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# Implementing alternative models for introducing multiplication

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*The research literature provides plenty of examples of epistemological analyses of multiplication and descriptions of the complexity of the conceptual field of multiplication. Nevertheless, multiplication is often introduced as repeated addition, although decades of research have identified this pedagogical choice as leading to persistent problems in students' conceptualisation of multiplication. In this paper, we describe a teaching design that aims to implement theoretical and empirical research results regarding multiplication in classroom practice. Within our design, models in the form of iconic representations serve as a means for creating patterns that make multiplicative invariants and structures visible. The teaching we have designed is currently tested in a mid-scale randomized controlled trial and in a large-scale professional development project.*

*Keywords: Multiplication, primary school, models, implementation.*

## Introduction

Multiplicative structures constitute a vast part of compulsory school mathematics. The importance of the concept of multiplication for students' possibilities to develop mathematical competence cannot be overstated. When multiplication is introduced in school mathematics, children start a journey that should take them to advanced mathematical knowledge about fractions, ratio, rate, slope, similarity, trigonometry, probability and vector spaces. In addition, physics and social science are heavily reliant on multiplicative reasoning and proportional relationships, e.g., statements of equality of ratios  $a/b = c/d$ . Hence, multiplication is a fundamental concept, and it is therefore crucial to learn from educational research and to implement gained insights into classroom practice.

Up until today, research has gathered extensive knowledge on students' understanding of multiplication (e.g., Clark & Kamii, 1996; Fischbein, Deri, Nello & Marino, 1985; Harel & Confrey, 1994; Larsson, 2016; Thompson & Saldanha, 2003). We know that concepts preferably develop over long periods, through experience of a large number of situations (Vergnaud, 1988). We also know that the "whole number bias" leads students to a solid belief that *multiplication makes bigger* and that multiplication is synonymous with *repeated addition*. However, we do not yet know how to transform research results about students' understanding of the properties of multiplication into classroom practice in order to obviate the whole number bias. In this paper, we will describe an ongoing project, aiming to bridge the gap between research findings and teaching practice through an innovation (Century & Cassata, 2016): a carefully designed teaching sequence implementing alternative models for the introduction of multiplication. The implementation is carried out in Sweden, where multiplication traditionally is introduced in school year 2, at age 8–9 years. Research on the effects of the implementation is an integral part of the project.

## **The whole number bias**

The whole number bias is “a tendency to use the single-unit counting scheme applied to whole numbers to interpret instructional data on fractions” (Ni & Zhou, 2005, p. 27). This bias may be a result of an unchallenged presentation of multiplication as repeated addition with whole numbers. The consequences of a biased conceptualisation have been long known. Bachelard (1938) concluded that knowledge that has been formed and tested for validity in action tends to form a consolidated belief concerning the nature of the concept. When the concept is later challenged in new situations, assimilation might be more complex than if initial knowledge was absent. This is why an introduction of multiplication as repeated addition may become an epistemological obstacle for students. Fischbein et al. (1985) conclude that:

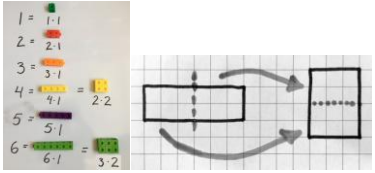
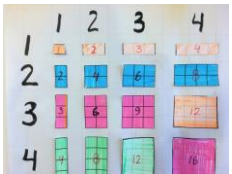
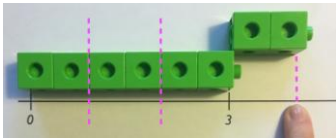
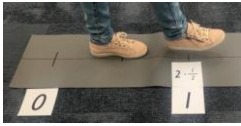
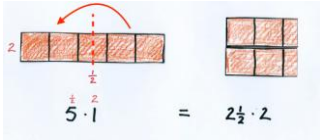
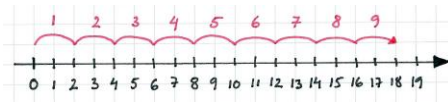
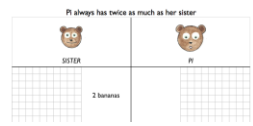
The initial didactical models seem to become so deeply rooted in the learner’s mind that they continue to exert control over mental behavior even after the learner has acquired formal mathematical notions that are solid. (p. 16)

There are two misconceptions of the properties of multiplication specifically linked to the whole number bias: conceptualising multiplication as an additive relation rather than a multiplicative relation, and a belief that a product will always be bigger than each of its factors. As long as multiplication is limited to whole numbers, these misconceptions will not constrain children’s reasoning, and are therefore unlikely to be challenged. However, as soon as children are introduced to multiplication with fractions and decimals their conceptualisation will no longer be sufficient. Products such as  $\frac{1}{2}$  times 5 cannot be calculated as repeated addition, and the law of commutativity, which could be helpful in this situation, is difficult to explain in terms of repeated addition. Furthermore, a firm belief that multiplication makes bigger will undermine the children’s possibilities to interpret situations with operations like  $0.22 \cdot 1.20$  (Bell, Swan & Taylor, 1981), since the answer is less than 1.20, which is not compatible with a conceptualisation of multiplication built on additive reasoning with whole numbers. In summary, conceptualising multiplication as repeated addition works when dealing with whole numbers but will become a misunderstanding of multiplicative properties when extended to rational and real numbers. This is a strong argument for finding alternative ways to introduce multiplication, and thereby avoiding the whole number bias.

## **Alternative models for introducing multiplication**

The complexity of multiplication is reflected in the body of research, and might be one reason why innovative designs are scarce in both research and practice. The efforts to describe dimensions and aspects of multiplicative knowledge are diverse, and show less agreement than, for example, the literature on additive knowledge (Nunes & Bryant, 1996). In Vergnaud’s (1988) terms, multiplicative structures can be seen as a conceptual field, consisting of situations, invariants and symbolic representations. Compared to additive structures, multiplication embraces a much more diverse family of situations. Already in the simplest case, with two factors, multiplication unifies qualitatively different situations, which can be both one-dimensional, like scaling, and two-dimensional, like one-to-many correspondence, area, intensity and Cartesian products (Harel & Confrey, 1994; Nunes & Bryant, 1996; Vergnaud, 1988). In line with the theory of conceptual

fields, our teaching design comprises both activities focused on situations and activities focused on invariants, with care taken to link established symbolic notation to both types of activities (see Table 1).

Week 1		Factorization (multiplicative grouping) of numbers with lattice models. Regrouping of lattices.														
Week 2		Connecting lattices to symbolic notation for multiplication and division. Constructing the multiplication table by sorting lattices by factors.														
Week 3		Placing numbers on number lines by measuring with skips and cubes. Discovering proportional relationships.														
Week 4		Measuring number lines with halves. Equating different numeric expressions for multiple halves.														
Week 5		Regrouping lattices by halving and doubling. Multiplication with half-integers.														
Week 6	<table border="1" data-bbox="339 1245 756 1346"> <tr> <td>FOLDS IN HALF</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>6</td> </tr> <tr> <td>NUMBER OF PARTS</td> <td>2</td> <td>4</td> <td>8</td> <td>16</td> <td>32</td> <td>64</td> </tr> </table>	FOLDS IN HALF	1	2	3	4	5	6	NUMBER OF PARTS	2	4	8	16	32	64	Exponential relationships and associativity in the context of repeated folding (inspired by Empson & Turner, 2006).
FOLDS IN HALF	1	2	3	4	5	6										
NUMBER OF PARTS	2	4	8	16	32	64										
Week 7		Equal grouping situations: partition, quotation and one-to-many correspondences.														
Week 8		Proportions: double, half, times four and fourth.														

**Table 1: Overview of the sequence of multiplicative activities. Week 1–5 focus on investigating multiplicative patterns and structure, while week 6–8 focus on modelling multiplicative situations**

However, since classifications of multiplicative situations, including examples of classroom activities, have been thoroughly described elsewhere, we will focus this paper on the use of models in the form of iconic representations, and how such models may aid the discovery of invariants. Since multiplicative situations can be both one- and two-dimensional, we have chosen to use both one-dimensional number lines and two-dimensional lattices as models.

A coherent conceptualisation of multiplication requires an appreciation of multiplicative invariants across situations. In other words, one has to recognise patterns in different situations as manifestations of the same structure. Within the implementation, patterns and structure (Mulligan & Mitchelmore, 2013) has proven to be a useful terminology for conveying the importance of invariants to teachers, and for describing the purpose of activities where patterns are created and described. Seeing patterns calls for experience of a breadth of cases, rather than focusing on one particular situation at the time. Therefore, the first five weeks focus on exploring specific models of numbers (see Table 1).

From the perspective of embodied cognition, Lakoff and Núñez (2000) argue that arithmetic is a conceptual blend, resting on four conceptual mappings from concrete situations. These mappings, or metaphors, each contribute to the concepts of number and arithmetic in their own way, implying that no single metaphor can reflect these concepts in full. Our lattice models are related to the first two of the grounding metaphors: object collection and object construction, while number lines are related to the latter two: measuring with a stick and movement along a path. There is a particularly close relationship between multiplication and measuring. Both simple proportions and area build on two important ideas: unitizing and norming (Lamon, 1994). Unitizing is described as the ability to construct a reference unit or a unit whole, and norming is then to interpret a situation in terms of that unit. In other words: creating a measure and then using it for measurement. This idea is also mirrored in verbal expressions on the form “two threes”, where the number three becomes the unit six is measured in.

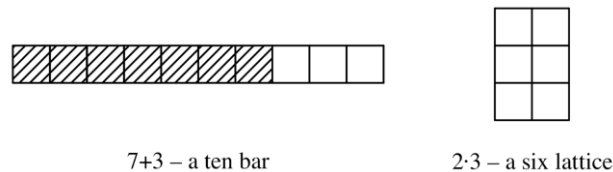
We will now describe how lattices, number lines and verbal expressions are used as models in our teaching design. In particular, we will detail how these models convey important properties of multiplication.

### **Lattice models**

Lattice models are in many situations exchangeable with arrays, but they do have some advantages. In arrays, each unit is separated from the others and pictured as indivisible. In lattices, the units are placed in a continuum, which makes the idea of dividing units by adding new lines attainable. Furthermore, open arrays are visually closer to lattices than arrays. Within our teaching design, both interlocking cubes and drawn grids are used to create lattice models. The activities using interlocking cubes could often be done in the same way with arrays of tickers or other objects, but that would make the step to drawn grids larger. By using cubes and grids of the same size ( $2 \cdot 2$  cm), the two representations are easily compared, and lattices of both kinds can be used in the same systematic arrangements.

Lattices can be seen as ‘bar models for multiplication’, in that they provide a way to represent different multiplicative groupings of numbers, just as bars can represent different additive groupings. Within the design, the class has previously worked with finding number bonds to ten by building ten bars and naming them “\_ + \_”. Multiplication can then be seen as a *different* way to group numbers (see Figure 1). Our choice of introducing multiplication by lattices hence introduce multiplication not as an additive relation, but as something different than addition. The criteria for multiplicative groupings are easily established by discussion in class, distinguishing proper lattices

from other shapes. The existence of different lattices for a number calls for expressions of the form “ $\_ \cdot \_$ ” to be introduced as names for lattices. Finding different lattices for each number then becomes an explorative activity, leading to discussions of which lattices are the same (commutativity), and why some numbers have more lattices than others (primes and composite numbers) (Week 1, Table 1). Hence, important multiplicative invariants, which are hidden when multiplication is conceptualised as repeated addition, are made visible already in the first activity.



**Figure 1: Bars and lattices as two different ways of grouping numbers**

Arranging lattices according to products reveal that multiplicative groupings do not form a pattern as simple as for additive groupings. Finding patterns in lattices therefore requires more careful instruction. Directing attention to lattices of twos leads to the discovery of even numbers and could be generalised to multiples of other numbers. This idea will inspire to a different arrangement of the lattices, focusing on factors rather than products. Building lattices of twos, threes, fours, etc. and ordering them in rows will result in the multiplication table (Week 2, Table 1). The table will spark a new exploration of patterns in the products, and the lattices will provide a means for explaining those patterns. For example, you can see two copies of the product in the second row in the corresponding product in the fourth row (a basis for the distributive law and powers), and that the products in the second column are the same as the products in the second row, because the lattices in the column are just rotated versions of the lattices in the row (commutativity).

Using interlocking cubes facilitate successive adding and subtracting of columns and rows, which are mainly additive strategies. However, they also allow for multiplicative regrouping of numbers. When comparing the two twelve lattices  $6 \cdot 2$  and  $3 \cdot 4$ , you can split one of the lattices and rearrange the parts, in order to see how one lattice can be regrouped into the other (Week 1, Table 1). Paper grids enable generalisation of such multiplicative regroupings to half-integers (numbers of the form  $n + \frac{1}{2}$ ), since paper can be cut and arranged to illustrate e.g.,  $2 \cdot 5 = 4 \cdot 2.5$ . This immediately gives rise to multiplications where the product is smaller than one of the factors, such as  $4 = 8 \cdot 0.5$  (Week 5, Table 1). The model hence enables the discovery of cases where multiplication does not make bigger, within the first weeks of introducing multiplication.

### Number lines

Number lines are important tools for arithmetic reasoning. Approximate use of empty number lines can both aid and communicate reasoning even in complex arithmetic problems. However, for empty number lines to be meaningful, one needs to be aware of the properties of linearity, closely linked to multiplication. Within our design, the class will use a variation of number lines, which put different properties of linearity in the fore.

During choral counting, a number line with markings and numbers, placed on the wall, is often used for pointing at numbers while skip counting (e.g., 0, 2, 4, 6, etc.). In some activities, a floor number

line is used. This number line has evenly spaced markings, with every fifth marking a little longer than the others, and a collection of number cards which can be placed in different ways in different activities. Every placement of two numbers raises the question of where other numbers should be and leads to discussions of equal spacing of numbers. In week 3 this is done with 0, and 1 or 2 in different positions (see Table 1). When starting with 1, the situation is a pure one-to-many correspondence, while starting with 2 requires more complex proportional reasoning. When justifying a proposed placement of numbers, or figuring out a placement, the children walk along the line, taking e.g., two-steps or three-steps. When documenting the work, teachers use verbal expressions and symbolic notation e.g., “you take three two-steps, so it’s three twos, we write  $3 \cdot 2$ ”. A similar activity is done with semi-empty number lines on paper, where only markings for specific numbers are present, e.g., 0 and 3. The task is to place markings for the missing numbers between 0 and 6, using interlocking cubes as a measuring tool. This activity is also provided at different levels, with some restricted to pure one-to-many correspondences.

As in the case with the lattice model, these activities are easily extended to fractions, starting with halves. Walking in regular or half-steps to different numbers creates an opportunity to introduce multiplication of fractions in an intuitive context. Since multiple double or triple steps were called twos or threes and represented as  $n \cdot 2$  or  $n \cdot 3$ , it makes perfect sense to call multiple half-steps halves and represent them as  $n \cdot \frac{1}{2}$ . Documenting the number of each type of step it takes to reach different numbers will result in a pattern with a proportional structure: the number of half-steps is always twice the number of regular steps (Week 4, Table 1). After whole class exploration of these patterns on the floor number line, children work in pairs to place half-integers on empty number lines, using interlocking cubes as a measuring tool, this time with 1 cube =  $\frac{1}{2}$ .

### **Verbal expressions**

In relation to both lattices and the number line, care is taken to use sustainable verbal expressions of the form “two threes” and “five halves”. As emphasized by Thompson and Saldanha (2003), there is a principal difference between “two threes” and “add three two times”. It is possible to generalise the first expression beyond integers, to e.g., “one and a half threes” or “one and a half third”, while “adding three one and a half time” is harder to conceptualise. Phrases of the first form mimic phrases used in measurement, making a number the measuring unit: “one and a half threes” – “one and a half meter”. While such phrases are used to discuss both models, they relate to each model a bit differently. Each phrase, e.g., “three twos”, labels a certain way of walking to a number or a series of skips to a number, which is different from what “two threes” labels. In contrast, both expressions can be linked to the same lattice, and the corresponding ways of seeing the lattice are made explicit, pointing out the three twos and the two threes in this six lattice.

### **Implementation of the alternative models for multiplication**

Taken together, our design covers a variety of advanced aspects and properties within the first weeks of the introduction of multiplication. The handling of advanced content is made possible by the highly structured material organising the teaching of each week in a cycle with six phases, including whole class discussions as well as pair work and individual documentation. This cyclic teaching structure has been previously described and tested in preschool (Sternier & Helenius,

2015). The material developed for the implementation of the design includes explicit instructions on what the teacher should do and say in each phase, and in reaction to different student actions and reasoning. Each activity also contains a detailed description of the mathematical concepts and procedures involved, as well as the purpose of each activity in relation to children's learning. Activities, as well as instructions, were tried and evaluated by small groups of teachers during the development of the material, increasing the quality and functionality of the design. Thorough testing of the activities is important for the teachers' trust in the material, which is particularly important due to the high level of prescription in the design.

The teaching design has been implemented in an intervention research project covering 15 weeks in grade 2, where the multiplicative part covers 8–9 weeks. The project is a randomized controlled trial with 28 participating teachers in grade 2 (14 using our material, 14 in a control group). Data from the pre- and post-tests are currently being analysed. In a parallel project we are developing similar teaching sequences for grades 1 and 3. Including sequences dealing with additive structures and number, the complete material will cover most of the area of number and operations for grades 1–3. These sequences will also be tried and evaluated by small groups of teachers during development, together with auxiliary professional development material. In collaboration with 5 municipalities, we will implement the teaching design together with professional development for around 500 teachers starting in the fall of 2020.

### **Closing reflection**

The literature on the downsides of relying on multiplication as repeated addition is substantial. Still, teaching experiments with alternative models for multiplication are sparse, one example being Nunes, Bryant, Evans, and Bell (2010), who report on two studies covering only a few sessions. Also among teachers and textbook authors the dominance of repeated addition as the model for introducing multiplication persists. Fishbein et al. (1985) suggest that this model is chosen because it fits the mental requirements of children in the first years of schooling. It is simply hard for teachers to resist building on addition, since it is a concept that children are already familiar with at the time when multiplication is introduced. However, as seen above, multiplication in fact has several complementary roots that are not additive, such as measuring and one-to-many correspondence (Lamon, 1994; Nunes & Bryant, 1996). Many of these roots are intuitive for children and can hence be used as alternative bases for introducing multiplication.

According to Century and Cassata (2016), there are three key foci for implementation research: the innovation, the aligned outcomes, and the influential factors. This paper has described the innovation of an ongoing, large scale implementation project: a teaching sequence introducing multiplication by means of deliberately chosen models for making multiplicative patterns visible and comparable, which in turn inspire discussions where multiplicative structures are explicated. The analysis of test results in the randomised controlled trial will reveal to what extent this innovation achieves the desired outcomes, while the grade 1–3 project will provide opportunities to study the influential factors, as well as to further develop the innovation.

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