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# Prospective primary teachers' knowledge about the mathematical practice of defining

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*This study concerns the knowledge of two prospective primary teachers about the mathematical practice of defining. Starting with their analysis of a video recording of a primary lesson on constructing a definition of a polygon, the trainees discussed what it means to define in mathematics. Through analysis of the content of this discussion, we identified what they considered to be the nature of this practice and the features they felt a good definition should embody. The findings demonstrate that students tend to consider demonstrations as the touchstone of mathematical practice, and that they do not consider the use of examples as a valid element of formal mathematics. At the same time, they recognize that definitions must meet the criteria of hierarchy and minimality but see no need for the criteria of existence and arbitrariness.*

*Keywords: Teachers' knowledge, mathematical practice, defining, student teacher.*

## Introduction

Defining is a common practice in mathematics classes at all levels but is something not always dealt with in all its facets: establishing the necessary and sufficient conditions of a concept, preferably the minimum necessary, using only previously defined concepts without utilising the word assigned to the concept in the definition itself (Zazkis & Leikin, 2008).

What is more, definitions can be arbitrary, in the sense that concepts can be defined in different ways according to the particular aspect one wishes to foreground. This arbitrariness can disconcert students (Vinner, 1991), and it is the teacher's job to manage such uncomfortable facts of mathematical life and decide when his or her students are ready to meet them.

Involving students in the process of defining promotes the construction of knowledge (Edwards & Ward, 2008; Leikin, Berman & Zaslavsky, 1998; Villiers, 1988). Regardless of whether the teacher decides to tackle certain aspects of defining (such as their arbitrariness) with their students at a particular moment, if he or she wants them to generate definitions for themselves rather than memorise given definitions, they will necessarily have to learn how to define in mathematics. Hence, achieving a certain depth of understanding regarding the meaning of the practice of defining, the characteristics of a good definition and the role it plays in mathematics learning enables the teacher to base their classroom practice on sound foundations. In this respect, Zazkis and Leikin (2008) regard teachers' conceptions of the notion of defining as a fundamental requirement for successful teaching.

In this paper, we describe two prospective primary teachers' knowledge about the notion of defining in mathematics, based on their discussion of a video. This understanding forms a part of their knowledge of practices in mathematics, as we explain in the theoretical framework below.

## **Theoretical framework**

Defining, like demonstrating or arguing, is one of the basic elements of the work of mathematics. We can say, then, that these elements form a part of mathematics practice, in the sense that they allow us to do mathematics. Knowledge of practices in mathematics supposes knowledge of the syntax of mathematics consistent with Schwab (1978). A teacher's knowledge of practices in mathematics (henceforth KPM), together with their knowledge of topics and the structure of mathematics constitute the mathematical knowledge which in the *Mathematics Teacher's Specialised Knowledge* model (MTSK) is considered necessary for teaching the subject (Carrillo-Yañez et al., 2018). In addition to mathematical knowledge, the model takes account of the domains of pedagogical content knowledge and beliefs about mathematics and mathematics teaching and learning. However, in this study, we deal with neither prospective teachers' beliefs nor pedagogical content knowledge.

Understanding the notion of defining involves recognition of the role that definitions play in mathematics and the characteristics they embody, or at least it is desirable for them to embody (Zaslavsky & Shir, 2005). Zaslavsky & Shir (2005) suggest the following features as characteristic of a good definition:

... a mathematical definition must be *noncontradicting* (i.e., all conditions of a definition should coexist) and *unambiguous* (i.e., its meaning should be uniquely interpreted). In addition, there are some features of a mathematical definition that are imperative only when applicable: A mathematical definition must be *invariant* under change of representation; and it should also be *hierarchical*, that is, it should be based on basic or previously defined concepts, in a *noncircular* manner. (Zaslavsky & Shir 2005, p. 319)

Van Dormolen and Zaslavsky (2003) offer logically necessary criteria for a good definition, among others, the criterion of hierarchy (it should be described as a special case of a previously determined more general concept). In contrast to Zaslavsky and Shir (2005), the former regard hierarchy as a necessary characteristic. In addition to this feature, they include the criteria of existence (there must be proof of the existence of at least one instance of the new concept), equivalence (where more than one definition exists, they must be demonstrated to be equivalent), and axiomatization (it should fit into a deductive system such that when one reaches the point where the criterion of hierarchy can no longer be applied, we can resort to axioms or postulates).

There are other features for which there is a lack of agreement as to whether they are necessary or not. Foremost among these is the criterion of minimality.

A definition is considered minimal if it is economical, with no superfluous unnecessary conditions or information. That is, a minimal definition should consist only of information that is strictly necessary for identifying the defined concept. (Zaslavsky & Shir, 2005, p. 320)

A study by Leikin and Winicki-Landman (2000) involving secondary teachers highlights the rigour of a definition, although from a pedagogic point of view, other features are underlined, such as the definition being intuitive, being written clearly enough for the students to grasp with ease, combining what they already know with what they need to know, being appropriate for solving problems and enabling generalisations to be made. Relaxing the rigour with which one applies the criterion of minimality tends to be motivated by pedagogical considerations.

There have been various studies into teachers' and prospective teachers' notions of what constitutes a mathematical definition. These have found, among other things, that prospective teachers do not always know the difference between a theorem and a definition (Leikin & Zazkis, 2010) and are sometimes unaware of the arbitrariness of definitions in geometry (Linchevsky, Vinner & Karsenty, 1992).

Escudero, Gavilán and Sánchez-Matamoros (2014), in their study about prospective primary teachers, describes various ways in which they understand what a mathematical definition is and tracks the changes in this understanding. From this progression, from naming to giving a list, from naming to giving a list of minimal features, and from giving a list of features to equivalent definitions, the authors infer the assimilation of some of the criteria of mathematical definitions discussed above. In particular, the recognition of minimality and equivalence is indicative of a more advanced understanding of what constitutes a mathematical definition. Elsewhere, Sánchez and García (2014), again with prospective primary teachers, find potential points of conflict between criteria associated with mathematical definitions (understood within the study as mathematical norms) and sociomathematical norms. For example, minimality comes into conflict with the sociomathematical norm which requires students to explain themselves as fully as possible in response to a classroom task, or that they should make use of all the data and properties in a figure. Conflicting requirements such as these could account for the difficulty which students experience with some of the mathematical attributes.

## **Data collection and analysis**

In Spain, the initial education for teachers on Primary Education corresponds to a four-year degree. Then they can obtain a general teaching graduate or specific on sports, music, foreign language or special education. At the University of Huelva, for each year, a compulsory Mathematic subject is planned. These subjects deal with content knowledge and pedagogical content knowledge about problem solving (first year), numbers and arithmetics (second year), statistics and measurements (third year) or geometry (fourth year). In these subjects, some concepts are reviewed in depth with regards to the Knowledge of Practices in Mathematics, like heuristics to solve Mathematical problems, proofs, types of proofs, and language and notation role.

This study takes the form of a one-on-one teaching experiment (Cobb & Gravemeijer, 2008) in which three prospective primary teachers trialled activities ultimately intended for use with a group of primary trainees on the subject Mathematics Education: Geometry. The participants' selection was intentional because of their particular disposition towards reflection on mathematics teaching and learning, and their sound knowledge of mathematics. We make use of only two participants' data in this paper; they will be called Ismael and Ramón. At the time of the study, they both were in

the third year of their degree, so they have not taken the subject on Geometry. The purpose behind the choice of these particular students was to measure the development potential of the activities in a favourable environment.

Here we focus on the first part of the teaching experiment, in which the prospective teachers individually watched and analysed a video recording of an actual primary lesson. After writing their individual impressions, which they handed in to the teacher educators, the group came together to discuss the lesson, with one of the educators/researchers in the role of moderator.

The video analysed by the trainees featured a year 5 primary group (11 years old), in which their teacher introduced the topic of defining polygons by showing the pupils a bag with polygons and non-polygons shaped cards. The lesson was conducted as a whole-class activity, with pupils going up to the board one by one to take a figure from the bag and place it in one of two groups (they could create two groups, without being given any previous criteria). The teacher expected the pupils to apply the criteria “is a polygon”. The classification task formed the basis of the follow-up activity in which the class constructed a definition of polygon together. This video has been chosen as it exhibits the mathematical practice of defining and it allows us to show teacher’s knowledge about this practice. One of the objectives of this activity (the analysis of the Primary education class), is that future teachers must do a reflection about what is a definition within a mathematical context and which characteristics a mathematical definition could have.

The research question we are intending to respond to, is which knowledge about the mathematical practice of defining both prospective primary teachers (henceforth PPTs) show.

Data to solve the research question come from the individual analysis document of the PPTs and the transcription of the oral discussion (audio recorded). The units we will show belong to the discussion transcription. In this discussion and also for everything concerning to knowledge about the practice of defining, the main objective of the researcher and moderator is, for the first instance, to make the PPTs to glimpse their ideas. During the individual PPTs analysis, some comments came up concerning to characteristics of the definition built in the Primary lesson (e.g. “they have repeated elements”). The most frequent questions to the PPTs (during the discussion of their individual analysis) focused on their interpretation of the definition practice. Consequently, the responses showed which characteristics the PPTs linked to a mathematical definition.

The group discussion of the video was itself audio-recorded and subsequently transcribed. Below we analyse the transcript in terms of the elements discussed in the theoretical framework, essentially the features of a good mathematical definition. Therefore, we made a content analysis in which we selected those fragments in which PPTs considered the mathematical practice of defining developed in the Primary lesson and they related the features associated to a mathematics definition. We connected these fragments to the characteristics of a mathematics definition from the theoretical framework. This analysis is first undertaken by each researcher individually and afterwards jointly agreed. This process let us build our common explanation from our observation.

## Knowledge of the mathematical practice of defining

One of the aspects which two of the prospective teachers focus on in their analysis is mathematical definitions, and in particular the role of examples in constructing a definition:

- Educator: What do you think is the aim of the first task?
- Ismael: ...what he's looking for is the definition, what's happening is that he's starting from examples so that they see the differences between what it [a polygon] is and what it isn't.
- Educator: What do you think about the fact that he starts from examples to get there, to get them to construct the definition of a polygon?
- Ismael: As an item of mathematics, it is not mathematically correct to construct a definition from examples. [...] Maybe it's good in terms of getting the pupils to understand, but not in terms of mathematics.
- Ramón: Is it more correct or less correct in mathematical terms? Yeah, starting from examples could be less correct, but for primary children it's an introduction. And besides, in our lessons when we do demonstrations what do we do first? Try a few numbers to see and then afterwards do the demonstration well.

Ismael appeals to the argument of *rigour* which features in the literature in contradistinction to pedagogical considerations (Leikin & Winicki-Landman, 2000; Zazkis & Leikin, 2008). In his view, the use of examples does not meet the standards of rigour demanded by mathematics, but justifies their use for the pedagogical advantage they afford in making a mathematical concept accessible to the primary pupils. Ramón concurs, criticising the use of examples to construct a definition. Both prospective teachers settle on demonstrations as a prototypical practice of formal mathematics, and, aware that in formal mathematics it is not considered valid to perform a demonstration via examples, they extrapolate the proscription to definitions. The use of examples is not intrinsic to formal mathematics. It can serve to make a mathematical object accessible, such as the rubric in demonstrations, but it is not a rigorous procedure.

Both the prospective primary teachers (henceforth PPTs) seem to dispense with the *criterion of existence* for definitions, whereby examples serve as proof of the existence of at least one mathematical object fulfilling the definition.

However, at another point in the debate, Ramón appraise the use of examples to define a mathematical concept, in reference to the examples given by the cardboard cut-out figures:

- Ramón: If the pupils are given a figure with edges, how are they going to know what is inside and what is outside?

The PPTs insist that a new mathematical concept should be generated on the basis of differentiating the properties of pre-existing concepts (*the criterion of hierarchy*), while also making a contrast with properties the concepts do not have. In other words, it is not sufficient to present images (examples) which illustrate the properties of a concept, rather it is necessary to include images which illustrate these very properties by their absence. In this way, a deeper understanding is

achieved of the concept being defined. This appraisal might also be associated by the PPTs with the pedagogical advantages it affords more than with mathematical practice itself.

The *criterion of arbitrariness* does not seem to be considered by the prospective teachers, who, it would seem, believe that mathematical concepts can have only one formal definition. Accordingly, in the following turn, Ramón gets impressed by how the definition built in the primary school classroom is so exact (to which he believes is the unique right definition):

Ramón: [...] the idea of the activity is that they do the classifying, as if polygons had never been classified before and they were doing it for the first time, so that they realise that the classification system they come up with is similar to the actual one. [...] It surprised me how well they managed it, coming up with a definition from examples, that primary children can make such identical definition simply from examples.

On the other hand, the prospective teachers do recognize the *criterion of minimality* as a desirable attribute of a mathematical definition, when they come to consider the wording of the definition the class finally agree on (a polygon is *a flat shape, it has angles, it has vertices, it has straight sides and no curves; all the sides are joined at the end*):

Ramón: It made me laugh when they put in the definition “it has straight sides and no curves,” it’s the same thing.

Educator: Would you have cut that out?

Ramón: At the end of the recording, one pupil says that not having any curved lines is the same as having all the sides straight, and it wasn’t necessary to say both things.

Ismael: I think that if that’s the first time they’ve done this and the children have reached that conclusion, then it’s fine, even if it’s redundant. I’d leave it like that, and then next time we did the topic I’d suggest we cut out that bit. You work on it a bit so it looks a bit more like the definition we have.

## **Discussion and conclusions**

The prospective teachers involved in this study seem to identify formal mathematics practices with the practice of demonstrating. Their touchstone in this regard is mathematical rigour, which exerts its influence over the practice of defining. In particular, the use of examples falls short of the demands of rigour, and can be justified only on pedagogical grounds or as a means of making a mathematical object more accessible before defining it formally. They recognise that a mathematical definition should meet the criteria of hierarchy and minimality, whilst they do not see the need for the criteria of existence or arbitrariness. This lack of recognition of the criterion of arbitrariness mirrors the results of other studies (Linchevsky, Vinner & Karsenty, 1992). The prospective teachers also tend to regard the practice of defining as supplying a list of minimal features, a result consistent with the findings in Escudero, Gavilán and Sánchez-Matamoros (2014).

In the course of the PPTs’ debate about the essential features of a good definition, they frequently consider pedagogical criteria over mathematical considerations about a good definition. This

posture is consistent with the professional profile of primary teacher and coincides with practices observed at other levels of education (Leikin & Winicki-Landman, 2000; Zazkis & Leikin, 2008).

These findings are particularly relevant to the context of developing teaching strategies for training prospective teachers in terms of their knowledge of practices in mathematics, specifically in this instance the nature of definitions. In particular, with respect to the initial training provided for prospective primary teachers at the university in which the study was carried out, there is cause for rethinking the role given to demonstrations, as their pre-eminence on the syllabus vis-à-vis other practices seems to be leading students to regard it as the touchstone of mathematical practices.

At the same time, before their initial training, the prospective teachers have little knowledge of formal mathematics as the mathematical knowledge they know is with regards to scholar mathematics. Taking the opportunity presented by their training to approach the nature of different mathematical practices should enable them to develop a more advanced view of what constitutes mathematics, building a bridge between school and advanced mathematics (issue that is also addressed in Crisan, 2019, February). The transition, however, from knowledge based on intuitions to that based on formal definitions and their features is not without its challenges (Bampili, Zachariades & Sakonidis, 2017).

In this paper, we have focused our analysis on the knowledge of two prospective primary teachers about the mathematical practice of defining, although data also reveal their conceptions about mathematics and about teaching and learning. The video recording was used as a starting point to reflect on the mathematical practice of defining and to validate this practice inside school mathematics. It might be considered the relationship between the selected video and the knowledge that prospective primary teachers have shown, for example, the fact that the polygon definition was built from polygon and non-polygon examples in the context of a primary school classroom. For future work, we will investigate on these issues. Also, we will consider extending this study to prospective kindergarten teacher and secondary teacher, in order to design task to strengthen their knowledge about the mathematical practice of defining.

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