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# Knowledge of a mathematician to teach divisibility to prospective secondary school teachers

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*Mathematics teachers' knowledge has been studied extensively in the last decades, especially in research building on Lee Shulman's work. However, the same emphasis has not been placed on research on the mathematics teacher educator's (MTE) knowledge. Notwithstanding, the concern to characterize the knowledge of these educators has been emphasized in recent studies of mathematics education and models for knowledge of MTEs have appeared in literature. In this perspective, we present an episode which occurred in a Number Theory undergraduate classroom, where a mathematician, who acts in teacher preparation, demonstrates Euclid's division algorithm theorem. The data, which is part of a case study, is analyzed with the objective of identifying indicators of knowledge of MTEs. Among the results, knowledge of MTEs emerge in relation to knowledge of topics, knowledge of the structure of mathematics and knowledge of practices in mathematics.*

*Keywords: Mathematics teacher educator, mathematician, number theory, Euclid's division algorithm theorem, Mathematics Teachers' Specialized Knowledge.*

## Introduction

One of the roles of a Mathematics Teacher Educator (MTE) is to promote Prospective Mathematics Teachers (PMTs) knowledge in order to make them capable of establishing connections between teacher education and their practice. According to Jaworski (2008, pp. 1), MTEs "are professionals who work with practicing teachers and/or prospective teachers to develop and improve the teaching of mathematics". Considering that the knowledge of Mathematics Teachers (MT) is specialized, regarding the perspective of the Mathematics Teachers' Specialized Knowledge - MTSK (Carrillo-Yañez et al., 2018), the work of the MTE is even more important. In this sense, this work intends to contribute to research about the knowledge of the MTE and its role in teacher education, particularly, in a Number Theory course for PMTs.

Even if Number Theory has many connections with school algebra, many MTs understand this topic as being unrelated to their pedagogical practice (Smith, 2002). The theme divisibility, for example, is frequently treated by PMTs as being a trick or a procedure to be memorized, rather than a relation between integer numbers (Zazkis, Sinclair, & Liljedahl, 2003).

The topic divisibility is present from the earliest years of the schooling, including division of natural numbers for example. The integer numbers are gradually introduced in the mathematics school curriculum and some divisibility criteria are presented. In this context, there is a natural underlying question: Why is Euclid's division algorithm valid? This question is answered in a Number Theory course for PMTs, where the Euclid's Division Algorithm Theorem is presented.

In Brazilian universities, mathematicians are mostly responsible for the mathematical preparation of PMTs. These professionals “act as teacher educators de facto, without explicitly identifying themselves in this role” as claimed by Leikin, Zazkis and Meller (2017, pp. 2). In this scenario, our foci of research is the knowledge these professionals reveal in their teaching. These mathematicians, who are eventually in the role of preparing PMTs, have a solid knowledge in the scientific field of mathematics and aim to develop research in this field and, on the other hand, their pedagogical content knowledge arises from practice (Fiorentini, 2004).

This paper is part of a broader research project which aims to understand and characterize, in the scope of Number Theory, the specialized knowledge of those who act as mathematics teacher educators. In this paper we address the particular research question: *What elements characterize the specialized knowledge of a mathematics teacher educator in relation to Euclid's division algorithm theorem?*

## **Literature review**

The knowledge of the MTE is different than both the knowledge of the PMT and the knowledge of the MT (Jaworski, 2008; Zopf, 2010; Contreras et al. 2017). Jaworski (2008) called this knowledge Mathematics Teacher Educator Knowledge, which has particular aspects as well as common points with both the knowledge of the PMT and the knowledge of the MT. In the intersection, they need to know: mathematics, the pedagogy related to mathematics, and the curriculum which the mathematics teacher based their work. Furthermore, the MTE also needs to know: both the professional and the research literature linked to the teaching and learning of mathematics, to know teaching and learning theories, and to know research methodologies that investigate teaching and learning on schools/educational systems.

Zopf (2010) observes that the difference between the knowledge of the MTE and the knowledge of the MT lies in the mathematical content. While the teacher teaches mathematics, the MTE teaches the knowledge to teach mathematics. The teaching purposes are also different, since the children learn mathematics for themselves, while the teachers learn mathematics for teaching their students. Therefore, Zopf (2010) proposes the Mathematical Knowledge for Teaching Teachers (MKTT) in order to describe the knowledge of the MTEs, which includes the knowledge necessary for teaching.

Building on Shulman's work, Carrillo-Yañez et al. (2018) divide the knowledge of the mathematics teacher into two domains: Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). Thereafter, Contreras et al. (2017) state that the knowledge mobilized by MTEs and teachers present differences when MK and PCK are considered. The differences in MK are related with the fact that the knowledge of the MTE is larger in terms of reach and depth, that is, the mathematical knowledge of the MTE has a more coherent and solid theoretical structure, besides the MTE has more experience with the validation/construction of the mathematical knowledge. On the other hand, PCK contains knowledge about the characteristics of learning of the PMTs, knowledge about how to teach the content of the teacher education and knowledge of different ways to organize the content of teacher education. In this paper we will focus on the MK of the teacher educator participant.

## **Teacher educator's knowledge: theoretical perspective**

In Carrillo-Yañez et al. (2018), the authors discuss their Mathematics Teachers' Specialized Knowledge (MTSK) model. In this model, it is considered that the teacher's knowledge to teach is specialized and that the MK is subdivided into three subdomains: the Knowledge of Topics (KoT), the Knowledge of the Structure of Mathematics (KSM) and the Knowledge of Practices in Mathematics (KPM). On the other hand, the PCK is also subdivided into three subdomains: the Knowledge of Mathematics Teaching (KMT), the Knowledge of Features of Learning Mathematics (KFLM) and the Knowledge of Mathematics Learning Standards (KMLS). At the center of the model, are the domain of the teachers' beliefs, which are related to all subdomains. In this paper, because we are interested in the knowledge of a mathematician who works in teacher education, we will focus on his Mathematical Knowledge.

KoT includes knowledge of procedures, definitions and properties, representations and models, registers of representations and applications. The KSM subdomain includes knowledge of connections between mathematical items, such as connections based on simplification, connections based on increased complexity, auxiliary connections and transverse connections. In its turn, KPM includes knowledge about demonstrating, justifying, defining, making deductions and inductions, giving examples and understanding the role of counterexamples. In the context of Number Theory, particularly in the scope of the Euclid's division algorithm theorem, the MTE knowledge includes, for example:

KoT – To know definitions and results that compose the proof of the Euclid's division algorithm theorem, such as the definition of absolute value and the well-ordering principle.

KSM – To establish connections between the Euclid's division algorithm theorem and posterior topics in the Number Theory course, such as linear congruence.

KPM – To know different types of proofs, such as the proof by contradiction that justifies the fact that the remainder is less than the divisor, in proof of the Euclid's division algorithm theorem.

## **Context and methods**

Our investigation had a qualitative approach. In particular, we adopted the instrumental case study (Stake, 2006) as the research method, looking for information about the subject's knowledge that can be included in the theory about the MTE knowledge. In order to answer the research question, we discuss a classroom episode of a Number Theory course for secondary PMTs, where the MTE aims to present Euclid's division algorithm theorem as well as its proof. The participant Andre, a pseudonym, has Graduation, Master degree and PhD in Mathematics. Since the Master, his research interests lie in Algebra and Geometry. Andre has been teaching at the mentioned university for five years, where he teaches for students of different undergraduate courses, such as Mathematics, Physics and Chemistry. In the period which his classes were observed, Andre was teaching the Number Theory course to undergraduate students in mathematics for the second time in his academic career.

The case study of Andre is part of a larger research study which aims to understand what are the knowledge of MTEs, in particular the ones that teach Number Theory to prospective secondary

school mathematics teachers. The results reported in this paper are exclusively based on this participant.

The aforesaid Number Theory course is a 15 week-long course, which is offered once in each semester as a common discipline for prospective teachers and bachelors of mathematics. Furthermore, the PMTs, are oriented to take these classes in the 6th semester of their undergraduate. The course includes standard contents of a first course in Number Theory, such as divisibility, prime numbers, linear congruence, Diophantine equations and primitive roots.

The data collection occurred in the period between March and July of 2018, in a Brazilian university, comprising class recordings and field notes from the researcher. The classes of this Number Theory course were observed and recorded. Starting from the transcript of the recording of the subject's classes, we divided each class into episodes and chose the episode in which Andre demonstrates Euclid's Division Algorithm Theorem to present and discuss the knowledge revealed by the MTE in this episode, using the MTSK categories.

## Analysis and discussion

### The class episode and its discussion

Andre started the topic of divisibility at the end of the previous class, when he presented the definition and some basic properties of divisibility. The episode that we analyze here is part of a class, which the teacher educator started defining prime numbers. Thereafter he proved the existence part of the Fundamental Theorem of Arithmetic<sup>1</sup>. In the sequence, he proved that there are infinite prime numbers, and he defined Greatest Common Factor<sup>2</sup> as well as he proved some properties<sup>3</sup>. Then, in the episode that follows, the MTE introduces and proves Euclid's Division Algorithm Theorem (EDAT): Considering  $a \in \mathbb{Z}$  and  $b \in \mathbb{Z}_+$ , there are unique integers  $r$  and  $q$  such that  $a = bq + r$ , where  $0 \leq r < b$ .

The episode begins with Andre enunciating the EDAT (Figure 1) and drawing the attention of the students to the connections between this result and linear congruence, which will be introduced later in the course. Andre also notes that the proof of the EDAT must be done in two parts: existence and uniqueness. Considering the limited number of pages allowed in this paper, we approach in our analysis the existence proof only. His proof starts considering the set  $S$  of all possible non-negative remainders of the division of  $a$  by  $b$  (Figure 2). Naturally, the first step is to prove that  $S$  is not empty, thereunto Andre discusses with the students to find an integer  $x$  such that the expression  $a - bx$  is non-negative. The conversation continues for some time and the students do not find this particular  $x$ . One of the students apologizes for his incorrect answer and Andre discusses the importance of the students asking questions as well as the need for observing the details of the enunciation of the theorem. Thereafter, Andre provides the sought  $x$  (Figure 3). Since  $S$  is a non-empty set of non-negative integers, thus  $S$  has a minimal element. He denotes this minimum by  $r$

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<sup>1</sup> Any integer greater than 1 can be written as a finite product of prime numbers.

<sup>2</sup> The Greatest Common Factor of two integer numbers is the largest positive integer number that divides each one of these integers.

<sup>3</sup> Such as "If  $d = gcd(a, b) \Rightarrow gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ ".

and in the sequence he proves that this element satisfies the theorem conditions (Figure 4). The existence of  $r$  implies the existence of  $q$ .

Andre: Let's see the night star! The Euclid's Division Algorithm Theorem. Then we consider two integers  $a$  and  $b$ . Actually, I'm going to get  $b$  as positive so I do not have problems. Then there exist, and are unique, integers  $q$  and  $r$  such that  $a$  is equal to  $b$  times  $q$ , plus  $r$ , with  $r$  being positive, but strictly smaller than  $b$ . Ok?

EUCLID'S DIVISION ALGORITHM

Let  $a, b \in \mathbb{Z}$ , with  $b > 0$ , so EXIST and are UNIQUE  $q, r \in \mathbb{Z}$  such that  $a = b \cdot q + r$  with  $0 \leq r < b$ .

Figure 1: Euclid's division algorithm theorem written on the blackboard

The choice  $b > 0$  in the enunciation of the EDAT means that Andre is enunciating a particular version of that result. In the general version, the only condition is that  $b$  to be a non-zero integer. Probably, Andre regards this option to save time, since the proof considering  $b \neq 0$  is divided into more cases. This is an observation that is not mentioned by Andre to the students. He knows how to prove the EDAT (KoT) and he considers  $b > 0$  in order to save time in this class.

Andre: So, in a few classes, which I'm not sure exactly when it is going to be, the division algorithm will be a direct consequence of the congruencies when we study the arithmetic modulo  $n$ . But in this moment we will prove (the EDAT) with the tools that we have.

In the above transcript, Andre establishes a connection between the EDAT and the linear congruence, which is a later topic in the course (KSM). He also establishes a simplification connection in the moment that he states that the EDAT can be seen as a consequence of the modular arithmetic (KSM), and it seems that he attempts to promote this connection in the students.

Andre: It is saying there [pointing to the content in the Figure 1] that my proof must to be written in two parts. Firstly, I must to prove that they exist ( $q$  and  $r$ ) and then I must to prove that they are unique. In this point, we should have understood that the most difficult part (of the proof) is the existence. Regarding the uniqueness, let us suppose that there exist two and we will see that they are the same. Actually, there is no secret about how to prove the uniqueness, but we will begin by demonstrating the existence. OK.

When he states that the proof should be done in two parts (existence and uniqueness), Andre demonstrates knowledge about proof techniques (KPM), and about how to demonstrate the existence and uniqueness of  $q$  and  $r$  (KPM).

Andre: First part. I will consider this set (Figure 2). I take all the sets (integers) of the form  $a - xb$ , where  $a$  and  $b$  are the numbers that I gave at the beginning,  $x$  is an integer and  $a - xb$  is non-negative. OK? I am getting this subset of integers.

Proof. Existence

To consider  $S = \{a - xb | x \in \mathbb{Z} e a - xb \geq 0\}$

**Figure 2: Set  $S$  written on the blackboard**

In this part, it is possible to identify a heuristic strategy to this particular topic: the choice of an appropriate set  $S$  of natural numbers to approach a property of that set (KPM), namely, the existence of a minimum element in the set.

Andre: I would like to prove that it ( $S$ ) is not empty. Because it is not just a subset of integers. It is a subset of non-negative integers. This is one of my hypothesis, that these integers are non-negative. We know that a non-empty subset of this  $\mathbb{Z}_{\geq 0}$  always admits a minimum. I will play with this. Firstly, I have to prove that it is not empty. To prove that it is not empty, it is enough to show an element in there. Am I right? What element will I get?

To prove that the set  $S$  is non-empty, Andre notes that it is a subset of non-negative integers and thus  $S$  admits a minimal element (KoT). Then, he remembers the fact of all set composed by non-negative integers has a minimum, that is, he refers to the well-ordering principle, which demonstrates that he knows this result (KoT). When he observes that to demonstrate  $S \neq \emptyset$  is necessary just to exhibit one of its elements, Andre gives the way to verify if a set is non-empty (KoT).

Andre: No, that is okay. Do not apologize. Do not apologize, this question allows us (to see) the details, that every detail that is written is important. It is not  $a$  and it is not  $x = 0$ . What is the number that we know that is positive? It is  $b$ . So to get around this, I would put the minus in  $a$  to obtain a positive sum. The only problem is that I do not know if the  $a$  is positive or negative. [...] So I get minus the absolute value of  $a$ . So I have no problem. Because I have the absolute value of  $a$ , I will get  $a$  plus the absolute value of  $a$  multiplied by  $b$ . The  $b$ , yes, it is strictly positive from this one here (initial condition of the theorem). So, this means that there are at least one. Then this value is greater than or equal to zero.

$x = - a $	$\Rightarrow$	$S \neq \emptyset$	admits minimum
$a +  a b \geq 0$			I call it $r \geq 0$

**Figure 3: Proof that  $S$  is non-empty written on the blackboard**

When Andre discusses the importance of the student questions, he exposes his beliefs about the need to perceive and to consider all the conditions of the enunciation theorem. In addition, he demonstrates to understand that this is an aspect to be developed together with his students.

Andre: Then  $S$  is not empty. If  $S$  non-empty is a subset of it ( $\mathbb{Z}_{\geq 0}$ ),  $S$  admits a minimal element. If there is a minimal element, I must to call it of something. I call it by  $r$ . Then let us see if this  $r$  is exactly what I want. OK! First property. The  $r$  which I already know is greater than or equal to zero. Since  $r$  is an element of this form, the minimum of this set (refers to the set  $S$ ), all elements that lie in that set are non-negative, in particular  $r$  is also non-negative. [...] What is the set that I considered? I chose exactly all the integers on the type  $a - xb$  equal to an integer.

This means that I am considering all relations such that  $a$  is equal to  $x$  times  $b$  plus one integer. Then, this set  $S$  is chosen exactly to satisfy this relationship here. Am I right? So, I am defining the remainders, I am defining the smallest of the remainders and I want to see that the smallest of the remainders fits here [he refers to  $a = bq + r$ , in the theorem].

After calling  $r$  the minimal element of the set  $S$ , Andre tries to demonstrate that this  $r$  is in the conditions of the theorem. Thus, after showing that if  $r$  is an element of the set  $S$  then  $r \geq 0$  (KoT), Andre intends to prove the fact that the remainder is less than  $b$ , revealing his knowledge of different types of proofs (KPM), such as the proof by contradiction.

I want to prove  $r < b$ .

By contradiction, I will assume that  $r \geq b$ . Then,  $0 \leq r - b = a - bq - b = a - (q + 1)b$ .

I name  $q = x$  such that  $a - bq = r$

$\Rightarrow r - b \in S$

$r - b < r$ .

**Figure 4: Proof that  $r$  is strictly less than  $b$  written on the blackboard**

Andre: It means that this integer here is on the type  $a$  minus one integer times  $b$ . And the integers of the type  $a$  minus another integer times  $b$  are, from definition, the elements of  $S$ . Because also ... they are non-negative. This implies that  $r$  minus  $b$  is an element in  $S$ , because it is an element that is exactly written in the form that the elements of  $S$  were defined and it is non-negative. It satisfies both the conditions, then it is an element within  $S$ . Being an element of  $S$ , and being strictly small than  $r$ , there is a contradiction. Why? Because by definition,  $r$  is the minimum. So there can not exist another element strictly small than  $r$  within the set  $S$ . To come in a contradiction means that the hypothesis that I started all this is absurd. Then, it is impossible  $r$  to be greater than  $b$ , this implies that  $r$  must to be strictly small than  $b$ . Thereat, we finish the existence proof. Why? Because I prove that there are those integers  $q$  and  $r$  that satisfy what I want. I wondered two numbers  $q$  and  $r$  such that  $a$  is equal to  $b$  times  $q$ , plus  $r$ .

This kind of proof by contradiction used by Andre (KPM) is recurrent in algebra. When the thesis is contested, a conflict arises in relation to the minimality of an element.

### Some final comments

In this paper, we analyzed the mathematical knowledge of a mathematician, who teaches for PMTs. The Mathematics Teachers' Specialized Knowledge applies to the analysis of this MTE's knowledge because Andre is teaching mathematics to PMTs. In order to characterize the knowledge of this MTE from the demonstration of Euclid's division algorithm theorem, we find evidence of knowledge of topics, knowledge of the structure of mathematics and knowledge of practices in mathematics. However, this knowledge of the MTE about Euclid's Division Algorithm Theorem is



different from the expected knowledge of PMTs and MTs in the same topic, considering that they will not teach this theorem.

The focus is not to evaluate or to prescribe which should be the knowledge of MTEs. Our interest is to investigate which is the existing knowledge in MTEs who participates in our case study, considering the particular Brazilian teacher education context and the role of mathematicians in this context. In this sense, our findings can aid in the elaboration of a model for the specialized knowledge of MTE, as proposed by Contreras et al. (2017), and may also contribute to the investigations into the knowledge of the MTEs. Furthermore, we propose to investigate indications of what are the fundamental knowledges of these MTEs.

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