + \bar{u}, b_L + \bar{b}, J_L + \bar{J})$ is a solution of (1.10) in

$$\left(L^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1}) \cap L^1(\dot{B}_{p,1}^{\frac{3}{p}+1}) \right) \times \left(L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}) \right) \times \left(L^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1}) \cap L^1(\dot{B}_{q,1}^{\frac{3}{q}+1}) \right)$$

and satisfies (2.4).

The continuity of (u, B, J) is straightforward. Indeed, the right-hand sides of (4.1) belong to $L_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})$, $L_T^1(\dot{B}_{q,1}^{\frac{3}{q}-1})$, $L_T^1(\dot{B}_{q,1}^{\frac{3}{q}-1})$ respectively.

4.3. Uniqueness. Let (u_1, b_1, J_1) and (u_2, b_2, J_2) be two solutions of (4.1) in

$$E_p \times E_q \times E_q$$

with same initial data $(u_0, b_0, J_0) \in \dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{q,1}^{\frac{3}{q}-1} \times \dot{B}_{q,1}^{\frac{3}{q}-1}$. Without loss of generality, we assume (u_2, b_2, J_2) is the solution that constructed in the previous steps (in fact, one only needs J_2 is small in E_q).

Set $\delta u := u_2 - u_1$, $\delta b := b_2 - b_1$, and $\delta J := J_2 - J_1$, we see that $(\delta u, \delta b, \delta J)$ satisfies

$$\begin{cases} \partial_t \delta u - \mu \Delta \delta u = Q_a(\delta b, b_1) + Q_a(b_2, \delta b) - Q_a(\delta u, u_1) - Q_a(u_2, \delta u), \\ \partial_t \delta b - \nu \Delta \delta b = Q_b(u_1 - \varepsilon J_1, \delta b) + Q_b(\delta u - \varepsilon \delta J, b_2), \\ \partial_t \delta J - \nu \Delta \delta J = \nabla \times Q_b(u_1 - \varepsilon J_1, \text{curl}^{-1} \delta J) + \nabla \times Q_b(\delta u - \varepsilon \delta J, \text{curl}^{-1} J_2), \\ (\delta u(0, x), \delta b(0, x), \delta J(0, x)) = (0, 0, 0). \end{cases}$$

With our assumptions on two solutions, one can verify that the right-hand sides of above system belong to $L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})$, $L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})$, $L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})$ respectively, thus by

Lemma 3.11, Lemma 3.8 (take $\theta = \frac{3}{p} - \frac{3}{q}$) and product laws (4.10)-(4.13), one has

$$\begin{aligned}
& \|\delta u(t)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|(\delta b, \delta J)(t)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} + \int_0^t \left(\|\delta u(\tau)\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} + \|(\delta b, \delta J)(\tau)\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}} \right) d\tau \\
& \lesssim \int_0^t \left(\|(b_1, b_2)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\delta b\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} + \|(b_1, b_2)\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|\delta b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|(u_1, u_2)\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\delta u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \right. \\
& \quad + (\|u_1\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \|J_1\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) \|\delta b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|b_2\|_{\dot{B}_{q,1}^{\frac{3}{q}}} (\|\delta u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \|\delta J\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}) \\
& \quad + (\|u_1\|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \|(J_1, J_2)\|_{\dot{B}_{q,1}^{\frac{3}{q}}}) \|\delta J\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + (\|u_1\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} + \|(J_1, J_2)\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}}) \|\delta J\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \\
& \quad \left. + \|\delta u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|J_2\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|\delta u\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} \|J_2\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \right) d\tau, \\
& \lesssim \int_0^t \Omega(\tau) (\|\delta u(\tau)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|(\delta b, \delta J)(\tau)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}) d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\Omega(\tau) := & \left(\|(u_1, u_2)(\tau)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + 1 \right) \|(u_1, u_2)(\tau)\|_{\dot{B}_{p,1}^{\frac{3}{p}+1}} \\
& + \left(\|(b_1, b_2, J_1, J_2)(\tau)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} + 1 \right) \|(b_1, b_2, J_1, J_2)(\tau)\|_{\dot{B}_{q,1}^{\frac{3}{q}+1}}.
\end{aligned}$$

It is clear that our assumptions ensure $\Omega \in L^1(\mathbb{R}_+)$, Gronwall lemma then enables us to conclude that $(\delta u, \delta b, \delta J) \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}^3$.

For completing the proof of the existence for the original Hall-MHD system, we have to check that $J_0 = \nabla \times b_0$ implies $J = \nabla \times b$, so that (u, b) is indeed a distributional solution of (1.1)–(1.3). Actually, it is easy to see that $\nabla \times b_L = J_L$, thus we only need to check $\bar{J} = \nabla \times \bar{b}$. Noticing that

$$(\partial_t - \Delta)(\nabla \times \bar{b} - \bar{J}) = \nabla \times Q_b(u - \varepsilon J, \text{curl}^{-1}(\nabla \times \bar{b} - \bar{J})).$$

Hence, using Lemma 3.11 and product law (4.11), one gets for all $t \geq 0$,

$$\begin{aligned}
& \|(\nabla \times \bar{b} - \bar{J})(t)\|_{\dot{B}_{q,1}^{\frac{3}{q}-2}} + \int_0^t \|\nabla \times \bar{b} - \bar{J}\|_{\dot{B}_{q,1}^{\frac{3}{q}}} d\tau \\
& \leq C \int_0^t \|u - \varepsilon J\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|\nabla \times \bar{b} - \bar{J}\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} d\tau.
\end{aligned}$$

Then (2.3) combined with interpolation inequality and Gronwall lemma enables that $\nabla \times \bar{b} - \bar{J} \equiv 0$ on $\mathbb{R}_+ \times \mathbb{R}^3$.

A slight modification on the proof could yields local well-posedness by assuming only $\|\nabla \times b_0\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}$ is small enough and in addition

$$-\frac{1}{3} < \frac{1}{q} - \frac{1}{q}.$$

In fact, by Lemma 3.10 one can guarantee (4.15) and (4.16) are satisfied with sufficient small time $T > 0$. And Lemma 3.11 implies that

$$\|J_L\|_{L_T^\infty(\dot{B}_{q,1}^{\frac{3}{q}-1})} \lesssim \|\nabla \times b_0\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \lesssim \varepsilon_0.$$

We omit another details here and thus the proof of Theorem 2.1 is completed. \square

5. THE PROOF OF THEOREM 2.2

In this section, we devote to proving the decay estimates of the solution provided by Theorem 2.1. Started with the data (u_0, b_0) satisfies (2.2), we know that the solution $(u, b) \in E_p \times E_q$ such that $\nabla \times b \in E_q$ and satisfies (2.4).

For any fixed $m \geq 1$, let $T \geq 0$ be the largest t such that

$$W(t) := \sup_{0 \leq \tau \leq t} \tau^{\frac{m}{2}} \left(\|D^m u(\tau)\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|D^m b(\tau)\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \right) \leq C_0 \varepsilon_0,$$

where C_0 will be chosen later.

5.1. Decay estimates for velocity fields. Applying $\dot{\Delta}_j$ to the equation (1.1) and taking D_x^α ($|\alpha| = m$) on the resulting equation leads to

$$\partial_t \dot{\Delta}_j D_x^\alpha u - \mu \Delta \dot{\Delta}_j D_x^\alpha u = \dot{\Delta}_j D_x^\alpha \mathcal{P} \operatorname{div} (b \otimes b) - \dot{\Delta}_j D_x^\alpha \mathcal{P} \operatorname{div} (u \otimes u).$$

Then

$$\dot{\Delta}_j D_x^\alpha u = e^{t\mu\Delta} \dot{\Delta}_j D_x^\alpha u_0 + \int_0^t e^{(t-s)\mu\Delta} \mathcal{P} \left(\dot{\Delta}_j D_x^\alpha \operatorname{div} (b \otimes b) - \dot{\Delta}_j D_x^\alpha \operatorname{div} (u \otimes u) \right) ds.$$

Lemma 3.3 thus implies that

$$\begin{aligned} \|\dot{\Delta}_j D_x^\alpha u\|_{L^p} &\leq C e^{-c\mu 2^{2j}t} \|\dot{\Delta}_j D_x^\alpha u_0\|_{L^p} \\ &\quad + C \int_0^t e^{-c\mu 2^{2j}(t-s)} \left(\|\dot{\Delta}_j D_x^\alpha \mathcal{P} \operatorname{div} (b \otimes b)\|_{L^p} \right. \\ &\quad \left. + \|\dot{\Delta}_j D_x^\alpha \mathcal{P} \operatorname{div} (u \otimes u)\|_{L^p} \right) ds \\ &\leq C e^{-c\mu 2^{2j}t} \|\dot{\Delta}_j D_x^\alpha u_0\|_{L^p} + A_1 + A_2 + A_3, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} A_1 &:= C \int_0^{\frac{t}{2}} e^{-c\mu 2^{2j}(t-s)} \left(\|\dot{\Delta}_j D_x^\alpha \mathcal{P} \operatorname{div} (b \otimes b)\|_{L^p} + \|\dot{\Delta}_j D_x^\alpha \mathcal{P} \operatorname{div} (u \otimes u)\|_{L^p} \right) ds, \\ A_2 &:= C \int_{\frac{t}{2}}^t e^{-c\mu 2^{2j}(t-s)} 2^j \|\dot{\Delta}_j D_x^{\alpha-1} \mathcal{P} \operatorname{div} (b \otimes b)\|_{L^p} ds, \\ A_3 &:= C \int_{\frac{t}{2}}^t e^{-c\mu 2^{2j}(t-s)} 2^j \|\dot{\Delta}_j D_x^{\alpha-1} \operatorname{div} (u \otimes u)\|_{L^p} ds. \end{aligned}$$

Noticing there exists a constant $\tilde{c} > 0$ such that

$$e^{-c\mu 2^{2j}t} 2^j k \leq e^{-\tilde{c}\mu 2^{2j}t} t^{-\frac{k}{2}}, \quad \text{for any } k \geq 0. \tag{5.2}$$

By employing Proposition 3.2, a straightforward calculation shows that

$$\begin{aligned} A_1 &\leq C t^{-\frac{m}{2}} \int_0^{\frac{t}{2}} e^{-\tilde{c}\mu 2^{2j}(t-s)} \left(\|\dot{\Delta}_j \operatorname{div} (b \otimes b)\|_{L^p} + \|\dot{\Delta}_j \operatorname{div} (u \otimes u)\|_{L^p} \right) ds \\ &\leq C g_j t^{-\frac{m}{2}} 2^{-(\frac{3}{p}-1)j} \left(\|\operatorname{div} (b \otimes b)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} + \|\operatorname{div} (u \otimes u)\|_{L^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \right) \\ &\leq C g_j t^{-\frac{m}{2}} 2^{-(\frac{3}{p}-1)j} \left(\|u\|_{E_p}^2 + \|b\|_{E_q}^2 \right) \\ &\leq C g_j t^{-\frac{m}{2}} 2^{-(\frac{3}{p}-1)j} \left(1 + \frac{1}{\mu} + \frac{1}{\nu} \right) \varepsilon_0^2, \end{aligned} \tag{5.3}$$

where $\{g_j\}_{j \in \mathbb{Z}} \in \ell^1$ and $\|\{g_j\}\|_{\ell^1} \leq 1$.

Thanks to (3.3), we have

$$\begin{aligned}
\|\dot{\Delta}_j D_x^{\alpha-1} \mathcal{P} \operatorname{div}(b \otimes b)\|_{L^p} &\leq C 2^j \|\dot{\Delta}_j (D_x^{\alpha-1} b \otimes b)\|_{L^p} \\
&\leq C g_j 2^{-\left(\frac{3}{q}-1\right)j} \|D_x^{\alpha-1} b \otimes b\|_{\dot{B}_{p,1}^{\frac{3}{q}}} \\
&\leq C g_j 2^{-\left(\frac{3}{q}-1\right)j} \|D_x^{\alpha-1} b\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \\
&\quad + C g_j 2^{-\left(\frac{3}{q}-1\right)j} \|D_x^{\alpha-1} b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}}.
\end{aligned}$$

By means of interpolation, we get

$$\begin{aligned}
&\|D_x^{\alpha-1} b\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|D_x^{\alpha-1} b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{q,1}^{\frac{6}{q}-\frac{3}{p}}} \\
&\lesssim \left(\|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}^{1-r} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}^r \right) \left(\|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{1}{m}-\frac{3}{q}-1}}^{\frac{1}{m}} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}^{1-\frac{1}{m}} \right) \\
&\quad + \left(\|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \right) \left(\|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{1}{m}-r}}^{\frac{1}{m}-r} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}^{1-\frac{1}{m}+r} \right) \\
&\lesssim \|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{1}{m}-r}}^{1+\frac{1}{m}-r} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}^{1-\frac{1}{m}+r},
\end{aligned}$$

where $r := \frac{3}{m} \left(\frac{1}{p} - \frac{1}{q} \right) \leq \frac{1}{m}$.

Since $\frac{3}{p} - \frac{3}{q} < 1$, (5.2) and Hölder inequality imply that

$$\begin{aligned}
A_2 &\leq C g_j 2^{-\left(\frac{3}{p}-1\right)j} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\frac{3}{q})} \|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{1}{m}-r}}^{1+\frac{1}{m}-r} \|b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}}^{1-\frac{1}{m}+r} ds \\
&\leq C g_j 2^{-\left(\frac{3}{p}-1\right)j} \left(\frac{t}{2}\right)^{-\frac{m}{2}(1+\frac{1}{m}-r)} W^{1+m-r} \varepsilon_0^{1-\frac{1}{m}+r} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\frac{3}{q})} ds \quad (5.4) \\
&\leq 2^{\frac{m}{2}+1} C g_j 2^{-\left(\frac{3}{p}-1\right)j} t^{-\frac{m}{2}} W^{1+\frac{1}{m}-r} \varepsilon_0^{1-\frac{1}{m}+r}.
\end{aligned}$$

Thanks to (4.10), we have

$$\begin{aligned}
\|\dot{\Delta}_j D_x^{\alpha-1} \mathcal{P} \operatorname{div}(u \otimes u)\|_{L^p} &\leq C 2^j \|\dot{\Delta}_j (D_x^{\alpha-1} u \otimes u)\|_{L^\infty(L^p)} \\
&\leq C g_j 2^{-\left(\frac{3}{p}-1\right)j} \|D_x^{\alpha-1} u \otimes u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \\
&\leq C g_j 2^{-\left(\frac{3}{p}-1\right)j} \|D_x^{\alpha-1} u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}}.
\end{aligned}$$

By means of interpolation,

$$\begin{aligned}
&\|D_x^{\alpha-1} u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \\
&\lesssim \|D_x^\alpha u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} \|D_x^\alpha u\|_{\dot{B}_{p,1}^{\frac{1}{m}-\frac{3}{p}-1}}^{\frac{1}{m}} \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}^{1-\frac{1}{m}}.
\end{aligned}$$

Thus, (5.2) and Hölder inequality imply that

$$\begin{aligned}
A_3 &\leq C g_j 2^{-\left(\frac{3}{p}-1\right)j} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|D_x^\alpha u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}^{1+\frac{1}{m}} \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}^{1-\frac{1}{m}} ds \\
&\leq C g_j 2^{-\left(\frac{3}{p}-1\right)j} \left(\frac{t}{2}\right)^{-\frac{m+1}{2}} W^{1+\frac{1}{m}} \varepsilon_0^{1-\frac{1}{m}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} ds. \quad (5.5) \\
&\leq 2^{\frac{m}{2}+1} C g_j 2^{-\left(\frac{3}{p}-1\right)j} t^{-\frac{m}{2}} W^{1+\frac{1}{m}} \varepsilon_0^{1-\frac{1}{m}}.
\end{aligned}$$

Putting (5.3), (5.4) and (5.5) together, one has

$$t^{\frac{m}{2}} \|D^m u\|_{\dot{B}_{p,1}^{\frac{3}{q}-1}} \lesssim \varepsilon_0 + \left(1 + \frac{1}{\mu} + \frac{1}{\nu}\right) \varepsilon_0^2 + W^{1+\frac{1}{m}-r} \varepsilon_0^{1-\frac{1}{m}+r} + W^{1+\frac{1}{m}} \varepsilon_0^{1-\frac{1}{m}}. \quad (5.6)$$

5.2. Decay estimates for Magnetic fields. Applying $\dot{\Delta}_j$ to equation (1.3) and taking D^α on the resulting equation leads to

$$\partial_t \dot{\Delta}_j D^\alpha b - \nu \Delta \dot{\Delta}_j D^\alpha b = \dot{\Delta}_j D^\alpha \nabla \times ((u - \varepsilon J) \times b). \quad (5.7)$$

Then

$$\dot{\Delta}_j D_x^\alpha b = e^{t\nu\Delta} \dot{\Delta}_j D_x^\alpha b_0 + \int_0^t e^{(t-s)\nu\Delta} \left(\dot{\Delta}_j D_x^\alpha \nabla \times (u \times b) - \varepsilon \dot{\Delta}_j D_x^\alpha \nabla \times (J \times b) \right) ds.$$

Lemma 3.3 thus implies that

$$\begin{aligned} \|\dot{\Delta}_j D_x^\alpha b\|_{L^q} &\leq C e^{-c\nu 2^{2j}t} \|\dot{\Delta}_j D_x^\alpha b_0\|_{L^q} \\ &\quad + C \int_0^t e^{-c\nu 2^{2j}(t-s)} \left(\|\dot{\Delta}_j D_x^\alpha \nabla \times (u \times b)\|_{L^p} \right. \\ &\quad \quad \left. - \|\dot{\Delta}_j D_x^\alpha \nabla \times (\varepsilon J \times b)\|_{L^q} \right) ds \\ &\leq C e^{-c\nu 2^{2j}t} \|\dot{\Delta}_j D_x^\alpha b_0\|_{L^q} + A_4 + A_5 + A_6, \end{aligned} \quad (5.8)$$

where

$$\begin{aligned} A_4 &:= C \int_0^{\frac{t}{2}} e^{-c\nu 2^{2j}(t-s)} \left(\|\dot{\Delta}_j D_x^\alpha \nabla \times (u \times b)\|_{L^q} + \|\dot{\Delta}_j D_x^\alpha \nabla \times (\varepsilon J \times b)\|_{L^q} \right) ds, \\ A_5 &:= C \int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^j \|\dot{\Delta}_j D_x^{\alpha-1} \nabla \times (u \times b)\|_{L^q} ds, \\ A_6 &:= C \int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^j \|\dot{\Delta}_j D_x^{\alpha-1} \nabla \times (\varepsilon J \times b)\|_{L^q} ds. \end{aligned}$$

Similar with the method of getting estimates (5.3) and (5.5), one can easily show that

$$\begin{aligned} A_4 &\leq C t^{-\frac{m}{2}} \int_0^{\frac{t}{2}} e^{-\tilde{c}\mu 2^{2j}(t-s)} \left(\|\dot{\Delta}_j \nabla \times (u \times b)\|_{L^q} + \|\dot{\Delta}_j \nabla \times (\varepsilon J \times b)\|_{L^q} \right) ds \\ &\leq C g_j t^{-\frac{m}{2}} 2^{-(\frac{3}{q}-1)j} \left(\|\nabla \times (u \times b)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} + \|\nabla \times (\varepsilon J \times b)\|_{L^1(\dot{B}_{q,1}^{\frac{3}{q}-1})} \right) \\ &\leq C g_j t^{-\frac{m}{2}} 2^{-(\frac{3}{q}-1)j} \left(\|u\|_{E_p}^2 + \|(b, J)\|_{E_q}^2 \right) \\ &\leq C g_j t^{-\frac{m}{2}} 2^{-(\frac{3}{q}-1)j} \left(1 + \frac{1}{\mu} + \frac{1}{\nu}\right) \varepsilon_0^2, \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} A_5 &\leq C g_j 2^{-(\frac{3}{q}-1)j} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|D_x^{\alpha-1}(u \times b)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} ds \\ &\leq C g_j 2^{-(\frac{3}{q}-1)j} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \left(\|D_x^{\alpha-1} u \times b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u \times D_x^{\alpha-1} b\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \right) ds \\ &\leq C g_j 2^{-(\frac{3}{q}-1)j} W^{1+\frac{1}{m}} \varepsilon_0^{1-\frac{1}{m}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}(1+m)} ds \\ &\leq 2^{\frac{m}{2}+1} C g_j 2^{-(\frac{3}{q}-1)j} t^{-\frac{m}{2}} W^{1+\frac{1}{m}} \varepsilon_0^{1-\frac{1}{m}}. \end{aligned} \quad (5.10)$$

Because $\operatorname{div} b = 0$, one can rewrite

$$\nabla \times (\varepsilon J \times b) = \varepsilon \nabla \times \left(\operatorname{div} (b \otimes b) - \nabla \left(\frac{|b|^2}{2} \right) \right) = \varepsilon \nabla \times (\operatorname{div} (b \otimes b)),$$

then, Hölder inequality yields

$$\begin{aligned} A_6 &\leq C\varepsilon \int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^3 j \|\dot{\Delta}_j D_x^{\alpha-1} b \otimes b\|_{L^q} ds \\ &\leq C\varepsilon 2^{-(\frac{3}{q}-1)j} g_j \int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^{2j} \|D_x^{\alpha-1} (b \otimes b)\|_{\dot{B}_{q,1}^{\frac{3}{q}}} ds \\ &\leq C\varepsilon 2^{-(\frac{3}{q}-1)j} g_j \int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^{2j} \|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \|\nabla \times b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} ds \quad (5.11) \\ &\leq C\varepsilon \varepsilon_0 2^{-(\frac{3}{q}-1)j} g_j \left(\frac{t}{2}\right)^{-\frac{m}{2}} \int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^{2j} s^{\frac{m}{2}} \|D_x^\alpha b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} ds \\ &\leq 2^{\frac{m}{2}+1} C \frac{\varepsilon}{\nu} 2^{-(\frac{3}{q}-1)j} g_j t^{-\frac{m}{2}} W(T) \varepsilon_0, \end{aligned}$$

here we use the fact that

$$\int_{\frac{t}{2}}^t e^{-c\nu 2^{2j}(t-s)} 2^{2j} ds \leq \frac{2}{c\nu}.$$

Putting (5.9), (5.10) and (5.11) together, one has

$$t^{\frac{m}{2}} \|D^m b\|_{\dot{B}_{q,1}^{\frac{3}{q}-1}} \lesssim \varepsilon_0 + \left(1 + \frac{1}{\mu} + \frac{1}{\nu}\right) \varepsilon_0^2 + W^{1+\frac{1}{m}} \varepsilon_0^{1-\frac{1}{m}} + \frac{\varepsilon}{\nu} W \varepsilon_0.$$

This combine with (5.6) implies that

$$W(T) \leq C\varepsilon_0 + C \left(1 + \frac{1}{\mu} + \frac{1}{\nu} + \frac{\varepsilon}{\nu} C_0 + C_0^{1+\frac{1}{m}-r} + C_0^{1+\frac{1}{m}}\right) \varepsilon_0^2.$$

Thanks to (4.17), one can take suitable C_0 such that $W(T) < \frac{1}{2} C_0 \varepsilon_0$. By the continuous induction, we have $W(t) \leq C_0 \varepsilon_0$ for all $t \geq 0$. It completes the proof of Theorem 2.2. \square

6. THE PROOF OF THEOREM 2.3

In order to prove the theorem ($\mu = \nu$ is assumed), we need to notice from [10] that if (u, b) is a solution of Hall-MHD system (1.1)-(1.3) in the sense of distribution, then the so-called *velocity of electron* $v := u - \varepsilon J$ satisfies:

$$\begin{aligned} \partial_t v - \mu \Delta v &= \mathcal{P}(\operatorname{div} (b \otimes b) - \operatorname{div} (u \otimes u)) - \varepsilon \nabla \times ((\nabla \times v) \times b) \\ &\quad + \nabla \times (v \times u) + 2\varepsilon \nabla \times (v \cdot \nabla b). \end{aligned} \quad (6.1)$$

The equation (6.1) is still quasi-linear compare to the equation of current J . However, owing to

$$((\nabla \times (\nabla \times v)) \times b, v)_{L^2} = ((\nabla \times v) \times b, \nabla \times v)_{L^2} = 0,$$

the most nonlinear term cancels out when performing an energy method. Thus, contrasting to the uniqueness part of Theorem 2.1, it will help us to release the smallness assumption on current J .

We now focusing to the proof of Theorem 2.3. Since $(u_{0,2}, b_{0,2})$ satisfies the initial conditions in [10] Theorem 2.2 about the local well-posedness of Hall-MHD.

Thus, supplemented with initial data $(u_{0,2}, b_{0,2})$ there exists a solution (u_2, b_2) on the maximal time interval $[0, T^*)$ fulfilling

$$(u_2, b_2, \nabla \times b_2) \in E_2(t),$$

for all $t < T^*$.

Define $v_i := u_i - \varepsilon J_i$, ($i = 1, 2$). It is then convenient to consider the difference $(\tilde{u}, \tilde{b}, \tilde{v}) := (u_1 - u_2, b_1 - b_2, v_1 - v_2)$, which satisfies:

$$\begin{cases} \partial_t \tilde{u} - \mu \Delta \tilde{u} := d_1, \\ \partial_t \tilde{b} - \mu \Delta \tilde{b} := d_2, \\ \partial_t \tilde{v} - \mu \Delta \tilde{v} := d_1 + d_3 + d_4 + d_5, \\ (\tilde{u}, \tilde{B}, \tilde{v})|_{t=0} = (u_{0,1} - u_{0,2}, b_{0,1} - b_{0,2}, v_{0,1} - v_{0,2}), \end{cases} \quad (6.2)$$

where

$$d_1 := \mathcal{P}(-\operatorname{div}(\tilde{b} \otimes \tilde{b}) + \operatorname{div}(\tilde{b} \otimes b_1) + \operatorname{div}(b_1 \otimes \tilde{b}) + \operatorname{div}(\tilde{u} \otimes \tilde{u}) - \operatorname{div}(\tilde{u} \otimes u_1) - \operatorname{div}(u_1 \otimes \tilde{u})),$$

$$d_2 := \nabla \times (-\tilde{v} \times \tilde{b} + v_1 \times \tilde{b} + \tilde{v} \times b_1),$$

$$d_3 := -\varepsilon \nabla \times (-\nabla \times \tilde{v}) \times \tilde{b} + (\nabla \times v_1) \times \tilde{b} + (\nabla \times \tilde{v}) \times b_1,$$

$$d_4 := \varepsilon \nabla \times (-\tilde{v} \times \tilde{u} + v_1 \times \tilde{u} + \tilde{v} \times u_1),$$

$$d_5 := 2\varepsilon \nabla \times (-\tilde{v} \cdot \nabla \tilde{b} + v_1 \cdot \nabla \tilde{b} + \tilde{v} \cdot \nabla b_1).$$

We know that $(\tilde{u}, \tilde{b}, \tilde{v}) \in L_t^\infty(\dot{B}_{2,1}^{\frac{1}{2}}) \cap L_t^1(\dot{B}_{2,1}^{\frac{5}{2}})$ since both (u_i, b_i, v_i) belong to that space. Now, we shall estimate the difference $(\tilde{u}, \tilde{b}, \tilde{v})$ in the space $\dot{B}_{2,1}^{\frac{1}{2}}$, one thus has to verify d_1 to d_5 live in the space $L_T^1(\dot{B}_{2,1}^{\frac{1}{2}})$ firstly, which is quite easy. A standard energy method gives that for all $t \in [0, T^*)$,

$$\begin{aligned} \|(\tilde{u}, \tilde{B}, \tilde{v})(t)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \mu \int_0^t \|(\tilde{u}, \tilde{B}, \tilde{v})\|_{\dot{B}_{2,1}^{\frac{5}{2}}} d\tau \lesssim \int_0^t (\|(d_1, d_2, d_4, d_5)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} \\ + \|\nabla \times ((\nabla \times v_1) \times \tilde{b})\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \sum_{j \in \mathbb{Z}} 2^{\frac{3j}{2}} \|[\Delta_j, (b_1 + \tilde{b}) \times] (\nabla \times \tilde{v})\|_{L^2}) d\tau. \end{aligned} \quad (6.3)$$

Using the fact that $\dot{B}_{2,1}^{\frac{1}{2}}$ is an algebra and

$$\|b\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \sim \|(u, v)\|_{\dot{B}_{2,1}^{\frac{1}{2}}}, \quad \|\nabla b\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \sim \|(u, v)\|_{\dot{B}_{2,1}^{\frac{3}{2}}}.$$

one has

$$\begin{aligned} \|d_1\|_{\dot{B}_{2,1}^{\frac{1}{2}}} &\lesssim \|\tilde{b}\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^2 + \|\tilde{b}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|b_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^2 + \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|u_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}}, \\ \|d_2\|_{\dot{B}_{2,1}^{\frac{1}{2}}} &\lesssim \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\tilde{b}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|v_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\tilde{b}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|b_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}}, \\ \|d_4\|_{\dot{B}_{2,1}^{\frac{1}{2}}} &\lesssim \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|v_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\tilde{u}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|u_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}}, \\ \|d_5\|_{\dot{B}_{2,1}^{\frac{1}{2}}} &\lesssim \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla \tilde{b}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla b_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|v_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla \tilde{b}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \\ &\lesssim \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|(\tilde{u}, \tilde{v})\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|(\tilde{u}, \tilde{v})\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|(u_1, v_1)\|_{\dot{B}_{2,1}^{\frac{3}{2}}}, \end{aligned}$$

and

$$\|\nabla \times ((\nabla \times v_1) \times \tilde{b})\|_{\dot{B}_{2,1}^{\frac{1}{2}}} \lesssim \|v_1\|_{\dot{B}_{2,1}^{\frac{5}{2}}} \|\tilde{b}\|_{\dot{B}_{2,1}^{\frac{1}{2}}}.$$

Thanks to the commutator estimate (see [8])

$$\sum_{j \in \mathbb{Z}} 2^{\frac{3j}{2}} \|[\dot{\Delta}_j, w]z\|_{L^2} \lesssim \|\nabla w\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|z\|_{\dot{B}_{2,1}^{\frac{1}{2}}},$$

we have

$$\sum_{j \in \mathbb{Z}} 2^{\frac{3j}{2}} \|[\dot{\Delta}_j, (b_1 + \tilde{b}) \times](\nabla \times \tilde{v})\|_{L^2} \lesssim \|(\tilde{u}, \tilde{v}, u_1, v_1)\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\tilde{v}\|_{\dot{B}_{2,1}^{\frac{3}{2}}}.$$

Hence, by interpolation and Young's inequality, inequality (6.3) becomes

$$\begin{aligned} & \|(\tilde{u}, \tilde{b}, \tilde{v})(t)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \mu \int_0^t \|(\tilde{u}, \tilde{b}, \tilde{v})(\tau)\|_{\dot{B}_{2,1}^{\frac{5}{2}}} d\tau \\ & \leq \|(\tilde{u}, \tilde{b}, \tilde{v})(0)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \int_0^t (\tilde{\Omega}(\tau) + C\|(\tilde{u}, \tilde{b}, \tilde{v})\|_{\dot{B}_{2,1}^{\frac{5}{2}}}) \|(\tilde{u}, \tilde{b}, \tilde{v})(\tau)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} d\tau \end{aligned} \quad (6.4)$$

with $\tilde{\Omega}(t) := C(\|u_1, b_1, v_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}}^2 + \|v_1\|_{\dot{B}_{2,1}^{\frac{5}{2}}})$.

Now, one needs to prove the following bootstrap argument (see similar result in [11]).

Lemma 6.1. *Let X, D, W be three nonnegative measurable functions on $[0, T]$. Assume that there exists a nonnegative real constant C such that for any $t \in [0, T]$,*

$$X(t) + \mu \int_0^t D(\tau) d\tau \leq X(0) + \int_0^t (\tilde{\Omega}(\tau)X(\tau) + CX(\tau)D(\tau)) d\tau. \quad (6.5)$$

If, in addition,

$$2CX(0) \exp\left(\int_0^T \tilde{\Omega}(\tau) d\tau\right) < \mu, \quad (6.6)$$

then, for any $t \in [0, T]$, one has

$$X(t) + \frac{\mu}{2} \int_0^t D d\tau \leq X(0) \exp\left(\int_0^t \tilde{\Omega} d\tau\right). \quad (6.7)$$

Proof. Let \tilde{T} be the largest $t \leq T$ such that

$$2C \sup_{0 \leq t' \leq t} X(t') \leq \mu. \quad (6.8)$$

Then, (6.5) implies that for all $t \in [0, \tilde{T}]$, we have

$$X(t) + \frac{\mu}{2} \int_0^t D d\tau \leq X(0) + \int_0^t \tilde{\Omega}(\tau)X(\tau) d\tau. \quad (6.9)$$

By Gronwall lemma, this yields for all $t \in [0, \tilde{T}]$,

$$X(t) + \frac{\mu}{2} \int_0^t D(\tau) d\tau \leq X(0) \exp\left(\int_0^t \tilde{\Omega}(\tau) d\tau\right).$$

Hence, it is clear that if (6.6) is satisfied, then (6.8) is satisfied with a strict inequality. A continuity argument thus ensures that we must have $\tilde{T} = T$ and thus (6.7) on $[0, T]$. \square

Noticing our assumptions on (u_1, b_1) ensure that $\tilde{\Omega} \in L^1(\mathbb{R}_+)$. By virtue of (2.6), let η satisfies

$$2C\eta \exp(\|\tilde{\Omega}\|_{L^1(\mathbb{R}_+)}) < \mu,$$

and apply Lemma 6.1 to inequality (6.4), we have for any $t \in [0, T^*)$,

$$\|(\tilde{u}, \tilde{B}, \tilde{v})(t)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \frac{\mu}{2} \int_0^t \|(\tilde{u}, \tilde{B}, \tilde{v})(\tau)\|_{\dot{B}_{2,1}^{\frac{5}{2}}} d\tau \leq \eta \exp(\|\tilde{\Omega}\|_{L^1(\mathbb{R}_+)}).$$

The above inequality ensures that $(\tilde{u}, \tilde{b}, \tilde{v}) \in L^\infty(0, T^*; \dot{B}_{2,1}^{\frac{1}{2}}) \cap L^1(0, T^*; \dot{B}_{2,1}^{\frac{5}{2}})$ and so does (u_2, b_2, v_2) , thus we conclude by classic arguments that (u_2, b_2, v_2) can continued beyond T^* , which finally implies that $T^* = \infty$. This completes the proof of Theorem 2.3. \square

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