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3-DIMENSIONAL SKEW-SYMMETRIC ALGEBRAS AND THE VARIETY OF HOM-LIE ALGEBRAS

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ABSTRACT. An algebra is called skew-symmetric if its multiplication operation is a skew-symmetric bilinear application. We determine all these algebras in dimension 3 over a field of characteristic different from 2. As an application, we determine the subvariety of 3-dimensional Hom-Lie algebras. For this type of algebras, we study also the dimension 4.

Key Words: 3-dimensional skew-symmetric algebras. Hom-Lie algebras Variety.

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INTRODUCTION. An algebra over a field \mathbb{K} is a \mathbb{K} -vector space equipped with a bilinear product. The multiplication operation in an algebra may or may not be associative, leading to the notions of associative algebras and nonassociative algebras. If we denote by μ this multiplication, we do not assume that μ satisfies any quadratic, ternary, or n -ary relation. When we consider only the finite dimensional framework, that is, when the \mathbb{K} -vector space is of finite dimension, one of the first natural problems arising to determine all these algebras. For example, the classification of algebras up to isomorphism for a given dimension seems interesting and looks at first easy to solve. It is strange that this problem was solved only in dimension 2 (see the works of Goze-Remm, Peterssen or Bekbaev [1, 10, 9]). For special classes of algebras, this classification work was carried on for greater dimensions, but always relatively small; for example, associative algebras are classified up to dimension 6 and Lie algebras up to dimension 7 ([2]). For nonassociative algebras (see for example [3]) there are few results. Of course the classification is complete for some special classes, for example the simple Lie algebras and the simple associative algebras. But for the general case, we have not much information, probably because we know very few invariants in multilinear algebra. Furthermore, when the classification exists (always with the fear that this classification could be incomplete), it is often difficult to use. Consider for example the classification of the complex nilpotent Lie algebras of dimension 7. We can think that this list is complete. In this case there are five one-parameter families and more than one hundred of algebras (this number depends on the way the authors classify and it is not an invariant). To recognize among this list a given Lie algebra it is often difficult especially when it is not written in a basis respecting the invariants used to obtain the classification. Moreover it is not convenient to determine subclasses, for example to determine the contact nilpotent Lie algebras. To test each algebra of this list is more than boring. That is why, as presented in our previous works, we prefer to determine classes invariant by isomorphism and minimal in the sense that we specify.

Date:

In this work we are interested in the 3-dimensional algebras over a field \mathbb{K} with skew-symmetric multiplication. We suppose that \mathbb{K} is an arbitrary field of characteristic different from 2. The paper is organized as follows: In the first part we study the automorphism group of a 3-dimensional algebra. In particular, we characterize among all these algebras those which are Lie algebras by studying the dimension of the automorphism group. In the second part we classify the nilpotent skew-symmetric algebras by showing that all these algebras are Lie algebras (but it is not the case in higher dimensions). Next we classify the solvable skew-symmetric algebras and, to end this classification, the non-solvable case. The last section is essentially dedicated to the study of Hom-Lie algebras, which are a particular class of skew-symmetric algebras. We show that any skew-symmetric algebra of dimension 3 is Hom-Lie. This is no longer true in dimension 4 where we study the algebraic variety of Hom-Lie algebras.

1. THE AUTOMORPHISM GROUP OF A 3-DIMENSIONAL SKEW-SYMMETRIC ALGEBRA

1.1. Generalities. An algebra over a field \mathbb{K} is a \mathbb{K} -vector space V with a multiplication operation given by a bilinear map

$$\mu : V \times V \rightarrow V.$$

Such a multiplication is a linear tensor on V of type $(2, 1)$, that is 2-times contravariant and 1-time covariant. Throughout this paper, the vector space V is finite-dimensional and fixed. Consider for example consider $V = \mathbb{K}^n$. We denote by $A = (V, \mu)$ a \mathbb{K} -algebra with multiplication μ . We also assume that the field \mathbb{K} is of characteristic zero and that the multiplication μ is skew-symmetric, that is

$$\mu(X, Y) = -\mu(Y, X).$$

In this case, we shall say that A is a skew-symmetric algebra, written as *ss*-algebra to short. Two \mathbb{K} -algebras $A = (V, \mu)$ and $A' = (V, \mu')$ are isomorphic if there is a linear isomorphism

$$f : V \rightarrow V$$

such that

$$f(\mu(X, Y)) = \mu'(f(X), f(Y))$$

for all $X, Y \in V$ and we denote by $Aut(A)$ or $Aut(\mu)$ the automorphism group of the algebra $A = (V, \mu)$.

Now assume that $\dim V = 3$. Let $\{e_1, e_2, e_3\}$ be a fixed basis of V . A general skew-symmetric bilinear map μ has the following expression

$$(1) \quad \begin{cases} \mu(e_1, e_2) = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3, \end{cases}$$

and the set of these applications is a 9-dimensional vector space $ssAlg_3$ parametrized by the structure constants $\alpha_i, \beta_i, \gamma_i, i = 1, 2, 3$, identifying μ with its structure constants. The linear group $GL(V)$ acts on $ssAlg_3$ by :

$$(g, \mu) \in GL(V) \times ssAlg_3 \rightarrow \mu_g \in ssAlg_3$$

with

$$\mu_g(X, Y) = g^{-1} \mu(g(X), g(Y))$$

for any $X, Y \in V$. Let $\mathcal{O}(\mu)$ be the orbit of μ corresponding to this action. It is a nonsingular algebraic subvariety of $ssAlg_3$ (see for example [7]). To compute the dimension of the subvariety $\mathcal{O}(\mu)$, it is sufficient to compute the dimension of its tangent space at the point μ . Since $ssAlg_3$ is linear, its tangent space $T_\mu(ssAlg_3)$ at μ is identified to itself and it is constituted by skew-symmetric bilinear map $\psi \in ssAlg_3$. The tangent space $T_\mu(\mathcal{O}(\mu))$ at the point μ to the orbit is a linear subspace of $T_\mu(ssAlg_3) = ssAlg_3$, whose elements are the skew-symmetric applications of type $\psi = \delta_\mu f$ with $f \in End(V)$ and

$$\delta_\mu f(X, Y) = \mu(f(X), Y) + \mu(X, f(Y)) - f(\mu(X, Y))$$

for any $X, Y \in V$. Recall also that f is a derivation of the algebra $A = (V, \mu)$ if $\delta_\mu(f) = 0$. The set $Der(\mu)$ of derivations of A is a Lie algebra, more specifically, the Lie algebra of the algebraic group $Aut(\mu)$.

1.2. The matrix M_μ associated to μ . To compute the dimension of $T_\mu(\mathcal{O}(\mu))$, we consider the following matrix M_μ associated to μ , which will give a linear representation of μ . Recall that we have fixed $\{e_i, 1 \leq i \leq 3\}$ a basis of V and identified the skew-symmetric applications on V with its structure constants related to this basis. Let f be in $gl(V)$. We denote also by f the matrix of f related to the fixed basis. If $f = (f_{ij})$, we consider the vector v_f of \mathbb{K}^9 :

$$v_f = (f_{1,1}, f_{2,1}, f_{3,1}, f_{1,2}, \dots, f_{2,3}, f_{3,3}).$$

We have

$$\delta f(e_i, e_j) = \mu(f(e_i), e_j) + \mu(e_i, f(e_j)) - f(\mu(e_i, e_j))$$

for $1 \leq i < j \leq 3$ and we still denote $\delta f(e_i, e_j)$ the column matrix of its components in the basis $\{e_1, e_2, e_3\}$. The vector

$$\begin{pmatrix} \delta f(e_1, e_2) \\ \delta f(e_1, e_3) \\ \delta f(e_2, e_3) \end{pmatrix}$$

is then a matrix in $\mathcal{M}_{9,1}$. It corresponds to a matrix product:

$$M_\mu \cdot {}^t v_f.$$

This matrix M_μ is a square matrix of order 9 and the map

$$\mu \rightarrow M_\mu$$

gives a linear representation of μ .

If we consider the structure constants of μ related to the basis $\{e_i, 1 \leq i \leq 3\}$ given in (1), then M_μ is equal to

$$M_\mu = \begin{pmatrix} 0 & 0 & -\alpha_3 & -\beta_1 & \alpha_1 & \alpha_2 & -\gamma_1 & 0 & 0 \\ \beta_1 & -\alpha_1 & -\beta_3 & 0 & 0 & \beta_2 & 0 & -\gamma_1 & 0 \\ \gamma_1 & 0 & -\gamma_3 - \alpha_1 & 0 & \gamma_1 & \gamma_2 - \beta_1 & 0 & 0 & -\gamma_1 \\ 0 & \alpha_3 & 0 & -\beta_2 & 0 & 0 & -\gamma_2 & \alpha_1 & \alpha_2 \\ \beta_2 & -\alpha_2 + \beta_3 & 0 & 0 & -\beta_2 & 0 & 0 & -\gamma_2 + \beta_1 & \beta_2 \\ \gamma_2 & \gamma_3 & -\alpha_2 & 0 & 0 & -\beta_2 & 0 & \gamma_1 & 0 \\ -\alpha_3 & 0 & 0 & \alpha_2 - \beta_3 & \alpha_3 & 0 & -\alpha_1 - \gamma_3 & 0 & \alpha_3 \\ 0 & -\alpha_3 & 0 & \beta_2 & 0 & 0 & -\beta_1 & -\gamma_3 & \beta_3 \\ 0 & 0 & -\alpha_3 & \gamma_2 & \gamma_3 & -\beta_3 & -\gamma_1 & 0 & 0 \end{pmatrix}$$

Proposition 1. *For any μ , the matrix M_μ is singular. Moreover, there exists μ such that $\text{Rank}(M_\mu) = 8$.*

Proof. The first part is a direct computation of the determinant. We can see also that 0 is always an eigenvalue. Now we consider μ given by $(\beta_1 = 1, \gamma_2 = 2, \alpha_3 = 1)$. For this algebra the kernel of M_μ is of dimension 1 and constituted of the matrices

$$f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}$$

and $\text{Rank}(M_\mu) = 8$. For a generic algebra, that is, without algebraic conditions between the structure constants, the kernel of M_μ is a 1-dimensional vector space generated by the vector

$$\begin{aligned} &(-\alpha_1\alpha_2\beta_1 + \alpha_1^2\beta_2 - \alpha_2^2\gamma_1 + \beta_3^2\gamma_1 + \alpha_1\alpha_2\gamma_2 - \beta_1\beta_3\gamma_3 + \beta_3\gamma_2\gamma_3 - \beta_2\gamma_3^2, \\ &\alpha_2\beta_1^2 + \alpha_1\beta_1\beta_2 - 2\beta_2\gamma_1(\alpha_2 + \beta_3) + \alpha_2\beta_1\gamma_2 + \alpha_1\beta_2\gamma_2 + \beta_1\beta_3\gamma_2 - \beta_3\gamma_2^2 + \beta_1\beta_2\gamma_3 + \beta_2\gamma_2\gamma_3, \\ &2\beta_2\gamma_1(\alpha_1 - \gamma_3) - \alpha_2\beta_1\gamma_1 + \beta_1\beta_3\gamma_1 - \alpha_1\beta_1\gamma_2 - \alpha_2\gamma_1\gamma_2 + \beta_3\gamma_1\gamma_2 + \alpha_1\gamma_2^2 - \beta_1^2\gamma_3 + \beta_1\gamma_2\gamma_3, \\ &\alpha_1\alpha_3\beta_1 - \alpha_1^2\beta_3 + 2\alpha_3\gamma_1(\alpha_2 + \beta_3) - \alpha_1\alpha_3\gamma_2 - \alpha_1\alpha_2\gamma_3 - \alpha_3\beta_1\gamma_3 - \alpha_1\beta_3\gamma_3 + \alpha_3\gamma_2\gamma_3 - \alpha_2\gamma_3^2, \\ &\alpha_3\beta_1^2 - \alpha_1\beta_1\beta_3 - \alpha_2^2\gamma_1 + \beta_3^2\gamma_1 + \alpha_1\alpha_2\gamma_2 - \alpha_3\gamma_2^2 - \beta_1\beta_3\gamma_3 + \alpha_2\gamma_2\gamma_3, \\ &\alpha_1\beta_3\gamma_1 - \alpha_1\alpha_2\gamma_1 - 2\alpha_3\gamma_1(\beta_1 + \gamma_2) + \alpha_1^2\gamma_2 + \alpha_1\beta_1\gamma_3 + \alpha_2\gamma_1\gamma_3 - \beta_3\gamma_1\gamma_3 + \alpha_1\gamma_2\gamma_3 + \beta_1\gamma_3^2, \\ &\alpha_2\alpha_3(\beta_1 - \gamma_2) - 2\alpha_3\beta_2(\alpha_1 - \gamma_3) + \alpha_1\alpha_2\beta_3 + \alpha_3\beta_1\beta_3 - \alpha_1\beta_3^2 - \alpha_3\beta_3\gamma_2 + \alpha_2\gamma_3(\alpha_2 - \beta_3), \\ &\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1 + \alpha_2\beta_1\beta_3 + \alpha_1\beta_2\beta_3 - 2\alpha_3\beta_2(\gamma_2 - \beta_1) + \alpha_2\beta_3\gamma_2 - \beta_3^2\gamma_2 + \alpha_2\beta_2\gamma_3 + \beta_2\beta_3\gamma_3, \\ &\alpha_1\alpha_2\beta_1 + \alpha_3\beta_1^2 - \alpha_1^2\beta_2 - \alpha_1\beta_1\beta_3 - \alpha_3\gamma_2^2 + \alpha_2\gamma_2\gamma_3 - \beta_3\gamma_2\gamma_3 + \beta_2\gamma_3^2). \end{aligned}$$

Corollary 2. *For any 3-dimensional ss-algebra $A = (V, \mu)$ over \mathbb{K} , the dimension of the automorphism group $\text{Aut}(\mu)$ is greater than or equal to 1.*

Proof. In fact, $\text{Aut}(\mu)$ is an algebraic group whose Lie algebra is isomorphic to $\text{Der}(\mu)$, the Lie algebra of derivations of μ , that is the subspace of $\text{gl}(V)$ whose elements f satisfy $\delta(f) = 0$ or equivalently such that the vector ${}^t v_f$ is in the kernel of M_μ . Since the rank of this matrix is smaller than 8, its kernel is bigger than 1.

Proposition 3. *The algebra $A = (V, \mu)$ is a Lie algebra if and only if there exists a nonzero vector X in V such that the linear endomorphism $L_X : Y \rightarrow \mu(X, Y)$ is in the kernel of M_μ .*

Proof. If $f \in \text{End}(A)$ then, considering the basis $\{e_i, 1 \leq i \leq 3\}$, we identify f with the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and we associate it with the vector

$$v_f = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3).$$

If we take $X = e_1$, then

$$L_{e_1} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & \beta_1 & \beta_2 \\ 0 & \gamma_1 & \gamma_2 \end{pmatrix}$$

and $M_\mu \cdot {}^t v_{L_{e_1}} = 0$ is equivalent to

$$\begin{cases} -\alpha_1\beta_3 + \alpha_3\beta_1 - \gamma_3\alpha_2 + \alpha_3\gamma_2 = 0, \\ \beta_2\alpha_1 - \beta_1\alpha_2 - \gamma_3\beta_2 + \beta_3\gamma_2 = 0, \\ \gamma_2\alpha_1 + \gamma_3\beta_1 - \beta_3\gamma_1 - \gamma_1\alpha_2 = 0, \end{cases}$$

which corresponds to Jacobi's conditions.

Proposition 4. *If the ss-algebra $A = (V, \mu)$ is a Lie algebra, then the rank of M_μ is smaller than or equal to 6.*

Proof. In fact, since $A = (V, \mu)$ is a Lie algebra, the endomorphisms $L_X(Y) = \mu(X, Y)$ are derivations and the vectors v_{L_X} are in the kernel of M_μ . If $L_{e_1}, L_{e_2}, L_{e_3}$ are linearly independent, then $\ker M_\mu \geq 3$ and the rank of M_μ is smaller than 6. If these vectors are linearly dependent, we have a non trivial linear combination between these vectors and without loss of generality, we can assume that $L_{e_1} = 0$. But $v_{L_{e_1}} = (0, 0, 0, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3)$ and $v_{L_{e_1}} = 0$ implies that

$$M_\mu = \begin{pmatrix} 0 & 0 & 0 & -\beta_1 & \alpha_1 & 0 & -\gamma_1 & 0 & 0 \\ \beta_1 & -\alpha_1 & 0 & 0 & 0 & 0 & 0 & -\gamma_1 & 0 \\ \gamma_1 & 0 & -\alpha_1 & 0 & \gamma_1 & -\beta_1 & 0 & 0 & -\gamma_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_1 & 0 & 0 \end{pmatrix}$$

which rank is less or equal to 3.

The converse of this proposition is not true. We can find non-Lie algebras whose associated matrix M_μ is of rank 6. For example we consider the algebra

$$\begin{cases} \mu(e_1, e_2) = e_2, \\ \mu(e_1, e_3) = \gamma_2 e_3, \\ \mu(e_2, e_3) = e_1. \end{cases}$$

This algebra is a Lie algebra if and only if $\gamma_2 = -1$. In this case the rank is 6 and the kernel is composed of the endomorphisms

$$\begin{pmatrix} 0 & -a_3 & a_2 \\ -a_2 & b_2 & 0 \\ a_3 & 0 & -b_2 \end{pmatrix}.$$

If $\gamma_2 = 1$, the algebra is not a Lie algebra but the rank is also 6. In this case the kernel is composed of endomorphisms

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & -b_2 \end{pmatrix}.$$

While the last algebra is not a Lie algebra, we can embed the algebra in a larger class of algebras such as algebras of Lie type (see [6]) or Hom-Lie algebras. We shall see that in the last section.

1.3. What happen in dimension 4? If $A = (V_4, \mu)$ is a 4-dimensional skew-symmetric \mathbb{K} -algebra, then M_μ is a non square matrix of order 24×16 . It is not too much complicated to write this matrix, but we do not make it here. If we identified A with μ and μ with its structure constants related with a fixed basis $\{e_1, e_2, e_3, e_4\}$ of V , the set of 4-dimensional ss-algebras is a 24-dimensional vector space denoted $ssAlg_4$.

Proposition 5. *There exists a Zariski-open set in the affine variety $ssAlg_4$ constituted of ss-algebras (V, μ) whose automorphism group $Aut(\mu)$ is of dimension 0.*

Proof. In fact, for a generic algebra of $ssAlg_4$, there are no relations between the structure constants and the corresponding matrix M_μ is of rank 16. Then, any derivation is trivial and the algebraic group $Aut(\mu)$ is of dimension 0. For example the following algebra

$$\left\{ \begin{array}{l} \mu(e_1, e_2) = e_1 + e_4, \\ \mu(e_1, e_3) = e_2 + e_3, \\ \mu(e_1, e_4) = e_3 + e_4, \\ \mu(e_2, e_3) = e_1 + e_3, \\ \mu(e_2, e_4) = e_2, \\ \mu(e_3, e_4) = e_2 + e_3 \end{array} \right.$$

satisfies this property.

Let us note that $ssAlg_4$ is an affine algebraic variety isomorphic to \mathbb{K}^{24} and the Zariski open set constituted of algebras $A = (V_4, \mu)$ whose automorphism group $Aut(\mu)$ is of dimension 0 is a finite union of algebraic components where each one is the complementary of an hypersurface, then they are algebraic subvarieties of $ssAlg_4$.

Remark: Two-dimensional case. Before studying the dimension 3, we have naturally studied the 2-dimensional case in [10]. In this case, if $\mu(e_1, e_2) = \alpha e_1 + \beta e_2$, then

$$M_\mu = \begin{pmatrix} 0 & 0 & -\beta & \alpha \\ \beta & -\alpha & 0 & 0 \end{pmatrix}.$$

The rank of M_μ is 2 as soon as $\alpha \neq 0$ or $\beta \neq 0$ and in this case $\ker(M_\mu)$ is of dimension 2. This implies that the automorphism group of $A = (V, \mu)$ is an algebraic group of dimension 2. We have also noted that any ss-algebra of dimension 2 is a Lie algebra.

1.4. Deformations and rigidity. A deformation of μ , in the Gerstenhaber's sense, is given by a tensor $(2, 1)$ in $\mathbb{K}[t] \otimes ssAlg_3$. By and large, the notion of deformation permits to describe a neighborhood of μ in $ssAlg_3$. There exists a cohomological approach of the notion of deformations. In our situation, the complex of cohomology is well known. The space of p -cochains is the space of p -linear skew symmetric applications, the first space of cohomology corresponds to the kernel of M_μ , and the second space is a factor space isomorphic to $T_\mu(ssAlg_3)/ImM_\mu$. This complex is well described in the context of operads. We denote by $Sign$ the quadratic operad encoding the category of skew-symmetric algebras. Recall its construction (see also [8]). Let $\mathbb{K}[\Sigma_2]$ be the group algebra of the symmetric group of degree 2. We have $\mathbb{K}[\Sigma_2] = \mathbb{1}_2 \oplus sgn_2$. Considered as a Σ_2 -module, where $\mathbb{1}$ is the 1-dimensional representation and sgn_2 the 1-dimensional signum representation. We then consider $\Gamma(sgn_2)$ the free operad generated by a skew symmetric operation. In particular for this operad, we have $\Gamma(sgn_2)(0) = 0$, $\Gamma(sgn_2)(1) \simeq \mathbb{K}$, $\dim \Gamma(sgn_2)(2) = 1$ because $\Gamma(sgn_2)(2)$ is the $\mathbb{K}[\Sigma_2]$ -module generated by $\{x_1x_2 = -x_2x_1\}$, $\dim \Gamma(sgn_2)(3) = 3$, $\Gamma(sgn_2)(3)$ is the $\mathbb{K}[\Sigma_2]$ -module

generated by $\{(x_1x_2)x_3, (x_2x_3)x_1, (x_3x_1)x_2\}$ and $\dim \Gamma(\text{sgn}_2)(4) = 15$ because $\Gamma(\text{sgn}_2)(2)$ is the $\mathbb{K}[\Sigma_2]$ -module generated by

$$\begin{aligned} & \{(x_1x_2)(x_3x_4), (x_1x_3)(x_2x_4), (x_1x_4)(x_2x_3), ((x_1x_2)x_3)x_4, ((x_1x_2)x_4)x_3, ((x_1x_3)x_2)x_4, \\ & ((x_1x_3)x_4)x_2, ((x_1x_4)x_2)x_3, ((x_1x_4)x_3)x_2, ((x_2x_3)x_1)x_4, ((x_2x_3)x_4)x_1, ((x_2x_4)x_1)x_3, \\ & ((x_2x_4)x_3)x_1, ((x_3x_4)x_1)x_2, ((x_3x_4)x_2)x_1\}. \end{aligned}$$

This operad is a Koszul operad, then the cohomology which parametrizes the deformations is the operadic cohomology.

Recall also the notion of rigidity which traduces the fact that any deformation of μ is isomorphic to μ .

Definition 6. *A skew-symmetric algebra $A = (V, \mu)$ is rigid if the orbit $\mathcal{O}(\mu)$ is Zariski-open in the algebraic variety $ssAlg_3$.*

This topological notion can be replaced by an algebraic condition. If we denote by $H_S^*(\mu, \mu)$ the complex of deformations, an element $\mu \in Alg_3$ is rigid if and only if $H_S^2(\mu, \mu) = 0$ since the variety Alg_3 is an affine reduced variety. We deduce from the previous results

Proposition 7. *For any $\mu \in ssAlg_3$, we have*

$$\dim \mathcal{O}(\mu) \leq 8.$$

In particular none of the 3-dimensional ss-algebras is rigid in $ssAlg_3$.

Proof. Since $ssAlg_3$ is a linear plane and then a reduced algebraic variety, and since $\dim ssAlg_3 - \dim \mathcal{O}(\mu) \geq 1$, none of these algebras is rigid in $ssAlg_3$.

This notion of rigidity, which concerns an element, can be extended to parametrized families of algebras.

Definition 8. *Let F_{t_1, \dots, t_k} be a family of algebras of $ssAlg_3$ parametrized by t_1, \dots, t_k . This family is rigid if its orbit by the action of the group $GL(V)$ is Zariski open in $ssAlg_3$.*

If $\mathcal{O}(F_{t_1, \dots, t_k})$ is this orbit, its rigidity implies that $\overline{\mathcal{O}(F_{t_1, \dots, t_k})}$ (the closure in the Zariski sense) is an algebraic component of $ssAlg_3$. But $ssAlg_3$ is connected so if F_{t_1, \dots, t_k} is rigid, $\overline{\mathcal{O}(F_{t_1, \dots, t_k})} = ssAlg_3$.

Proposition 9. *The family $F_{\beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3}$ whose elements are the algebras*

$$\begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3, \end{cases}$$

is rigid in $ssAlg_3$.

In fact any deformation of any element of this family is isomorphic to an element of the same family.

2. NILPOTENT CASE

Let (A, μ) be a \mathbb{K} -algebra. We consider the descending central series

$$\mathcal{C}^0(\mu) = A, \mathcal{C}^1(\mu) = \mu(A, A), \mathcal{C}^k(\mu) = \mu(\mathcal{C}^{k-1}(\mu), A), \quad k \geq 2.$$

The algebra A is nilpotent if there exists k such that $\mathcal{C}^k(\mu) = 0$. The smallest k such that $\mathcal{C}^k(\mu) = 0$ is the nilindex. For a 3-dimensional nilpotent algebra (A, μ) , we have only the following sequences:

- (1) $A \supset \mathcal{C}^1(\mu) = 0$,
- (2) $A \supset \mathcal{C}^1(\mu) \supset \mathcal{C}^2(\mu) = 0$,
- (3) $A \supset \mathcal{C}^1(\mu) \supset \mathcal{C}^2(\mu) \supset \mathcal{C}^3(\mu) = 0$.

It is obvious that the last nontrivial term $\mathcal{C}^{k-1}(\mu)$ is contained in the center. Thus the first case corresponds to abelian case, the last is impossible because it would imply that $\dim \mathcal{C}^1(\mu) = 2$. Hence $A \supset \mathcal{C}^1(\mu) \supset \mathcal{C}^2(\mu) = 0$ remains. We consider an adapted basis $\{e_1, e_2, e_3\}$. It satisfies

$$(2) \quad \begin{cases} \mu(e_i, e_3) = 0, \\ \mu(e_1, e_2) = \gamma_1 e_3, \end{cases}$$

with $\gamma_1 \neq 0$. We deduce

Proposition 10. *Any 3-dimensional nilpotent algebra is a Lie algebra. It is isomorphic to the abelian 3-dimensional Lie algebra or to the Heisenberg algebra \mathfrak{h}_3 .*

Note that in the nonabelian case the rank of M_μ is equal to 3.

Remark: What happen in dimension greater than or equal to 4? We have seen that in dimension 3, any nilpotent algebra is a Lie algebra. This property doesn't extend to higher dimensions. Let us consider for example the filiform case, that is nilpotent algebras such that

$$A \supset \mathcal{C}^1(\mu) \supset \dots \supset \mathcal{C}^k(\mu) = 0,$$

with $\dim A/\mathcal{C}^1(\mu) = 2$ and $\dim \mathcal{C}^i(\mu)/\mathcal{C}^{i+1}(\mu) = 1$, $i = 1, \dots, k-1$, where k is the nilindex of A . If the dimension is less than or equal to 4, then we can conclude that these algebras are also Lie algebras, in dimension 5 (or greater) we can find filiform non-Lie algebras. Consider the family

$$(3) \quad \begin{cases} \mu(e_1, e_i) = e_{i+1}, \quad i = 2, 3, 4, \\ \mu(e_2, e_3) = ae_4 + be_5, \\ \mu(e_2, e_4) = ce_5, \\ \mu(e_3, e_4) = de_5. \end{cases}$$

Such an algebra is not a Lie algebra as soon as $a - c \neq 0$ or $d \neq 0$. But this algebra is provided with a Hom-Lie algebra structure (see Section 5).

3. SOLVABLE CASE

Let (A, μ) a \mathbb{K} algebra. We consider the derived series:

$$\mathcal{D}^0(\mu) = A, \mathcal{D}^1(\mu) = \mu(A, A), \mathcal{D}^k(\mu) = \mu(\mathcal{D}^{k-1}(\mu), \mathcal{D}^{k-1}(\mu)), \quad k \geq 2.$$

The algebra A is solvable if there exists k such that $\mathcal{D}^k(\mu) = 0$. The smallest integer k such that $\mathcal{D}^k(\mu) = 0$ is the solvindex. For a 3-dimensional solvable algebra (A, μ) , we have only the following sequences:

- (1) $A \supset \mathcal{D}^1(\mu) = 0$,
- (2) $A \supset \mathcal{D}^1(\mu) \supset \mathcal{D}^2(\mu) = 0$,

$$(3) \quad A \supset \mathcal{D}^1(\mu) \supset \mathcal{D}^2(\mu) \supset \mathcal{D}^3(\mu) = 0.$$

In the first case A is abelian. In the second case, if $\dim A/\mathcal{D}^1(\mu) = 2$, we consider an adapted basis $\{e_1, e_2, e_3\}$ and the hypothesis imply that

$$(4) \quad \mu(e_i, e_j) = a_{ij}e_3, \quad 1 \leq i < j \leq 3.$$

Not to consider again the nilpotent case, we consider $a_{13} \neq 0$ or $a_{23} \neq 0$. This algebra is a solvable Lie algebra for any a_{ij} and the rank of $M_\mu = 5$. In particular the automorphism group is of dimension 4. Let us note that any algebra of (4) is isomorphic to

$$\mu(e_1, e_3) = e_3,$$

and in this case, the identity component of this group is constituted of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ x & y & 0 \\ z & 0 & t \end{pmatrix}$$

with $yt \neq 0$.

Assume now that $\dim A/\mathcal{D}^1(\mu) = 1$ and consider an adapted basis $\{e_1, e_2, e_3\}$. This means that $\{e_2, e_3\}$ is a basis of $\mathcal{D}^1(\mu)$. By hypothesis $\mu(e_2, e_3) = 0$. Then μ satisfies

$$(5) \quad \begin{cases} \mu(e_1, e_2) = \beta_1 e_2 + \gamma_1 e_3, \\ \mu(e_1, e_3) = \beta_2 e_2 + \gamma_2 e_3, \\ \beta_1 \gamma_2 - \beta_2 \gamma_1 \neq 0. \end{cases}$$

Such algebra is also a Lie algebra. Recall that the classification, up to isomorphism, of 3-dimensional solvable Lie algebras is given in the book of Jacobson ([5]). The rank of M_μ is smaller or equal to 5. For example, it is equal to 5 for

$$\begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = e_2, \end{cases}$$

and equal to 3 for

$$\begin{cases} \mu(e_1, e_2) = e_2, \\ \mu(e_1, e_3) = e_3. \end{cases}$$

Let the descending sequence be filiform, that is $A \supset \mathcal{D}^1(\mu) \supset \mathcal{D}^2(\mu) \supset \mathcal{D}^3(\mu) = 0$. Assume that $\{e_1, e_2, e_3\}$ is an adapted basis to this flag. We have

$$\mu(e_2, e_3) = \gamma_3 e_3, \quad \mu(e_1, e_2) = \beta_1 e_2 + \gamma_1 e_3, \quad \mu(e_1, e_3) = \beta_2 e_2 + \gamma_2 e_3,$$

with $\gamma_3 \neq 0$ and $\beta_1 \neq 0$ or $\beta_2 \neq 0$. Such an algebra is never a Lie algebra. Let us note that since $\gamma_3 \neq 0$, giving $e'_2 = \gamma_3^{-1} e_3$, we can consider that $\gamma_3 = 1$.

Proposition 11. *Any 3-dimensional solvable algebra is a Lie algebra or is isomorphic to the non-Lie algebra given by*

$$(6) \quad \begin{cases} \mu(e_1, e_2) = \beta_1 e_2 + \gamma_1 e_3, \\ \mu(e_1, e_3) = \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = e_3, \end{cases}$$

with $\beta_1 \neq 0$ or $\beta_2 \neq 0$.

We have

$$\text{Rank}(M_\mu) = 8$$

as soon as $\beta_2 \neq 0$ and the kernel is generated by

$$(-\beta_2, \beta_1 + \gamma_2, -\beta_1^2 - 2\beta_2\gamma_1 + \beta_1\gamma_2, 0, 0, \beta_1, 0, 0, \beta_2).$$

If $\beta_2 = 0$, then $\text{Rank}(M_\mu) = 7$ and the kernel is generated by

$$\{(0, 0, -\gamma_1, 0, 0, 0, 0, 0, 1), (0, 0, -\beta_1 + \gamma_2, 0, 0, 1, 0, 0, 0)\}.$$

It would be interesting to present this parameter β_2 as an invariant which is done in the following proposition:

Proposition 12. *The Killing symmetric form $K_\mu(X, Y) = \text{tr}(L_X \circ L_Y)$ where $L_U(V) = \mu(U, V)$ of an algebra (6) is nondegenerate if and only $\beta_2 \neq 0$.*

Proof. The matrix of K_μ in the basis $\{e_1, e_2, e_3\}$ is

$$\begin{pmatrix} \beta_1^2 + 2\beta_2\gamma_1 + \gamma_2^2 & \gamma_2 & -\beta_2 \\ \gamma_2 & 1 & 0 \\ -\beta_2 & 0 & 0 \end{pmatrix}$$

and its determinant is equal $-\beta_2^2$. Let us note that (A, μ) is not a simple algebra as $I = \mathbb{K}\{e_2, e_3\}$ is an ideal of A , but its Killing form is nondegenerate as soon as $\beta_2 \neq 0$. Let us note also that K_μ is, in this case, a not invariant pseudo scalar product.

4. NON-SOLVABLE CASE

If (A, μ) is not solvable, there exists k with

$$\mathcal{D}^k(\mu) = \mathcal{D}^{k-1}(\mu) \neq 0.$$

We have the following possibilities:

- (1) $A = \mathcal{D}^1(\mu)$,
- (2) $A \supset \mathcal{D}^1(\mu) = \mathcal{D}^2(\mu) \neq 0$,
- (3) $A \supset \mathcal{D}^1(\mu) \supset \mathcal{D}^2(\mu) = \mathcal{D}^3(\mu) \neq 0$.

The sequence $A \supset \mathcal{D}^1(\mu) \supset \mathcal{D}^2(\mu) = \mathcal{D}^3(\mu) \neq 0$ is impossible because this implies that $\dim \mathcal{D}^2(\mu) = 1$ and $\mu(\mathcal{D}^2(\mu), \mathcal{D}^2(\mu)) \neq 0$, which is impossible since μ is skew-symmetric. The same is true for the sequence $A \supset \mathcal{D}^1(\mu) = \mathcal{D}^2(\mu) \neq 0$. Thus, if (A, μ) is not solvable, we have

$$A = \mathcal{D}^1(\mu)$$

that is

$$\begin{cases} \mu(e_1, e_2) = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3 \end{cases}$$

with

$$\det \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \neq 0.$$

Lemma 13. *There exists X, Y in A such that X, Y and $\mu(X, Y)$ are independent.*

Proof. Assume that this property is not true. In any basis $\{e_1, e_2, e_3\}$ we must have

$$\begin{cases} \mu(e_1, e_2) = \alpha_1 e_1 + \beta_1 e_2, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \beta_3 e_2 + \gamma_3 e_3 \end{cases}$$

with

$$\det \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & 0 & \gamma_2 \\ 0 & \beta_3 & \gamma_3 \end{pmatrix} = -\alpha_1 \beta_3 \gamma_2 - \alpha_2 \beta_1 \gamma_3 \neq 0.$$

Let $e'_2 = ae_2 + be_3$. Therefore, we see that $\text{rank}\{e_1, e'_2, \mu(e_1, e'_2)\} = 2$ implies $\beta_1 = \gamma_2$. Likewise $\text{rank}\{e_2, e'_1 = ae_1 + be_3, \mu(e'_1, e_2)\} = 2$ implies $-\gamma_3 = \alpha_1$ and $\text{rank}\{e_3, e'_2 = ae_1 + be_2, \mu(e_3, e'_2)\} = 2$ implies $\beta_3 = \alpha_2$. We deduce

$$-\alpha_1 \beta_3 \gamma_2 - \alpha_2 \beta_1 \gamma_3 = -\alpha_1 \alpha_2 \beta_1 + \alpha_2 \beta_1 \alpha_1 = 0.$$

This gives a contradiction.

From this lemma we can find a basis $\{e_1, e_2, e_3\}$ of A such that

$$\begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3, \end{cases}$$

and $\alpha_2 \beta_3 - \alpha_3 \beta_2 \neq 0$. We shall show that this family can be reduced to a family with only 5 parameters. This condition implies that α_2 or α_3 is non zero. We can assume that $\alpha_3 \neq 0$. In fact, if $\alpha_3 = 0$ and $\alpha_2 \neq 0$ we have our result. If $\alpha_2 \neq 0$, then the change of basis $e'_1 = e_1 - \alpha_2/\alpha_3 e_2$ and $e'_i = e_i$ for $i = 2, 3$ permits to consider $\alpha_2 = 0$. We deduce:

Proposition 14. *Let (A, μ) a 3-dimensional nonsolvable algebra. Then this algebra is isomorphic to the algebra*

$$(7) \quad \begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3, \end{cases}$$

with $\alpha_3 \beta_2 \neq 0$, or

$$(8) \quad \begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \beta_3 e_2 + \gamma_3 e_3, \end{cases}$$

with $\alpha_2 \beta_3 \neq 0$.

The algebras (7) are Lie algebras if and only if $\beta_3 = \gamma_2 = \gamma_3 = 0$, that is if we have

$$\begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = \beta_2 e_2, \\ \mu(e_2, e_3) = \alpha_3 e_1, \end{cases}$$

with $\alpha_3 \beta_2 \neq 0$; for the algebras (8), if and only if $\gamma_2 = \gamma_3 = 0, \beta_3 = -\alpha_2$, that is,

$$\begin{cases} \mu(e_1, e_2) = e_3, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \beta_2 e_2, \\ \mu(e_2, e_3) = -\alpha_2 e_2 \end{cases}$$

with $\alpha_2 \neq 0$.

For any generic algebra μ belonging to the family (7), the rank of M_μ is 8, the kernel is generated by the vector

$$(-\beta_3^2 - \beta_3\gamma_2\gamma_3 + \beta_2\gamma_3^2, 2\beta_2\beta_3 + \beta_3\gamma_2^2 - \beta_2\gamma_2\gamma_3, -\beta_3\gamma_2 + 2\beta_2\gamma_3, -2\alpha_3\beta_3 - \alpha_3\gamma_2\gamma_3, \beta_3^2 - \alpha_3\gamma_2^2, 2\alpha_3\gamma_2 + \beta_3\gamma_3, \alpha_3\beta_3\gamma_2 - 2\alpha_3\beta_2\gamma_3, 2\alpha_3\beta_2\gamma_2 + \beta_3^2\gamma_2 - \beta_2\beta_3\gamma_3, \alpha_3\gamma_2^2 + \beta_3\gamma_2\gamma_3 - \beta_2\gamma_3^2)$$

and of rank 6 in case of a Lie algebra, that is if $(\gamma_2, \beta_3, \gamma_3) = (0, 0, 0)$.

We have a similar result for the family (8), but in this case the kernel is generated by

$$(\beta_3\gamma_2\gamma_3 - \beta_2\gamma_3^2, -\beta_3\gamma_2^2 + \beta_2\gamma_2\gamma_3, 0, \alpha_2\gamma_3^2, -\alpha_2\gamma_2\gamma_3, 0, \alpha_2^2\gamma_3 - \alpha_2\beta_3\gamma_3, \alpha_2\beta_3\gamma_2 - \beta_3^2\gamma_2 + \alpha_2\beta_2\gamma_3 + \beta_2\beta_3\gamma_3, \alpha_2\gamma_2\gamma_3 + \beta_3\gamma_2\gamma_3 - \beta_2\gamma_3^2).$$

Proposition 15. *Any 3-dimensional nonsolvable algebra is a simple algebra and belongs to the family (7) or (8). For all these algebras, the automorphism group is of dimension 1 except if the algebra is a Lie algebra, in this case it is of dimension 3.*

5. APPLICATIONS: HOM-LIE ALGEBRAS

5.1. Hom-Lie algebras. The notion of Hom-Lie algebras was introduced by Hartwig, Larson and Silvestrov in [4]. Their principal motivation concerns deformations of the Witt algebra. This Lie algebra is the complexification of the Lie algebra of polynomial vector fields on a circle. A basis for the Witt algebra is given by the vector fields

$$L_n = -z^{n+1} \frac{\partial}{\partial z}$$

for any $n \in \mathbb{Z}$. The Lie bracket is given by

$$[L_m, L_n] = (m - n)L_{m+n}.$$

The Witt algebra is also viewed as the Lie algebra of derivations of the ring $\mathbb{C}[z, z^{-1}]$. Recall that a derivation on an algebra with product denoted by ab is a linear operator satisfying $D(ab) = D(a)b + aD(b)$. The Lie bracket of two derivations D and D' is $[D, D'] = D \circ D' - D' \circ D$. We can also define a new class of linear operators which generalize derivations, the Jackson derivate, given by

$$D_q(f)(z) = \frac{f(qz) - f(z)}{qz - z}.$$

It is clear that D_q is a linear operator, but its behavior on the product is quite different as the classical derivative:

$$D_q(fg(z)) = g(z)D_q(f(z)) + f(qz)D_q(g(z)).$$

The authors of [4] interpret this relation by putting

$$(9) \quad D_q(fg) = gD_q(f) + \sigma(f)D_q(g)$$

where σ is given by $\sigma(f)(z) = f(qz)$ for any $f \in \mathbb{C}[z, z^{-1}]$. Starting from (9) and for a given σ , one defines a new space of "derivations" on $C[z, z^{-1}]$ constituted of linear operators D satisfying this relation. With the classical bracket we obtain is a new type of algebras so called σ -deformations of the Witt algebra. This new approach naturally leads to consider the space of σ -derivations, that is, linear operators satisfying (9), and to provide it with the multiplication associated with a bracket. This new algebra is not a Lie algebra because the bracket doesn't satisfies the Jacobi conditions. The authors shows that this bracket satisfies a "generalized Jacobi condition", and they have called this new class of algebras the class of

Hom-Lie algebras. Since then this notion was developed through numerous studies and was also generalized in other directions (see [11]).

We denote by \mathcal{HLie}_n the subset $ssAlg_n$ whose elements are n -dimensional Hom-Lie algebras. We have seen that $ssAlg_n$ is an affine variety isomorphic to $\mathbb{K}^{n^2(n-1)/2}$. We will study in particular the case $n = 3$ and $n = 4$, proving that in dimension 3 any skew-symmetric algebra is a Hom-Lie algebra and in dimension 4, \mathcal{HLie}_4 is an algebraic hypersurface in $ssAlg_4$. We end this work by the determination of binary quadratic operads whose associated algebras are Hom-Lie.

Definition 16. *A Hom-Lie algebra structure on the vector space V is a triple $A = (V, \mu, \alpha)$ consisting of a skew-bilinear map $\mu : V \times V \rightarrow V$ and a linear space homomorphism $f : V \rightarrow V$ satisfying the Hom-Jacobi identity*

$$\circlearrowleft_{x,y,z} \mu(\mu(x, y), f(z)) = 0$$

for all x, y, z in A , where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutations on x, y, z .

For example, a Hom-Lie algebra whose endomorphism f is the identity is a Lie algebra. We deduce, since any 2-dimensional skew-symmetric algebra (the multiplication μ is a skew-symmetric bilinear map) is a Lie algebra, that any 2-dimensional algebra is a Hom-Lie algebra.

In what follows we are interested in the determination of all Hom-Lie algebras for small dimensions.

5.2. Hom-Lie algebras of dimension 3. Any 3-dimensional skew-symmetric \mathbb{K} -algebra $A = (V, \mu)$ is defined by its structure constants $\{\alpha_i, \beta_i, \gamma_i\}_{i=1,2,3}$ with respect to a given basis $\{e_1, e_2, e_3\}$:

$$\begin{cases} \mu(e_1, e_2) = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, \\ \mu(e_1, e_3) = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3, \\ \mu(e_2, e_3) = \alpha_3 e_1 + \beta_3 e_2 + \gamma_3 e_3. \end{cases}$$

Let f be an element of $gl(3, \mathbb{K})$ and consider its matrix under the same basis $\{e_1, e_2, e_3\}$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

We then define the vector

$$v_f = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3).$$

For such an algebra we associate the following matrix, HL_μ , belonging to $\mathcal{M}(3, 9)$ the space of matrices of order (3×9) and given by

$$\begin{pmatrix} \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \\ \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \\ \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 \end{pmatrix},$$

where

$$\begin{aligned} v_1 &= (-\beta_3, \alpha_3, 0, \beta_2, -\alpha_2, 0, -\beta_1, \alpha_1, 0), \\ v_2 &= (-\gamma_3, 0, \alpha_3, \gamma_2, 0, -\alpha_2, -\gamma_1, 0, \alpha_1), \\ v_3 &= (0, -\gamma_3, \beta_3, 0, \gamma_2, -\beta_2, 0, -\gamma_1, \beta_1). \end{aligned}$$

Using the notation $ij.k$ in place of $\mu(\mu(e_i, e_j), e_k)$, we have

$$HL_\mu = (23.1 \quad 23.2 \quad 23.3 \quad 31.1 \quad 31.2 \quad 31.3 \quad 12.1 \quad 12.2 \quad 12.3)$$

with

$$\begin{aligned} 23.1 &= \begin{pmatrix} -\alpha_1\beta_3 - \alpha_2\gamma_3 \\ -\beta_1\beta_3 - \beta_2\gamma_3 \\ -\gamma_1\beta_3 - \gamma_2\gamma_3 \end{pmatrix}, \quad 23.2 = \begin{pmatrix} \alpha_1\alpha_3 - \alpha_3\gamma_3 \\ \beta_1\alpha_3 - \beta_3\gamma_3 \\ \gamma_1\alpha_3 - \gamma_3\gamma_3 \end{pmatrix}, \quad 23.3 = \begin{pmatrix} \alpha_2\alpha_3 + \alpha_3\beta_3 \\ \beta_2\alpha_3 + \beta_3\beta_3 \\ \gamma_2\alpha_3 + \gamma_3\beta_3 \end{pmatrix} \\ 31.1 &= \begin{pmatrix} \alpha_1\beta_2 + \alpha_2\gamma_2 \\ \beta_1\beta_2 + \beta_2\gamma_2 \\ \gamma_1\beta_2 + \gamma_2\gamma_2 \end{pmatrix}, \quad 31.2 = \begin{pmatrix} -\alpha_1\alpha_2 + \alpha_3\gamma_2 \\ -\beta_1\alpha_2 + \beta_3\gamma_2 \\ -\gamma_1\alpha_2 + \gamma_3\gamma_2 \end{pmatrix}, \quad 31.3 = \begin{pmatrix} -\alpha_2\alpha_2 - \alpha_3\beta_2 \\ -\beta_2\alpha_2 - \beta_3\beta_2 \\ -\gamma_2\alpha_2 - \gamma_3\beta_2 \end{pmatrix} \\ 12.1 &= \begin{pmatrix} -\alpha_1\beta_1 - \alpha_2\gamma_1 \\ -\beta_1\beta_1 - \beta_2\gamma_1 \\ -\gamma_1\beta_1 - \gamma_2\gamma_1 \end{pmatrix}, \quad 12.2 = \begin{pmatrix} \alpha_1\alpha_1 - \alpha_3\gamma_1 \\ \beta_1\alpha_1 - \beta_3\gamma_1 \\ \gamma_1\alpha_1 - \gamma_3\gamma_1 \end{pmatrix}, \quad 12.3 = \begin{pmatrix} \alpha_2\alpha_1 + \alpha_3\beta_1 \\ \beta_2\alpha_1 + \beta_3\beta_1 \\ \gamma_2\alpha_1 + \gamma_3\beta_1 \end{pmatrix}. \end{aligned}$$

Theorem 17. *Any skew-symmetric 3-dimensional algebra is a Hom-Lie algebra.*

Proof. Consider a basis $\{e_1, e_2, e_3\}$ of the algebra (A, μ) and let $\alpha_i, \beta_i, \gamma_i$ for $i = 1, 2, 3$ be its structure constants defined previously. Let f be in $gl(3, \mathbb{K})$ and let us consider the associated vector

$$v_f = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3).$$

The endomorphism $f \in gl(3, \mathbb{K})$ satisfies the Hom-Jacobi condition if and only if its corresponding vector $v_f = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)$ is in the kernel of the matrix HL_μ . But this matrix belongs to $\mathcal{M}(3, 9)$ and represents a linear morphism

$$t : \mathbb{K}^9 \rightarrow \mathbb{K}^3.$$

From the rank theorem we have

$$\dim \text{Ker } t = 9 - \dim \text{Im } t \geq 6.$$

Then this kernel is always non trivial and for any algebra μ , there exists a non trivial element in the kernel. Consequently this algebra always admits a non trivial Hom-Lie structure.

5.3. Classification of Hom-Lie algebras of dimension 3. We have determined the 3-dimensional skew-symmetric \mathbb{K} -algebras in the above argument. Since any Hom-Lie algebra is skew-symmetric, we deduce the classification of Hom-Lie algebras. Moreover, for a given skew-symmetric algebra, we can calculate the endomorphisms f associated with the Hom-Lie-Jacobi condition solving the linear system:

$$(23.1 \quad 23.2 \quad 23.3 \quad 31.1 \quad 31.2 \quad 31.3 \quad 12.1 \quad 12.2 \quad 12.3) {}^t v_f = 0.$$

For example, the identity map whose associated vector is

$$v_{Id} = (1, 0, 0, 0, 1, 0, 0, 0, 1)$$

is in the kernel of HL_μ if and only if μ satisfies

$$(23)1 + (31)2 + (12)3 = 0$$

that is if it is a Lie algebra. We deduce

Proposition 18. *If the Hom-Lie algebras $A = (V, \mu)$ and $A' = (V, \mu')$ are isomorphic, then the kernels of the associated matrices HL_μ and $HL_{\mu'}$ are isomorphic.*

5.4. **The case of dimension 4.** Let $A = (V, \mu)$ a 4-dimensional skew-symmetric \mathbb{K} -algebra. Let us choose a basis $\{e_1, e_2, e_3, e_4\}$ of V and let us consider the corresponding structure constants of μ :

$$\mu(e_i, e_j) = \sum_{k=1}^4 C_{i,j}^k e_k, \quad 1 \leq i, j \leq 4.$$

This algebra is a Hom-Lie algebra if there exists a linear endomorphism f satisfying the Hom-Lie Jacobi equations. The endomorphism f is represented in the basis $\{e_1, e_2, e_3, e_4\}$ by a square matrix $(a_{i,j})$ of order 4. As in dimension 3, we consider the vecteur $v_f \in \mathbb{K}^{16}$

$$v_f = (a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}, a_{1,2}, a_{2,2}, \dots, a_{3,4}, a_{4,4}).$$

Then f satisfies the Hom-Lie conditions if and only if v_f is solution on the linear system

$$HL_\mu \cdot {}^t v_f = 0$$

where HL_μ is the square matrix of order 16:

$$HL_\mu = \begin{pmatrix} 23.i & 31.i & 12.i & 0 \\ 24.i & 41.i & 0 & 12.i \\ 34.i & 0 & 41.i & 13.i \\ 0 & 34.i & 42.i & 23.i \end{pmatrix}, \quad i = 1, 2, 3, 4,$$

and $ij.k$ is the matrix

$$ij.k = \begin{pmatrix} \sum_l C_{i,j}^l C_{l,k}^1 \\ \sum_l C_{i,j}^l C_{l,k}^2 \\ \sum_l C_{i,j}^l C_{l,k}^3 \\ \sum_l C_{i,j}^l C_{l,k}^4 \end{pmatrix}.$$

The kernel of HL_μ is not trivial if and only if the rank of HL_μ is smaller or equal to 15. We deduce that $A = (V, \mu)$ is provided with a Hom-Lie structure if the structure constants $C_{i,j}^k$ satisfy the homogeneous polynomial equation of degree 16:

$$\det(HL_\mu) = 0.$$

Proposition 19. *The set \mathcal{HL}_4 of 4-dimensional \mathbb{K} -Hom-Lie algebras is provided with a structure of algebraic variety embedded in \mathbb{K}^{24} .*

For any $A = (V, \mu) \in \mathcal{HL}_4$ we consider the vector space $\text{Ker } HL_\mu$. We thus define a singular vector bundle $K(\mathcal{HL}_4)$ whose fiber over $A = (V, \mu)$ is $\text{Ker } HL_\mu$. This fiber corresponds to the set of Hom-Lie structure which can be defined on a given 4-dimensional algebra.

Remarks.

- (1) In dimension 3, \mathcal{HL}_3 is the affine variety $ssAlg_3$ which is isomorphic to the affine space \mathbb{K}^9 . In this case, the fibers of $K(\mathcal{HL}_3)$ are vector spaces of dimension greater or equal to 6.
- (2) In dimension 4, we are confronted with the resolution of the equation of degree 16 with 24 variables $\det(HL_\mu) = 0$. If \mathbb{K} is algebraically closed, we can simplify this problem because the endomorphism f admits a general reduced form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ 0 & a_{2,2} & a_{2,3} & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}$$

In this case, we need only to consider the reduced matrix HL'_μ of order 16×7 :

$$\begin{pmatrix} 23.1 & 31.1 & 31.2 & 12.2 & 12.3 & 0 & 0 \\ 24.1 & 41.1 & 41.2 & 0 & 0 & 12.3 & 12.4 \\ 34.1 & 0 & 0 & 41.2 & 41.3 & 13.3 & 13.4 \\ 0 & 34.1 & 34.2 & 42.2 & 42.3 & 23.3 & 23.4 \end{pmatrix}$$

It remains to verify that there exists at least one point of $ssAlg_4$ which do not belong to \mathcal{HL}_4 . Let us consider, for example, the following algebra:

$$\begin{cases} \mu(e_1, e_2) = e_2 + 2e_3 - e_4, \\ \mu(e_1, e_3) = e_1 + 2e_2 - e_3, \\ \mu(e_1, e_4) = 2e_1 - e_2 + e_4, \\ \mu(e_2, e_3) = -e_1 + e_3 + 2e_4, \\ \mu(e_2, e_4) = e_1 + 2e_2 - e_3 + 3e_4, \\ \mu(e_3, e_4) = -2e_1 - e_2 + e_3 + 2e_4. \end{cases}$$

The associated matrix is

$$\begin{pmatrix} -5 & -1 & 3 & -4 & -1 & 1 & 1 & -6 & 0 & 3 & -3 & -3 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & -1 & -2 & -4 & 6 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & -1 & 1 & 5 & -3 & -1 & 3 & 0 & -3 & 2 & 1 & 0 & 0 & 0 & 0 \\ -2 & -9 & -4 & 1 & -2 & -1 & -4 & -5 & -2 & -1 & 4 & 7 & 0 & 0 & 0 & 0 \\ -5 & -4 & 4 & 6 & 2 & 1 & -5 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & -3 \\ 3 & 3 & 4 & 4 & -2 & 0 & -5 & 4 & 0 & 0 & 0 & 0 & 6 & 2 & -1 & 0 \\ -5 & 6 & 1 & -3 & -2 & -5 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & -3 & 2 & 1 \\ -1 & -8 & -3 & 5 & 2 & 5 & 4 & 1 & 0 & 0 & 0 & 0 & -2 & -1 & 4 & 7 \\ -5 & -1 & 3 & -7 & 0 & 0 & 0 & 0 & 2 & 1 & -5 & -3 & 1 & -1 & -1 & 6 \\ 1 & -6 & -2 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & -5 & 4 & 0 & 1 & 2 & 4 \\ 3 & -3 & -1 & 2 & 0 & 0 & 0 & 0 & -2 & -5 & 4 & -1 & -5 & 3 & 1 & -3 \\ -3 & -6 & -6 & -3 & 0 & 0 & 0 & 0 & 2 & 5 & 4 & 1 & 2 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & -5 & -1 & 3 & -7 & 5 & 4 & -4 & -6 & -5 & -1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 1 & -6 & -2 & -1 & -3 & 5 & -4 & -4 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 & -1 & 2 & 5 & -6 & -1 & 3 & 1 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & -3 & -6 & -6 & -3 & 1 & 8 & 3 & -5 & -2 & -9 & -4 & 1 \end{pmatrix}$$

and its determinant is non zero. This algebra is not Hom-Lie. As a consequence we deduce that there exists an open set in $ssAlg_4$ whose elements are not Hom-Lie algebras.

Proposition 20. *In the affine plane $ssAlg_4$, there is a Zariski open set whose elements are 4-dimensional algebras without Hom-Lie structure.*

6. ALGEBRAS OF LIE TYPE OF DIMENSION 3

Since the multiplication μ is skew-symmetric, any quadratic relation concerning μ can be reduced to a relation of type

$$h_1(X, Y, Z)\mu(\mu(X, Y), Z) + h_2(X, Y, Z)\mu(\mu(Y, Z), X) + h_3(X, Y, Z)\mu(\mu(Z, X), Y) = 0$$

where h_1, h_2, h_3 are functions $h_i : V^3 \rightarrow V$. In particular, we have the class of algebras of Lie type ([6]) given by the relation

$$\mu(\mu(X, Y), Z) + a(X, Y, Z)\mu(\mu(Y, Z), X) + b(X, Y, Z)\mu(\mu(Z, X), Y) = 0$$

with $a \neq 0$. For example, In the first section, we have considered the non Lie algebra

$$\begin{cases} \mu(e_1, e_2) = e_2, \\ \mu(e_1, e_3) = e_3, \\ \mu(e_2, e_3) = e_1. \end{cases}$$

This algebra satisfies the identity

$$\mu(\mu(X, Y), Z) + \mu(\mu(Y, Z), X) + b(X, Y, Z)\mu(\mu(Z, X), Y) = 0$$

where b is a trilinear form satisfying $b(e_1, e_2, e_3) = -b(e_3, e_1, e_2) = 1$. It is an algebra of Lie type. An important study of the automorphism group of these algebras is presented in [6]. In this example, $\dim Aut(\mu) = 3$ and the identity component is the group of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

with $ad - bc = 1$. It is isomorphic to $SO(2)$. For all the non Lie algebras determined in the above sections, let us determine the algebras of Lie type.

- (1) Algebras (6). Such algebras are of Lie type if $\beta_2 = \beta_3 = 0$, that is if it is also a Lie algebra.
- (2) Algebras (7). Such algebras are never algebras of Lie type.
- (3) Algebras (8). Such an algebra is a Lie algebra and only if it is a Lie algebra $(\gamma_2, \beta_3, \gamma_3) = (0, 0, 0)$. Assume $\alpha_2 = 1, \beta_2 = 0$. We have a Lie algebra structure if and only if $(\gamma_2, \beta_3, \gamma_3) = (0, -1, 0)$. We have a complex structure of algebra of Lie type, which is not a Lie algebra if and only if $\gamma_2 = \gamma_3 = 0$ and β_3 is a complex root of -1 (we assume here that $\mathbb{K} = \mathbb{C}$) that is $\beta_3 = -j$ or $-j^2$. In this case the relation is

$$\mu(\mu(X, Y), Z) + j\mu(\mu(Y, Z), X) + j^2\mu(\mu(Z, X), Y) = 0$$

or

$$\mu(\mu(X, Y), Z) + j^2\mu(\mu(Y, Z), X) + j\mu(\mu(Z, X), Y) = 0.$$

Moreover in this case M_μ is of rank 8 and the automorphism group is of dimension 1 and generated by the automorphisms

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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