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# The problem of $0.999\dots$ : Teachers' school-related content knowledge and their reactions to misconceptions

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*The identity  $0.999\dots=1$  connects to a variety of mathematical concepts and therefore allows for an investigation of teachers' school-related content knowledge (SRCK). This is a facet of their content knowledge (CK) that emphasizes links between school mathematics and the mathematics acquired in tertiary education. Some numerous reported misconceptions and potentially conflicting ideas on  $0.999\dots$  were presented to secondary school teachers in interviews. Especially in the case of conflicting ideas the relation between SRCK and teachers' reactions as part of their pedagogical content knowledge (PCK) was examined. The results showed that SRCK, as well as CK in general, is a prerequisite to successfully applying PCK. Further they lead to a discussion of the SRCK model's use in qualitative research.*

*Keywords: Qualitative research, Mathematics teachers, Secondary school mathematics, SRCK, Misconceptions*

## Introduction

Many research projects throughout the last four decades have shown that  $0.999\dots < 1$  is a common misconception among students from secondary school and university, especially among pre-service mathematics teachers. Studies report that between a quarter and half of pre-service mathematics teachers agree that  $0.999\dots < 1$  (Bauer, 2011; Buchholtz et al., 2012). Even among those who choose  $0.999\dots = 1$ , only a minority is able to give appropriate reasons. Among German pre-service mathematics teachers, the most common argument for equality refers to a rounding process (Buchholtz et al., 2012). According to Bauer (2011), there seems to be no difference between high school students' and pre-service teachers' answers, and Yopp, Burroughs and Lindaman (2011) suggest that this robust misconception is found among primary level in-service teachers as well.

Nevertheless, there is a need for teachers to deal with  $0.999\dots$ : In the federal state of Lower Saxony (Germany) the local curriculum explicitly requires that high school students after grade 10 "explicate the equality  $0.999\dots = 1$  as a result of an infinite process" (Niedersächsisches Kultusministerium [NK], 2015, p. 26, translated by the authors). Answering questions on  $0.999\dots$  requires well-founded knowledge of its nature. The mathematical background links to notions of infinity, convergence of series and the non-uniqueness of decimal representations. The concept of school-related content knowledge (SRCK), recently introduced by Dreher, Lindmeier, Heinze, and Niemand (2018), emphasizes such links between less formal school mathematics and formal academic mathematics.

Throughout this paper, we will focus on teachers' SRCK regarding the identity  $0.999\dots = 1$  and their reactions to students' erroneous comments. In line with Yopp et al. (2011), Buchholtz et al. fear that

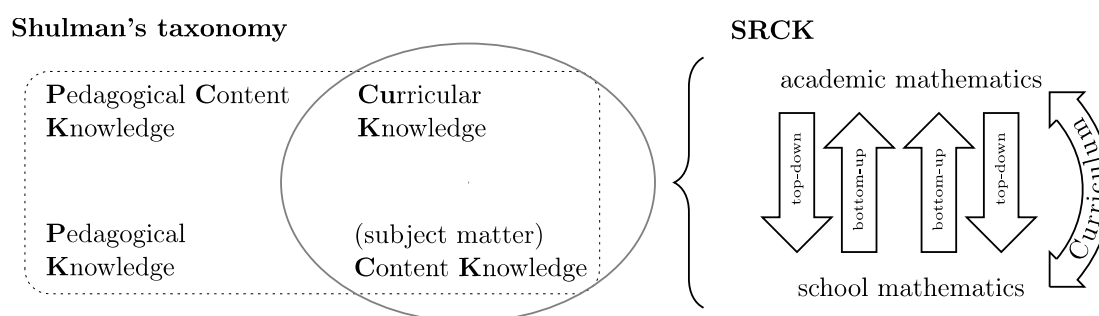
the academic knowledge in this field [0.999..., authors' remark] is not active, and so no substantial link between university and school mathematics can be made. (Buchholtz et al., 2012, p.116)

In that case, the high potential of cognitive conflicts that is intrinsic in this identity could be left unused: Teachers' deep understanding of the topic is necessary for successful teaching towards conceptual changes (Scott, Asoko, & Driver, 1991; Tall & Vinner, 1981). If it is missing, "teachers have the potential of undermining student understanding of important concepts" (Yopp et al., 2011, p. 313). To our knowledge, there is neither a study addressing in-service secondary school teachers' understanding of 0.999... nor addressing their teaching practices regarding that topic. This paper tries to step into the gap by an explorative interview study with eight in-service mathematics teachers from secondary schools.

## Theoretical background

Recently, Dreher et al. (2018) conceptualized problems around the old question of which knowledge is necessary for teachers with a new model of teacher knowledge emphasizing links between school mathematics and related university concepts. Their model refines Shulman's taxonomy as sketched in Figure 1. Dreher et al. (2018) split content knowledge (CK) into formal academic mathematics and more intuitive school mathematics.

In the German context, those two areas are linked by the secondary schools' curriculum, in which "fundamental ideas" (Dreher et al., 2018, p. 326; Kuntze et al., 2011) represent the structure of academic mathematics in school. They organize the curriculum's content and are revised by the students throughout the grades in a spiraled structure: Under the idea of "Numbers", Number systems develop from natural (grade 4) to nonnegative rational (grade 6) to rational (grade 8) to real numbers (grade 10). "Functions" start in verbal descriptions of dependencies (grade 6) and continue over tabular, graphical and formal descriptions of simple functions (grade 8) to more and more complicated ones (NK, 2015).



**Figure 1: Model of secondary school teachers' knowledge (inspired by Dreher et al., 2018, p. 330)**

In addition to curricular links between mathematics at school and in academics, Dreher et al. (2018) consider top-down relations – for example, during lesson planning, when teachers break down their academic knowledge to school level – and bottom-up relations that e.g. link students' "creative questions and answers" (Dreher et al., 2018, p. 328) to underlying mathematical concepts. Basing the links in teaching situations and the curriculum, the SRCK model is strongly related to the classroom. The representation of academic mathematics in the curriculum leads us to understand curricular knowledge as part of SRCK.

Dreher et al. (2018) assessed SRCK using a questionnaire and showed that the concept is empirically separable from general CK and from PCK. Yet they left it open to identify cognitive processes of SRCK as well as how to identify them in qualitative ways. As we will see, the identity  $0.999\dots=1$  inherits a broad variety of connections to academic mathematics. Therefore, analyzing the mathematics in teachers' reactions to students' responses on this topic seems a suitable way to observe SRCK. As students' erroneous statements are clearly part of PCK (Chick & Baker, 2005), this approach needs to separate knowledge about misconceptions from knowledge about the mathematics behind. We think that this is possible and, even more so, that the reactions to misconceptions linked to big ideas of mathematics reveal both SRCK and PCK and allow insight into their connections (Kuntze et al., 2011). So far, it seems that on the one hand the width and depth of CK predetermine how teachers implement their PCK in the classroom. On the other hand, without PCK, CK cannot be applied at all (Baumert & Kunter, 2013).

### **The identity of $0.999\dots=1$ in academic mathematics and school mathematics**

The mathematical symbol  $0.999\dots$  has a variety of meanings: It is a recurring decimal number as well as a sequence of finite decimals or of partial sums (in the dynamic sense of Weigand (2016)) or the limit of this sequence (in the static sense of Weigand (2016)) and a geometric series. These meanings cannot be clearly separated and may occur simultaneously (Bauer, 2011).

In school,  $0.999\dots$  is first met when working on decimal representations for fractions, which is one of the early experiences of infinity (Gardiner, 1985). Generalizing procedures like subtracting from finite decimals to their infinite counterparts at this stage happens unquestioned and is often used to show that  $0.999\dots=1$  (Bauer, 2011; Eisenmann, 2008). Yet Eisenmann (2008) points out that, for secondary school students, the decimal  $0.999\dots$  is very different from  $0.111\dots$  in its nature, as the latter one appears as a result of the (illegitimate but widely spread) written division algorithm for  $1:9$ . Never may  $0.999\dots$  be constructed as a result of  $1:1$ ,  $2:2$  etc., which might cause the doubts about  $0.999\dots=1$ , as reported in the introduction. Other reasons for the widely spread misconception that  $0.999\dots<1$  might be found in the language of limits, where terms like "approach" mislead to "but never reaches", or in the understanding of decimal expansions, including the representation's non-uniqueness (Tall & Vinner, 1981; Weller, Arnon, Dubinsky, 2009).

In academic mathematics, the interpretation of  $0.999\dots$  as the limit of a geometric series is the most present one (Buchholtz et al., 2012). Thus, the question why  $0.999\dots=1$  is related to the topology of the real numbers and in nonstandard analysis is no longer true (Richman, 1999). In standard analysis, proving  $0.999\dots=1$  requires an understanding of converging sequences and series, including links to the big idea of "dealing with infinity" (Kuntze et al., 2011).

### **Teachers' reactions to cognitive conflicts**

Secondary school students might know from the introduction of decimal numbers in grade 6 that  $0.999\dots=1$ . This can later contrast their concept image of convergence of the series  $0.999\dots$  towards 1, as it is reported in some students' answers in Bauer (2011):

Mathematically it is 1, nevertheless I think there is always something remaining, even if it is however small. (Bauer, 2011, p. 90, translated by the authors)

Here, the student openly presents what Tall and Vinner (1981) call “potential conflict factors”: His concept image does not allow a limit to be actually reached, which leads to  $0.999\dots < 1$  (Tall & Vinner, 1981). This is closely related to a dynamic limit concept emphasizing potential and neglecting actual infinity (Weigand, 2016). Yet the student knows and states the mathematically correct answer. Though it cannot be seen from the quote whether he feels the conflict and a need to settle it, the potential for a cognitive conflict is evident. Despite a number of publications on how to evoke cognitive conflicts in the classroom, much less seems to be known on how teachers can help their students overcome them (Scott et al., 1991).

## Research questions and study design

The large number of possible connections from  $0.999\dots$  to academic mathematics and the lack of knowledge about teachers’ reactions to cognitive conflicts leads to three research questions: How do secondary school teachers react when students bring up misconceptions or potentially conflicting factors concerning the identity  $0.999\dots = 1$ ? How do they refer to academic mathematics in their answers? And how do the reactions and the academic knowledge shown relate to each other?

In this explorative study, we report on interviews with eight teachers from different secondary schools in Lower Saxony. According to the curriculum (NK, 2015, pp. 26 and 58), all participants were expected to have experience in teaching  $0.999\dots = 1$ . The interviews were conducted in an eased situation without any time pressure. First the teachers were asked about their contact with the topic in school. Then student’s response A (see Figure 2, responses taken from Bauer (2011)) was handed out in printed form. The participants were asked to comment on it and to explain how they would react. Then student’s response B – containing potential conflict factors as the quote above – was treated the same way. Finally, the teachers were asked to comment on a quote by Gardiner (1985; see Figure 2), especially on difficulties in putting his suggestion into practice.

<p>Instead of <math>0.99999\dots</math> (and so forth) we also write <math>0.\bar{9}</math>. Which of the following statements is true? Tick and give a reason.</p> <p><input type="checkbox"/> <math>0.\bar{9} &lt; 1</math>  <input type="checkbox"/> <math>0.\bar{9} = 1</math>  <input type="checkbox"/> <math>0.\bar{9} &gt; 1</math></p>	<p>Student’s response A:</p> <p><math>0.\bar{9}</math> is unimaginably close to 1, but never reaches 1</p>	<p>Student’s response B:</p> <p>Mathematically, <math>0.\bar{9} = 9/9 = 1</math>. But as you can put infinitely many digits of 9 behind, this number is always smaller than 1.</p>	<p>Quotation from Gardiner (1985), University of Warwick:</p> <p>No sensitive teacher would try to banish such ideas [misconceptions]: they are the raw material on which we must build.</p>
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**Figure 2: Material given to the teachers (translated by the authors unless originally English)**

The interviews were recorded, fully transcribed and analyzed using an inductive qualitative content analysis (Mayring, 2015). The transcripts were coded openly in two separate rounds: first for any eye-catching differences and for teachers’ reactions, second with focus on SRCK. In both rounds, the relevant teachers’ phrases were paraphrased in a common linguistic style. Then the paraphrases were shortened and rewritten more abstractly to receive possible categories. Those were unified and reduced to the most important categories, which were finally sorted into dimensions. In both rounds, coding manuals were written and given to an independent analyzer who recoded all transcripts. In cases of diverging ratings, consensus was reached. If necessary, examples were added to the manual or categories were renamed carefully.

## Results

Tables 1-3 show most of the categories found in the material in the left column. On the right, they provide the results of the rating. Dimensions are written in bold text, categories normally.

### Teachers' SRCK on 0.999...

From the broad range of links to academic mathematics, only a few were actually made by the teachers. All of them understood  $0.999\dots$  as a limit process in the dynamic sense. Only two teachers showed understanding of a static limit concept as well. Surprisingly, no teacher referred to a geometric series. Probably they had in mind that series are not taught in Lower Saxony schools.

**Table 1: SRCK dimensions shown by the teachers**

	T1	T2	T3	T4	T5	T6	T7	T8
<b>Nature of 0.999...</b>								
is an object different from 1				X	X			X
is a different representation for the same number as 1	X						X	X
is a dynamic limit	X	X	X	X	X	X	X	X
is a static limit	X					X		
<b>The identity <math>0.999\dots=1</math></b>								
knows proof(s) for the identity	X		X	X			X	X
understands the presented proof	X	X		X		X	X	X
doubts the presented proof			X		X			X
... is a definition								X
... is the result of a rounding process					X		X	
... does not hold					X			

One teacher denied that  $0.999\dots=1$  and one read the equality as a definition. Both of them saw no need in spending much time on the students' responses. Further, two teachers referred to rounding. Five teachers knew a proof for  $0.999\dots=1$ , four of them using  $0.999\dots=9/9=1$  or  $0.999\dots=3/3=1$  and one subtracting  $0.999\dots$  from 1. All of them applied techniques from finite decimal representations to infinite ones without justification. Three teachers doubted student B's claim that  $0.999\dots=9/9$ , though two of them claimed that  $0.999\dots=3/3$ . Teacher T8 later accepted the equality.

### Reactions to the students' responses

	T1	T2	T3	T4	T5	T6	T7	T8
<b>Dealing with student A's answer</b>								
link to underlying concepts			X	X				
arithmetical proofs	X	X	X	X		X	X	
correcting the fact		X					X	X
ignore							X	X
no specific reaction mentioned					X		X	X
<b>Dealing with student B's answer</b>								
link to underlying concepts	X		X			X	X	
arithmetical proofs		X		X		X		
no specific reaction mentioned					X			X

**Table 2: Teachers' reactions to both students' answers (shortened by non occurring)**

Inductively, we found five categories of teachers' reactions that applied to reactions on both students' responses. Most of the teachers responded to student's response A by showing an arithmetical proof for  $0.999\dots=1$ . Only two teachers went beyond by stressing notions of infinity or the static limit concept. Three pointed out the importance of correcting the student so that he would not memorize something wrong. The fourth reaction was to ignore, which two of the teachers intended. Teacher T7 named a broad variety of strategies depending on the students' grade.

As we comment on the teachers' reactions to student's response B, it should be noted that not all teachers noticed the conflicting factors in it (see Table 3). Only two teachers clearly named them. Both reacted by stressing the decimal representations' non-uniqueness; especially, they changed their approach compared to how they dealt with response A.

	T1	T2	T3	T4	T5	T6	T7	T8
Contact with $0.999\dots=1$ in school	sometimes	never	often	often	sometimes	rarely	-	rarely
Noticed the potential conflict factors	fully	no	no	partially	no	partially	fully	partially

**Table 3: Further four dimensions in teachers' responses**

Three teachers felt some conflict in the student's text but could not make sense of it. Some claimed that the student B had already disproved himself. Two of them (and another teacher who did not see any conflict at all) indicated that stressing the proof again should convince the student that  $0.999\dots=1$ . One wanted to use the proof to focus on a static limit concept, thus falling into both categories *link to underlying concepts* and *arithmetical proof*. Another link was made to the notion of infinity.

## Discussion

Regarding to our first two research questions, we need to conclude: Even though the topic of  $0.999\dots$  offers many possible links to academic knowledge, only the two concepts of non-unique representation and static limits were actually, and rarely, named by this study's participants. Further, several teachers' answers showed inconsistencies in their SRCK: One teacher claimed that  $0.999\dots$  and 1 are two representations for the same number and nevertheless referred to the process as rounding. Some claimed that  $0.999\dots=3/3$  but would not equal  $9/9$ . This strengthens our doubts about the teachers' access to academic mathematical knowledge, though all successfully finished at least two semesters of advanced mathematics at university. It especially raises the question on how to teach those links between school and academic mathematics to (pre- and in-service) teachers. An example of model-based task design for teacher education is given in Montes, Climent, Carillo, and Contreras (2019) in these proceedings; adopting this approach to the SRCK-model seems fruitful for both evaluating the task design process and improving teacher education programs.

In general, the teachers tend to respond to the misconception  $0.999\dots<1$  by showing an arithmetic proof. Some of them believed so much in the convincing power of proofs that they reacted to student B by restating and emphasizing the proof the student had already given. Only two participants fully recognized the potential conflict in student's response B; their common way of reacting was to establish a link to the decimal representations' non-uniqueness.

### Linking SRCK to the teachers' reactions

Except for T8, all teachers who showed at least one link to academic mathematics in their SRCK chose to talk about underlying concepts when confronted with the conflicting factors. This suggests that SRCK provides a most convenient way to react to cognitive conflicts, and thereby – as already known for CK in general – is a preliminary to successfully applying PCK (Baumert & Kunter, 2013). Yet in T3, we found a teacher who focused on the very general topic of (actual) infinity, thus establishing a link to one of the big ideas in mathematics (Kuntze et al., 2011) without showing much academic knowledge on  $0.999\dots=1$ .

Teachers who have more contact with the topic 0.999... in school more often relate it to underlying concepts. Whether experience mediates between SRCK and the reactions shown (Yopp et al., 2011) or whether teachers with higher SRCK spend more time and effort on 0.999... and thus gain more experience, cannot be answered from the data. Similarly, fully noticing the cognitive conflict seems to be linked to an approach of relating to underlying concepts as well as linked to high SRCK.

### **Using the SRCK model in qualitative research**

In our study, we used the SRCK model and qualitative content analysis to create detailed, comparable insight into teachers' knowledge. Yet the identity  $0.999...=1$  reveals boundaries of the distinction between academic and school mathematics: NK (2015) requires understanding limit processes without ever mentioning infinity. So does understanding infinity belong mainly to school or to academic mathematics? In the first case, the reactions of T3 that linked to this notion could form a new category *horizontal links within school mathematics*, whereas the other links are *vertical links to academic concepts*. This distinction would also foster independence of the curricular – hence cultural – background of our study: All mathematical links could be recorded before splitting them into horizontal and vertical ones depending on the local curriculum. This opens the door to an international curriculum-sensitive, yet comparable approach to SRCK.

Inductively developing dimensions brought to light some aspects of beliefs, for example, on the functions and nature of proofs. In subsequent studies, a multilayer model of teachers' knowledge including convictions – as suggested by Kuntze et al. (2011) – should be taken into account. Furthermore, studies like ours may help understand which of the many possible links from the curricular content to underlying concepts are a substantial part of SRCK in teachers' opinions.

The dimensions and categories developed for teachers' reactions allowed a clear distinction with broad consensus among all analyzers. In studies of a larger sample, they should be refined; especially, questions on their impact on students' learning and on their adequacy throughout different school grades arise.

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