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► **To cite this version:**

Thomas Hausberger, Frédéric Patras. The didactic contract and its horizon of expectation. *Educere et Educare*, 2019, 14 (33), 10.17648/educare.v15i33.22457 . hal-02429977

**HAL Id: hal-02429977**

**<https://hal.science/hal-02429977>**

Submitted on 2 Mar 2021

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## **The didactic contract and its horizon of expectation**

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### **Abstract:**

The aim of the present article is to investigate the meaning of the didactic contract from the point of view of a key philosophical concept, originated largely in Husserl's phenomenology, namely the notion of horizon. We feature in particular the notion of horizon of expectation, as developed by H. R. Jauss and the Constance School. The core of the article is to explicit the ideas that result from the confrontation of the principles governing the notion of didactic contract with the idea of horizon of expectation. This theoretical perspective is illustrated with two case studies: the analysis of a dialog between two graduate students who are working on a mathematical problem that has been elaborated with the methodology of didactic engineering and an instance of oral communication between professional mathematicians in the context of a research seminar in mathematics. We conclude by reflexive comments about the nature of interactions between the fields of philosophy and mathematics education produced in our research and lastly comment on the fertility of networking approaches from didactics of mathematics and hermeneutics. Namely, we underline that hermeneutics and phenomenology may be applied in the context of mathematics education but also, conversely, that didactical contexts and theoretical constructs may enrich philosophical accounts.

**Key-words:** Phenomenology, horizon of expectation, hermeneutics, didactic contract.

### **Introduction : didactic contract as hermeneutical contract.**

The notion of didactic contract has been introduced in mathematics education by Brousseau (1997) to designate the “system of reciprocal obligation” that determines “explicitly to some

extent, but mainly implicitly - what each partner, the teacher and the student, will have the possibility for managing and, in some way or another, be responsible to the other person for” (Brousseau 1997, p. 31). The starting point of this article is the idea that this largely implicit system of rules shares many features with phenomena at the core of modern hermeneutics. According to the latter, the reading of a poetry, of a novel, the contemplation of a piece of art, are largely governed by the expectations of the reader or of the spectator. These expectations are driven by various factors related for example to what the reader think a poetry should be, to the versification rules it expects the poetry to follow (or not), and so on. Similarly, according to Brousseau, the student’s reception of a lecture, of the text of an exercise, is driven by his preconceptions on what he believes the teacher to expect.

The analogy may seem limited at first sight, but we will try to show that it can give rise to a research program at the interface of didactics and philosophy, that would consist in adapting various fundamental concepts and techniques of hermeneutics to the didactical context in an attempt to augment Brousseau’s theory with new epistemological insights, besides creating a possibly fruitful dialog between didactics and a central piece of contemporary theories in aesthetics.

Concretely, the aim of the present article is to investigate the meaning of the didactic contract from the point of view of a key philosophical concept, originating largely in Husserl’s phenomenology, namely the notion of horizon. We will feature in particular the notion of horizon of expectation, as developed by H. R. Jauss and the Constance School.

The article is organized as follows. We recall first the leading principles of Brousseau’s theory. We present then a brief overview of Husserl’s notion of horizon emphasizing its phenomenological content. The next section introduces Jauss’ horizon of expectation. Some of

its features are relevant in the context of the analysis of literature, the initial purpose of Jauss' investigations, but less interesting for our purposes. We limit therefore essentially our account to the components of the theory that we believe to make sense for didactics -giving deliberately a limited account. We turn then to the core of the article: expliciting the ideas that result from the confrontation of the principles governing the notion of didactic contract with the idea of horizon of expectation. This confrontation is developed along two sections, focusing respectively on the objective knowledge that results from the two approaches and on the dynamical structure of both the didactic contract and the horizon. We illustrate these theoretical ideas with two case studies: the analysis of a dialog between two graduate students who are working on a mathematical problem that has been elaborated with the methodology of didactic engineering (Artigue, 2009); an instance of oral communication between professional mathematicians in the context of a research seminar in mathematics. We conclude by reflexive comments about the nature of interactions between the fields of philosophy and mathematics education produced in our research and lastly comment on the fertility of networking approaches from didactics of mathematics and hermeneutics.

### **1. Didactic contract according to Brousseau.**

Didactics consists in the study of teaching-learning phenomena through the investigation and organization of didactical systems. On first hand, such educational entities are modelled by means of three sub-systems: the teacher(s), the student(s), and the target knowledge. The dynamics of any didactical system is governed by a didactic contract. According to Sarrazy (1995, p. 86), the origin of this central notion introduced by Brousseau was the observation of strategies developed by students to solve tasks. Brousseau interpreted instances of learning dysfunctionality by the argument that students were focusing on finding out what the teacher was

expecting them to do (in other words, they were trying to uncover the implicit contract) rather than developing adequate understanding of the knowledge aimed at. For instance, when asked questions like ‘why did you add these two numbers?’, which didn’t make sense in the given context, Gaël would invariably answer: ‘because this is what the teacher said that we have to do’, ‘this is how I was taught’. Brousseau thus described this phenomenon in the sociological and cultural terms of didactic contract rather than in psychological terms. This contributed to provide a scientific basis for didactics as an emerging discipline, thanks to the explanatory scope of this notion and its use as a lever to facilitate learning.

This was also in tune with the spirit of the times. According to Sarrazy, the epistemological context that accompanied the birth of the didactic contract was a shift from structuralist views inherited from Bourbaki and implemented in the 1970 modern (or new) math reform to a more subject-centered paradigm. The interactionist paradigm of sociology which was spreading in the second half of the 1970s certainly played a role. Goffman (1974), for instance, described various ‘contracts’ that bind our interactions with other people in everyday and professional lives. He called them ‘frames’ (kinds of scenarios) that may be played in different ‘keys’ (e.g. as comedies or tragedies).

Brousseau’s notion of didactic contract appeared together with the birth of the Theory of Didactical Situations (Brousseau, 1997). This theory sees learning as an adaptation to a milieu organized by the teacher (this is precisely his duty). The role of didactical studies is thus to study conditions of success (and failure) of this epistemic game between the teacher and the student-milieu system. Particular attention must be paid to the rules and strategies of the game which are specific to the target knowledge, in other words to the didactic contract. Let us also point out that this notion must be distinguished from the pedagogical contract, which designates the general

social contract, independent of the target knowledge, that binds the actors of the teaching institution.

What is a bit paradoxical about the didactic contract is that, contrary to chess or other games, the contract is not explicit and can vary from classroom to classroom, culture to culture, and according to the knowledge to be taught. The contract cannot be that a teacher simply tells students the method to solve the assigned task and what the right answer is. In order for the learning to be effective, the student must make choices between different strategies and interpret the task. This explains why part of the contract remains implicit and must be uncovered by students, while part of the contract is more or less explicit and gives a frame to the assigned task.

## **2. Phenomenological idea of horizon**

The notion of horizon plays a central role in Husserl's phenomenology. We follow here its exposition in the *Cartesian Meditations* (Husserl, 1950, Sect. 19, Actuality and potentiality of intentional life and ff.). Recall first that intentionality refers to the fact that our consciousness is always directed towards its contents, whatever their nature. This is particularly true in the context of theoretical endeavours, where we aim at grasping ideas, contents, truths.

What Husserl observes first is that multiplicity is inherent to intentionality, among others because synthesis always drives the unity of consciousness (Husserl 1950, Sect. 18, Identification, fundamental form of synthesis. The universal synthesis of transcendental time). The key point for our forthcoming investigations is that "this multiplicity is not exhausted by the description of actual *cogitata*". Indeed, each actual *cogitatum* has its own potentialities that, "far from being undetermined are, as far as their content is concerned, intentionally pre-traced in the current state itself". To establish a first connexion with our previous discussion of Brousseau, students trying to solve an exercise have to perform various syntheses: they have to gather, for

example, the various symbols and notions in the exercise into a whole to make sense of its meaning and understand the question. Understanding this synthesis is however not enough to account for what happens. Indeed, the synthesis is organized implicitly, among others, according to “rules of the game” and these rules, although not proper *cogitata* (the student is usually not aware of the way they contribute to drive his attempts to solve the exercise) have to be taken into account to give a complete access to the understanding of the learning process and its outcomes.

The notion of *horizon* was introduced in phenomenology precisely to account for this particular structure of intentionality: each state of consciousness has an horizon that accounts for the potentialities of consciousness. In our previous example, the expected ability of the students to solve the exercise is connected to and could not be understood without the existence of an horizon of their understanding of the content of the questions they have to solve. More generally, in mathematics, these phenomena relate to the fact that, besides being directed towards problems, objects, proofs, our consciousness is also shaped implicitly by the structural properties of the horizon in which they happen to be embedded.

Interestingly, Husserl chooses to illustrate the phenomenon in the *Cartesian Meditations* by an example carrying a mathematical as well as a perceptive and material content: the cube. The horizon is never completely given, it always carries some indetermination (otherwise it would be a proper *cogitatum*). In spite of this, it always has a certain “*structure of determination*” (*Struktur der Bestimmtheit*). For example, “the cube -viewed from one side- does not ‘tell’ anything on the concrete determination of its hidden faces. However, it is ‘grasped’ in advance as a cube, and then in particular as colored, rough, and so on, each of these determinations leaving always other ones undetermined.”

### **3. Horizon of expectation**

The notion of horizon of expectation, central in this article, builds on the general idea of horizon by putting forward some specific features, particularly relevant when it comes to analyze aesthetic and cognitive phenomena. It has been developed largely in the hermeneutical context<sup>1</sup>. The idea of linking phenomenology with hermeneutics owes much to Gadamer (1960), one of the most prominent theorists of philosophical hermeneutics who, as a student of Heidegger, added ontological features to the husserlian phenomenological idea of horizon<sup>2</sup>. Our interest will however focus primarily on another theorist of hermeneutics, Jauss (1970-72).

The work of Jauss and of the Constance School to which he belongs, contributed to put forward the idea that literature cannot be understood without taking into account the point of view of the reader. In other terms, the reader contributes to define the meaning of a poetry, a novel, an essay, and so on. To indicate how we intend to transport Jauss' ideas in the didactical context, we quote and translate him into English (from the French edition) adding inside brackets analogical statements that refer to mathematical education.

“Even when it appears, a literary work [A mathematical lecture or exercise] does not present itself as an absolute novelty emerging out of a desert of information; there is a full game of announces, signals -patent or latent-, of implicit references, of familiar characteristics, that predispose its public to a certain mode of reception [...]. At this first stage of the aesthetical [didactical] experience, the psychological process of reception of a text does not reduce itself to the contingent succession of simple subjective impressions; this is a guided perception that proceeds according to a well-determined indicative scheme [...]” (Jauss (1970-72), Sect. VII (French edition)).

The notion of horizon of expectation allows to describe and formalizes this phenomenon. The

1 On hermeneutics and mathematics in general, we refer to (Patras, 2013).

2 On Jauss and Gadamer, see for instance the Preface by J. Starobinski to Jauss' *Pour une Esthétique de la réception* (1970-72).



reader (the student) grasps a new text (a new lecture, a new exercise, and so on) with various expectations and according to rules to which he has been introduced and has gained familiarity and confidence through his past experiences.

To take an example from another field than literature or mathematics, we expect a sonata to be shaped according to certain rules. This is not so surprising for the theorist of music who knows that these rules have been largely codified. However, most of us have not learned these rules, our only access to them is through sensitivity and intuition. We will often expect therefore a theme to reappear, most probably in various tonalities, although without ever being able to explain *why* we carry such expectations. We refer to Meyer (2008) for various analysis along such lines. There is a kind of analogous « musicality » in mathematics: a student solving a long problem, divided into several parts, will usually expect that the answers to first questions will reappear later as tools to solve more advanced ones, especially if these first questions appear to be clearly instrumental (that is, without a proper cognitive interest in themselves). The more gifted students will grasp the whole structure of the reasoning and be able to exploit this knowledge.

The notion of horizon of expectation has of course a much wider spectrum of applications than what this example might suggest. Focussing on the didactic contract, we feel it is the origin of essential components of horizons of expectations in mathematics. Brousseau's point of departure, his observation of learning dysfunctionalities due to the focus of students on the didactic contract - and their misunderstanding thereof in the context of given tasks – provides a paradigmatic example of a situation where the scope of hermeneutics would meet the one of didactics: unraveling the structures underlying the reception/interpretation of these tasks.

#### **4. Towards objectivity**

“[The objective meaning of the horizon] never presents itself as *forever given* ; it comes into

light only when the horizon and the new horizons (that are however pre-traced) become explicit” (Husserl, 1950, Sect. 19). In fact, neither the didactic contract nor the horizon of a student’s intentional act can be fully made explicit. It belongs to their very nature to remain partially undetermined, fluctuant, allowing to the student the freedom to find his own interpretation of the rules and the way they have to be used. According to Brousseau (1997, p. 32), “A totally explicit contract is doomed to failure”, since learning requires that students “revolt, negotiate, search for a new contract which depends on the new ‘state’ of knowledge, acquired and desired”.

This underdetermination raises however a methodological issue: aren’t the contract and the horizon necessarily individual, subjective experiences of which no theoretical account could be given? Brousseau, Husserl, Jauss and their heirs, all agree on the fact that this is not the case and that the didactic contract, respectively the horizon of an intentional act, have objective features that can be described by the didactician or the philosopher.

In Husserl’s phenomenology, the components of the horizon are indeed not themselves *cogitata* but they may, together with the very shape that underlies them, be given objectivity and become themselves proper *cogitata*. In Husserl’s words: “We can question each horizon on ‘what is implied in it’ that we can explicit, unraveling the possible potentialities of psychic life. That way, we also unravel its *objective meaning* which is only indicated in the actual *cogito* and is only present implicitly” (Husserl, 1950, Sect. 19). Similarly, in Jauss, the horizon of expectation of the reader is not the mere effect of subjectivity and can be understood by a proper inquiry on the structural elements surrounding the reading -for example the understanding of what an essay should be, at a given moment in history.

In a didactical context this means that various rules, beliefs, or expectations can be accounted for, although they remain implicit in the students’ consciousness. A key argument in favor of

such objective components in didactic contracts and horizons of expectations is their intersubjective character. In practice, many implicit expectations can be observed to be shared by a group of students, especially if they have the same backgrounds. Students arriving in a given course with different cultures will instead possibly have different expectations and will obey different rules when interpreting the tasks they face, leading possibly to learning dysfunctionalities.

### **5. Dynamical structures**

“The theoretical concept in didactics is not the contract (the good, the bad, the true, or the false contact), but the hypothetical *process of finding a contract*. It is this process which represents the observations and must model and explain them.” (Brousseau, 1997, p32)

The existence of natural links between Brousseau’s theory and the phenomenological theory of horizons is further demonstrated by considering another central argument to both of them: the necessity of taking into consideration processes, dynamical structures.

In Phenomenology, the horizon of an intentional act is not only underdetermined, it is constantly changing and evolving. According to the *Cartesian Meditations*, it is an essential feature of consciousness, as consciousness of something, that it can transform itself into new modes of consciousness and be however always directed towards the same intentional object. In such a situation, the object remains the same but the horizon of the intentional act is evolving, and this evolution can be analyzed -for instance, because implicit components of the initial horizon can be grasped in the new one, as it happens in mathematics when our knowledge of an object is progressively augmented<sup>3</sup>.

In Hermeneutics, “the relationship of an isolated text to the paradigm, to the series of prior texts that constitute a literary genre, is also established according to a permanent creation and

modification process of an horizon of expectation. The new text evokes to the reader (or the listener) a whole set of expectation and rules of the game to which he has been familiarized by prior texts and that can be, along the reading, modulated, corrected, modified, or simply reproduced.” (Jauss (1970-72) p 56).

Although a different process than the aesthetic one encountered in literature, learning mathematics in a didactical relationship also implies the dynamics of creation, modification, even rupture of the didactic contract as necessary steps for the learner to achieve the expected rearrangement of knowledge required by the new target knowledge. In this process, “the teacher’s work consists of proposing a learning situation to the student in such a way that she produces her knowing as a personal answer to a question and uses it or modifies it in order to satisfy the constraints of the milieu and not just the teacher’s expectation” (Brousseau, 1997, p. 228). In hermeneutical and didactical terms, the teacher partly shapes the horizon of expectation of the student by the negotiation of a (didactic) contract in a phase called in didactics the *devolution* of the problem. This doesn’t mean that the contract will remain stable: “It is in fact the breaking of the contract that is important [...] Knowledge will be exactly the thing that will solve the crisis caused by such breakdowns” (Brousseau, 1997, p32). This is quite a tricky game: the milieu, through conflicting aspects of the horizon, should be potent enough to produce the necessity of a new opening in the form of the new knowledge which enters the horizon of the solution of the problem in the student’s consciousness.

In further reference to Jauss’ quote, one should observe that the relationship to the paradigms has specific features in a didactical context, due to the normative side of teaching. The paradigm (the official knowledge of the teaching institution organized according to certain standards) is ultimately established by the teacher: it is his duty, at the appropriate moment of the learning

process, to express that learning has been achieved (or not) and designate as a reference what needs to be retained of the properties of objects that have been encountered. This fundamental social phenomenon is called in didactics the *institutionalization* of knowledge.

## **6. Our first case study: the theory of banquets, a didactic engineering**

As a piece of didactic engineering (Artigue, 2009), the theory of banquets was built (by the first author of this paper) on the basis of an epistemological analysis of mathematical structuralism and in particular of the meta-concept of “structure” (Hausberger, 2016b). The structure of banquets is therefore an invented structure (a didactical creation), which bears some similarities with Group Theory but is much simpler and therefore allows an in-class discussion of the structuralist methodology through reflexive thinking on the assigned tasks. It must be taught after a course in Group Theory, so that students have already developed techniques to classify finite groups of small orders up to isomorphism, techniques which may be thematized in the context of banquets.

A banquet is a set  $E$  endowed with a binary relation  $R$  which satisfies the following axioms:

- A1. No element of  $E$  satisfies  $xRx$ .*
- A2. If  $xRy$  and  $xRz$  then  $y = z$ .*
- A3. If  $yRx$  and  $zRx$  then  $y = z$ .*
- A4. For all  $x$ , there exists at least one  $y$  such that  $xRy$ .*

In part 1 of the worksheet, students were asked the following questions:

- 1 a. Coherence: is it a valid (non-contradictory) mathematical theory? In other words, does there exist a model?*
- 1 b. Independence: is any axiom a logical consequence of others or are all axioms mutually independent?*
- 2 a. Classify all banquets of order  $n \leq 3$*
- 2 b. Classify banquets of order 4*
- 2 c. What can you say about  $\mathbb{Z}/4\mathbb{Z}$  endowed with  $xRy \Leftrightarrow y = x+1$ ?*

The next sections of the worksheet were dedicated to the further development of the theory: notions of sub-banquet, irreducible banquet, structure theorem (a banquet is the disjoint union of

tables) which corresponds to the well-known theorem of canonical cycle-decomposition of a permutation.

The theory of banquets carries several phenomenological aspects, starting with its very name that brings an intuitive background and draws on the mental image of guests sitting around tables for a meal. This approach thus meets Freudenthal's (1983) point of view that mathematical structures organize phenomena and should be developed together with mental images and representations. It also connects with Patras' (2001) critique, in the tradition of Husserl, of the gap between axiomatic presentations of mathematical theories in modern papers (and most textbooks on abstract algebra) and their underlying intuitive contents, which results in a loss of meaning in contexts of communicating, teaching or learning mathematics.

Those phenomenological aspects are discussed extensively in Hausberger (2017, section 3). In the sequel, we will restrict to what relates to the didactic contract and horizon of expectation and focus on the dialogue between two graduate students who worked on the tasks described above. Didactical analyses of excerpts of the transcript may be found in Hausberger (2016a) and a full transcript (in French) in Hausberger (2016b). The novelty here lies in the interrelation with hermeneutics.

The students' worksheet begins in fact with two quatrains from the poem *Palace* by Apollinaire, followed by introductory comments that also break with standard teaching practises: "The theory of banquets won't be found in Algebra textbooks: it is a didactical invention. Its aim is to provide an adequate context to discuss, on a simple example, how a mathematical structuralist theory works...". Whereas devolution always amounts to the negotiation of a new contract, these lines illustrate a case where this is made explicit to students. This is all the more necessary as the mathematical content does not belong to the official knowledge, and meta-mathematical

knowledge (knowledge on the mathematical activity itself) is the didactical stake. Of course, the pertinence of poetry to facilitate the rupture with standard mathematical didactic contracts -but negotiate a path towards epistemology of mathematics- is not clear. Returning to Jauss, it is quite an “isolated text with respect to the (teaching) paradigm”, so that the structure of the horizon of its reception by the readers (students) cannot be taken for granted. Further actions of the teacher will be required to further refine the didactic contract, as an evolving process.

We now focus on the work of the two graduate students who will be called Guy and Hans in this account. “So, what does this structure look like?” asks Hans. “The ordering on the real numbers looks like this... the fact that  $\mathbf{R}$  is archimedean... no, it’s not” replies Guy. The students then interpret the given example (question 2c, see above) as a “kind of a shift on  $\mathbf{Z}/n\mathbf{Z}$ ”. Their attempt to identify a form under the system of axioms leads to the drawing on a sheet of paper of the following diagrams:

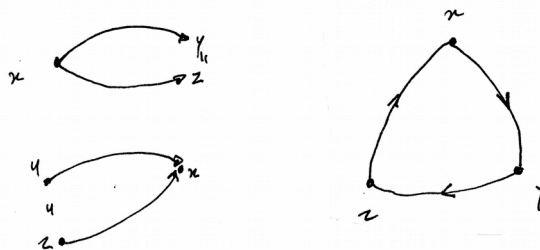


Figure 1: semiotic representations spontaneously produced by the students to make sense of the axioms of a banquet

Hans explains the top left drawing: “Globally, we have a point  $x$  that leads to  $y$  and to  $z$ , by necessity we have an equality”. These ideas further lead to the drawing given on the right of figure 1 as a representation of a banquet of order 3.

At this point, the teacher has chosen to enter the game in order to clarify the status of these representations borrowed from the semiotic register of representations of graph theory.

Teacher: What is, for you, the status of these drawings?

Hans: These two aim to make relations more explicit, I mean axioms A2 and A3, and this one

(pointing to the drawing on the right) is a means for us to get an idea of a model that would resemble to this (pointing now at the axiomatic of banquets).

Guy: In the 3-case, rather.

Teacher: Do you know any mathematical domain in which similar representations are used?

Guy: Graphs

Teacher: Can we consider that this graph is a model of banquet constructed inside graph theory?

Guy: I don't see why it shouldn't be one.

Hans: a priori yes.

Guy: In the 3-case, yes.

Hans: Let's look at the case of 4.

The structure of the horizon of expectation of the students thus contained the graph viewed as an "idea of a model" rather than a real model in the sense of model theory. This may be related to the general status of graphical representations in standard mathematical didactic contracts: they are often regarded as means to help conduct abstract reasonings but they are not granted the status of genuine mathematical objects. This is what happens here, the didactic contract specific to graph theory is not applied. The intervention of the teacher will be needed to legitimate the use of graphs and link them to the notion of model. This is a phase of partial institutionalization that allows to renegotiate the didactic contract, structure further the horizon of expectation and facilitate the development in the direction of a specific theoretical horizon. It is worth noting that students have firsthand adopted a scientific yet somewhat doubtful attitude ("don't see", "shouldn't", "a priori"). The success of the intervention can only be asserted when they engage further in the classification task by explicitly using the graphs' repertoire.

### **7. Second cases study: using diagrams in a mathematical seminar**

"Mathematicians don't communicate their results in the form in which they discover them; they re-organize them, they give them as general a form as possible. Mathematicians perform a 'didactical practice' which consists of putting knowledge into a communicable, decontextualized, depersonalized, detemporalized form. The teacher first undertakes the opposite action; a recontextualization and a repersonalization of knowledge." (Brousseau, 1997, p. 227).



In this quote, Brousseau is alluding to the phenomenon called didactical transposition, which is studied methodologically by didacticians when questioning the origin of the official knowledge contained in a syllabus. By communication, he means the standard context of a lecture. But the phenomenon that is described also applies to the case of communication among mathematicians, which has been the focus of very few didactical studies.

In this last section, we develop such an example: a research seminar where the speaker wants to convey algebraic and combinatorial ideas using diagrammatic representations. The example is based on empirical evidence and methodological reflexions by the second author, both as a speaker having to present certain definitions and ideas and as a listener in others' talks where similar notions and techniques were presented. We limit the discussion to the (elementary but typical in a certain class of combinatorial constructions and problems) definition of a certain class of partitions, but similar arguments would hold for more advanced objects, properties, or results.

We recall first some definitions. Set partitions are fundamental objects in combinatorics, that are met from the very first steps in the theory. Noncrossing partitions form an interesting subclass, which has applications in particular in probability and quantum field theory (in so-called planar theories) and has been studied intensively during the last decade. Classical references on the subject are the articles by Kreweras (1972) and Simion (2000). The context of the seminars were results in free probability theory such as Nica et al. (2006) or Ebrahimi-Fard et al. (2016).

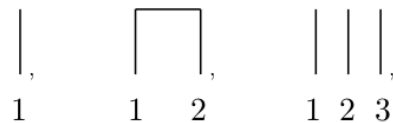
Recall that a partition  $L$  of the set  $[n] := \{1, \dots, n\}$  is a set of non-empty subsets  $L = \{L_1, \dots, L_k\}$  of  $[n]$ , called blocks, mutually disjoint ( $L_i \cap L_j = \emptyset$  for all  $i \neq j$ ), and whose union is  $[n]$ . A partition  $L$  of  $[n]$  is said to be noncrossing if there are no four-tuples  $(p, q, r, s)$  of elements of  $[n]$  with  $p, q$  in a block and  $r, s$  in another block of  $L$  with furthermore  $1 \leq p < r < q < s \leq n$ .

The definition just given of noncrossing partitions, in spite of its simplicity, does not immediately lead to an intuitive understanding of its content. Conveying this intuitive understanding to listeners that are not already familiar with the notion can be achieved in several ways. The most usual one would be to give the above formal definition (that would become for the listeners the object of an intentional act) and illustrate then this definition with statements such as “for example, the partition  $\{\{1, 3\}, \{2, 4\}\}$  is not noncrossing, whereas  $\{\{1,4,5,7\}, \{2,3\}, \{6\}\}$  is”.

We would like to discuss here another approach, often followed by the second author and other speakers on the subject or similar ones. We argue here that this empirical didactical practice can be understood under the light of the didactic contract and the notion of horizon of expectation. The starting point of the analysis is that a research seminar follows different implicit rules than a lecture or a seminar in which students solve tasks under the supervision of a teacher. The speaker is not necessarily expected to make all arguments explicit, nor to attain a systematic rigor. In practice, he wouldn't be able to do so, due to time-constraints, but neither is he willing to: most of the time in such situations, the key point is not so much achieving precision than introducing to ideas and results to which the listener wishing to do so can get a rigorous access by reading the corresponding formalized texts. Expressions often used spontaneously to describe the goals of a seminar such as “to give a picture”, “to show results” and even sometimes “to tell a story” are clearly the indicator of a quite specific form of didactic contract where creating an horizon of expectation and understanding (the belief that “something interesting happens here” that is worth inquiring further) is often more important than attaining certainty.

To turn back to the example of noncrossing partitions, it is natural for these reasons, when introducing the notion in the context of a seminar to give, rather than the definition itself, a

characteristic example from which the definition can be abstracted. To do this, another strategy, commonly used, is to represent partitions in a diagrammatic form, which amounts to enforce a particular semantics. This idea is achieved in the following way. The set  $\{1, \dots, n\}$  can be represented by a sequence of aligned vertical segments, and its subsets by drawing segments joining their upper extremities. For example,  $L = \{\{1\}\}; \{\{1,2\}\}; \{\{1\},\{2\},\{3\}\}$  will be represented by



and, forgetting the labels, the other partitions of  $[3]$ ,  $\{\{1\},\{2,3\}\}; \{\{1,2\},\{3\}\}; \{\{1,3\},\{2\}\}; \{\{1,2,3\}\}$  by



all of which are noncrossing. With 4 to 6 elements we find for example



for the partitions,  $\{\{1, 4\}, \{2, 3\}\}, \{\{1, 3\}, \{2, 4\}\}; \{\{1, 5\}, \{2\}, \{3\}, \{4\}\}, \{\{1, 5\}, \{2\}, \{3, 4\}\}$  and  $\{\{1, 6\}, \{2\}, \{3, 5\}, \{4\}\}$ , where only the second one is non noncrossing. This gives the general rule: a partition is noncrossing whenever it can be represented graphically (as above) in such a way that no segments are crossing. Thus, the partition  $\{\{1, 4\}, \{2, 3\}, \{5, 6, 7\}\}$



is noncrossing. Going back to the idea of introducing the general notion of noncrossing partitions through meaningful examples, the structure of this partition is too simple to give access to the general rule. One would prefer therefore to draw, as characteristic, a partition with a richer

implicit combinatorics such as



Such an example is often enough to allow trained mathematicians to guess automatically what would be the abstract definition encoding the corresponding structure. It is therefore most often useless to state the abstract definition, but there is more to it.

What follows describes a deliberate didactical choice by the second author of this article in such a context (whose relevance can be discussed, but this is not the point here). There is some advantage to keep certain definitions such as the one of noncrossing partitions underdetermined. The chosen example becomes immediately for the listener the object of an intentional act and creates an horizon of expectation: it is indeed now part of the didactic contract that the example points out towards a general definition that can be abstracted out of it. Two phenomena will occur simultaneously (this is what one experiences when being a listener in such a situation and trying to analyze what happens). First, the listener will try to perform the act that he is expected to do: abstract the general definition from the example -at least intuitively, that is by getting the intimate conviction that he could do so if he wished to. Performing this act will transform the horizon of the intentional act, allowing the listener at later stages of the seminar to be convinced that he understands what is at stake when confronted with general statements on noncrossing partitions.

Second, and this second phenomenon is more interesting from a didactical and phenomenological point of view, there is some advantage in maintaining in the listener a doubt on the fact that he truly masters the general notion. This doubt will indeed force him to reexamine, during all the seminar, the validity of his understanding of the phenomena by inquiring whether or not the statements of the speakers, the examples he develops, are

compatible with it. If they are not, he will have to revise his views and, in this process, will acquire a more robust understanding of the materials than he would have by sticking to a formal and rigorous but unintuitive definition.

Developing these ideas would bring us outside the scope of the present article and relate to ideas such as the ones of Lakatos (1976) on the role of errors in the discovery process. We only underline that such a didactical strategy, whatever its general meaningfulness, points out at the possibility of figuring out manifold types of didactic contracts and expectation horizons.

### **Conclusion**

This article originated in the observation of deep similarities between the theoretical construct of didactic contract in didactics of mathematics and the notion of horizon of expectations in hermeneutics and phenomenology. Its first aim was to provide evidence for the existence of common interests and views, and for the fruitfulness of this point of view.

We argue that it allows to augment the theory of didactic contracts with phenomenological insights and techniques. As pointed out by Sarrazy (1995, p. 94), there is a tendency, particularly in interventionist studies, to push for the explicitation of the didactic contract which is misinterpreted as a set of didactical rules, thus conventions. The hermeneutical point of view gives new tools to focus on what is left implicit -on purpose- and needs to be transformed through its journey in the horizon of intentionality of students. The description of the structure of the horizon of expectation that the didactical situation contributes to organize is a way to grasp the necessary negotiations and renegotiations of the didactic contract as a dynamic process that evolves through the learning phases. Phenomenology and hermeneutics offer a new language and methodology to explore the rules of this game played inside the triangulation of the teacher-students-milieu.

Conversely, didactics appears to be a quite natural field of investigation for hermeneutics and phenomenology. The way in which the didactic contract shapes an horizon of expectation has several features that make it particularly relevant to such studies. Indeed, contrary to what happens elsewhere, it is precisely a direct scope of the teacher to shape and engineer the expectation horizon of a lecture, of a given assignment to the students. We face therefore a situation where horizons are not a mere abstract view on intentionality and cognitive processes, but (although implicitly) a key component of a theoretical and practical endeavour.

As a conclusion, this study provides a concrete example of the fertility of a close interplay between philosophy and mathematics education. Ernest (2016) highlighted in his synthesis three distinct directions: philosophy applied to or of mathematics education; philosophy of mathematics applied to mathematics education; philosophy of education applied to mathematics education. In this study, we underlined how hermeneutics and phenomenology may be applied in the context of mathematics education but we also showed, conversely, how didactical contexts and theoretical constructs may enrich philosophical accounts. Although not surprising, this is a fourth case of interaction that was not envisaged by Ernest. We indeed expect and look forward to fruitful outcomes of interdisciplinary researches at the interface of didactics, hermeneutics, phenomenology and philosophy of mathematics.

## **Bibliography**

- Artigue, M. (2009). Didactical design in mathematics education. In C. Winsløw (Ed.), *Proceedings of NORMA08 Nordic Research in Mathematics Education* (pp. 7–16). Rotterdam: Sense Publishers.
- Brousseau, G. (1997). *The theory of didactical situations in mathematics*. Dordrecht: Kluwer.
- Ebrahimi-Fard K. and Patras F. (2016) The Splitting Process in Free Probability Theory, *International Mathematics Research Notices*, No. 9, pp. 2647-2676.
- Ernest, P. (2016). An overview of the Philosophy of Mathematics Education. *Revista Eletrônica*

- de Educação Matemática*. Florianópolis (SC), v.11, Ed. Filosofia da Educação Matemática, 3-20.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht : Reidel
- Gadamer, H. G. (1960). *Wahrheit und Methode. Grundzüge einer philosophischen Hermeneutik*.
- Hausberger, T. (2016a). Abstract algebra, mathematical structuralism and semiotics. In K. Krainer & N. Vondrová (Ed.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education*. Prague: Faculty of Education, Charles University, 2145-2151.
- Hausberger, T. (2016b). Enseignement et apprentissage de l'algèbre abstraite à l'université et premiers éléments d'une didactique du structuralisme algébrique : études croisées en didactique et épistémologique des mathématiques. *Note de synthèse pour l'habilitation à diriger des recherches*. Retrieved from <https://hal.archives-ouvertes.fr/tel-01408565>
- Hausberger, T. (2017). Enseignement et apprentissage de l'algèbre abstraite à l'université : éléments pour une didactique du structuralisme algébrique. In Barrier, T. & Chambris, C. (Eds.), *Actes du séminaire national de didactique des mathématiques 2017*. Paris : IREM de Paris 7 & ARDM, 78-98. Retrieved from <https://hal.archives-ouvertes.fr/hal-02001693/document>
- Husserl, E., Breda, H. L., IJsseling, S., & Boehm, R. (1950). *Husserliana: Cartesianische Meditation und Pariser Vorträge* (Vol. 1). M. Nijhoff.
- Husserl, E., & Derrida, J. (1962). *L'origine de la géométrie*. Paris: Presses universitaires de France.
- Goffman, E. (1974). *Frame Analysis: An Essay on the Organization of Experience*. London: Harper and Row.
- Jauss, H. R., (1970-72) *Literaturgeschichte als Provokation*. Suhrkamp Verlag, Frankfurt am Main 1970. *Kleine Apologie der ästhetischen Erfahrung*. Universitätsverlag Konstanz, Konstanz 1972. Trad. franç. *Pour une esthétique de la réception*. Gallimard, 1978.
- Kreweras, G. (1972) Sur les partitions non croisées d'un cycle, *Discrete Mathematics*, volume 1, number 4, pages 333–350.
- Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge university press.

- Meyer, L. B. (2008). *Emotion and meaning in music*. University of Chicago Press.
- Nica, A. and Speicher, R. (2006). *Lectures on the combinatorics of free probability* (Vol. 13). Cambridge University Press.
- Patras, F. (2001). *La pensée mathématique contemporaine*. Paris : Presses Universitaires de France.
- Patras, F. (2013). Mathématiques et herméneutique. *Archives de Philosophie*, 76(2), 217-238.
- Sarrazy, B. (1995): Le contrat didactique. *Revue Française de Pédagogie* 112, 85-118.
- Simion, R. (2000) Noncrossing partitions, *Discrete Mathematics*, volume 217, numbers 1–3, pages 367-409.