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### ► To cite this version:

Veronika Hubeňáková, Ingrid Semanišínová. Preservice teachers' mathematical knowledge for teaching combinatorial thinking. Eleventh Congress of the European Society for Research in Mathematics Education, Utrecht University, Feb 2019, Utrecht, Netherlands. hal-02430472

**HAL Id: hal-02430472**

**<https://hal.science/hal-02430472>**

Submitted on 7 Jan 2020

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# Preservice teachers' mathematical knowledge for teaching combinatorial thinking

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*In the paper, we conceptualize Mathematical Knowledge for Teaching as introduced and elaborated by Ball et al. (2008) for the topic of combinatorics using Lockwood (2013) model of combinatorial thinking. We developed test consisting of three combinatorial problems and 6 connected tasks (for each problem) focused on the Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) to answer our research question: What are the differences in the combinatorial PCK between preservice teachers with regard to their combinatorial CK? We analyse and discuss data collected from 10 preservice teachers concerning one of the combinatorial problems and its connected tasks.*

*Keywords: Preservice teachers, combinatorics, pedagogical content knowledge, content knowledge.*

## Introduction and background

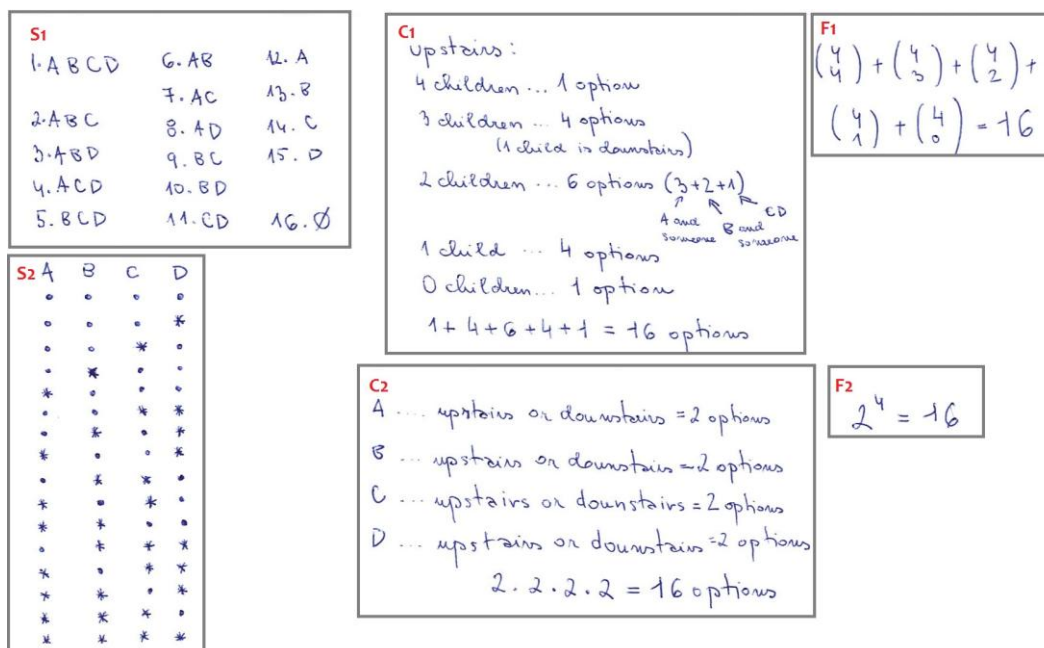
Combinatorial topics form an important part of the Slovak mathematics curriculum at middle and high school levels, as they are thought to provide a rich environment for the development of problem-solving skills (Rosenstein et al., 1997). However, Slovak curriculum documents and textbooks lead students to value formulas more than the ideas. Our aim in teaching preservice teachers (PSTs) is to broaden their knowledge to enable them to scaffold their future students' combinatorial thinking instead of the blind use of formulas. The motivation for our research is to inform our practice by both understanding more deeply how knowledge necessary to teach combinatorics is learnt and scaffolding such learning during the university studies of PSTs.

## Theoretical framework

In this chapter, we are going to frame topic-specific (combinatorial) preservice teacher mathematical knowledge. We build on the theoretical framework of Mathematical Knowledge for Teaching (MKT) as introduced and elaborated by Ball et al. (2008). More specifically, we focused on CCK, SCK, KCS and KCT. We use Lockwood (2013) model of combinatorial thinking to specify combinatorial MKT. We are going to explain the framework using the following combinatorial task (Batanero, 1997): Four children: Anna, Barbara, Cyril and Daniel went to spend the night at their grandparents' house. Their grandparents have two separate bedrooms for them (one downstairs and another upstairs). In how many different ways can the grandparents assign children to bedrooms? For example: Anna, Barbara, Cyril and Daniel will sleep in the room upstairs and nobody will sleep downstairs.

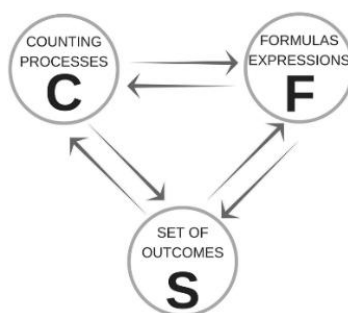
The task can be solved in at least six different ways as depicted at the Figure 1 (see the next page). We can see two different ways how to group them. In the first way, there are three groups: S1-S2, C1-C2, F1-F2. We can see different types of solutions here. The first group, there are *sets of outcomes* (S), collections of the objects being counted. In the second group, *counting process* (C) is

clearly elaborated. In the third group, the solution stands on the using of the *formula* (F). The second way for grouping the solutions is the following: S1-C1-F1 and S2-C2-F2. Each of the group of the solutions can be characterized by the same mathematical idea and they differ by the level of abstraction.



**Figure 1: Six different solutions of the combinatorial problem**

Lockwood model highlights the importance of developing the relationships between sets of outcomes, counting processes, and formulas/expressions. “For a given counting problem, a student may work with one or more of these components and may explicitly or implicitly coordinate them.” (Lockwood, 2013, p. 253).



**Figure 2: Model of combinatorial thinking (adapted from Lockwood, 2013)**

Now, we are ready to frame combinatorial MKT. **CCK:** PST can solve a combinatorial problem using one direction in Lockwood model, usually it is  $S \rightarrow C$  or  $C \rightarrow F$  and/or perceive some direction from the proposed correct solution, usually it is  $F \rightarrow C$  and/or  $C \rightarrow S$ . **SCK:** PST can solve a combinatorial problem in at least two different ways and perceives (not necessarily suggests) all directions of the Lockwood model. **KCS:** PST realizes that student solving the problem in the way S1-C1-F1 can have difficulties to understand the solutions S2-C2-F2 and perceives different levels of abstraction. Moreover, PST is familiar with possible student combinatorial misconceptions.

**KCT:** PST's response to the student's solution is scaffolding in the sense of the Lockwood model, meaning that it shifts students to the proximal level of abstraction.

### **Context and research question**

The teacher education at bachelor's level (3 year program) at our university is, concerning MKT, currently focused on advanced mathematics knowledge, with SCK partially addressed. At the master's level (2 year program), the goals are framed mostly in terms of KCS and KCT development. This model is in line with the claim of Depaepe, Verschaffel and Kelchtermans (2013) from their literature review, which says: "Based on studies that use distinct test items to measure CK and PCK it is concluded that both knowledge components are positively correlated and that CK is a necessary, though not sufficient, condition for PCK." (Depaepe, Verschaffel, & Kelchtermans, 2013, p. 21) However, none of the papers these authors reviewed concerned combinatorics. What is more, the profound understanding of specific mathematical content is one of the possible points of focus for the research in teacher knowledge suggested at CERME 9 (Ribeiro et al., 2015). Therefore, we are interested, in accordance with the theoretical underpinnings and the needs of the field, to find the answer to the research question: What are the differences in the combinatorial PCK between PSTs with regard to their combinatorial CK?

### **Research Methodology**

#### **Participants**

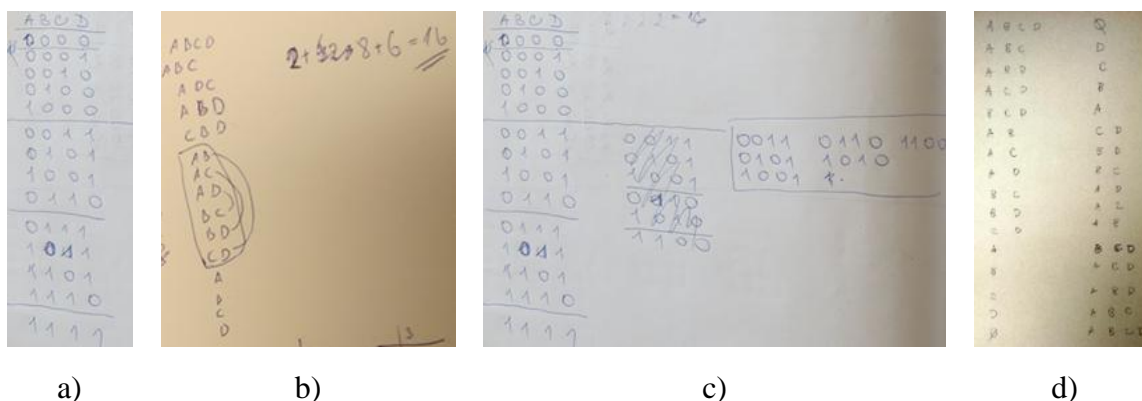
12 PSTs in the academic year 2017/2018 were finishing their master mathematics teaching programme, 10 of them participated in the data collection. They had already passed all mandatory courses, including *Discrete mathematics*, which aimed to develop advanced content knowledge concerning combinatorics, *Methods of mathematics problem solving* where among other things SCK was addressed and *Didactics of mathematics*. Therefore, we chose these participants to gain insight into our current practice of teacher education. However, their PCK could be developed outside of their mandatory courses. Thus, we needed to know whether the PSTs used other options to develop their PCK. Based on the short interviews, we found out that the university courses and mandatory school practices were more or less the only experience with mathematics education for 3 out of the 10 PSTs (*no additional experience*), 3 of them provided regular tutoring and/or lecturing (*narrow additional experience*), 4 out of the 10 PSTs, furthermore, have written diploma thesis in mathematics education (*broad additional experience*).

#### **Test design and its evaluation**

We developed a research tool to capture PSTs' CK and PCK in domain of combinatorics in accordance with our theoretical perspective. As explained by Depaepe, Verschaffel and Kelchtermans (2013), if cognitive perspective to teacher knowledge is taken in account (what is the case here), then test is very relevant tool to measure it. The research tool consisted of three combinatorial problems with six connected tasks each. The first and the third connected task to the combinatorial problem measured the level of CK (Lockwood, 2013). The remaining connected tasks measured PSTs' level of PCK (Ball et al., 2008). All connected tasks were open-ended purposely, to let the PSTs display their CK and PCK without any restrictions or guidance.

All PSTs solved one standard combinatorial problem and one of two, randomly chosen, non-standard combinatorial problems. In the paper, we analyse 10 PSTs' solutions of the already introduced combinatorial problem (Batanero, 1997) and the following connected tasks (time allocated for the task is given in the brackets):

1. Solve the problem in several different ways (at least two). (8')
2. Which of your solutions is mathematically the most valuable? Explain why. (2')
3. There is solution  $2^4$  stated in the textbook. What set of outcomes corresponds with this solution? Write several elements of this set of outcomes. Explain. (5')
4. Imagine, this problem was solved by your students. Write down two examples of students' solutions containing mistakes which are expected by you. Explain them. (10')
5. Imagine, you are a high school teacher and you are solving this problem during the lesson. There are following students' solutions stated in their notebooks (in the Figure 3a and 3b). Explain these solutions and write down, how would you react. (5')
6. Look at the following students' solutions (see Figure 3b, 3c and 3d). Order them according to the level of abstraction (1 – the most abstract solution, 3 – the least abstract solution). Explain. (5')



**Figure 3: Examples of students' solutions for the connected tasks 5 and 6**

The test was firstly verified by 6 PhD students in mathematics (5 of them had already finished their mathematics teaching programme) in order to prescribe the sufficient time for each individual task, to refine formulations, and suggest the coding. Table 1 briefly summarizes the coding. The authors coded PSTs solutions separately, afterwards the coding was compared and few inconsistencies were discussed and resolved. According to CK (see Table 2), we categorized PSTs into three groups. The low CK group included the PSTs who demonstrated neither CCK nor SCK. The medium CK group included the PSTs who demonstrated only CCK. Finally, the high CK group included the PSTs who demonstrated both CCK and SCK.

<b>PCK</b>			
<b>Task</b>	<b>Aim</b>	<b>What PCK components were coded?</b>	
<b>KCS</b>	Task 4	Formulation of misconceptions	Is misconception typical and connected to combinatorics? Is explanation of the misconception focused on student's thinking?
	Task 6	Level of	Are solutions ordered according to level of abstraction?

	abstraction	Are the reasons for ordering well explained?
<b>KCT</b>	Task 5	Response on student's solution
		Would the response which was suggested by PST scaffold student's learning?

**Table 1: Evaluation of PCK**

CK		
	CCK	SCK
<b>Conditions</b>	Problem solved correctly and/or understanding of the standard correct solution.	Problem solved at least in two different ways and correct mathematical argumentation.
<b>Combinatorial thinking</b>	One direction from Lockwood's model	All directions: $S \rightarrow C$ ; $C \rightarrow F$ ; $C \rightarrow S$ ; $F \rightarrow C$ ; $S \rightarrow F$ ; $F \rightarrow S$

**Table 2: Evaluation of CK**

## Findings & discussion

First, we describe the distribution of the participants into different categories. Table 3 illustrates that our research sample covered most of the possible categories. Only the category with low CK is not fully covered, however as the assigned combinatorial problem was standard, it is not surprising. Given the distribution into coded categories and subject numbers, we will analyse these data in qualitative and interpretative way.

Content Knowledge \ Additional Experience	None	Narrow	Broad
	Low	1	0
Medium	1	2	1
High	1	1	3

**Table 3: Numbers of PSTs grouped in categories**

Second, we take a deeper look into each of the PCK components given in the Table 1. We will provide examples of PSTs responses to explain how they were coded.

### Formulation of possible student misconceptions<sup>1</sup>

#### Example A

Maybe they say: I choose who out of 4 children will be downstairs.  
and they write down (4) out of 4  
But they figure out it is  $4 \neq 1$  and thus it is nonsense.

#### Example B

Children will be in the rooms like this:

1 <sup>st</sup> room	2 <sup>nd</sup> room	} 5 options
4 children	0 children	
3 children	1 child	
2 children	2 children	
1 child	3 children	
0 children	4 children	

**Figure 4: Examples of misconceptions**

<sup>1</sup> We translated PSTs answers from Slovak language and re-wrote it to preserve the structure used by PSTs.

As we can see, in the Example A (Figure 4), PST tried to formulate misconception, however it is artificial and not connected to combinatorial idea. On the other hand, Example B (Figure 4) is very common one, it is called confusing the type of object (e.g. Batanero, 1997), it is clearly connected to combinatorics.

### **Level of abstraction**

#### *Example C*

“Solution 3 (Figure 3d) seems the least abstract to me, because the student is writing down the set of outcomes from which it is clear what is happening in each of the rooms. It is obvious that he can imagine what is going on in there. Solution 2 (Figure 3c) is the most abstract because the student is grouping those people, however he probably does not really understand what he is doing.”

#### *Example D*

“In the solution 3 (Figure 3d), there is clearly written which child is in which room ... students can immediately imagine – from the representation – how the children are split into two rooms. In the solution 1 (Figure 3b), there is some intermediate step from the specific solution to the abstract one, or an attempt to simplify, to shorten process of writing down the set of outcomes. The solution 2 (Figure 3c) – lot of children would be probably lost in such kind of solution. It is logical, however quite symbolic. Unlike the other two solutions, here are rooms assigned to children not children to rooms.”

In both examples, PSTs ordered the solutions in the correct way. The difference is, the PST from the Example C, did not notice different mathematical idea behind the solution 2 compared to solutions 1 and 3. On the other hand, PST from the Example D explains this difference.

### **Response on student’s solution (depicted in the Figure 2a)**

#### *Example E*

“I would call 4 volunteers, and I would ask the student with this solution to arrange them in all the possible ways when two of them are in the same room.”

#### *Example F*

“I would probably give 50% for understanding and coding. I want the student to realize that the set of outcomes is not the most adequate solution.”

#### *Example G*

“I would ask the student to check the set of outcomes whether it is complete. I would praise him for the representation of set of outcomes he came up with.”

Student’s solution in the Figure 2a is the set of outcomes connected with the formula  $2^4$ . The activity suggested in the Example E is not related to this formula; therefore, it would probably not be scaffolding. In the Example F, PST would prompt student to use formula, which would not necessarily scaffold the student’s combinatorial thinking if the student is not ready to see the relevance of formula to the situation. In the Example G, the PST recognized that she cannot orient

the student to the counting process or formula without the student first comprehending how the set of outcomes is generated, thus it was coded as scaffolding response.

Third, we present which of the coded PCK components, were (+) or were not (-) observed in the particular categories of PSTs (see Table 4, the next page). In the Table 4, results of PSTs with medium and high CK are presented, the one PST with low CK demonstrated no PCK.

		None experience		Narrow experience		Broad experience	
		No SCK	SCK	No SCK	SCK	No SCK	SCK
KCS	Misconceptions	+	-	+	-	+	+
	Level of abstraction	-	-	-	+	-	+
KCT	Understanding	+	+	+	+	+	+
	Scaffolding response	-	-	-	-	-	+

**Table 4: PCK components demonstrated / missing in particular PSTs categories**

**Formulation of possible student misconceptions** as part of the KCS was missing only for PSTs with high CK without *broad additional experience*. These PSTs did not have an opportunity to learn from their own mistakes and the lack of experience was not supporting them to learn from mistakes of others. This could be the reason why they were not able to estimate what kind of misconceptions are common.

**Level of abstraction** as part of the KCS was observed only when PSTs had high CK and at least *narrow additional experience*. These differences can possibly be explained by the role of SCK as we defined it in combinatorics. If PSTs perceive all connections between possible solutions as suggested in the Lockwood’s model, they are able to understand how abstract particular student’s solution is.

**Understanding of the student’s solutions** was managed by all PSTs except the one with the low CK. Obviously, the PST with low CK could not distinguish the nature of the error correctly, also she had troubles to reason the correct one. While the error was quite familiar, and the solution was the standard one, the other PSTs were able to explain the student’s solution using only CCK, it was not necessary to activate SCK here.

Concerning **scaffolding response**, only the group of PSTs with high PCK and *broad additional experience* was able to provide scaffolding reactions. Moreover, the scaffolding reaction was suggested only if the level of abstraction was perceived and explained clearly.

## Conclusions

Let us return to the research question: What are the differences in the combinatorial PCK between PSTs with regard to their combinatorial CK? On the one hand, PSTs with low or medium CK displayed less PCK. On the other hand, when it comes to formulation of misconceptions, they outperformed those, who had high PCK and *none or narrow additional experience*. Thus, the high CK did not automatically meant good combinatorial PCK. It seems that the other necessary condition for combinatorial PCK development was *broad additional experience*. It supported PCK



development especially when the PST had high level of CK. This is important, because PST who becomes a novice teacher is expected to learn mostly through their own teaching experience.

Back to our first motivation for the research, the findings will feed back to our research and practice in the following ways: **(1)** we need to inquire and specify how to develop combinatorial SCK effectively during the bachelor's level programme. **(2)** We need to identify the key factors of the *additional experience*, which scaffolded PSTs learning and subsequently design the PST education to promote these factors. Therefore, we need to focus on the task construction and design similarly to Policastro, Mellone, Ribeiro and Fiorentini (2019) and Montes, Climent, Carrillo and Contreras (2019).

Moreover, we will reframe the research with the model of Mathematics Teachers' Specialized Knowledge (Carrillo, et al., 2018) which address the limitations we experienced using MKT model (Ball, et al., 2008). We suggest that such reframing and the subsequent comparison could bring more clarity into mathematical teacher knowledge needed for teaching combinatorics.

### **Acknowledgements**

This work was partially supported by the Slovak grant VEGA 1/0265/17 and by the National project IT Academy – Education for the 21st Century.

### **References**

- Batanero, C., Navarro-Pelayo, V., & Godino, J. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, 32, 181–199.
- Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for Teaching. *Journal of Teacher Education*, 59(5), 389-407.
- Carrillo, J., Climent, N., Montes, M., Contreras, L.C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, A., Ribeiro, M., & Muñoz-Catalán, M.C. (2018). The Mathematics Teacher's Specialised Knowledge (MTSK) model. *Research in Mathematics Education*. (Online). DOI 10.1080/14794802.2018.1479981.
- Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education*, 34, 12-25.
- Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior*, 32, 251– 265.
- Montes, M., Climent, N., Carrillo, J. & Contreras, L.C. (2019). Constructing tasks for primary teacher education from the perspective of Mathematics Teachers' Specialised Knowledge. Manuscript in preparation.
- Policastro, M., Mellone, M., Ribeiro, M, & Fiorentini, D. (2019). Conceptualising tasks for teacher education: from a research methodology to teachers' knowledge development. Manuscript in preparation.

- Ribeiro, M., Aslan-Tutak, F., Charalambous, Ch., & Meinke, J. (2015). Introduction to the papers of TWG20: Mathematics teacher knowledge, beliefs, and identity: Some reflections on the current state of the art. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9, 4-8 February 2015)* (pp. 3177-3183). Prague: Charles University in Prague, Faculty of Education and ERME.
- Rosenstein, J. G., Franzblau, D. S., & Roberts, F. S. (1997). *Discrete mathematics in the schools* (Vol. 36, Discrete Mathematics and Theoretical Computer Science). Providence, RI: American Mathematical Society, National Council of Teachers of Mathematics.