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# Stability analysis of piecewise affine discrete-time systems\*

Leonardo B. Groff

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**Abstract**—This paper presents an implicit model for piecewise affine functions as the interconnection of a linear function and ramp nonlinearities. We show how to verify positive-definiteness of piecewise quadratic functions by exploiting sector properties of the ramp function. These properties are characterized by a set of identities instead of the usual sector inequalities adopted in the study of Lurie systems. Based on this setup, we then formulate conditions for stability of discrete-time piecewise systems using piecewise quadratic Lyapunov functions in terms of linear matrix inequalities. The results are illustrated in numerical examples.

## I. INTRODUCTION

Models for Piecewise Affine (PWA) systems have been proposed to study engineered systems such as nonlinear circuits [1], [2] and hybrid systems, where simple piecewise affine nonlinearities may lead to complex behavior. In the context of control systems early studies trace back to [3] where an explicit representation was introduced. Moreover, static nonlinearities such as saturation or deadzone can also be studied in this framework since these functions are indeed piecewise affine.

Another example of practical interest of piecewise affine, continuous, functions in discrete-time systems appears in the context of Receding Horizon Optimal Control [4] in which multi-parametric linear or quadratic programs can be solved offline to obtain piecewise control laws together with its partition on the state space, thus resulting a PWA closed loop system. These piecewise continuous functions are referred to as *explicit Model Predictive Control*. The control strategies based on explicit MPC have the advantage of avoiding online

solution of Quadratic Programs. On the other hand, their implementation demands the solution to a location problem for gain selection. Although efficient strategies for the *point location* problem based on binary tree searches have been proposed [5] (see also the review in [6]), this is still a bottleneck of the approach.

Stability of PWA systems has been studied with Lyapunov inequalities. Early theorems have proposed the use of simple quadratic functions as Lyapunov function (LF) candidates. Refinements of these sufficient conditions have been proposed by considering piecewise quadratic LF [7], [8]. In the context of explicit MPC, piecewise affine LF has been studied in [9] and in the context of conewise linear systems, in [10]. A drawback of the results using a different quadratic function defined for each set in the partition appears when assessing the decrease of the LF during a transition. Indeed, one needs to evaluate and enumerate all the possible transitions between partitions.

In this paper we present an implicit representation of PWA functions. Based on the use of ramp functions, the proposed representation allows to avoid some shortcomings of the explicit representations. In particular, by adopting the proposed representation one can easily parametrize continuous piecewise quadratic Lyapunov functions by considering a generalized quadratic form involving ramp functions. Moreover it is possible to assess the stability of PWA systems by evaluating Lyapunov inequalities through linear matrix inequalities (LMI) tests.

This paper is organized as follows: Section II presents the proposed implicit representation for PWA functions that rely on ramp functions. Section III characterizes ramp function properties in terms of quadratic identities and inequalities and presents conditions for verifying positivity of piecewise quadratic forms. In Section IV we apply the positivity verification to devise conditions for stability of discrete-time PWA systems using PWQ Lyapunov functions. Finally we illustrate the obtained results in numerical examples in Section V and present concluding remarks and perspectives in Section VI.

**Notation** Let  $\mathbb{R}^{n \times m}$  denote the set of matrices with real coefficients of dimension  $n$  by  $m$ . Let  $M_{(i,j)}$  denote the element in the  $(i,j)$  entry of matrix  $M$ , define  $\mathbb{D}^n = \{M \in \mathbb{R}^{n \times n} \mid M_{(i,j)} = 0, i \neq j\}$ ,  $\mathbb{S}^n = \{M \in \mathbb{R}^{n \times n} \mid M = M^\top\}$ , and  $\mathbb{P}^{n \times m} = \{M \in \mathbb{R}^{n \times m} \mid M_{(i,j)} \geq 0, \forall i, j\}$ . For  $M \in \mathbb{R}^{n \times n}$  we define  $He(M) := M + M^\top$ . For  $\Omega \subseteq \mathbb{R}$ ,  $\mathbf{1}_\Omega$  is the indicator function of the set  $\Omega$ , that is  $\mathbf{1}_\Omega(\theta) = 1$  if  $\theta \in \Omega$ , and  $\mathbf{1}_\Omega(\theta) = 0$  if  $\theta \in \Omega^c$ , with  $\Omega^c = \mathbb{R} \setminus \Omega$ .

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The function  $\text{sat}_{[\underline{\mu}, \bar{\mu}]} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  denotes the decentralized asymmetric saturation function, of which the  $i$ -th element is defined by

$$\text{sat}_{[\underline{\mu}, \bar{\mu}]}(\theta) := \begin{cases} \underline{\mu}_i & \text{if } \theta_i < \underline{\mu}_i \\ \theta_i & \text{if } \underline{\mu}_i \leq \theta_i \leq \bar{\mu}_i, \\ \bar{\mu}_i & \text{if } \theta_i > \bar{\mu}_i \end{cases}$$

where  $\theta_i$  is the  $i$ -th element of vector  $\theta$ .

## II. IMPLICIT REPRESENTATION OF CONTINUOUS PWA FUNCTIONS

Consider the mapping  $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n_f}$  defined by

$$f(x) = F_1 x + F_2 \phi(y(x)) \quad (1a)$$

$$y(x) = F_3 x + F_4 \phi(y(x)) + f_5 \quad (1b)$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^{n_y}$ ,  $F_1 \in \mathbb{R}^{n_f \times n}$ ,  $F_2 \in \mathbb{R}^{n_f \times n_y}$ ,  $F_3 \in \mathbb{R}^{n_y \times n}$ ,  $F_4 \in \mathbb{R}^{n_y \times n_y}$ ,  $f_5 \in \mathbb{R}^{n_y}$ , and the vector function  $\phi : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_y}$ , which is defined elementwise by the ramp function given by

$$\phi_i(y) = r(y_i) := \begin{cases} 0 & \text{if } y_i < 0 \\ y_i & \text{if } y_i \geq 0 \end{cases}, i = 1, \dots, n_y. \quad (2)$$

We use (1)-(2) as a model for continuous PWA functions thus avoiding the explicit definition of partitions and the corresponding affine functions, as in the standard PWA function representation [3], i.e.

$$f(x) = A_i x + b_i \quad \forall x \in \Gamma_i. \quad (3)$$

Note that with (1)-(2), it is the vector function  $\phi(y(x))$  and the regions where its arguments are not negative that implicitly define the PWA partition of  $\mathbb{R}^n$ . Also, thanks to the continuity of  $\phi$  we have that  $f(x)$  is continuous. Below, we illustrate the representation (1) with two examples.

**Example 1** Consider (1) with

$$\begin{aligned} n = 2, \quad n_y = 3, \quad n_f = 1 \\ F_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \\ F_3 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad f_5 = 0_{3 \times 1}. \end{aligned} \quad (4)$$

The corresponding partition of  $\mathbb{R}^2$  in this case is given by the following sets

$$\begin{aligned} \Gamma_1 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) \geq 0, \phi_2(y(x)) = \phi_3(y(x)) = 0\} \\ \Gamma_2 &= \{x \in \mathbb{R}^2 | \phi_2(y(x)) > 0, \phi_1(y(x)) = \phi_3(y(x)) = 0\} \\ \Gamma_3 &= \{x \in \mathbb{R}^2 | \phi_3(y(x)) > 0\}, \end{aligned}$$

and is depicted in Figure 1. An explicit equivalent representation for  $f(x)$  as in (3) is given by

$$f(x) = \begin{cases} x_1 & \text{if } x \in \Gamma_1 = \{x \in \mathbb{R}^2 | x_1 \geq 0; x_2 \leq x_1\} \\ -x_1 & \text{if } x \in \Gamma_2 = \{x \in \mathbb{R}^2 | x_1 < 0; x_2 \leq -x_1\} \\ x_2 & \text{if } x \in \Gamma_3 = \{x \in \mathbb{R}^2 | x_2 > 0; -x_2 < x_1 < x_2\}. \end{cases} \quad (5)$$

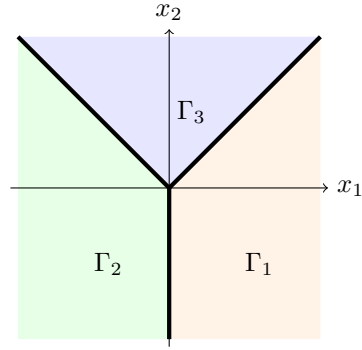


Fig. 1: Partition of  $\mathbb{R}^2$  for  $f(x)$  defined in (5).

**Example 2** Given a matrix  $K \in \mathbb{R}^{n_f \times n}$ , and vectors  $\bar{\mu} \in \mathbb{R}^{n_f}$  and  $\underline{\mu} \in \mathbb{R}^{n_f}$ , the PWA function  $f(x) = \text{sat}_{[\underline{\mu}, \bar{\mu}]}(Kx)$  can be expressed in form (1) with matrices

$$\begin{aligned} F_1 &= K, \quad F_2 = \begin{bmatrix} -I_{n_f} & I_{n_f} \end{bmatrix} \\ F_3 &= \begin{bmatrix} K \\ -K \end{bmatrix}, \quad F_4 = 0_{n_y \times n_y}, \quad f_5 = \begin{bmatrix} -\bar{\mu} \\ \underline{\mu} \end{bmatrix}. \end{aligned} \quad (6)$$

For instance, with  $n = 2$ ,  $n_f = 1$ ,  $K = \begin{bmatrix} -1 & 1 \end{bmatrix}$ ,  $\underline{\mu} = -1$  and  $\bar{\mu} = 2$  we obtain the following partition of  $\mathbb{R}^2$  in terms of  $\phi$

$$\begin{aligned} \Gamma_1 &= \{x \in \mathbb{R}^2 | \phi_2(y(x)) > 0\} \\ \Gamma_2 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) = \phi_2(y(x)) = 0\} \\ \Gamma_3 &= \{x \in \mathbb{R}^2 | \phi_1(y(x)) > 0\}, \end{aligned}$$

which is depicted in Figure 2. An explicit representation for  $f(x)$  is given by

$$f(x) = \begin{cases} -1 & \text{if } x \in \Gamma_1 = \{x \in \mathbb{R}^2 | Kx < -1\} \\ Kx & \text{if } x \in \Gamma_2 = \{x \in \mathbb{R}^2 | -1 \leq Kx \leq 2\} \\ 2 & \text{if } x \in \Gamma_3 = \{x \in \mathbb{R}^2 | 2 < Kx\}. \end{cases}$$

In this paper we use (1) as a model to study stability of discrete-time PWA systems. The main feature of (1) that will be exploited in the formulation of stability conditions in terms of Lyapunov inequalities is the sector properties of the ramp function. These sector properties will be key to obtain numerically tractable conditions for the verification of piecewise quadratic inequalities. The

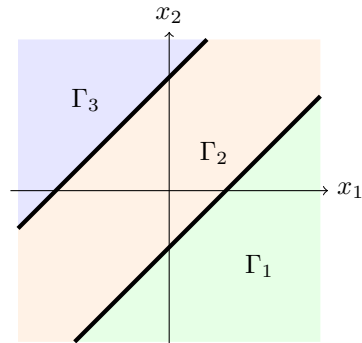


Fig. 2: Partition of  $\mathbb{R}^2$  for  $f(x)$  defined in (6).

implicit representation also simplifies the analysis since the partitions and possible transitions between sets of the partition do not have to be explicitly accounted for in the Lyapunov inequalities.

Moreover, handling uncertainties in the partition induced by (1) can be simpler than with an explicit representation since these uncertainties can be cast as uncertainty on the matrices  $F_3$ ,  $F_4$ , and  $f_5$ , which can be described by matrix sets such as polytopic or norm-bounded ones [11].

#### A. Conditions for well-posedness

Below we provide a condition for the well-posedness of the implicit equation (1b) for all  $x \in \mathbb{R}^n$ , that is, a solution to

$$\tilde{f}(y(x)) = y(x) - F_4\phi(y(x)) = F_3x + f_5. \quad (7)$$

In [12, Proposition 2] it is shown that a locally Lipschitz function  $\tilde{f}(y(x))$  of which the Jacobian satisfies  $J_y\tilde{f}(y) \in \mathcal{M} \subset \mathbb{R}^{n_y \times n_y}$  for almost all  $y \in \mathbb{R}^{n_y}$  where  $\mathcal{M}$  is a compact, convex set, with each of its elements non-singular, implies that there exists a unique globally Lipschitz function  $y(\xi)$  satisfying  $\tilde{f}(y) = \xi$ . Such a result is used in [12] to obtain a condition for the well-posedness of an algebraic loop involving saturation and deadzone functions.

Using the definition of the ramp function in (2), we have that the Jacobian with respect to  $y$  of  $\tilde{f}$  in (7) is given by  $J_y\tilde{f}(y) = (I - F_4\Delta)$  with  $\Delta \in \mathcal{D} = \{\Delta \in \mathbb{D}^n | \Delta_{(i,i)} \in \{0,1\}\}$ , which is a compact and convex set. Thus, following [12, Proposition 2] a unique solution to (7) exists if  $(I - F_4\Delta)$  is non-singular for all  $\Delta \in \mathcal{D} = \{\Delta \in \mathbb{D}^n | \Delta_{(i,i)} \in \{0,1\}\}$ . A condition for the well-posedness is then cast as an LMI constraint (see [13], [12]) as in the proposition below.

*Proposition 1 ([12, Proposition 1]):* If there exist a matrix  $W \in \mathbb{D}^{n_y}$  such that  $-2W + WF_4 + F_4^\top W < 0$  then  $(I - F_4\Delta)^{-1}$  exists  $\forall \Delta \in \mathcal{D}$ .

In the following, we will assume that the condition for well-posedness of (7) of above Proposition 1 holds.

Note that in Example 1 above, (7) has an explicit solution thanks to the structure of  $F_4$  in (4). Indeed, one obtains  $y_1$  and  $y_2$  directly from the term  $F_3x$ , giving  $y_1 = x_1$  and  $y_2 = -x_1$ , which are used to obtain  $y_3 = r(y_1) - r(y_2)$ . Similarly, for Example 2 the solution to equation (7) is explicit since  $F_4 = 0$ , it is then straightforward to compute  $f(x)$  using  $y = F_3x + f_5$ .

### III. PRELIMINARY RESULTS

#### A. Sector Properties of Ramp Functions

Several results to verify the positivity of generalized quadratic forms involving sector bounded nonlinearities rely on sector inequalities that hold either globally or locally [14], [15]. In the following, we introduce properties of ramp functions (2), which are given by sector identities rather than sector inequalities. These identities will be

instrumental to verify the positivity of generalized quadratic forms containing ramp functions.

*Lemma 1:* The ramp function  $r$  satisfies the identity

$$r(\theta)(r(\theta) - \theta) = 0. \quad (8)$$

*Proof:* If  $\theta < 0$ , we have  $r(\theta) = 0$ . If  $\theta \geq 0$  we have that  $(r(\theta) - \theta) = 0$ , thus (8) follows. ■

*Lemma 2:* The ramp function  $r$  satisfies the identity

$$\theta - (r(\theta) - r(-\theta)) = 0. \quad (9)$$

*Proof:* If  $\theta < 0$ , we have  $r(\theta) = 0$  and  $r(-\theta) = -\theta$ . Thus, for  $\theta < 0$ ,  $\theta - r(\theta) + r(-\theta) = \theta - 0 - \theta = 0$ . If  $\theta > 0$  we have  $r(\theta) = \theta$  and  $r(-\theta) = 0$ , thus, for  $\theta > 0$ ,  $\theta - r(\theta) + r(-\theta) = \theta - \theta + 0 = 0$ . ■

*Remark 1:* Note that the identities in (8)-(9) also hold if  $r$  is replaced by any function of the class

$$\rho(\Omega, \theta) = \mathbf{1}_\Omega(\theta)\theta \quad (10)$$

where  $\Omega \in \{\mathcal{I} \subseteq \mathbb{R} | \theta \in \mathcal{I}, -\theta \notin \mathcal{I}\}$  (for instance the ramp function is obtained with  $\Omega = [0, \infty)$ , i.e.  $r(\theta) = \rho([0, \infty), \theta)$  and  $r(-\theta) = \rho((-\infty, 0), -\theta)$ ). Indeed,

$$\begin{aligned} \rho(\Omega, \theta) (\rho(\Omega, \theta) - \theta) &= \mathbf{1}_\Omega(\theta)\theta (\mathbf{1}_\Omega(\theta)\theta - \theta) \\ &= \mathbf{1}_\Omega(\theta)\theta (\mathbf{1}_{\Omega^c}(\theta)\theta) = 0 \end{aligned}$$

generalizing (8). We also have

$$\begin{aligned} \theta - (\rho(\Omega, \theta) - \rho(\Omega^c, -\theta)) &= \theta - (\mathbf{1}_\Omega(\theta)\theta - (-1)\mathbf{1}_{\Omega^c}(\theta)\theta) \\ &= \theta - (\theta) = 0, \end{aligned}$$

thus generalizing (9). ▽

Following the above remark, even though (8) is an identity, this relation is valid for a larger class of functions which includes the ramp function. To exclude other functions in this class, we have to include a set of inequalities that hold only for ramp functions, that is only for (10) with  $\Omega = [0, \infty)$ . This set of inequalities is the following

$$r(\theta) \geq 0 \quad (11a)$$

$$r(-\theta) \geq 0 \quad (11b)$$

$$r(\theta)r(\eta) \geq 0 \quad (11c)$$

$$r(\theta)r(-\eta) \geq 0 \quad (11d)$$

$$r(-\theta)r(-\eta) \geq 0. \quad (11e)$$

Using the definition of  $\phi$  in (2) and the identities related to the ramp function in the above lemmas element-wise in  $\phi$  we obtain the lemmas below.

*Lemma 3:* For any  $T_1 \in \mathbb{D}^{n_y}$  the function  $\phi$  defined in (2) satisfies the identity

$$s_1(T_1, \phi(y), y) := \phi^\top(y)T_1(\phi(y) - y) = 0. \quad (12)$$

$\forall y \in \mathbb{R}^{n_y}$ .

*Lemma 4:* For any vector  $\zeta \in \mathbb{R}^{n_c}$  and  $R \in \mathbb{R}^{n_c \times n_y}$  the function  $\phi$  defined in (2) satisfies the identity

$$s_2(R, \zeta, \phi(y), y) := \zeta^\top R(y - (\phi(y) - \phi(-y))) = 0. \quad (13)$$

$\forall y \in \mathbb{R}^{n_y}$ .

And finally, using the inequalities in (11), we obtain the lemma below.

*Lemma 5:* For any matrix  $M \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$  the function  $\phi$  in (2) satisfies the inequality

$$s_3(M, \phi(y)) := \begin{bmatrix} 1 \\ \phi(y) \\ \phi(-y) \end{bmatrix}^\top M \begin{bmatrix} 1 \\ \phi(y) \\ \phi(-y) \end{bmatrix} \geq 0. \quad (14)$$

$\forall y \in \mathbb{R}^{n_y}$ .

### B. Conditions for Positivity of Extended Quadratic Forms

In this section we use the above lemmas to set conditions to verify the positivity of generalized quadratic forms of the type

$$h(x) = \begin{bmatrix} 1 \\ x \\ \phi(y(x)) \\ \phi(-y(x)) \end{bmatrix}^\top H \begin{bmatrix} 1 \\ x \\ \phi(y(x)) \\ \phi(-y(x)) \end{bmatrix} = \chi(x)^\top H \chi(x). \quad (15)$$

*Proposition 2:* Given a generalized quadratic form  $h(x)$  as in (15), if there exist matrices  $T_1 \in \mathbb{D}^{n_y}$ ,  $T_2 \in \mathbb{D}^{n_y}$ ,  $R \in \mathbb{R}^{1+n+2n_y \times n_y}$ ,  $M \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$  such that

$$h(x) + s_1(T_1, \phi(y(x)), y(x)) + s_1(T_2, \phi(-y(x)), -y(x)) + s_2(R, \chi, \phi(y(x)), y(x)) - s_3(M, \phi(y(x))) \geq 0 \quad (16)$$

then

$$h(x) \geq 0 \quad \forall x \in \mathbb{R}^n. \quad (17)$$

*Proof:* From Lemmas 3 and 4, which hold for all  $y(x)$ , and from (16) it follows that

$$h(x) \geq s_3(M, \phi(y(x))) \quad \forall x \in \mathbb{R}^n.$$

Then, using Lemma 5 we obtain

$$h(x) \geq 0 \quad \forall x \in \mathbb{R}^n. \quad \blacksquare$$

Setting conditions to verify the non-negativity of a generalized quadratic form as (15) by solving the inequality (16) makes possible to solve Lyapunov inequalities related to the positivity of piecewise quadratic functions. These inequalities are studied in the next section. Moreover if matrix  $H$  has an affine dependence on unknown variables, the inequality (16) can be cast as an LMI, therefore yielding constraints of a semi-definite program, which can be solved with freely available optimization software [16]. In this case, the general LMI corresponding to (16) is presented in the appendix.

## IV. STABILITY ANALYSIS OF PWA SYSTEMS WITH PWQ LYAPUNOV FUNCTIONS

In this section we apply the results for the verification of non-negativity of generalized quadratic forms presented in the previous section to study stability of discrete-time systems defined by the implicit representation of piecewise affine functions given in (1).

Consider discrete time systems of the form

$$x^+ = f(x), \quad (18)$$

where  $x \in \mathbb{R}^n$  is the state at instant  $k \in \mathbb{N}$ ,  $f(x)$  is defined by matrices  $F_i$  as in (1) and  $x^+$  is the value of the state at the instant  $k+1$ . From (1b) we have  $y^+ = F_3x^+ + F_4\phi(y^+) + f_5$ . We assume that  $f(0) = 0$ , i.e. the origin is an equilibrium point.

*Remark 2:* If  $F_2$  has full column rank  $f(0) = 0$  implies  $\phi(y(0)) = 0$ , and the equation (1b) thus become  $y = f_5$ . Since  $\phi(y) = 0$ , this last relation imposes that  $f_{5i} \leq 0$ ,  $i = 1, \dots, n_y$ .  $\lrcorner$

Several results in the literature have studied the class of PWA systems using the explicit representation or alternatives as detailed in [17]. Regarding stability analysis of (18), piecewise quadratic Lyapunov functions have been considered and the resulting inequalities often require a first evaluation of the possible transitions between sets of the partitions when casting the inequality related to the decrease of the function [8], [10], [18].

Here we propose a continuous Piecewise quadratic Lyapunov function which is a generalized quadratic form on  $x$  and the function  $\phi(y(x))$  thus not requiring an explicit quadratic form in each set of the partition nor enumerating all the possible transitions. We consider Lyapunov candidate functions  $V: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ ,  $V(0) = 0$  given by

$$V(x) = \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}^\top P \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}. \quad (19)$$

To obtain a quadratic bound for function  $V(x)$ , let  $P_1 \in \mathbb{R}^{n \times n}$ ,  $P_2 \in \mathbb{R}^{n \times n_y}$  and  $P_3 \in \mathbb{R}^{n_y \times n_y}$  such that

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix}.$$

We have

$$V(x) \leq \|P_1\| \|x\|^2 + 2 \|P_2\| \|x\| \|\phi\| + \|P_3\| \|\phi\|^2. \quad (20)$$

To obtain an upper-bound for  $\|\phi\|$  in terms of  $\|x\|$  let  $\bar{y} := y - f_5$  and use (1b) to obtain

$$\bar{y} = F_3x + F_4\phi(\bar{y} + f_5).$$

Since, from Remark 2,  $f_{5i} \leq 0$ ,  $i = 1, \dots, n_y$ , we have  $r(\bar{y}_i + f_{5i}) = \delta_i \bar{y}_i$  with  $\delta_i \in [0, 1]$ . Note that in case  $f_{5i} > 0$  for some  $i$  then it is not possible to obtain  $r(\bar{y}_i + f_{5i}) = \delta_i \bar{y}_i$  with  $\delta_i \in [0, 1]$ . We thus have  $\phi(\bar{y} + f_5) = \Delta \bar{y}$  for  $\Delta \in \mathcal{D}$  (using the notation introduced in Subsection II-A). From the well-posedness assumption,  $(I - F_4\Delta)$  is invertible for all  $\Delta \in \mathcal{D}$ , thus we obtain  $\bar{y} = (I - F_4\Delta)^{-1} F_3x$  and

$$\phi(y) = \phi(\bar{y} + f_5) = \Delta \bar{y} = \Delta(I - F_4\Delta)^{-1} F_3x,$$

which gives

$$\|\phi(y)\| \leq \left( \max_{\Delta \in \mathcal{D}} \|\Delta(I - F_4\Delta)^{-1} F_3\| \right) \|x\| = \sigma \|x\|.$$

From (20) we have  $V(x) \leq \epsilon_2(P) \|x\|^2$ , with

$$\epsilon_2(P) = \|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\|. \quad (21)$$

The theorem below presents condition for the global stability of the origin of (18) using (19) as a Lyapunov function candidate.

*Theorem 1:* If there exist  $P \in \mathbb{S}^{(n+n_y) \times (n+n_y)}$ , matrices  $T_1 \in \mathbb{D}^{n_y}$ ,  $T_2 \in \mathbb{D}^{n_y}$ ,  $R_1 \in \mathbb{R}^{1+n+2n_y \times n_y}$ ,  $M_1 \in \mathbb{P}^{(1+2n_y) \times (1+2n_y)}$  and a positive scalar  $\epsilon$  such that

$$\begin{aligned} & (V(x) - \epsilon x^\top x) + s_1(T_1, \phi(y(x)), y(x)) \\ & + s_1(T_2, \phi(-y(x)), -y(x)) + s_2(R_1, \chi, \phi(y(x)), y(x)) \\ & - s_3(M_1, \phi(y(x))) \geq 0 \end{aligned} \quad (22)$$

and matrices  $T_3 \in \mathbb{D}^{2n_y}$ ,  $T_4 \in \mathbb{D}^{2n_y}$ ,  $R_2 \in \mathbb{R}^{1+n+4n_y \times 2n_y}$ ,  $M_2 \in \mathbb{P}^{(1+4n_y) \times (1+4n_y)}$  and a scalar  $\eta \in (0, 1)$  such that

$$\begin{aligned} & -(V(x^+) - (1 - \eta)V(x)) + s_1(T_3, \phi(\tilde{y}), \tilde{y}) \\ & + s_1(T_4, \phi(-\tilde{y}), -\tilde{y}) + s_2(R_2, \tilde{\chi}, \phi(\tilde{y}), \tilde{y}) \\ & - s_3(M_2, \phi(\tilde{y})) \geq 0 \end{aligned} \quad (23)$$

with  $\tilde{\chi} = [1 \ x^\top \ \phi(\tilde{y})^\top \ \phi(-\tilde{y})^\top]^\top$  and  $\tilde{y} = [y^\top \ y^{+\top}]^\top$  then the origin of (18) is exponentially stable.

*Proof:* Following Proposition 2 we respectively have that if (22) and (23) hold then

$$\epsilon_1 x^\top x \leq V(x)$$

$$V(x^+) \leq (1 - \eta)V(x).$$

From (19) and (21) we have that  $V(x) \leq \epsilon_2 x^\top x$ . Then one obtains  $\|x(k)\| \leq \epsilon_3 e^{\delta k} \|x(0)\|$  with  $\epsilon_3 = (\frac{\epsilon_2}{\epsilon_1})^{\frac{1}{2}}$ ,  $\delta = \ln(\sqrt{1 - \eta})$ . ■

Note that (22) and (23) can be cast in form (16) with appropriate functions  $h(x)$ , that depend affinely on the elements of  $P$ . Thus, they can be expressed as an LMI on the decision variables  $P$ ,  $T_i$ ,  $i = 1, \dots, 4$ ,  $M_j$  and  $R_j$ ,  $j = 1, 2$  (see Appendix).

## V. NUMERICAL EXAMPLES

In this section, we illustrate the results of Theorem 1 with two numerical examples. In the first, we demonstrate the global stability of a piecewise linear system, and in the second one, we analyze the global stability of a system subject to actuator saturation.

*Example I.* Consider a piecewise linear system given by (18) with

$$F_1 = \begin{bmatrix} 0.5 & 0.1 \\ -1 & 0.5 \end{bmatrix} \quad F_2 = \kappa \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and  $F_3$ ,  $F_4$  and  $f_5$  as in (4).

Applying Theorem 1, we can show that the system is globally stable for  $\kappa = 0.41$ , and (19) is a Lyapunov function for the system with

$$P = \begin{bmatrix} -2.9402 & 0.6029 & 1.2960 & -1.7078 & -7.7380 \\ 0.6029 & 1.7008 & 1.6769 & -2.9945 & -2.9163 \\ 1.2960 & 1.6769 & 10.3397 & 9.0159 & 9.2738 \\ -1.7078 & -2.9945 & 9.0159 & 13.8520 & 11.8024 \\ -7.7380 & -2.9163 & 9.2738 & 11.8024 & 14.5700 \end{bmatrix}.$$

Note that the matrix  $P$  is not positive definite. Indeed this is not imposed by the conditions in Theorem 1.

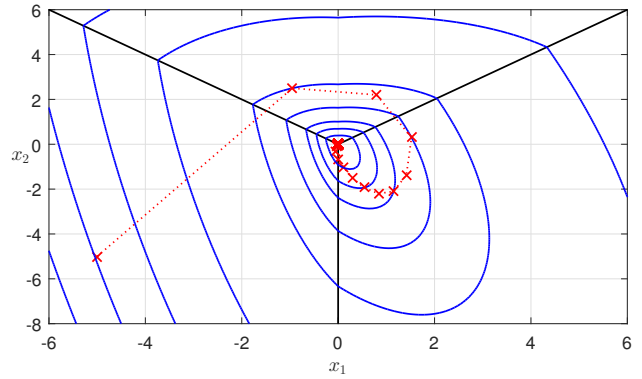


Fig. 3: System trajectory and Lyapunov function level sets for Example I.

However, since (22) holds we have that the Lyapunov function is guaranteed to be positive definite. A trajectory of the system is shown in Figure 3, along with the level sets of the decreasing Lyapunov function. For comparison, the dual problem presented in [8, Section II] demonstrate that there does not exist a quadratic Lyapunov function, that is  $V(x) = x^\top P_1 x$ , with  $P_1 \in \mathbb{R}^{n \times n}$ , that certifies the stability for  $\kappa \geq 0.357$ .

*Example II.* Consider the following system subject to asymmetric actuator saturation inspired from [19, Section V.A] and considering a model obtained by discretizing the corresponding linearized system with a sampling period of 100ms,

$$x^+ = Ax + B \text{sat}_{[-1,15]}(Kx)$$

where

$$A = \begin{bmatrix} 0.9464 & 0.0957 \\ -0.9568 & 0.9033 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0049 \\ 0.0959 \end{bmatrix}, \\ K = [9.9000 \quad 0.4950].$$

Using (6) we have that the right hand side of the above system is written as (18) with  $f(x)$  defined by

$$F_1 = A + BK, \quad F_2 = [-B \quad B]$$

and  $F_3$ ,  $F_4$  and  $f_5$  as in (6).

A solution to the inequalities in Theorem 1, shows that the system is globally stable and (19) is a Lyapunov function for the system with

$$P = \begin{bmatrix} 0.1372 & 0.1684 & -0.0030 & -0.0241 \\ 0.1684 & 1.0349 & -0.0241 & 0.0668 \\ -0.0030 & -0.0241 & 0.1042 & -0.0073 \\ -0.0241 & 0.0668 & -0.0073 & 0.0934 \end{bmatrix}.$$

Figure 4, depicts a trajectory of the system that enters level sets of decreasing values of the obtained Lyapunov function. The proposed framework clearly shows that one can directly obtain asymmetric Lyapunov functions.

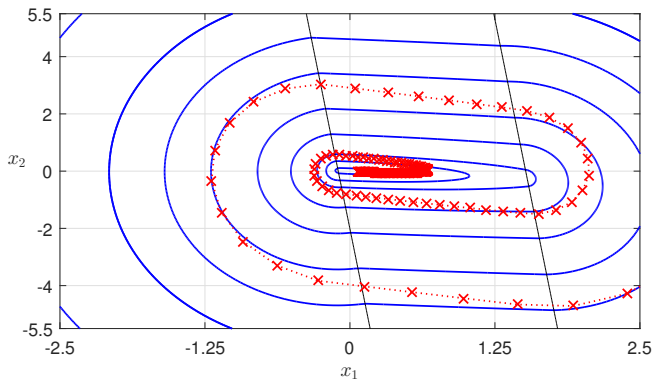


Fig. 4: System trajectory and Lyapunov function level sets for Example II.

## VI. CONCLUSION AND FUTURE WORK

We have presented an implicit representation for PWA functions and have used it as a model for piecewise affine discrete time systems. We also presented a condition for an algebraic equation to be well posed thus guaranteeing that the implicit function is well defined. Such an implicit representation simplifies the analysis of continuous piecewise affine systems since casting Lyapunov inequalities using PWQ Lyapunov functions does not require enumerating all possible transitions between partitions (as e.g. in [8]) and the stability tests require the verification of only two linear matrix inequalities.

The solution to the inequalities, which are given by generalized quadratic forms rely on the characterization of a set of nonlinearities by using sector identities and a set of inequalities that apply only to this function, being less conservative than generic sector bounded conditions.

We are now investigating how to approach the local (regional) stability case and the synthesis of stabilizing PWA control laws. Future work also includes the proposition of stability conditions for continuous-time systems.

## APPENDIX

### A. LMI in Proposition 2

The inequality (16) in Proposition 2 can be verified by solving the LMI in (24).

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$$\begin{aligned}
 & \text{Let } M = \text{He} \left( \begin{bmatrix} \frac{1}{2}M_{11} & M_{12} & M_{13} \\ 0 & \frac{1}{2}M_{22} & M_{23} \\ 0 & 0 & \frac{1}{2}M_{33} \end{bmatrix} \right) \\
 & H + \text{He} \left( \begin{bmatrix} -\frac{1}{2}M_{11} & 0 & -M_{12} & -M_{13} \\ 0 & 0 & 0 & 0 \\ -T_1 f_5 & -T_1 F_3 & (T_1(I - F_4)) - \frac{1}{2}M_{22} & -M_{23} \\ T_2 f_5 & T_2 F_3 & 0 & (T_2(I + F_4)) - \frac{1}{2}M_{33} \end{bmatrix} + \begin{bmatrix} f_5^\top \\ F_3^\top \\ F_4^\top - I \\ I \end{bmatrix} R^\top \right) \geq 0 \quad (24)
 \end{aligned}$$