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Regularity structures and paracontrolled calculus

I. BAILLEUL & M. HOSHINO

Abstract. We prove a general equivalence statement between the notions of models and modelled distribution over a regularity structure, and paracontrolled systems indexed by the regularity structure. This takes in particular the form of a parametrisation of the set of models over a regularity structure by the set of reference functions used in the paracontrolled representation of these objects. The construction of a modelled distribution from a paracontrolled system is explicit, and takes a particularly simple form in the case of the BHZ regularity structures used for the study of singular stochastic partial differential equations.

1 – Introduction

Two different sets of tools for the study of singular stochastic partial differential equations (PDEs) have emerged recently, under the form of Hairer's theory of regularity structures [9, 6, 7, 5] and paracontrolled calculus [8, 2, 3], after Gubinelli, Imkeller and Perkowski' seminal work. While Hairer's theory has now reached the state of a ready-to-use black box for the study of singular stochastic PDEs, like Cauchy-Lipschitz well-posedness theorem for ordinary differential equations, the task of giving a self-contained treatment of renormalisation matters within paracontrolled calculus remains to be done. It happens nonetheless to be possible to compare the two languages, independently of their applications to the study of singular stochastic PDEs. This task was initiated in our previous work [4], where we proved that the set of admissible models $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ over a concrete regularity structure equipped with an abstract integration map is parametrised by a paracontrolled representation of $\mathsf{\Pi}$ on the set of trees with non-positive homogeneity. Such a statement is concerned with models on regularity structures associated with singular stochastic PDEs. We step back in the present work and prove a general result giving a parametrization of any model $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ on any reasonable concrete regularity structure, in terms of representations of the maps g and $\mathsf{\Pi}$ by paracontrolled systems. (All the words will be explained below.) Being reasonable means here satisfying assumptions **(A-C)** from Section 3 and Section 4. The result takes the following form. Given a concrete regularity structure

$$\mathscr{T} = \big((T^+, \Delta^+), (T, \Delta)\big),$$

denote by $\mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$ the space of models on $\mathbb{R}^d$ decreasing rapidly at infinity. Given $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi}) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$, denote by $\mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g})$ the space of modelled distributions taking values in the vector space T, with regularity exponent γ. (All function spaces are defined in Section 2.)

Theorem 1. *Let $\mathscr{T}$ be a concrete regularity structure satisfying assumptions* **(A-C)**. *Then $\mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$ is homeomorphic to the product space*

$$\prod_{\sigma \in \mathcal{G}^+_\circ} C^{|\sigma|}_{\mathrm{rap}}(\mathbb{R}^d) \times \prod_{\tau \in \mathcal{B}_\bullet,\, |\tau| < 0} C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d).$$

The set $\mathcal{B}_\bullet$ above parametrizes part of a linear basis of the vector space T, while the set $\mathcal{G}^+_\circ$ parametrizes part of a linear basis of the vector space T^+. Assumption **(A)** is a harmless requirement on how polynomials sit within T and T^+. Assumption **(B)** is a very mild requirement on the splitting map $\Delta : T \to T \otimes T^+$, and assumption **(C)** is a structure requirement on T^+ and Δ^+ that provides a fundamental induction structure. The three assumptions are met by all concrete regularity structures built for the study of singular stochastic PDEs.

Given a model $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ on a concrete regularity structure, natural regularity spaces are given by the Hölder-type spaces $\mathcal{D}^\gamma(T, \mathsf{g})$. The parametrization of $\mathcal{D}^\gamma(T, \mathsf{g})$ by data in paracontrolled representations of elements of that space requires in general a structure condition on these data reminiscent of a similar condition introduced by Martin and Perkowski in [10]; it is stated in Theorem 16. This non-trivial structure condition has a clear meaning in terms of an extension problem for the map g from the Hopf algebra T^+ to a larger Hopf algebra. It happens nonetheless to take a very simple form for special concrete regularity structures satisfying assumption **(D)**.

Theorem 2. *Let a concrete regularity structure $\mathscr{T}$ satisfy assumptions* **(A-D)**. *Pick $\gamma \in \mathbb{R}$, and $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi}) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$. Then $\mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g})$ is homeomorphic to the product space $\prod_{\tau \in \mathcal{B}_\bullet, |\tau| < \gamma} C^{\gamma - |\tau|}_{\mathrm{rap}}(\mathbb{R}^d)$.*

Unlike the other assumptions, assumption **(D)** is fundamentally a requirement on a linear basis of T, not on the concrete regularity structure itself. It may then happen that one basis of T satisfies it whereas another does not. Satisfying assumption **(D)** thus means the existence of a linear basis satisfying this assumption. It happens that the class of concrete regularity structures introduced by Bruned, Hairer and Zambotti in [6] for the study of singular stochastic PDEs all satisfy assumption **(D)**, despite the fact that their canonical bases do not satisfy it.

Theorem 3. *The* BHZ *concrete regularity structures satisfy assumptions* **(A-D)**.

Like in our previous work [4], we work here with the usual isotropic Hölder space rather than with anisotropic spaces. All results given here hold true in that more general setting, with identical proofs. The reader will find relevant technical details in the work [10] of Martin and Perkowski. The above statements have counterparts with functional spaces with polynomial growth at infinity, rather than with spaces with fast decrease at infinity; we let the reader prove these statements on the model of proofs of the present work.

Section 2 is dedicated to describing different functional spaces and operators. Section 3 is dedicated to giving paracontrolled representations of models and the reconstruction of modelled distributions in terms of data in paracontrolled systems, proving part of Theorem 1. The later is proved in Section 4, where the main work consists in providing a parametrization of g-maps by paracontrolled representations, Theorem 12. Theorem 2 and Theorem 3 are proved in Section 4.2 and 4.3, respectively. Appendix A gives back the setting of concrete regularity structures introduced in [4], while Appendix B gives a number of technical details that are variations on corresponding results from [4].

Notations • *We use exclusively the letters α, β, γ to denote real numbers, and use the letters σ, τ, μ, ν to denote elements of T or T^+.*

• *We agree to use the shorthand notation $\mathfrak{s}^{(+)}$ to mean both the statement $\mathfrak{s}$ and the statement $\mathfrak{s}^+$.*

• *We use the pairing notation $\langle \cdot, \cdot \rangle$ for duality between a finite dimensional vector space and its dual space.*

• *We adopt the notations and terminology of the work* [4]*, and write in particular $\mathsf{\Pi}^{\mathsf{g}}_x$ and $\widehat{\mathsf{g}_{yx}}$, for what is denoted by Π_x and Γ_{xy} in Hairer's terminology.*

2 – Functional setting

We describe in this section different function spaces we shall work with and introduce a modified paraproduct. For $x \in \mathbb{R}^d$, set

$$|x|_* := 1 + |x|, \quad x \in \mathbb{R}^d.$$

The weight function $|x|_*$ satisfies the inequalities

$$|x+y|_* \leqslant |x|_*|y|_*, \qquad |x/\lambda|_* \leqslant |x|_*,$$

for any $\lambda \geqslant 1$.

Let $(\rho_i)_{-1\leqslant i<\infty}$ be a dyadic decomposition of unity on $\mathbb{R}^d$, and let $\Delta_i f := \mathcal{F}^{-1}(\rho_i \mathcal{F} f)$. For $j \geqslant -1$, set

$$S_j := \sum_{i<j-1} \Delta_i.$$

Denote by Q_i and P_j the integral kernels associated with Δ_i and S_j

$$\Delta_i f(x) := \int_{\mathbb{R}^d} Q_i(x-y) f(y) dy, \qquad S_j f(x) := \int_{\mathbb{R}^d} P_j(x-y) f(y) dy.$$

- For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$, set

  $$\|f\|_{L^\infty_a(\mathbb{R}^d)} := \big\| |\cdot|_*^a f \big\|_{L^\infty(\mathbb{R}^d)},$$

  and define the corresponding space $L^\infty_a(\mathbb{R}^d)$ of functions with finite $\|\cdot\|_{L^\infty_a(\mathbb{R}^d)}$-norm. Set

  $$L^\infty_{\mathrm{rap}}(\mathbb{R}^d) := \bigcap_{a=1}^{\infty} L^\infty_a(\mathbb{R}^d), \qquad L^\infty_{\mathrm{poly}}(\mathbb{R}^d) := \bigcup_{a=1}^{\infty} L^\infty_{-a}(\mathbb{R}^d).$$

- For any distribution $\xi \in \mathcal{S}'(\mathbb{R}^d)$, set

  $$\|\xi\|_{C^\alpha_a(\mathbb{R}^d)} := \sup_{j\geqslant -1} 2^{j\alpha} \|\Delta_j \xi\|_{L^\infty_a(\mathbb{R}^d)}.$$

  and define the corresponding space $C^\alpha_a(\mathbb{R}^d)$ of functions with finite $\|\cdot\|_{C^\alpha_a(\mathbb{R}^d)}$-norm. We have $C^\alpha_0(\mathbb{R}^d) = C^\alpha(\mathbb{R}^d)$. Set

  $$C^\alpha_{\mathrm{rap}}(\mathbb{R}^d) := \bigcap_{a=1}^{\infty} C^\alpha_a(\mathbb{R}^d), \qquad C^\alpha_{\mathrm{poly}}(\mathbb{R}^d) := \bigcup_{a=1}^{\infty} C^\alpha_{-a}(\mathbb{R}^d).$$

- For any two-parameter function $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\alpha > 0$, set

  $$\|F\|_{\mathcal{C}^\alpha_a(\mathbb{R}^d\times\mathbb{R}^d)} := \sup_{x,y\in\mathbb{R}^d} \big(|x|_*^a \wedge |y|_*^a\big) \frac{|F(x,y)|}{|x-y|^\alpha}.$$

  and define the corresponding space $\mathcal{C}^\alpha_a(\mathbb{R}^d \times \mathbb{R}^d)$ of functions with finite $\|\cdot\|_{\mathcal{C}^\alpha_a(\mathbb{R}^d\times\mathbb{R}^d)}$-norm. Set also

  $$\mathcal{C}^\alpha(\mathbb{R}^d \times \mathbb{R}^d) := \mathcal{C}^\alpha_0(\mathbb{R}^d \times \mathbb{R}^d), \qquad \mathcal{C}^\alpha_{\mathrm{rap}}(\mathbb{R}^d \times \mathbb{R}^d) := \bigcap_{a=1}^{\infty} \mathcal{C}^\alpha_a(\mathbb{R}^d \times \mathbb{R}^d).$$

- For any $\mathbb{R}^d$-indexed family of distributions $\Lambda = (\Lambda_x)_{x\in\mathbb{R}^d} \subset \mathcal{S}'(\mathbb{R}^d)$ on $\mathbb{R}^d$, and $\alpha \in \mathbb{R}$, set

  $$\|\Lambda\|_{D^\alpha_a} := \sup_{x\in\mathbb{R}^d} \sup_{j\geqslant -1} |x|_*^a 2^{j\alpha} \big|\langle \Lambda_x, P_j(x-\cdot)\rangle\big|.$$

  Set

  $$D^\alpha_{\mathrm{rap}} := \bigcap_{a=1}^{\infty} D^\alpha_a.$$

(In Hairer' seminal work [9], models are assumed to satisfy a (λ, φ)-uniform regularity condition

$$\big|(\mathsf{\Pi}^{\mathsf{g}}_x \tau)(\varphi^\lambda_x)\big| \lesssim \lambda^{|\tau|},$$

locally uniformly in x. Requiring $(\mathsf{\Pi}^{\mathsf{g}}_x \tau)_{x\in\mathbb{R}^d} \in D^{|\tau|}$ is equivalent to the above uniform estimate – see Lemma 6.6 of Gubinelli, Imkeller and Perkowski' seminal work [8] on paracontrolled distributions.)

For any $f, g \in \mathcal{S}'(\mathbb{R}^d)$, we define the paraproduct

$$\mathsf{P}_f g := \sum_{j=1}^{\infty} (S_j f)(\Delta_j g),$$

and resonant operator

$$\Pi(f,g) := \sum_{|i-j|\leqslant 1} (\Delta_i f)(\Delta_j g).$$

For any $g \in \mathcal{S}'(\mathbb{R}^d)$, set

$$\mathcal{S}g := g - \mathsf{P}_1 g = (\Delta_{-1} + \Delta_0) g \in C^\infty(\mathbb{R}^d).$$

The following continuity result is an elementary variation on the classical continuity results for the paraproduct and resonant operators. We refer the reader to [1] for a reference.

Proposition 4. *Let $\alpha, \beta \in \mathbb{R}$, $a, b \in \mathbb{Z}$.*

- *If $\alpha \neq 0$, then $C^\alpha_a(\mathbb{R}^d) \times C^\beta_b(\mathbb{R}^d) \ni (f,g) \mapsto \mathsf{P}_f g \in C^{\alpha\wedge 0+\beta}_{a+b}(\mathbb{R}^d)$, is continuous.*
- *If $\alpha + \beta > 0$, then $C^\alpha_a(\mathbb{R}^d) \times C^\beta_b(\mathbb{R}^d) \ni (f,g) \mapsto \Pi(f,g) \in C^{\alpha+\beta}_{a+b}(\mathbb{R}^d)$, is continuous.*
- *If $\alpha, \beta \neq 0$ and $\alpha + \beta > 0$, then $C^\alpha_a(\mathbb{R}^d) \times C^\beta_b(\mathbb{R}^d) \ni (f,g) \mapsto f \cdot g \in C^{\alpha\wedge\beta}_{a+b}(\mathbb{R}^d)$, is continuous.*

As a consequence of the last item, the product fg, of $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in C^\alpha(\mathbb{R}^d)$, belongs to $C^\alpha_{\mathrm{rap}}(\mathbb{R}^d)$, for any $\alpha \in \mathbb{R}$ – so the space $C^\alpha_{\mathrm{rap}}(\mathbb{R}^d)$ is in particular not empty.

We use a modified paraproduct in Section 3.1.3. Note that

$$|\nabla|^m f := \mathcal{F}^{-1}\big(|\cdot|^m \mathcal{F} f\big),$$

for $m \in \mathbb{Z}$, is well-defined for functions $f \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform have support in an annulus. For $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, the map $|\nabla|^m$ sends continuously $C^\alpha_{\mathrm{rap}}(\mathbb{R}^d)$ into $C^{\alpha-m}_{\mathrm{rap}}(\mathbb{R}^d)$. For $m \in \mathbb{N}$, we define the **modified paraproduct**

$$\mathsf{P}^m_f g := |\nabla|^m \big(\mathsf{P}_f |\nabla|^{-m} g\big) = \sum_{j=1}^{\infty} |\nabla|^m \big(S_j f \cdot |\nabla|^{-m} \Delta_j g\big).$$

Note that $\mathsf{P}^0 = \mathsf{P}$. The first item of Proposition 4 also holds for the modified paraproduct P^m.

3 – From regularity structures and models to paracontrolled systems

We introduce in this section assumptions **(A)** and **(B)**, and show that they provide a framework where to represent models and reconstructions of modelled distributions by paracontrolled systems. We refer the reader to Appendix A and [4] for details on concrete regularity structures.

3.1 A basic assumption

Let $\mathscr{T} = \big((T^+, \Delta^+), (T, \Delta)\big)$ be a concrete regularity structure, with $T^+ = \bigoplus_{\alpha\in A^+} T^+_\alpha$ and $T = \bigoplus_{\beta\in A} T_\beta$. Write $\mathbf{1}_+$ for the unit of the algebra T^+. Set

$$\beta_0 := \min A.$$

Recall that we agree to use the shorthand notation $\mathfrak{s}^{(+)}$ to mean both the statement $\mathfrak{s}$ and the statement $\mathfrak{s}^+$.

Assumption (A) – *The spaces T^+ and T have linear bases $\mathcal{B}^+$ and $\mathcal{B}$, respectively, with the following properties.*

(a_1) *$\mathcal{B}^+$ is a commutative monoid freely generated by a finite set $\mathcal{B}_\circ^+$ and Taylor monomials $X_1, \dots, X_d$. Each element $\tau \in \mathcal{B}_\circ^+$ has a positive homogeneity. For general elements in $\mathcal{B}^+$, homogeneities are defined by $|X_i| = 1$, and multiplicativity*

$$|\tau\sigma| = |\tau| + |\sigma|.$$

(a_2) *The action of Δ^+ on polynomials is characterised by its action on the monomials*

$$\Delta^+ X_i = X_i \otimes \mathbf{1}_+ + \mathbf{1}_+ \otimes X_i,$$

that holds for all $1 \leqslant i \leqslant d$. Denote by $\mathcal{B}_X^+$ the submonoid generated by $X_1, \dots, X_d$, and define

$$T_X^+ := \mathrm{span}(\mathcal{B}_X^+).$$

For any $\tau \in \mathcal{B}_\circ^+$ and $k \in \mathbb{N}^d$, there is no term of the form $X^k \otimes X^\ell$, with $\ell \in \mathbb{N}^d$, in the expansion of $\Delta^+\tau$.

(b_1) *There exists a subset $\mathcal{B}_\bullet \subset \mathcal{B}$, such that $\mathcal{B}$ is in bijection with $\mathbb{N}^d \times \mathcal{B}_\bullet$. An element $(k, \sigma) \in \mathbb{N}^d \times \mathcal{B}_\bullet$, is denoted by $\underline{X}^k\sigma$, and assigned a homogeneity*

$$|\underline{X}^k\sigma| := |k| + |\sigma|.$$

(b_2) *If $\mathcal{B}_\bullet$ contains an element $\mathbf{1}$ with homogeneity 0, then it is unique and satisfies the identity*

$$\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}_+.$$

Write $\underline{X}^k$ for $\underline{X}^k\mathbf{1}$. Set

$$\mathcal{B}_{\underline{X}} := \{\underline{X}^k\}_{k\in\mathbb{N}^d} \subset \mathcal{B}.$$

The coproduct Δ on $\underline{X}^k$ is characterised by its action on the monomials

$$\Delta\underline{X}_i = \underline{X}_i \otimes \mathbf{1}_+ + \mathbf{1} \otimes X_i,$$

that holds for all $1 \leqslant i \leqslant d$, and by requiring multiplicativity on $\mathcal{B}_{\underline{X}}$. For general elements, one has the multiplicative formula

$$\Delta(\underline{X}^k\sigma) = (\Delta\underline{X}^k)\,(\Delta\sigma).$$

For any $\tau \in \mathcal{B}_\bullet$ and $k \in \mathbb{N}^d$, there is no term of the form $\underline{X}^k \otimes X^\ell$, with $\ell \in \mathbb{N}^d$, in the expansion of $\Delta\tau$.

For later use, denote by $\{\tau'\}_{\tau\in\mathcal{B}}$ the dual basis of $\mathcal{B}$. Following [4], for $\sigma, \tau \in \mathcal{B}^{(+)}$, write $\sigma \leqslant^{(+)} \tau$, if σ appears in the left hand side of the tensor products in the expansion of $\Delta^{(+)}\tau$, so we have the unique representation

$$\Delta^{(+)}\tau = \sum_{\substack{\sigma\in\mathcal{B}^{(+)} \\ \sigma\leqslant^{(+)}\tau}} \sigma \otimes (\tau/^{(+)}\sigma),$$

where $\tau/^{(+)}\sigma \in T^+$. Write $\sigma <^{(+)} \tau$, if $\sigma \leqslant^{(+)} \tau$ and $\sigma \neq \tau$. Write in particular, for $\tau \in \mathcal{B}^+$,

$$\begin{aligned}\Delta^+\tau &= \sum_{\sigma\in\mathcal{B}^+\backslash\mathcal{B}_X^+} \sigma \otimes (\tau/^+\sigma) + \sum_k X^k \otimes (\tau/^+X^k) \\ &=: \sum_{\sigma\in\mathcal{B}^+\backslash\mathcal{B}_X^+} \sigma \otimes (\tau/^+\sigma) + \sum_k \frac{X^k}{k!} \otimes D^k\tau\end{aligned}$$

The notations $\tau/^{(+)}\sigma$ and $\sigma <^{(+)} \tau$ are only used for τ and σ in $\mathcal{B}^{(+)}$. Be careful! The notations $\leqslant, <$, etc. are basis-dependent – like the matrix of a linear map. Extend by linearity the map D^k from T_α^+ to $T_{\alpha-|k|}^+$, for all $\alpha \in A$.

Lemma 5. *One has, for all $k, \ell \in \mathbb{N}^d$,*

(a) *$D^0\tau = \tau$,*

(b) *$D^kD^\ell\tau = D^{k+\ell}\tau$,*

(c) $D^kX^\ell = \frac{k!}{(\ell-k)!}\,X^{\ell-k}$,

(d) $D^k(\tau\sigma) = \sum_{k'}\binom{k}{k'}D^{k'}\tau D^{k-k'}\sigma$ – *Leibniz rule.*

Proof – Item *(b)* is a consequence of the coassociativity property

$$(\Delta^+\otimes\mathrm{Id})\Delta^+ = (\mathrm{Id}\otimes\Delta^+)\Delta^+$$

of the coproduct Δ^+. It gives indeed the identity

$$\Delta^+D^k\tau = D^k\tau\otimes\mathbf{1} + \sum_{\mu<^+\tau,\,\mu\notin\mathcal{B}_X^+} D^k\mu\otimes(\tau/^+\mu) + \sum_\ell\frac{X^\ell}{\ell!}\otimes D^{k+\ell}\tau, \tag{3.1}$$

We leave the proof of the other identities to the reader. ▷

3.2 From models to paracontrolled systems

We recall in this section some of the results proved in [4], stated here in the slightly more general setting of the present work. The proofs of these extensions are given in Appendix B.

Given Fréchet spaces E and F, denote by $L(E,F)$ the space of continuous linear maps from E into F. Recall G^+ stands for the set of characters of the Hopf algebra T^+. Given maps

$$\mathsf{g}:\mathbb{R}^d\to G^+,\quad \Pi\in L\big(T,\mathcal{S}'(\mathbb{R}^d)\big),$$

and $x,y\in\mathbb{R}^d$, set

$$\mathsf{g}_{yx} := (\mathsf{g}_y\otimes\mathsf{g}_x^{-1})\Delta^+\in G^+,$$

and

$$\Pi_x^{\mathsf{g}} := (\Pi\otimes\mathsf{g}_x^{-1})\Delta\in L\big(T,\mathcal{S}'(\mathbb{R}^d)\big).$$

Recall $T=\bigoplus_{\beta\in A}T_\beta$, and $\beta_0=\min A$.

Definition 6. *Let a concrete regularity structure $\mathscr{T}$ satisfying assumption* **(A)** *be given. We denote by*

$$\mathscr{M}_{\mathrm{rap}}(\mathscr{T},\mathbb{R}^d),$$

the set of pairs of maps

$$\mathsf{g}:\mathbb{R}^d\to G^+,\quad \Pi\in L\big(T,\mathcal{S}'(\mathbb{R}^d)\big),$$

such that

(a) *one has $\mathsf{g}_x(X^k)=x^k$, for all $x\in\mathbb{R}^d, k\in\mathbb{N}^d$;*

(b) *for any $\tau\in\mathcal{B}_\circ^+$, the function $x\mapsto\mathsf{g}_x(\tau)$ belongs to $L^\infty_{\mathrm{rap}}(\mathbb{R}^d)$, and the function*

$$(x,y)\mapsto\mathsf{g}_{yx}(\tau),$$

belongs to $\mathcal{C}^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d\times\mathbb{R}^d)$;

(c) *one has $(\Pi\underline{X}^k\sigma)(x) = x^k(\Pi\sigma)(x)$ and $(\Pi\mathbf{1})(x)=1$;*

(d) *for any $\tau\in\mathcal{B}_\bullet\backslash\{\mathbf{1}\}$, one has $\Pi\tau\in C^{\beta_0}_{\mathrm{rap}}(\mathbb{R}^d)$, and the $\mathbb{R}^d$-indexed family of distributions $\{\Pi_x^{\mathsf{g}}\tau\}_{x\in\mathbb{R}^d}$ belongs to $D^{|\tau|}_{\mathrm{rap}}$.*

The pair (g,Π) is called a **rapidly decreasing model** *on the concrete regularity structure $\mathscr{T}$.*

This definition does not depend on the choice of bases for T^+ and T. Fix $r>|\beta_0\wedge 0|$. We define metrics on the space of rapidly decreasing models on $\mathscr{T}$ setting

$$\|\mathsf{g}\|_a := \sup_{\tau\in\mathcal{B}_\circ^+}\left(\|\mathsf{g}_\cdot(\tau)\|_{L^\infty_a(\mathbb{R}^d)} + \|\mathsf{g}_{\cdot\cdot}(\tau)\|_{\mathcal{C}^{|\tau|}_a(\mathbb{R}^d\times\mathbb{R}^d)}\right),$$

and

$$\|\Pi\|_a^{\mathsf{g}} := \sup_{\sigma\in\mathcal{B}_\bullet}\left(\|\Pi\sigma\|_{C_a^{\beta_0}(\mathbb{R}^d)} + \big\|(\Pi_\cdot^{\mathsf{g}}\sigma)(\cdot)\big\|_{D_a^{|\sigma|}}\right).$$

With a slight abuse of notations, we write

$$\mathsf{g}_x(\tau) \in L^\infty_{\mathrm{rap}}(\mathbb{R}^d), \quad \mathsf{g}_{yx}(\tau) \in \mathcal{C}^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d \times \mathbb{R}^d).$$

Condition *(b)* from Definition 6 does not hold for $\tau \in \mathcal{B}^+_X$, instead one has

$$\mathsf{g}_x(X^k) \in L^\infty_{\mathrm{poly}}(\mathbb{R}^d), \quad \mathsf{g}_{yx}(X^k) \in \mathcal{C}^{|k|}(\mathbb{R}^d \times \mathbb{R}^d).$$

Since one has $L^\infty_{\mathrm{poly}}(\mathbb{R}^d) \cdot L^\infty_{\mathrm{rap}}(\mathbb{R}^d) \subset L^\infty_{\mathrm{rap}}(\mathbb{R}^d)$ and $\mathcal{C}^\alpha(\mathbb{R}^d \times \mathbb{R}^d) \cdot \mathcal{C}^\beta_{\mathrm{rap}}(\mathbb{R}^d \times \mathbb{R}^d) \subset \mathcal{C}^{\alpha+\beta}_{\mathrm{rap}}(\mathbb{R}^d \times \mathbb{R}^d)$, for all non-negative α, β, condition *(b)* holds for any $\tau \in \mathcal{B}^+ \backslash \mathcal{B}^+_X$. The next statement is a variation on Proposition 12 of [4], where we use now the usual polynomials and polynomial weights, and the modified paraproducts P^m. Its proof is given in Appendix B.

Theorem 7. *Pick $m \in \mathbb{N}$. For any model $\mathsf{M} = (\mathsf{g}, \Pi) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$, there exists a family*

$$\left\{ \Big([\![\tau]\!]^{m,\mathsf{g}} \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d) \Big)_{\tau \in \mathcal{B}^+ \backslash \mathcal{B}^+_X}, \; \Big([\![\sigma]\!]^{m,\mathsf{M}} \in C^{|\sigma|}_{\mathrm{rap}}(\mathbb{R}^d) \Big)_{\sigma \in \mathcal{B} \backslash \mathcal{B}_{\underline{X}}} \right\}$$

such that one has, for any $\tau \in \mathcal{B}^+ \backslash \mathcal{B}^+_X$ and $\sigma \in \mathcal{B} \backslash \mathcal{B}_{\underline{X}}$, the identities

$$\mathsf{g}(\tau) = \sum_{\substack{\mathbf{1} <^+ \nu <^+ \tau \\ \nu \in \mathcal{B}^+ \backslash \mathcal{B}^+_X}} \mathsf{P}^m_{\mathsf{g}(\tau /^+ \nu)} [\![\nu]\!]^{m,\mathsf{g}} + [\![\tau]\!]^{m,\mathsf{g}}, \tag{3.2}$$

$$\Pi \sigma = \sum_{\substack{\mu < \sigma \\ \mu \in \mathcal{B} \backslash \mathcal{B}_{\underline{X}}}} \mathsf{P}^m_{\mathsf{g}(\sigma / \mu)} [\![\mu]\!]^{m,\mathsf{M}} + [\![\sigma]\!]^{m,\mathsf{M}}. \tag{3.3}$$

Moreover, the mapping

$$\mathsf{M} \mapsto \left\{ \Big([\![\tau]\!]^{m,\mathsf{g}} \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d) \Big)_{\tau \in \mathcal{B}^+ \backslash \mathcal{B}^+_X}, \; \Big([\![\sigma]\!]^{m,\mathsf{M}} \in C^{|\sigma|}_{\mathrm{rap}}(\mathbb{R}^d) \Big)_{\sigma \in \mathcal{B} \backslash \mathcal{B}_{\underline{X}}} \right\}$$

is continuous.

We write $[\![\tau]\!]^{\mathsf{g}}$ and $[\![\sigma]\!]^{\mathsf{M}}$ instead of $[\![\tau]\!]^{m,\mathsf{g}}$ and $[\![\sigma]\!]^{m,\mathsf{M}}$, when $m = 0$. Given a model $\mathsf{M} \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$ on a regularity structure $\mathscr{T}$, and $\gamma \in \mathbb{R}$, define the space $\mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g})$ of **rapidly decreasing modelled distributions** as the set of functions

$$\boldsymbol{f} : \mathbb{R}^d \mapsto \bigoplus_{\beta < \gamma} T_\beta,$$

such that, for each $\tau \in \mathcal{B}$, the function $\langle \tau', \boldsymbol{f}(\cdot) \rangle$ belongs to $L^\infty_{\mathrm{rap}}(\mathbb{R}^d)$, and the function

$$(x, y) \mapsto \big\langle \tau', \boldsymbol{f}(y) - \widehat{\mathsf{g}_{yx}} \boldsymbol{f}(x) \big\rangle$$

belongs to $\mathcal{C}^{\gamma - |\tau|}_{\mathrm{rap}}(\mathbb{R}^d \times \mathbb{R}^d)$. We denote by $\mathsf{R}\boldsymbol{f}$ the reconstruction of a modelled distribution $\boldsymbol{f} \in \mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g})$; if $\gamma > 0$, it is charcaterized by the condition

$$\Big(\mathsf{R}\boldsymbol{f} - \Pi^{\mathsf{g}}_x \boldsymbol{f}(x) \Big)_{x \in \mathbb{R}^d} \in D^\gamma_{\mathrm{rap}}.$$

The proper setting to get a paracontrolled representation of a modelled distribution is given by the following

Assumption (B) – *For each $\tau, \mu \in \mathcal{B}$ with $\tau < \mu$, either $\mu / \tau \in T^+_X$, or $\mu / \tau \in \mathrm{span}(\mathcal{B}^+ \backslash \mathcal{B}^+_X)$.*

The next statement was proved in [4], Theorem 14, in the unweighted setting; its extension to the present setting is given in Appendix B.

Theorem 8. *Let $\mathscr{T}$ be a regularity structure satisfying assumption* **(A)** *and assumption* **(B)**. *Let a regularity exponent $\gamma \in \mathbb{R}$ and a model $\mathsf{M} = (\mathsf{g}, \Pi) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$ on $\mathscr{T}$ be given. For any modelled distribution*

$$\boldsymbol{f} = \sum_{|\sigma| < \gamma} f_\sigma \sigma \in \mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g}),$$

each coefficient f_σ *has a paracontrolled representation*

$$f_\sigma = \sum_{\substack{\sigma<\mu \\ \mu/\sigma \in \mathrm{span}(\mathcal{B}^+\backslash\mathcal{B}_X^+)}} \mathsf{P}_{f_\mu}[\![\mu/\sigma]\!]^{\mathsf{g}} + [\![f_\sigma]\!]^{\mathsf{g}}, \tag{3.4}$$

where $[\![f_\sigma]\!]^{\mathsf{g}} \in C_{\mathrm{rap}}^{\gamma-|\sigma|}(\mathbb{R}^d)$*. Moreover, there exists a distribution* $[\![\boldsymbol{f}]\!]^{\mathsf{M}} \in C_{\mathrm{rap}}^{\gamma}(\mathbb{R}^d)$ *such that*

$$\mathsf{R}\boldsymbol{f} = \sum_{\sigma\in\mathcal{B}\backslash\mathcal{B}_X} \mathsf{P}_{f_\sigma}[\![\sigma]\!]^{\mathsf{M}} + [\![\boldsymbol{f}]\!]^{\mathsf{M}}. \tag{3.5}$$

The mapping

$$\Big(\boldsymbol{f} \in \mathcal{D}^\gamma(\mathscr{T},\mathsf{g})\Big) \mapsto \left(\Big([\![\boldsymbol{f}]\!]^{\mathsf{M}}, \big([\![f_\sigma]\!]^{\mathsf{g}}\big)_{\sigma\in\mathcal{B}}\Big) \in C_{\mathrm{rap}}^{\gamma}(\mathbb{R}^d) \times \prod_{\sigma\in\mathcal{B}} C_{\mathrm{rap}}^{\gamma-|\sigma|}(\mathbb{R}^d)\right)$$

is continuous.

A similar statement with P^m used in place of P holds true. We end this section with three useful formulas involving g, that will be used in the proof of Theorem 12. The reader can skip this statement now and come back to it at the moment where it is needed. Recall $D^k\tau = 0$, for $|k| > |\tau|$. Let $P_X : T^+ \to T_X^+$, stand for the canonical projection map on T_X^+, and set

$$\begin{aligned} \mathsf{f}_x(\tau) &:= -(\mathsf{g}_x \otimes \mathsf{g}_x^{-1})(P_X\Delta^+\tau) \\ &= -\sum_\ell \frac{x^\ell}{\ell!}\, \mathsf{g}_x^{-1}(D^\ell\tau). \end{aligned}$$

For $\tau \neq \mathbf{1}$, we also have

$$\begin{aligned} \mathsf{f}_x(\tau) &:= (\mathsf{g}_x \otimes \mathsf{g}_x^{-1})\big((\mathrm{Id}-P_X)\Delta^+\tau\big) \\ &= \sum_{\sigma\leqslant^+\tau, \sigma\notin\mathcal{B}_X} \mathsf{g}_x(\sigma)\, \mathsf{g}_x^{-1}(\tau/^+\sigma). \end{aligned}$$

Lemma 9. *For any* $\tau \in \mathcal{B}^+\backslash\mathcal{B}_X^+$*, we have*

$$\mathsf{g}_x(D^k\tau) = \sum_{\sigma\leqslant^+\tau,\, \sigma\notin\mathcal{B}_X^+} \mathsf{g}_x(\tau/^+\sigma)\mathsf{f}_x(D^k\sigma). \tag{3.6}$$

and

$$\mathsf{g}_{yx}(D^k\tau) = \sum_{\sigma\leqslant^+\tau,\, \sigma\notin\mathcal{B}_X^+} \mathsf{g}_{yx}(\tau/^+\sigma)\mathsf{f}_y(D^k\sigma) - \sum_\ell \frac{(y-x)^\ell}{\ell!}\mathsf{f}_x(D^{k+\ell}\tau), \tag{3.7}$$

and

$$\begin{aligned} \mathsf{f}_x(D^k\tau) &= \partial_y^k\Big\{(\mathsf{g}_y \otimes \mathsf{g}_x^{-1})\big((\mathrm{Id}-P_X)\Delta^+\big)\tau\Big\}\Big|_{y=x} \\ &= \partial_y^k\left\{\sum_{\sigma\leqslant^+\tau,\, \sigma\notin\mathcal{B}_X^+} \mathsf{g}_y(\sigma)\, \mathsf{g}_x^{-1}(\tau/^+\sigma)\right\}\Bigg|_{y=x}. \end{aligned} \tag{3.8}$$

Note that one cannot interchange in (3.8) the derivative operator with the sum, as a given function $\mathsf{g}_y(\sigma)$ may not be sufficiently regular to be differentiated k times. Note that formula (3.7) does not have the classical feature of a Taylor-type expansion formula, which would rather involve an x-dependent term in front of $\mathsf{g}_{yx}(\tau/^+\sigma)$, in the first term of the right hand side.

Proof – • Note first that formula (3.1) for $\Delta^+(D^k\tau)$ gives

$$\mathsf{f}_x(D^k\tau) = \sum_{\nu\leqslant^+\tau, \nu\notin\mathcal{B}_X} \mathsf{g}_x(D^k\nu)\, \mathsf{g}_x^{-1}(\tau/^+\nu). \tag{3.9}$$

Formula (3.6) is an inversion formula for the preceding identity. One obtains the former from the latter by writing

$$\begin{aligned}
\sum_{\sigma\leqslant^+\tau,\,\sigma\notin\mathcal{B}_X^+} \mathsf{g}_x(\tau/^+\sigma)\mathsf{f}_x(D^k\sigma) &= \sum_{\nu\leqslant^+\sigma\leqslant^+\tau,\,\sigma,\nu\notin\mathcal{B}_X^+} \mathsf{g}_x(\tau/^+\sigma)\mathsf{g}_x^{-1}(\sigma/^+\nu)\mathsf{g}_x(D^k\nu)\\
&= \sum_{\nu\leqslant^+\sigma\leqslant^+\tau,\,\nu\notin\mathcal{B}_X^+} \mathsf{g}_x(\tau/^+\sigma)\mathsf{g}_x^{-1}(\sigma/^+\nu)\mathsf{g}_x(D^k\nu)\\
&= \sum_{\nu\leqslant^+\tau,\,\nu\notin\mathcal{B}_X^+} (\mathsf{g}_x^{-1}\otimes\mathsf{g}_x)(\tau/^+\nu)\mathsf{g}_x(D^k\nu)\\
&= \mathsf{g}_x(D^k\tau).
\end{aligned}$$

(In the second equality, we can remove the condition "$\sigma\notin\mathcal{B}_X^+$" because $\nu\leqslant^+ X^k$ implies that $\nu\in\mathcal{B}_X^+$. In the last equality, we use the property of the antipode.)

• Applying $\mathsf{g}_y\otimes\mathsf{g}_x^{-1}$ to (3.1), we have

$$\begin{aligned}
\mathsf{g}_{yx}(D^k\tau) &= \sum_{\mu\leqslant^+\tau,\,\mu\notin\mathcal{B}_X^+} \mathsf{g}_y(D^k\mu)\,\mathsf{g}_x^{-1}(\tau/^+\mu) + \sum_\ell \frac{y^\ell}{\ell!}\,\mathsf{g}_x^{-1}(D^{k+\ell}\tau)\\
&= \sum_{\mu\leqslant^+\nu\leqslant^+\tau,\,\mu\notin\mathcal{B}_X^+} \mathsf{g}_y(D^k\mu)\,\mathsf{g}_y^{-1}(\nu/^+\mu)\,\mathsf{g}_{yx}(\tau/^+\nu) - \sum_{\ell'}\frac{(y-x)^{\ell'}}{\ell'!}\,\mathsf{f}_x(D^{k+\ell'}\tau),
\end{aligned}\tag{3.10}$$

where we use the formula

$$\Delta^+(\tau/^+\mu) = \sum_{\mu\leqslant^+\nu\leqslant^+\tau} (\nu/^+\mu)\otimes(\tau/^+\nu)$$

in the expansion of $\mathsf{g}_x^{-1}(\tau/^+\mu)$. Identity (3.7) follows from (3.10) using (3.9). Note that $\mu\leqslant^+\nu$ and $\mu\notin\mathcal{B}_X^+$ implies that $\nu\notin\mathcal{B}_X^+$.

• Formula (3.8) comes from identity (3.9) by rewriting the terms $\mathsf{g}_x(D^k\nu)$ in an appropriate form. As a preliminary remark, notice that applying $\mathsf{g}_{yx}\otimes\mathsf{g}_x$ to the defining identity

$$\Delta^+\nu = \sum_{\sigma\leqslant^+\nu,\sigma\notin\mathcal{B}_X^+} \sigma\otimes(\nu/^+\sigma) + \sum_k \frac{X^k}{k!}\otimes D^k\nu,$$

for the $D^k\nu$, we have

$$\mathsf{g}_y(\nu) = \sum_{\sigma\leqslant^+\nu,\,\sigma\notin\mathcal{B}_X^+} \mathsf{g}_{yx}(\sigma)\,\mathsf{g}_x(\nu/^+\sigma) + \sum_k \mathsf{g}_x(D^k\nu)\,\frac{(y-x)^k}{k!}.$$

Since one has

$$\partial_y^k\mathsf{g}_{yx}(\sigma)\big|_{y=x} = 0,$$

for any $x\in\mathbb{R}^d$, whenever $|k|<|\sigma|$, one then has

$$\mathsf{g}_x(D^k\nu) = \mathbf{1}_{|k|<|\nu|}\,\partial_y^k\Big\{\mathsf{g}_y(\nu) - \sum_{\substack{\sigma<^+\nu,\,\sigma\notin\mathcal{B}_X^+\\ |\sigma|\leqslant|k|}} \mathsf{g}_{yx}(\sigma)\,\mathsf{g}_x(\nu/^+\sigma)\Big\}\Big|_{y=x}.\tag{3.11}$$

At the same time, for $\nu\notin\mathcal{B}_X^+$, one has

$$\mathsf{g}_y(\nu) = \sum_{\mu\leqslant^+\nu} (\mathsf{g}_x^{-1}*\mathsf{g}_x)(\nu/^+\mu)\,\mathsf{g}_y(\mu) = \sum_{\substack{\mu\leqslant^+\nu\\ \mu\notin\mathcal{B}_X^+}} (\mathsf{g}_x^{-1}*\mathsf{g}_x)(\nu/^+\mu)\,\mathsf{g}_y(\mu)$$

$$= \sum_{\substack{\mu\leqslant^+\sigma\leqslant^+\nu\\ \mu,\sigma\notin\mathcal{B}_X^+}} \mathsf{g}_x(\nu/^+\sigma)\,\mathsf{g}_x^{-1}(\sigma/^+\mu)\mathsf{g}_y(\mu),$$

since $\mu \leqslant^+ \sigma$ and $\mu \notin \mathcal{B}_X^+$ implies $\sigma \notin \mathcal{B}_X^+$. Furthermore, since $\mu \leqslant^+ \sigma \notin \mathcal{B}_X^+$ and $|\sigma| \leqslant |k|$ implies $\mu \notin \mathcal{B}_X^+$ or $\mu <^+ X^k$, we have

$$\sum_{\substack{\mu\leqslant^+\sigma\leqslant^+\nu\\ \mu,\sigma\notin\mathcal{B}_X^+,\,|\sigma|\leqslant|k|}} \mathsf{g}_x(\tau/^+\sigma)\,\mathsf{g}_x^{-1}(\sigma/^+\mu)\,\mathsf{g}_y(\mu) = \sum_{\substack{\sigma\leqslant^+\nu\\ \sigma\notin\mathcal{B}_X^+,\,|\sigma|\leqslant|k|}} \mathsf{g}_x(\nu/^+\sigma)\,\mathsf{g}_{yx}(\sigma) + p_{<k}(y),$$

where $p_{<k}$ is a polynomial of degree less than k, hence $\partial_y^k p_{<k} = 0$. We thus obtain from formula (3.11), that

$$\mathsf{g}_x(D^k\nu) = \partial_y^k\Bigg\{\sum_{\substack{\mu\leqslant^+\sigma\leqslant^+\nu\\ \mu,\sigma\notin\mathcal{B}_X^+,\,|\sigma|>|k|}} \mathsf{g}_x(\nu/^+\sigma)\,\mathsf{g}_x^{-1}(\sigma/^+\mu)\,\mathsf{g}_y(\mu)\Bigg\}\Big|_{y=x}.$$

Inserting this expression in formula (3.9) one gets, with $|k| < |\tau|$,

$$\begin{aligned}
\mathsf{f}_x(D^k\tau) &= \sum_{\mu\leqslant^+\tau,\,\mu\notin\mathcal{B}_X^+} \mathsf{g}_x^{-1}(\tau/^+\mu)\,\mathsf{g}_x(D^k\mu)\\
&= \partial_y^k\Bigg\{\sum_{\substack{\nu\leqslant^+\sigma\leqslant^+\mu\leqslant^+\tau\\ \nu,\sigma,\mu\notin\mathcal{B}_X^+,\,|\sigma|>|k|}} \mathsf{g}_x^{-1}(\tau/^+\mu)\,\mathsf{g}_x(\mu/^+\sigma)\,\mathsf{g}_x^{-1}(\sigma/^+\nu)\,\mathsf{g}_y(\nu)\Bigg\}\Big|_{y=x}\\
&= \partial_y^k\Bigg\{\sum_{\substack{\nu\leqslant^+\sigma\leqslant^+\tau\\ \nu,\sigma\notin\mathcal{B}_X^+,\,|\sigma|>|k|}} (\mathsf{g}_x * \mathsf{g}_x^{-1})(\tau/^+\sigma)\,\mathsf{g}_x^{-1}(\sigma/^+\nu)\,\mathsf{g}_y(\nu)\Bigg\}\Big|_{y=x}\\
&= \partial_y^k\Bigg\{\sum_{\nu\leqslant^+\tau,\,\nu\notin\mathcal{B}_X^+} \mathsf{g}_x^{-1}(\tau/^+\nu)\,\mathsf{g}_y(\nu)\Bigg\}\Big|_{y=x}.
\end{aligned}$$

▷

4 – From paracontrolled systems to models and modelled distributions

We prove the main results of this work in this section. Theorem 1 gives a parametrization of the space of models by data in paracontrolled representations. Its proof requires that we introduce assumption **(C)**, about the structure of the Hopf algebra (T^+, Δ^+). We prove Theorem 2 in Section 4.2 as a corollary of Theorem 12, giving a paracontrolled parametrization of g-maps. The case of BHZ regularity structures is investigated in Section 4.3.

4.1 From paracontrolled systems to models

The following claim is the same as Corollary 15 in [4], with the modified paraproduct P^m in the role of P. Recall from Proposition 7 the defnition of the reference distributions $[\![\sigma]\!]^{m,\mathsf{M}}$, in the paracontrolled representation of the Π operator of a model M, using the modified paraproduct P^m.

Proposition 10. *Pick $m \in \mathbb{N}$, and assume we are given a map $\mathsf{g} : \mathbb{R}^d \to G^+$, such that conditions (a) and (b) in Definition 6 are satisfied. Then for any family $\big\{[\![\tau]\!] \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)\big\}_{\tau\in\mathcal{B}_\bullet,|\tau|<0}$,* *there exists a unique model* $\mathsf{M} = (\mathsf{g}, \Pi) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$ *such that*

$$\Pi\tau = \sum_{\substack{\sigma<\tau\\ \sigma\in\mathcal{B}}} \mathsf{P}^m_{\mathsf{g}(\tau/\sigma)} [\![\sigma]\!]^{m,\mathsf{M}} + [\![\tau]\!], \qquad \forall\, \tau \in \mathcal{B}_\bullet,\ |\tau| < 0. \tag{4.1}$$

The map

$$\Big(\mathsf{g}, \Big\{ [\![\tau]\!] \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d) \Big\}_{\tau\in\mathcal{B}_\bullet, |\tau|<0} \Big) \mapsto \mathsf{M} \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$$

is continuous.

Note that the distribution $[\![\sigma]\!]^{m,\mathsf{M}}$ in (4.1) is a distribution recursively defined by the application of Theorem 7 to the subspace $\bigoplus_{\beta<|\tau|} T_\beta$. If $\sigma \in \mathcal{B}_\bullet$, then $[\![\sigma]\!]^{m,\mathsf{M}} = [\![\sigma]\!]$.

Proof – This is a consequence of Theorem 8 that can be proved as follows. For τ of negative homogeneity, we need to prove a uniform bound

$$\{\Pi^{\mathsf{g}}_x \tau\}_{x\in\mathbb{R}^d} \in D^{|\tau|}_{\mathrm{rap}}.$$

This is equivalent to saying that $\Pi\tau$ is a reconstruction of the modelled distribution $\boldsymbol{h}_\tau(x) := \sum_{\sigma<\tau} \mathsf{g}_x(\tau/\sigma)\sigma \in \mathcal{D}^{|\tau|}(\mathscr{T}, \mathsf{g})$ – as $|\tau| < 0$, the reconstruction is not unique. But Theorem 8 already provides us with a reconstruction of $\boldsymbol{h}_\tau$, of the form

$$\sum_{\sigma<\tau} \mathsf{P}^m_{\mathsf{g}(\tau/\sigma)} [\![\sigma]\!]^{m,\mathsf{M}} + [\![\boldsymbol{h}_\tau]\!]^{m,\mathsf{M}},$$

with $[\![\boldsymbol{h}_\tau]\!]^{m,\mathsf{M}} \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)$. Since the latter differs from $\Pi\tau$ by $\big([\![\tau]\!] - [\![\boldsymbol{h}_\tau]\!]^{m,\mathsf{M}}\big) \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)$, we conclude that $\Pi\tau$ is indeed another reconstruction of $\boldsymbol{h}_\tau$. We refer the reader to the end of the proof of Corollary 15 in [4] for the unique extension of Π to the whole of $\mathscr{T}$. (There is no other element than $\mathbf{1}$ of zero homogeneity in the present setting.) $\triangleright$

This proof makes it clear that the above parametrization of the set of Π maps is related to the non-uniqueness of the reconstruction map on the set of modelled distributions of negative regularity exponent. This statement leaves us with the task of giving a parametrization of the set of characters g on T^+ by their paracontrolled representation. We need for that purpose to make the following assumptions on the Hopf algebra (T^+, Δ^+) and the basis $\mathcal{B}^+$ of T^+. Recall that $D^k : T^+_\alpha \to T^+_{\alpha-|k|}$, is a linear map satisfying the recursive rules from Lemma 5.

Assumption (C)

(1) *There exists a finite subset* $\mathcal{G}^+_\circ$ *of* $\mathcal{B}^+_\circ$ *such that* $\mathcal{B}^+_\circ$ *is of the form*

$$\mathcal{B}^+_\circ = \bigsqcup_{\tau\in\mathcal{G}^+_\circ} \Big\{ D^k\tau \ ;\ k \in \mathbb{N}^d,\ |\tau| - |k| > 0 \Big\}.$$

For each $\alpha \in \mathbb{R}$, denote by $\mathcal{B}^+(\alpha^-)$ the submonoid of $\mathcal{B}^+$ generated by

$$\Big\{X_1, \dots, X_d\Big\} \cup \bigsqcup_{\sigma\in\mathcal{G}^+_\circ,\, |\sigma|<\alpha} \Big\{ D^k\sigma \ ;\ k \in \mathbb{N}^d,\ |\sigma| - |k| > 0 \Big\}.$$

(2) *For each* $\tau \in \mathcal{G}^+_\circ$*, the coproduct* $\Delta^+\tau$ *is of the form*

$$\Delta^+\tau = \tau \otimes \mathbf{1} + \sum_{\sigma<^+\tau,\, \sigma\notin\mathcal{B}^+_X} \sigma \otimes (\tau/^+\sigma) + \sum_k \frac{X^k}{k!} \otimes D^k\tau, \tag{4.2}$$

with $\sigma \in \mathcal{B}^+(|\tau|^-)$ *and* $\tau/^+\sigma \in \mathrm{span}\big(\mathcal{B}^+(|\tau|^-)\big)$*, for each* σ *in the above sum.*

Note the *disjoint* union in the description of $\mathcal{B}^+_\circ$. Assumption **(C-2)** provides a useful induction structure.

Lemma 11. *Formula* (4.2)*, with the constraints on* σ *and* τ/σ*, holds for any* $\tau \in \mathcal{B}^+$*.*

Proof – The proof is done by induction. Pick $\tau \in \mathcal{G}_\circ^+$, and assume that identity (4.2) holds for all $\tau' \in \mathcal{B}^+(|\tau|^-)$. By the recursive rules in Lemma 5, (4.2) holds for all the elements of the form

$$(D^k\tau)\nu,$$

where $k \in \mathbb{N}^d$ and $\nu \in \mathcal{B}^+(|\tau|^-)$. So (4.2) eventually holds for all $\tau' \in \mathcal{B}^+(\beta^-)$, where

$$\beta := \min\Big\{|\mu|; \mu \in \mathcal{G}_\circ^+, |\mu| > \alpha\Big\} > \alpha.$$

$\triangleright$

Recall from formula (3.11) that if we are given characters $(\mathsf{g}_x)_{x\in\mathbb{R}^d}$ on T^+ as in Definition 6, then

$$\mathsf{g}_x(D^k\tau) = \mathbf{1}_{|k|<|\tau|}\,\partial_y^k\Big\{\mathsf{g}_y(\tau) - \sum_{\substack{\sigma<^+\tau,\,\sigma\notin\mathcal{B}_X^+\\ |\sigma|\leqslant|k|}} \mathsf{g}_{yx}(\sigma)\,\mathsf{g}_x(\tau/^+\sigma)\Big\}\Big|_{y=x}. \tag{4.3}$$

The induction structure from assumption **(C-2)** restricts the above sum and shows that the family of all $\mathsf{g}_x(D^k\tau)$ is uniquely determined by the preceding formula. It follows then from assumption **(C-1)** that the character g on T^+ is entirely determined by the datum of the $\mathsf{g}(\tau)$, for $\tau \in \mathcal{G}_\circ^+$. Order the elements of $\mathcal{G}_\circ^+$ in non-decreasing order of homogeneity, so $\mathcal{G}_\circ^+ = \{\tau_1, \dots, \tau_N\}$, with $|\tau_1| \leqslant \cdots \leqslant |\tau_N|$. (An arbitrary order is chosen amongst those τ's with equal homogeneity.) We have in particular

$$\mathsf{g}_y(\tau_1) = \mathsf{g}_{yx}(\tau_1) + \sum_{|k|<|\tau_1|} \frac{(y-x)^k}{k!}\,\mathsf{g}_x(D^k\tau_1),$$

since $\mathcal{B}^+(|\tau_1|^-) = \mathcal{B}_X^+$, so for $|k| < |\tau_1|$, one has

$$\mathsf{g}_x(D^k\tau_1) = \partial_y^k\mathsf{g}_y(\tau_1)\big|_{y=x}, \tag{4.4}$$

and

$$\mathsf{f}_x(D^k\tau_1) = \mathsf{g}_x(D^k\tau_1),$$

and

$$\mathsf{g}_{yx}(D^k\tau_1) = \mathsf{g}_y(D^k\tau_1) - \sum_\ell \frac{(y-x)^\ell}{\ell!}\,\mathsf{g}_x(D^{k+\ell}\tau_1). \tag{4.5}$$

Recall that, given a concrete regularity structure $\mathscr{T}$,

$$\mathscr{T}^+ = \big((T^+,\Delta^+),(T^+,\Delta^+)\big)$$

is also a concrete regularity structure.

Theorem 12. *Let $\mathscr{T}$ stand for a concrete regularity structure satisfying assumption* **(A)** *and* **(B)**. *Assume that $\mathscr{T}^+$ satisfies assumption* **(C)**. *Then, for any family $\Big\{[\![\tau]\!] \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)\Big\}_{\tau\in\mathcal{G}_\circ^+}$, there exists a unique model $\mathsf{M}^{\mathsf{g}} = (\mathsf{g},\mathsf{g})$ on $\mathscr{T}^+$ such that*

$$\mathsf{g}(\tau) = \sum_{\substack{\sigma<^+\tau\\ \sigma\in\mathcal{B}^+\setminus\mathcal{B}_X^+}} \mathsf{P}_{\mathsf{g}(\tau/^+\sigma)}[\![\sigma]\!]^{\mathsf{M}^{\mathsf{g}}} + [\![\tau]\!], \qquad \forall\,\tau \in \mathcal{G}_\circ^+. \tag{4.6}$$

The map

$$\Big\{[\![\tau]\!] \in C^{|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)\Big\}_{\tau\in\mathcal{G}_\circ^+} \mapsto \mathsf{M}^{\mathsf{g}} \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}^+,\mathbb{R}^d)$$

is continuous.

The injectivity of the above map is elementary, so Theorem 12 and Proposition 10, with Theorem 7, prove all together Theorem 1.

The remaining of this section is dedicated to proving Theorem 12. The proof is done by induction on $i \in \{1,\dots,N\}$, where $\mathcal{G}_\circ^+ = \{\tau_1,\dots,\tau_N\}$, with $|\tau_1| \leqslant \cdots \leqslant |\tau_N|$.

Initialisation of the induction. Set

$$\mathsf{g}(\tau_1) := [\![\tau_1]\!],$$

and define $\mathsf{g}(D^k\tau_1)$ and $\mathsf{g}_{yx}(D^k\tau_1)$ by (4.4) and (4.5). It is clear on these formulas that they define elements of the spaces $C^{|\tau_1|-|k|}_{\text{rap}}(\mathbb{R}^d) \subset L^\infty_{\text{rap}}(\mathbb{R}^d)$ and $\mathcal{C}^{|\tau_1|-|k|}_{\text{rap}}(\mathbb{R}^d \times \mathbb{R}^d)$, respectively.

Induction step. Fix $\tau = \tau_n \in \mathcal{G}^+_\circ$, at the n^{th} step of the induction, and assume that g has been constructed on the submonoid $\mathcal{B}^+(|\tau|^-)$ as a smooth function of the bracket data – so all the functions $[\![\sigma]\!]^{\mathsf{M}^{\mathsf{g}}}$ and $\mathsf{g}(\tau/^+\sigma)$ make sense as elements of their natural spaces. Define $\mathsf{g}(\tau)$ by identity (4.6), and define $\mathsf{g}(D^k\tau)$ by (4.3), for all $k \in \mathbb{N}^d$ with $|k| < |\tau|$. The induction step consists in proving that $\mathsf{g}_x(D^k\tau) \in L^\infty_{\text{rap}}(\mathbb{R}^d)$ and $\mathsf{g}_{yx}(D^k\tau) \in \mathcal{C}^{|\tau|-|k|}_{\text{rap}}(\mathbb{R}^d \times \mathbb{R}^d)$, as one can use the inclusions (α, β non-negative)

$$L^\infty_{\text{poly}}(\mathbb{R}^d) \cdot L^\infty_{\text{rap}}(\mathbb{R}^d) \subset L^\infty_{\text{rap}}(\mathbb{R}^d)$$

and

$$\mathcal{C}^\alpha(\mathbb{R}^d \times \mathbb{R}^d) \cdot \mathcal{C}^\beta_{\text{rap}}(\mathbb{R}^d \times \mathbb{R}^d) \subset \mathcal{C}^{\alpha+\beta}_{\text{rap}}(\mathbb{R}^d \times \mathbb{R}^d),$$

to get the regularity properties of $\mathsf{g}_x(\mu\, D^k\tau)$ and $\mathsf{g}_{yx}(\mu\, D^k\tau)$, for $\mu \in \mathcal{B}^+(|\tau_n|^-)$.

We introduce for that purpose a regularity structure $\mathscr{T}^m(\tau)$ with Hopf algebra

$$T^+(|\tau|^-) := \operatorname{span}\big(\mathcal{B}^+(|\tau|^-)\big),$$

and T-space only made up of elements with negative homogeneity. We build a model $(\mathsf{g}, \mathsf{\Lambda})$ on $\mathscr{T}^m(\tau)$, from $\mathsf{g} : T^+(|\tau|^-) \to \mathbb{R}$ and $[\![\tau]\!]$, such that formula (3.8) giving $\mathsf{f}_x(D^k\sigma)$ can be interpreted in terms of that model, under the form of identities

$$\mathsf{f}_x(D^k\sigma) = \mathbf{J}^{k,m}\Big(\mathsf{\Lambda}^{\mathsf{g}}_x(\sigma^{(m)})\Big)(x)$$

for operators $\mathbf{J}^{k,m}$ on distributions – the symbols $\sigma^{(m)}$ are introduced below. The identity

$$\mathsf{\Lambda}^{\mathsf{g}}_x = \mathsf{\Lambda}^{\mathsf{g}}_y \circ \widehat{\mathsf{g}_{yx}}^{\,\delta}$$

is then used crucially to obtain estimates on $\mathsf{f}_x(D^k\sigma)$, that eventually give informations on $\mathsf{g}_x(D^k\tau)$ and $\mathsf{g}_{yx}(D^k\tau)$ via formulas (3.6) and (3.7).

Choose $m \in \mathbb{N}$, with $m > |\tau|$. Consider the formal symbols

$$\sigma^{(m)}$$

indexed by $\sigma \in \mathcal{B}^+ \backslash \mathcal{B}^+_X$, with homogeneity

$$\big|\sigma^{(m)}\big| := |\sigma| - m.$$

Set

$$T^m(\tau) := \operatorname{span}\Big(\sigma^{(m)}\ ;\ \sigma \in \mathcal{B}^+(|\tau|^-)\backslash\mathcal{B}^+_X \text{ such that } |\sigma| < |\tau|, \text{ or } \sigma = \tau\Big),$$

so all elements of $T^m(\tau)$ have negative homogeneity. We define a coassociative coproduct

$$\delta : T^m(\tau) \mapsto T^m(\tau) \otimes T^+(|\tau|^-)$$

setting

$$\delta(\sigma^{(m)}) := \sum_{\mu \leqslant^+ \sigma,\, \mu \notin \mathcal{B}^+_X} \mu^{(m)} \otimes (\sigma/^+\mu).$$

Assumption **(C-2)** ensures that

$$\Delta^+\big(T^+(|\tau|^-)\big) \subset T^+(|\tau|^-) \otimes T^+(|\tau|^-),$$

so

$$\mathscr{T}^m(\tau) := \Big((T^+(|\tau|^-), \Delta^+), (T^m(\tau), \delta)\Big)$$

is a concrete regularity structure. For $\mathsf{g} \in G^+$, set

$$\widehat{\mathsf{g}}^\delta := (\mathrm{Id} \otimes \mathsf{g})\delta$$

Let $|\nabla|^m$ be the Fourier multiplier operator

$$|\nabla|^m\zeta = \mathcal{F}^{-1}\big(|\cdot|^m\mathcal{F}\zeta\big).$$

We define an operator

$$\mathsf{\Lambda} : T^m(\tau) \mapsto \mathcal{S}'(\mathbb{R}^d)$$

setting

$$\mathsf{\Lambda}(\sigma^{(m)}) := |\nabla|^m\mathsf{g}(\sigma).$$

Lemma 13. *The pair* $(\mathsf{g},\mathsf{\Lambda})$ *is a rapidly decreasing model on the regularity structure* $\mathscr{T}^m(\tau)$.

Proof – Since we have the identity

$$\mathsf{\Lambda}(\sigma^{(m)}) = |\nabla|^m\mathsf{g}(\sigma) = \sum_{\mu<\sigma,\,\mu\notin\mathcal{B}_X^+}\mathsf{P}^m_{\mathsf{g}(\sigma/^+\mu)}|\nabla|^m[\![\mu]\!]^{\mathsf{g}} + |\nabla|^m[\![\sigma]\!]^{\mathsf{g}},$$

for all $\sigma\in\mathcal{B}^+(|\tau|^-)\backslash\mathcal{B}_X^+$ with $|\sigma|<|\tau|$, ot $\sigma=\tau$, from the intertwining relation defining P^m and the induction assumption, the operator $\mathsf{\Lambda}$ is the unique model on $\mathscr{T}^m(\tau)$ associated by Proposition 10 to the inputs

$$[\![\sigma^{(m)}]\!] := |\nabla|^m[\![\sigma]\!]^{\mathsf{g}} \in C^{|\sigma|-m}_{\mathrm{rap}}(\mathbb{R}^d),$$

since all elements of $T^m(\tau)$ have negative homogeneity. ▷

Note that it follows from identity (3.8) in Lemma 9 that the model $\mathsf{\Pi}$ and the function $\mathsf{f}(D^k\sigma)$ are related by the identity

$$\begin{aligned}
\mathsf{f}_x(D^k\sigma) &= \partial_y^k\Big\{\sum_{\mu\leqslant^+\sigma,\,\mu\notin\mathcal{B}_X^+}\mathsf{g}_x^{-1}(\sigma/^+\mu)\mathsf{g}_y(\mu)\Big\}\Big|_{y=x} \\
&= \partial_y^k\Big\{|\nabla_y|^{-m}\sum_{\mu\leqslant^+\sigma,\,\mu\notin\mathcal{B}_X^+}\mathsf{g}_x^{-1}(\sigma/^+\mu)\mathsf{\Lambda}\big(\mu^{(m)}\big)(y)\Big\}\Big|_{y=x} \\
&= \partial_y^k\Big\{|\nabla_y|^{-m}\mathsf{\Lambda}_x^{\mathsf{g}}\big(\sigma^{(m)}\big)(y)\Big\}\Big|_{y=x} \\
&=: \sum_j \mathbf{J}_j^{k,m}\Big(\mathsf{\Lambda}_x^{\mathsf{g}}\big(\sigma^{(m)}\big)\Big)(x),
\end{aligned} \tag{4.7}$$

where the operators $\mathbf{J}_j^{k,m}$ are defined by

$$\mathbf{J}_j^{k,m}(\zeta) := \partial^k|\nabla|^{-m}\Delta_j\zeta,$$

for an appropriate distribution $\zeta\in\mathcal{S}'(\mathbb{R}^d)$. If $j\geqslant 0$, since the Fourier transform of $\Delta_j\zeta$ is supported on an annulus, the function $\mathbf{J}_j^{k,m}(\zeta)$ is always well-defined; this is not the case of $\mathbf{J}_{-1}^{k,m}(\zeta)$. However, we only use in this section distributions ζ of the form $\zeta=|\nabla|^m\xi$ (where such ξ is unique in the class of rapidly decreasing functions), so $\mathbf{J}_{-1}^{k,m}(\zeta)=\partial^k\Delta_{-1}\xi$, in our setting.

Lemma 14. *For any* $\sigma\in\big(\mathcal{B}^+(|\tau|^-)\cup\{\tau\}\big)\backslash\mathcal{B}_X^+$, $k\in\mathbb{N}^d$, *and* $a\in\mathbb{N}$, *we have*

$$\begin{aligned}
\Big|\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(x)\Big| &\lesssim |x|_*^{-a}\,2^{-j(|\sigma|-|k|)}, \\
\Big|\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(y)\Big| &\lesssim |y|_*^{-a}\sum_{\mu\leqslant^+\sigma}|y-x|^{|\sigma|-|\mu|}2^{-j(|\mu|-|k|)}.
\end{aligned}$$

Consequently, $\mathsf{f}_x(D^k\sigma)\in L^\infty_{\mathrm{rap}}$.

Proof – For the first estimate, since $\mathbf{J}_{-1}^k\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(x)\in L^\infty_{\mathrm{rap}}$, by assumption, it is sufficient to consider the case $j\geqslant 0$. By the property of ρ_j, there exists a function $\widetilde{\rho}$ with Fourier transform of $\widetilde{\rho}$ supported on an annulus, and such that setting $\rho_j(\cdot) := \rho(2^{-j}\cdot)$, one has $\widetilde{\rho}_j\rho_j = \rho_j$. Set

$$\widetilde{Q}_j^{k,m} := \partial^k|\nabla|^{-m}(\mathcal{F}^{-1}\widetilde{\rho}_j),$$

and note the scaling property

$$\widetilde{Q}_j^{k,m}(\cdot) = 2^{j(d+|k|-m)}\widetilde{Q}_0^{k,m}(2^j\cdot).$$

We now use the fact that $(\mathsf{g},\mathsf{\Lambda})$ is a model to write

$$\begin{aligned}\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(x) &= \int \widetilde{Q}_j^{k,m}(x-y)\Delta_j\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(y)dy \\ &= \int \widetilde{Q}_j^{k,m}(x-y)\Delta_j\big(\mathsf{\Lambda}_y^{\mathsf{g}}\circ \widehat{\mathsf{g}_{yx}}^{\,\delta}(\sigma^{(m)})\big)(y)dy \\ &= \sum_{\mu\leqslant^+\sigma}\int \widetilde{Q}_j^{k,m}(x-y)\,\mathsf{g}_{yx}(\sigma/^+\mu)\,\Delta_j\big(\mathsf{\Lambda}_y^{\mathsf{g}}(\mu^{(m)})\big)(y)dy.\end{aligned}$$

Recall that $|x+y|_* \leqslant |x|_*|y|_*$, for all $x,y\in\mathbb{R}^d$. By Lemma 13, for any $a\in\mathbb{N}$ we have

$$\begin{aligned}|x|_*^a\Big|\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(x)\Big| &\lesssim \sum_{\mu\leqslant^+\sigma}\int |x-y|_*^a\big|\widetilde{Q}_j^{k,m}(x-y)\big||y-x|^{|\sigma|-|\mu|}\,|y|_*^a\,\Big|\Delta_j\big(\mathsf{\Lambda}_y^{\mathsf{g}}(\mu^{(m)})\big)(y)\Big|\,dy \\ &\lesssim \sum_{\mu\leqslant^+\sigma} 2^{-j(|\mu|-m)}\int |z|_*^a\big|\widetilde{Q}_j^{k,m}(z)\big||z|^{|\sigma|-|\mu|}\,dz \\ &\lesssim \sum_{\mu\leqslant^+\sigma} 2^{-j(|\mu|-m)}2^{j(|k|-m-|\sigma|+|\mu|)}\int |z|_*^a\big|\widetilde{Q}_0^{k,m}(z)\big||z|^{|\sigma|-|\mu|}\,dz \\ &\lesssim 2^{-j(|\sigma|-|k|)}.\end{aligned}$$

We get the second estimate from the first using once again the fact that $(\mathsf{g},\mathsf{\Lambda})$ is a model, writing

$$\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\sigma^{(m)})\big)(y) = \mathbf{J}_j^{k,m}\Big(\mathsf{\Lambda}_y^{\mathsf{g}}\big(\widehat{\mathsf{g}_{yx}}^{\,\delta}(\sigma^{(m)})\big)\Big)(y) = \sum_{\mu\leqslant^+\sigma}\mathsf{g}_{yx}(\sigma/^+\mu)\,\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_y^{\mathsf{g}}(\mu^{(m)})\big)(y).$$

$\rhd$

We can now prove that $\mathsf{g}_x(D^k\tau)\in L^\infty_{\mathrm{rap}}(\mathbb{R}^d)$ and $\mathsf{g}_{yx}(D^k\tau)\in \mathcal{C}^{|\tau|-|k|}_{\mathrm{rap}}(\mathbb{R}^d\times\mathbb{R}^d)$, and close the induction step. We use the formulas from Lemma 9 for that purpose. First, since

$$\mathsf{g}_x(D^k\tau) = \sum_{\sigma\leqslant^+\tau,\,\sigma\notin\mathcal{B}_X^+}\mathsf{g}_x(\tau/^+\sigma)\,\mathsf{f}_x(D^k\sigma),$$

with $\mathsf{g}_x(\tau/\sigma)\in L^\infty_{\mathrm{poly}}(\mathbb{R}^d)$ and $\mathsf{f}_x(D^k\sigma)\in L^\infty_{\mathrm{rap}}(\mathbb{R}^d)$, from Lemma 14, we have indeed $\mathsf{g}_x(D^k\tau)\in L^\infty_{\mathrm{rap}}(\mathbb{R}^d)$. Second, one can rewrite the identity

$$\mathsf{g}_{yx}(D^k\tau) = \sum_{\sigma\leqslant^+\tau,\,\sigma\notin\mathcal{B}_X^+}\mathsf{g}_{yx}(\tau/^+\sigma)\,\mathsf{f}_y(D^k\sigma) - \sum_\ell\frac{(y-x)^\ell}{\ell!}\,\mathsf{f}_x(D^{k+\ell}\tau),$$

from Lemma 9, using identity (4.7) for the f-terms. This gives

$$\begin{aligned}\mathsf{g}_{yx}(D^k\tau) &= \sum_j\left\{\sum_{\substack{\sigma\leqslant^+\tau\\|k|<|\sigma|}}\mathsf{g}_{yx}(\tau/^+\sigma)\,\mathbf{J}_j^{k,m}\big(\mathsf{\Lambda}_y^{\mathsf{g}}(\sigma^{(m)})\big)(y) - \sum_{|k+\ell|<|\tau|}\frac{(y-x)^\ell}{\ell!}\,\mathbf{J}_j^{k+\ell,m}\big(\mathsf{\Lambda}_x^{\mathsf{g}}(\tau^{(m)})\big)(x)\right\} \\ &=: \mathsf{g}_{yx}^j(D^k\tau).\end{aligned}$$

Given $x,y\in\mathbb{R}^d$, set $j_0=-1$, if $|y-x|\geqslant 2$, and pick otherwise $j_0\geqslant -1$ such that $|y-x|\simeq 2^{-j_0}$. One uses the first estimate from Lemma 14 to bound above the sum over $j\geqslant j_0$

$$|x|_*^a \sum_{j\geqslant j_0} \big|\mathsf{g}_{yx}^j(D^k\tau)\big| \lesssim \sum_{j\geqslant j_0} \sum_{\substack{\sigma\leqslant^+\tau\\ |k|<|\sigma|}} |y-x|^{|\tau|-|\sigma|}\, 2^{-j(|\sigma|-|k|)} + \sum_{j\geqslant j_0} \sum_{|k+\ell|<|\tau|} |y-x|^{|\ell|}\, 2^{-j(|\tau|-|k|-|\ell|)}$$

so

$$\begin{aligned} |x|_*^a \sum_{j\geqslant j_0} \big|\mathsf{g}_{yx}^j(D^k\tau)\big| &\lesssim \sum_{\substack{\sigma\leqslant^+\tau\\ |k|<|\sigma|}} |y-x|^{|\tau|-|\sigma|}\, 2^{-j_0(|\sigma|-|k|)} + \sum_{|k+\ell|<|\tau|} |y-x|^{|\ell|}\, 2^{-j_0(|\tau|-|k|-|\ell|)} \\ &\lesssim |y-x|^{|\tau|-|k|}. \end{aligned} \tag{4.8}$$

With no loss of generality, assume now that $|y-x|<2$. Then, since (g,Λ) is a model and

$$\Lambda_x^{\mathsf{g}}(\tau^{(m)}) = \Lambda_y^{\mathsf{g}}\big(\widehat{\mathsf{g}_{yx}}^{\,\delta}\tau^{(m)}\big) = \sum_{\sigma\leqslant^+\tau} \mathsf{g}_{yx}(\tau/^+\sigma)\Lambda_y^{\mathsf{g}}(\sigma^{(m)}),$$

we have for $\mathsf{g}_{yx}^j(D^k\tau)$ the formula

$$\begin{aligned} &\mathbf{J}_j^{k,m}\big(\Lambda_x^{\mathsf{g}}(\tau^{(m)})\big)(y) - \sum_{\substack{\sigma\leqslant^+\tau\\ |k|\geqslant|\sigma|}} \mathsf{g}_{yx}(\tau/^+\sigma)\mathbf{J}_j^{k,m}\big(\Lambda_y^{\mathsf{g}}(\sigma^{(m)})\big)(y) \\ &\qquad - \sum_{|k+\ell|<|\tau|} \frac{(y-x)^\ell}{\ell!}\mathbf{J}_j^{k+\ell,m}\big(\Lambda_x^{\mathsf{g}}(\tau^{(m)})\big)(x) \\ &= \lceil b\rceil \sum_{|k'|=\lceil b\rceil} \frac{(y-x)^{k'}}{k'!}\int_0^1 (1-t)^{\lceil b\rceil}\mathbf{J}_j^{k+k'}\big(\Lambda_x^{\mathsf{g}}(\tau^{(m)})\big)\big(x+t(y-x)\big)dt \\ &\qquad - \sum_{\substack{\sigma\leqslant^+\tau\\ |k|\geqslant|\sigma|}} \mathsf{g}_{yx}(\tau/\sigma)\,\mathbf{J}_j^{k,m}\big(\Lambda_y^{\mathsf{g}}(\sigma^{(m)})\big)(y), \end{aligned}$$

where $b := |\tau|-|k|$, by the multivariable Taylor remainder formula. Since $|y-x|<2$, $|x+t(y-x)|_* \simeq |x|_*$. It follows then from Lemma 14 that $\sum_{-1\leqslant j<j_0}\big|\mathsf{g}_{yx}^j(D^k\tau)\big|$ is bounded above by

$$\begin{aligned} &\sum_{j<j_0}\sum_{|k'|=\lceil b\rceil}\sum_{\sigma\leqslant^+\tau} |y-x|^{|k'|+|\tau|-|\sigma|}|x|_*^{-a}\, 2^{-j(|\sigma|-|k|-|k'|)} + \sum_{j<j_0}\sum_{\substack{\sigma\leqslant^+\tau\\ |k|\geqslant|\sigma|}} |y-x|^{|\tau|-|\sigma|}|y|_*^{-a}\, 2^{-j(|\sigma|-|k|)} \\ &\lesssim |x|_*^{-a}\sum_{\sigma\leqslant^+\tau} |y-x|^{|k'|+|\tau|-|\sigma|}\, 2^{-j_0(|\sigma|-|k|-|k'|)} + |y|_*^{-a}\sum_{\substack{\sigma\leqslant^+\tau\\ |k|\geqslant|\sigma|}} |y-x|^{|\tau|-|\sigma|}\, 2^{-j_0(|\sigma|-|k|)} \\ &\lesssim \big(|x|_*^{-a} + |y|_*^{-a}\big)|y-x|^{|\tau|-|k|}. \end{aligned}$$

Together with inequality (4.8), the preceding upper bound tells us that $\mathsf{g}_{yx}(D^k\tau)\in\mathcal{C}_{\mathrm{rap}}^{|\tau|-|k|}(\mathbb{R}^d\times\mathbb{R}^d)$. This closes the induction step.

Remark. *One can prove that Theorem 12 holds true in a parabolic setting $[0,T]\times\mathbb{R}^d$, with the paraproduct*

$$(\overline{\mathsf{P}}_f g)(t,x) = \left(\int_0^t e^{(t-s)\Delta}\Big(\mathsf{P}_f\big((\partial_t-\Delta)g\big)\Big)(s)\right)(x)$$

in place of P.

4.2 From paracontrolled systems to modelled distributions

We prove Theorem 2 in this section. Let $\mathscr{T}$ be a regularity structure satisfying assumptions **(A-C)**. Pick $\gamma \in \mathbb{R}$, and $\mathsf{M} = (\mathsf{g}, \Pi) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$.

The key observation is that proving Theorem 2 is equivalent to an extension problem for the map g. Consider indeed the commutative algebra $T^+_{\boldsymbol{F}}$ generated by $\mathcal{B}^+$ and new symbols

$$(\boldsymbol{F}_\tau)_{\tau\in\mathcal{B}, |\tau|<\gamma}.$$

Define the homogeneity of the symbol $\boldsymbol{F}_\tau$ by

$$|\boldsymbol{F}_\tau| := \gamma - |\tau|.$$

The coproduct $\Delta^+_{\boldsymbol{F}}$ on $T^+_{\boldsymbol{F}}$ extending Δ^+ and such that

$$\Delta^+(\boldsymbol{F}_\tau) = (\boldsymbol{F}_\tau) \otimes \mathbf{1} + \sum_{\tau \leqslant \mu} (\mu/\tau) \otimes (\boldsymbol{F}_\mu), \tag{4.9}$$

is coassociative and turns $T^+_{\boldsymbol{F}}$ into a Hopf algebra. It satisfies assumptions **(A-B)** with

$$\mathcal{B}^+_{\boldsymbol{F},\circ} := \mathcal{B}^+_\circ \cup \big\{ \boldsymbol{F}_\tau \,;\, |\tau| < |\gamma| \big\}$$

in the role of $\mathcal{B}^+_\circ$. Note that $T^+_{\boldsymbol{F}}$ does not satisfy assumption **(C)** in general, since the $D^k \boldsymbol{F}_\tau$ have no reason to be independent from the $\{\boldsymbol{F}_\mu\}_\mu$. The elementary proof of the next statement is left to the reader.

Lemma 15. *Given a family $(f_\tau)_{\tau\in\mathcal{B}}$ of continuous functions on $\mathbb{R}^d$, set*

$$\mathsf{g}_x(\boldsymbol{F}_\tau) := f_\tau(x).$$

Then

$$\big\langle \tau', \boldsymbol{f}(y) - \widehat{\mathsf{g}_{yx}} \boldsymbol{f}(x) \big\rangle = \mathsf{g}_{yx}(\boldsymbol{F}_\tau).$$

Defining a modelled distribution $\boldsymbol{f} \in \mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g})$ is thus equivalent to extending the map g from T^+ to $T^+_{\boldsymbol{F}}$ in such a way that the extended map on $(T^+_{\boldsymbol{F}}, \Delta^+_{\boldsymbol{F}})$ still satisfies the regularity constraints from Definition 6.

Recall from assumption **(B)** that either $\mu/\tau \in \mathrm{span}(\mathcal{B}^+ \backslash \mathcal{B}^+_X)$ or $\mu/\tau \in T^+_X$, for $\tau, \mu \in \mathcal{B}$. If $\mu/\tau \in T^+_X$, set

$$\mu/\tau =: \sum_k c^\mu_\tau(k)\, \frac{X^k}{k!},$$

and define

$$D^k \boldsymbol{F}_\tau := \sum_{\tau \leqslant \mu} c^\mu_\tau(k)\, \boldsymbol{F}_\mu. \tag{4.10}$$

Then we have

$$\Delta^+ \boldsymbol{F}_\tau = \boldsymbol{F}_\tau \otimes \mathbf{1} + \sum_{\tau\leqslant\mu,\, \mu/\tau \notin T^+_X} (\mu/\tau) \otimes \boldsymbol{F}_\mu + \sum_k \frac{X^k}{k!} \otimes D^k \boldsymbol{F}_\tau.$$

Theorem 16. *Let a concrete regularity structure $\mathscr{T}$ satisfying assumptions* **(A-C)** *be given, together with a family $\big\{ [\![f_\tau]\!] \in \mathcal{C}^{\gamma-|\tau|}_{\mathrm{rap}}(\mathbb{R}^d) \big\}_{\tau\in\mathcal{B},\, |\tau|<\gamma}$. Pick a model $(\mathsf{g}, \Pi) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T}, \mathbb{R}^d)$. Define*

$$f_\tau := \sum_{\substack{\tau\leqslant\mu,\, \mu/\tau\notin T^+_X \\ |\mu|<\gamma}} \mathsf{P}_{f_\mu} [\![\mu/\tau]\!]^{\mathsf{g}} + [\![\boldsymbol{f}_\tau]\!],$$

and

$$f_\tau^{(k)}(x) := \partial_y^k \Big\{ f_\tau(y) - \sum_{\substack{\tau\leqslant\mu,\, \mu/\tau\notin T_X^+ \\ |\mu|<\gamma,\, |\mu/\tau|\leqslant|k|}} \mathsf{g}_{yx}(\mu/\tau)\, f_\mu(x) \Big\}\Big|_{y=x}. \tag{4.11}$$

If the **structure conditions**

$$f_\tau^{(k)} = \sum_{\substack{\tau\leqslant\mu,\, \mu/\tau\in T_X^+ \\ |\mu|<\gamma}} c_\tau^\mu(k)\, f_\mu, \tag{4.12}$$

holds for any $\tau \in \mathcal{B}$ and $k \in \mathbb{N}^d$, then

$$\boldsymbol{f} = \sum_{\tau\in\mathcal{B}} f_\tau \tau \in \mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g}).$$

The structure condition is reminiscent of a condition introduced by Martin and Perkowski in [10] to give a characterisation of modelled distributions in terms of Besov type spaces. Given that we see f_τ as $\mathsf{g}(\boldsymbol{F}_\tau)$, formula (4.11) is nothing but a formula for $\mathsf{g}(D^k\boldsymbol{F}_\tau)$ – the analogue of formula (3.11) in the present setting.

Proof – Consider the extended Hopf algebra ${}^{\mathsf{free}}T^+_{\boldsymbol{F}}$ freely generated by the symbols

$$\mathcal{B}^+ \cup \Big\{ D^k(\boldsymbol{F}_\tau)\,;\, \tau\in\mathcal{B},\ \gamma > |\tau| + |k| \Big\}.$$

It satisfies assumptions **(A-C)**. By Theorem 12 giving a paracontrolled parametrization of the map g by its definition on the $\mathsf{g}(\tau)$, with $\tau \in \mathcal{G}^+_{\boldsymbol{F},\circ} := \mathcal{G}^+_\circ \cup \big\{\boldsymbol{F}_\tau\,;\, |\tau| < \gamma\big\}$, there exists a unique model g on ${}^{\mathsf{free}}T^+_{\boldsymbol{F}}$ that coincides with g on T^+, and such that

$$\mathsf{g}(\boldsymbol{F}_\tau) := \sum_{\substack{\tau\leqslant\mu \\ |\mu|<|\gamma|}} \mathsf{P}_{\mathsf{g}(\boldsymbol{F}_\mu)}[\![\mu/\tau]\!]^{\mathsf{g}} + [\![f_\tau]\!],$$

for all $\tau\in\mathcal{B}$ with $|\tau| < \gamma$. Since $T^+_{\boldsymbol{F}}$ is the quotient space of ${}^{\mathsf{free}}T^+_{\boldsymbol{F}}$ by the relations (4.10), and

$$\mathsf{g}\big(D^k\boldsymbol{F}_\tau\big) = \sum_{\substack{\tau\leqslant\mu,\, \mu/\tau\in T_X^+ \\ |\mu|<\gamma}} c_\tau^\mu(k)\, \mathsf{g}(\boldsymbol{F}_\mu),$$

from the structure condition (4.12), the map g is consistently defined on the quotient space, where it satisfies the estimates from Definition 6. ▷

One can get rid of the structure condition in some cases.

Assumption (D) – *For any $\tau \in \mathcal{B}_\bullet$, there is no term of the form $\sigma\otimes X^k$ with $k\neq 0$, in the formula for $\Delta\tau$.*

Under assumption **(D)**, given $\tau\in\mathcal{B}$, the only $\mu \geqslant \tau$ such that μ/τ has a non-null component on X^k is $\mu = X^k\tau$, so one has

$$D^k\boldsymbol{F}_\tau = k!\,\boldsymbol{F}_{X^k\tau},$$

and the structure condition (4.12) takes the simple form (4.13) below. Note that the data in the next statement is indexed by $\mathcal{B}_\bullet$, unlike in the general case of Theorem 16 where it is indexed by $\mathcal{B}$.

Corollary 17. *Let $\mathscr{T}$ be a regularity structure satisfying assumptions* **(A-D)***, and a family $\big([\![f_\tau]\!] \in C^{\gamma-|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)\big)_{\tau\in\mathcal{B}_\bullet,\, |\tau|<\gamma}$ be given. Pick a model $(\mathsf{g},\mathsf{\Pi}) \in \mathscr{M}_{\mathrm{rap}}(\mathscr{T},\mathbb{R}^d)$. Set, for $\tau\in\mathcal{B}_\bullet$*

with $|\tau| < \gamma$,

$$f_\tau := \sum_{\substack{\tau \leqslant \mu,\, \mu/\tau \notin T_X^+ \\ |\mu| < \gamma}} \mathsf{P}_{f_\mu} [\![\mu/\tau]\!]^{\mathsf{g}} + [\![f_\tau]\!],$$

and, for $\tau \in \mathcal{B}_\bullet, k \in \mathbb{N}^d \backslash \{0\}$ *with* $|k| + |\tau| < \gamma$,

$$f_{X^k\tau}(x) := \partial_y^k \Big\{ f_\tau(y) - \sum_{\substack{\tau \leqslant \mu,\, \mu/\tau \notin T_X^+ \\ |\mu/\tau| \leqslant |k|,\, |\mu| < \gamma}} \mathsf{g}_{yx}(\mu/\tau)\, f_\mu(x) \Big\}\Big|_{y=x}. \tag{4.13}$$

Then

$$\boldsymbol{f} := \sum_{\sigma \in \mathcal{B}, |\sigma| < \gamma} f_\sigma\, \sigma = \sum_{\substack{\tau \in \mathcal{B}_\bullet,\, k \in \mathbb{N}^d \\ |\tau| + |k| < \gamma}} f_{X^k\tau}\, X^k \tau \in \mathcal{D}^\gamma_{\text{rap}}(T, \mathsf{g}).$$

Note that assumption **(D)** is an assumption about the basis $\mathcal{B}$ of T we choose to work with, not about the regularity structure itself. It is thus possible that a given basis satisfies assumption **(D)** whereas another does not. This flexibility is at the heart of the proof of Theorem 3 in the next section.

4.3 Modelled distributions over BHZ regularity structures

Bruned, Hairer and Zambotti introduced in [6] class of regularity structures convenient for the study of singular stochastic PDEs. We call these structures **BHZ regularity structures**

$$\mathscr{T}_{\text{BHZ}} = \big((T^+_{\text{BHZ}}, \Delta^+_{\text{BHZ}}), (T_{\text{BHZ}}, \Delta_{\text{BHZ}}) \big).$$

Although the canonical basis of these concrete regularity structures do not satisfy assumption **(D)** the following result holds true.

Theorem 18. *One can construct a basis of* T_{BHZ} *that satisfies assumption* **(D)**.

The remaining of this section is dedicated to proving this statement. We recall first the elements of the construction of BHZ regularity structures that we need here. These concrete regularity structures are indexed by decorated rooted trees.

Any finite connected graph without loops and with a distinguished vertex is called a rooted tree. For any rooted tree τ, denote by N_τ the node set, by E_τ the edge set, by $\varrho_\tau \in N_\tau$ the distinguished vertex, called root of τ. Let also $\mathfrak{L}$ be a finite set of types. (Edges will be interpreted differently depending on their type, when given any model on $\mathscr{T}_{\text{BHZ}}$. Different types may for instance correspond to different convolution operators.) Let $\boldsymbol{\mathcal{B}}$ be the set of rooted decorated trees. Each $\tau \in \boldsymbol{\mathcal{B}}$ is a rooted tree equipped with the type map $\mathfrak{t} : E_\tau \to \mathfrak{L}$ and with the decorations

- $\mathfrak{n} : N_\tau \to \mathbb{N}^d$.
- $\mathfrak{o} : N_\tau \to \mathbb{Z}^d \oplus \mathbb{Z}(\mathfrak{L})$.
- $\mathfrak{e} : E_\tau \to \mathbb{N}^d$.

Equivalently, the set $\boldsymbol{\mathcal{B}}$ is generated recursively by the application of the following operations – see [6, Section 4.3].

- One has $X^k \in \boldsymbol{\mathcal{B}}$ for any $k \in \mathbb{N}^d$, where X^k is a tree with only one node $\bullet$, with $\mathfrak{n}(\bullet) = k$, and $\mathfrak{o}(\bullet) = 0 \oplus 0$.
- If $\tau, \sigma \in \boldsymbol{\mathcal{B}}$ then $\tau\sigma \in \boldsymbol{\mathcal{B}}$, where $\tau\sigma$ is called a *tree product*; $\tau\sigma$ is a graph $\tau \sqcup \sigma$ divided by the equivalence relation $\sim$ on $N_\tau \sqcup N_\sigma$, where $x \sim y$ means $x = y$ or $x, y \in \{\varrho_\tau, \varrho_\sigma\}$. On the root $\varrho_{\tau\sigma}$, the decorations $\mathfrak{n}(\varrho_{\tau\sigma}) = \mathfrak{n}(\varrho_\tau) + \mathfrak{n}(\varrho_\sigma)$ and $\mathfrak{o}(\varrho_{\tau\sigma}) = \mathfrak{o}(\varrho_\tau) + \mathfrak{o}(\varrho_\sigma)$ are given.

- For any $\mathfrak{t}\in\mathfrak{L}$ and $k\in\mathbb{N}^d$,
$$\tau\in\boldsymbol{\mathcal{B}}\quad\Rightarrow\quad I_k^{\mathfrak{t}}(\tau)\in\boldsymbol{\mathcal{B}},$$
where the tree $I_k^{\mathfrak{t}}(\tau)$ is obtained by adding on τ one distinguished node ϱ' and one edge $e=(\varrho_\tau,\varrho')$ of type $\mathfrak{t}$, with decorations $\mathfrak{e}(e)=k$ and $\mathfrak{o}(\varrho')=0\oplus 0$.
- For any $\alpha\in\mathbb{Z}^d\oplus\mathbb{Z}(\mathfrak{L})$, denote by R_α the operator on decorated rooted trees adding a value α on the decoration $\mathfrak{o}$ on ϱ_τ. Assume
$$\tau\in\boldsymbol{\mathcal{B}}\quad\Rightarrow\quad R_\alpha(\tau)\in\boldsymbol{\mathcal{B}}.$$
By applying the operator R_α with various α on each step as above, one can see that, if $\tau\in\boldsymbol{\mathcal{B}}$ then the same decorated tree with any other $\mathfrak{o}$-decorartion is also an element of $\boldsymbol{\mathcal{B}}$.

Each type $\mathfrak{t}$ is assigned a nonzero real number $|\mathfrak{t}|$. One assigns a homogeneity $|\mathfrak{n}|,|\mathfrak{o}|,|\mathfrak{e}|,|\mathfrak{t}|$ to the decorations and edge types of any decorated tree τ, and set
$$|\tau|:=|\mathfrak{n}|+|\mathfrak{o}|-|\mathfrak{e}|+|\mathfrak{t}|.$$
A noise-type object Θ is represented by $I_0^{\mathfrak{t}}(\tau)$, with $\mathfrak{t}$ of negative homogeneity.

With each subcritical singular stochastic PDE is associated a notion of conforming and strongly conforming decorated tree. The basis of $\mathcal{B}_{\mathrm{BHZ}}$ is made up of the set of elements of $\boldsymbol{\mathcal{B}}$ that strongly conforms with non-positive $\mathfrak{o}$-decorations, and one can identify T_{BHZ}^+ with a quotient of the algebra generated by the set of conforming trees with non-positive $\mathfrak{o}$-decorations by an equivalence relation. We do not need more details here and refer the interested reader to Section 5 of [6]. We do not describe in particular the details of the definition of the splitting maps Δ_{BHZ} and Δ_{BHZ}^+; we only record the following fact, where we write $\mathbf{1}$ for X^0, and X_i for X^{e_i}.

Proposition 19. [6, Proposition 4.17] *The coproduct $\Delta=\Delta_{\mathrm{BHZ}}:T_{\mathrm{BHZ}}\to T_{\mathrm{BHZ}}\otimes T_{\mathrm{BHZ}}^+$, satisfies the following identities*
$$\Delta\mathbf{1}=\mathbf{1}\otimes\mathbf{1},\quad \Delta X_i=X_i\otimes\mathbf{1}+\mathbf{1}\otimes X_i,\quad \Delta(\tau\sigma)=(\Delta\tau)(\Delta\sigma),$$
$$\Delta I_k^{\mathfrak{t}}(\tau)=(I_k^{\mathfrak{t}}\otimes\mathrm{Id})\Delta\tau+\sum_{|\ell|+|k|<|\tau|+|\mathfrak{t}|}\frac{X^\ell}{\ell!}\otimes I_{k+\ell}^{\mathfrak{t}}(\tau),\quad \Delta R_\alpha(\tau)=(R_\alpha\otimes\mathrm{Id})\Delta\tau.$$

The canonical bases $\mathcal{B}_{\mathrm{BHZ}}$ of BHZ concrete regularity structures do not satisfy assumption **(D)** since one has
$$\Delta I_0^{\mathfrak{t}}(X_i\Theta)=I_0^{\mathfrak{t}}(X_i\Theta)\otimes\mathbf{1}+I_0^{\mathfrak{t}}(\Theta)\otimes X_i+\sum_{|k|<|\Theta|+1+|\mathfrak{t}|}\frac{X^k}{k!}\otimes I_k^{\mathfrak{t}}(X_i\Theta),$$
for any edge type $\mathfrak{t}$ with positive homogeneity, but the second term in the right hand side contradicts to assumption **(D)**. Set
$$\boldsymbol{T}:=\mathrm{span}(\boldsymbol{\mathcal{B}}).$$
The tree product $(\tau,\sigma)\mapsto\tau\sigma$ and the operators $I_k^{\mathfrak{t}}$ and R_α are linearly extended to $\boldsymbol{T}$. For any $\mathfrak{t}\in\mathfrak{L}$ and $k,\ell\in\mathbb{N}^d$, we define the new operator ${}_\ell I_k^{\mathfrak{t}}:\boldsymbol{T}\to\boldsymbol{T}$, by
$${}_\ell I_k^{\mathfrak{t}}(\tau):=\sum_{m\in\mathbb{N}^d}\binom{\ell}{m}X^m(-1)^{\ell-m}I_k^{\mathfrak{t}}\big(X^{\ell-m}\tau\big).$$
(An operator ${}_\ell I_k$ represents the convolution with a kernel $x^\ell(\partial^k K)(x)$.) If τ is homogeneous, then ${}_\ell I_k^{\mathfrak{t}}(\tau)$ is also homogeneous and
$$\big|{}_\ell I_k^{\mathfrak{t}}(\tau)\big|=|\mathfrak{t}|-|k|+|\ell|+|\tau|.$$

Lemma 20. *Consider the subset $\widetilde{\boldsymbol{\mathcal{B}}}_\bullet\subset\boldsymbol{T}$ generated by the following rules.*

- $\mathbf{1} \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$.
- $\tau \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet \Rightarrow {}_\ell I_k^{\mathfrak{t}}(\tau) \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$.
- $\tau \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet \Rightarrow R_\alpha(\tau) \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$.
- $\tau, \sigma \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet \Rightarrow \tau\sigma \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$.

Set

$$\widetilde{\boldsymbol{\mathcal{B}}} := \Big\{ X^k \tau ; k \in \mathbb{N}^d, \tau \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet \Big\}.$$

Then $\widetilde{\boldsymbol{\mathcal{B}}}$ is a linear basis of $\boldsymbol{T}$, and there exists a basis $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_{\mathrm{BHZ}}$ of T_{BHZ} such that $\widetilde{\mathcal{B}} \subset \widetilde{\boldsymbol{\mathcal{B}}}$.

Proof – Assume that $\tau \in \boldsymbol{\mathcal{B}}$ is expanded by the basis $\widetilde{\boldsymbol{\mathcal{B}}}$, that is, τ is of the form

$$\tau = \sum_i a_i X^{k_i} \sigma_i$$

with $a_i \in \mathbb{R}$, $k_i \in \mathbb{N}^d$, and $\sigma_i \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$. Since the commutative property $R_\alpha(X^k \cdot) = X^k R_\alpha(\cdot)$ holds by the definition, $R_\alpha(\tau)$ is also expanded by $\widetilde{\boldsymbol{\mathcal{B}}}$. By the inversion formula

$$I_k^{\mathfrak{t}}\big(X^\ell \sigma\big) = \sum_{m \in \mathbb{N}^d} \binom{\ell}{m} X^m (-1)^{\ell - m} {}_{\ell - m} I_k^{\mathfrak{t}}(\sigma),$$

$I_k^{\mathfrak{t}}(\tau)$ is also expanded by $\widetilde{\boldsymbol{\mathcal{B}}}$. Certainly, if $\tau, \sigma \in \mathrm{span}(\widetilde{\boldsymbol{\mathcal{B}}})$, then $\tau\sigma \in \mathrm{span}(\widetilde{\boldsymbol{\mathcal{B}}})$. We can conclude that $\boldsymbol{T} = \mathrm{span}(\widetilde{\boldsymbol{\mathcal{B}}})$ by the induction on the number of edges on τ.
As in the definition of $\mathcal{B}_{\mathrm{BHZ}}$ from $\boldsymbol{\mathcal{B}}$, one obtains $\widetilde{\mathcal{B}}$ by keeping only those elements from $\widetilde{\boldsymbol{\mathcal{B}}}$ that strongly conform with non-positive $\mathfrak{o}$-decorations. ▷

The set $\widetilde{\boldsymbol{\mathcal{B}}}$ can be encoded as a set of rooted decorated trees using different decorations from the preceding decorations. Each $\tau \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$ is represented by a rooted tree with $\mathfrak{o}$ and $\mathfrak{e}$ decorations, together with a new decoration

$$\mathfrak{f} : E_\tau \to \mathbb{N}^d.$$

The map ${}_\ell I_k^{\mathfrak{t}} : \widetilde{\boldsymbol{\mathcal{B}}}_\bullet \to \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$, is defined as follows. For any $\tau \in \widetilde{\boldsymbol{\mathcal{B}}}_\bullet$ with root ϱ, the tree ${}_\ell I_k^{\mathfrak{t}}(\tau)$ is obtained by adding to τ one node ϱ' and one edge $e := (\varrho, \varrho')$, with decorations $\mathfrak{e}(e) = k$ and $\mathfrak{f}(e) = \ell$. Each $\tau = X^k \sigma \in \widetilde{\boldsymbol{\mathcal{B}}}$ is represented by a rooted tree with decorations $\mathfrak{n}, \mathfrak{o}, \mathfrak{e}, \mathfrak{f}$, where $\mathfrak{n}$ vanishes at any node except the root, where it is equal to k. We call this tree representation of elements of $\widetilde{\boldsymbol{\mathcal{B}}}$ the *non-canonical representation.*

Theorem 21. *The basis $\widetilde{\mathcal{B}}$ of T_{BHZ} satisfies assumption* **(D)**, *where $\widetilde{\mathcal{B}}_\bullet = \widetilde{\boldsymbol{\mathcal{B}}}_\bullet \cap \widetilde{\mathcal{B}}$.*

Proof – The proof is done by the induction on the number of edges on τ in its non-canonical representation. In fact, one can conclude a stronger claim; for any $\tau \in \widetilde{\mathcal{B}}_\bullet$, one has

$$\Delta \tau = \sum_{\sigma \in \widetilde{\mathcal{B}}_\bullet, \eta \notin X^k} c_{\sigma\eta}^{\tau} \sigma \otimes \eta. \tag{4.14}$$

It is sufficient to show that, if the coproduct of $\tau \in \widetilde{\mathcal{B}}_\bullet$ has such a form, then ${}_\ell I_k^{\mathfrak{t}}(\tau)$ also satisfies the same condition. To complete the proof, we compute explicitly the coproduct $\Delta({}_\ell I_k^{\mathfrak{t}}(\tau))$. Since

$$\begin{aligned}
\Delta I_k^{\mathfrak{t}}\big(X^a \tau\big) &= (I_k^{\mathfrak{t}} \otimes \mathrm{Id}) \Delta\big(X^a \tau\big) + \sum_\ell \frac{X^\ell}{\ell!} \otimes I_{k+\ell}^{\mathfrak{t}}\big(X^a \tau\big) \\
&= \sum_{\sigma \leqslant \tau,\, b \in \mathbb{N}^d} \binom{a}{b} I_k^{\mathfrak{t}}\big(X^b \sigma\big) \otimes X^{a-b} (\tau / \sigma) + \sum_{\ell \in \mathbb{N}^d} \frac{X^\ell}{\ell!} \otimes I_{k+\ell}^{\mathfrak{t}}\big(X^a \tau\big),
\end{aligned}$$

we have

$$
\begin{aligned}
\Delta\big({}_aI_k^{\mathrm{t}}(\tau)\big) &= \sum_{b\in\mathbb{N}^d}\binom{a}{b}\big(\Delta X^b\big)(-1)^{a-b}\Delta I_k\big(X^{a-b}\tau\big)\\
&= \sum_{\sigma\leqslant\tau,\, b,c,d\in\mathbb{N}^d}(-1)^{a-b}\binom{a}{b}\binom{b}{c}\binom{a-b}{d}X^cI_k^{\mathrm{t}}\big(X^d\sigma\big)\otimes X^{b-c}X^{a-b-d}(\tau/\sigma)\\
&\quad+\sum_{\ell,b,c\in\mathbb{N}^d}(-1)^{a-b}\binom{a}{b}\binom{b}{c}X^c\frac{X^\ell}{\ell!}\otimes X^{b-c}I_{k+\ell}^{\mathrm{t}}\big(X^{a-b}\tau\big)\\
&=: \text{(i)} + \text{(ii)}.
\end{aligned}
$$

The term (ii) does not contain any terms of the form $\sigma\otimes X^k$ with $k\neq 0$. The sum (i) is equal to

$$
\sum_{\substack{\sigma\leqslant\tau\\ a=c+c'+d+d'}}(-1)^{d+d'}\frac{a!}{c!c'!d!d'!}X^cI_k^{\mathrm{t}}\big(X^d\sigma\big)\otimes X^{c'}X^{d'}(\tau/\sigma)
$$

$$
\begin{aligned}
&= \sum_{\substack{\sigma\leqslant\tau\\ a=\alpha+\beta}}\frac{a!}{\alpha!\beta!}\left(\sum_{\alpha=c+d}(-1)^d\frac{\alpha!}{c!d!}X^cI_k^{\mathrm{t}}\big(X^d\sigma\big)\right)\otimes\left(\sum_{\beta=c'+d'}(-1)^{d'}\frac{\beta!}{c'!d'!}X^{c'}X^{d'}(\tau/\sigma)\right)\\
&= \sum_{\substack{\sigma\leqslant\tau\\ a=\alpha+\beta}}\binom{a}{\alpha}{}_\alpha I_k^{\mathrm{t}}(\sigma)\otimes(X-X)^\beta(\tau/\sigma)\\
&= \sum_{\sigma\leqslant\tau}{}_aI_k^{\mathrm{t}}(\sigma)\otimes(\tau/\sigma) = \big({}_aI_k^{\mathrm{t}}\otimes\mathrm{Id}\big)\Delta\tau.
\end{aligned}
$$

Since τ is assumed in the induction step to have a coproduct (4.14), hence $\Delta({}_\ell I_k^{\mathrm{t}}(\tau))$, enjoys the same property. $\rhd$

A – Concrete regularity structures

We recall in this appendix the setting of concrete regularity structures introduced in [4], and refer the reader to Section 2 of that work for motivations for the introduction of that setting.

Definition – *A* **concrete regularity structure** $\mathscr{T} = (T^+, T)$ *is the pair of graded vector spaces*

$$
T^+ =: \bigoplus_{\alpha\in A^+} T_\alpha^+, \qquad T =: \bigoplus_{\beta\in A} T_\beta
$$

such that the following holds.

- *The index set $A^+\subset\mathbb{R}_+$ contains the point 0, and $A^+ + A^+\subset A^+$; the index set $A\subset\mathbb{R}$ is bounded below, and both $A^+$and A have no accumulation points in $\mathbb{R}$. Set*

$$
\beta_0 := \min A.
$$

- *The vector spaces T_α^+ and T_β are finite dimensional.*
- *The set T^+ is an algebra with unit $\mathbf{1}$, with a Hopf structure with coproduct*

$$
\Delta^+ : T^+\to T^+\otimes T^+,
$$

such that $\Delta^+\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$, and, for $\tau\in T_\alpha^+$,

$$
\Delta^+\tau\in\left\{\tau\otimes\mathbf{1}+\mathbf{1}\otimes\tau+\sum_{0<\beta<\alpha}T_\beta^+\otimes T_{\alpha-\beta}^+\right\}, \tag{A.1}
$$

- *One has* $T_0^+ = \mathrm{span}(\mathbf{1})$, *and for any* $\alpha, \beta \in A^+$, *one has* $T_\alpha^+ T_\beta^+ \subset T_{\alpha+\beta}^+$.
- *One has a splitting map*
$$\Delta : T \to T \otimes T^+,$$
of the form
$$\Delta\tau \in \left\{ \tau \otimes \mathbf{1} + \sum_{\beta<\alpha} T_\beta \otimes T_{\alpha-\beta}^+ \right\} \tag{A.2}$$
for each $\tau \in T_\alpha$, *with the right comodule property*
$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta^+)\Delta. \tag{A.3}$$

Let $\mathcal{B}_\alpha^+$ *and* $\mathcal{B}_\beta$ *be bases of* T_α^+ *and* T_β, *respectively. We assume* $\mathcal{B}_0^+ = \{\mathbf{1}\}$. *Set*
$$\mathcal{B}^+ := \bigcup_{\alpha \in A^+} \mathcal{B}_\alpha^+, \quad \mathcal{B} := \bigcup_{\beta \in A} \mathcal{B}_\beta.$$
An element τ *of* $T_\alpha^{(+)}$ *is said to be* homogeneous *and is assigned* **homogeneity** $|\tau| := \alpha$. *The homogeneity of a generic element* $\tau \in T^{(+)}$ *is defined as* $|\tau| := \max\{\alpha\}$, *such that* τ *has a non-null component in* $T_\alpha^{(+)}$. *We denote by*
$$\mathscr{T} := \big((T^+, \Delta^+), (T, \Delta) \big)$$
a concrete regularity structure.

One of the elementary and important examples is the Taylor polynomial ring. Consider symbols $X_1, \dots, X_d$ and set
$$T_X := \mathbb{R}[X_1, \dots, X_d].$$
For a multi index $k = (k_i)_{i=1}^d \in \mathbb{N}^d$, we use the notation
$$X^k := X_1^{k_1} \cdots X_d^{k_d}.$$
We define the homogeneity $|X^k| = |k| := \sum_i k_i$, and the coproduct
$$\Delta X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i. \tag{A.4}$$
Then $\big((T_X, \Delta), (T_X, \Delta)\big)$ is a concrete regularity structure.

The set G^+ of nonzero characters $g : T^+ \mapsto \mathbb{R}$, forms a group with the convolution product
$$g_1 * g_2 := (g_1 \otimes g_2)\Delta^+.$$

B – Technical estimates

We provide in this appendix a number of technical estimates that are variations on the corresponding results from [4]. Proofs are given for completeness.

Lemma 22. *If* $\alpha \geqslant 0$ *and* $a \in \mathbb{Z}$, *then*
$$\int |P_i(x-y)||x-y|^\alpha |y|_*^{-a} dy \lesssim 2^{-i\alpha} |x|_*^{-a},$$
$$\int |Q_i(x-y)||x-y|^\alpha |y|_*^{-a} dy \lesssim 2^{-i\alpha} |x|_*^{-a}.$$

Proof – If $a \geqslant 0$,
$$|x|_*^a \int |P_i(x-y)||x-y|^\alpha |y|_*^{-a} dy \lesssim \int |P_i(x-y)||x-y|^\alpha |x-y|_*^a dy$$

$$= \int |P_i(y)||y|^\alpha |y|_*^a dy = \int |P_0(y)| \left|\frac{y}{2^i}\right|^\alpha \left|\frac{y}{2^i}\right|_*^a dy$$
$$\leqslant 2^{-i\alpha} \int |P_0(y)||y|^\alpha |y|_*^a dy \lesssim 2^{-i\alpha}.$$

If $a < 0$,

$$\int |P_i(x-y)||x-y|^\alpha |y|_*^{-a} dy \lesssim |x|_*^{-a} \int |P_i(x-y)||x-y|^\alpha |x-y|_*^{-a} dy$$
$$\lesssim 2^{-i\alpha}|x|_*^{-a}.$$

▷

Recall the two-parameter extension of the paraproduct, used in [4]. For any distribution Λ on $\mathbb{R}^d \times \mathbb{R}^d$, we define

$$(\mathbf{Q}_j\Lambda)(x) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} P_j(x-y)Q_j(x-z)\Lambda(y,z)dydz,$$
$$(\mathbf{P}\Lambda)(x) := \sum_{j \geqslant 1} (\mathbf{Q}_j\Lambda)(x).$$

If $\Lambda(y,z)$ is of the form $f(y)g(z)$, then $\mathbf{P}\Lambda = \mathsf{P}_f g$.

Proposition 23. [4, Proposition 8 (a)] *Let $a \in \mathbb{N}$.*

(a) *For any $\Lambda \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\|\mathbf{Q}_j\Lambda\|_{L_a^\infty(\mathbb{R}^d)} \lesssim 2^{-j\alpha}$ for all $j \geqslant 1$ and some $\alpha \in \mathbb{R}$, one has $\mathbf{P}\Lambda \in C_a^\alpha(\mathbb{R}^d)$ and*

$$\|\mathbf{P}\Lambda\|_{C_a^\alpha(\mathbb{R}^d)} \lesssim \sup_{j \geqslant 1} 2^{j\alpha} \|\mathbf{Q}_j\Lambda\|_{L_a^\infty(\mathbb{R}^d)}.$$

(b) *For any $\alpha > 0$ and $F \in \mathcal{C}_a^\alpha(\mathbb{R}^d \times \mathbb{R}^d)$, one has $\mathbf{P}F \in C_a^\alpha(\mathbb{R}^d)$ and*

$$\|\mathbf{P}F\|_{\mathcal{C}_a^\alpha(\mathbb{R}^d)} \lesssim \|F\|_{\mathcal{C}_a^\alpha(\mathbb{R}^d \times \mathbb{R}^d)}.$$

Proof – For (2), it is sufficient to show that $\|\mathbf{Q}_jF\|_{L_a^\infty(\mathbb{R}^d)} \lesssim 2^{-j\alpha}$. By Lemma 22,

$$|\mathbf{Q}_jF(x)| \lesssim \iint |P_j(x-y)Q_j(x-z)|(|y|_*^{-a} + |z|_*^{-a})|y-z|^\alpha dydz$$
$$\lesssim \iint |P_j(x-y)Q_j(x-z)|(|y|_*^{-a} + |z|_*^{-a})(|x-y|^\alpha + |x-z|^\alpha)dydz$$
$$\lesssim 2^{-j\alpha}|x|_*^{-a}.$$

▷

Recall from [3] the definition of the operator

$$\mathsf{R}^\circ(f,g,h) := \mathsf{P}_f\mathsf{P}_g h - \mathsf{P}_{fg}h.$$

This operator is continuous from $C^\alpha(\mathbb{R}^d) \times C^\beta(\mathbb{R}^d) \times C^\gamma(\mathbb{R}^d)$ into $C^{\alpha+\beta+\gamma}(\mathbb{R}^d)$, for any $\alpha, \beta \in [0,1]$ and $\gamma \in \mathbb{R}$ – see Proposition 14 therein.

Proposition 24. [4, Proposition 10] *Consider a function $f \in L_{\mathrm{poly}}^\infty(\mathbb{R}^d)$ and a finite family $(a_k, b_k)_{1 \leqslant k \leqslant N}$ in $L_{\mathrm{poly}}^\infty(\mathbb{R}^d) \times L_{\mathrm{poly}}^\infty(\mathbb{R}^d)$ such that*

$$f(y) - f(x) = \sum_{k=1}^N a_k(x)\big(b_k(y) - b_k(x)\big) + f_{yx}^\sharp, \quad x, y \in \mathbb{R}^d,$$

with a remainder $f_{yx}^\sharp$. Let $\alpha > 0$, $\beta \in \mathbb{R}$, and $a \in \mathbb{N}$. Assume that either of the following assumptions hold.

(a) *$f \in L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$, $a_kb_k \in L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$, $f^\sharp \in \mathcal{C}_{\mathrm{rap}}^\alpha(\mathbb{R}^d \times \mathbb{R}^d)$, and $g \in C_{\mathrm{poly}}^\beta(\mathbb{R}^d)$.*

(b) $f^\sharp \in \mathcal{C}^\alpha(\mathbb{R}^d \times \mathbb{R}^d)$ *and* $g \in C^\beta_{\mathrm{rap}}(\mathbb{R}^d)$.

Then one has the estimate

$$\sum_{k=1}^{N} \mathsf{R}^\circ\big(a_k, b_k, g\big) \in C^{\alpha+\beta}_{\mathrm{rap}}(\mathbb{R}^d).$$

Proof – As in the proof of Proposition 10 in [4], recall that

$$\sum_k \mathsf{R}^\circ(a_k, b_k, g) = -\mathscr{S}(\mathsf{P}_f g) + \mathsf{P}_f(\mathscr{S}g) - \sum_k \mathsf{P}_{a_k b_k}(\mathscr{S}g) - \mathbf{P}_{x,y}\Big(\big(\mathsf{P}_{f^\sharp_{\cdot x}} g\big)(y)\Big).$$

The first three terms belong to $C^\infty_{\mathrm{rap}}(\mathbb{R}^d)$, assuming either *(a)* or *(b)*. Consider the last term. Note that

$$\mathbf{Q}_j\Big(\big(\mathsf{P}_{f^\sharp_{\cdot x}} g\big)(y)\Big)(z) = \sum_{|i-j|\leqslant 4} \int P_j(z-x) Q_j(z-y)(S_i f^\sharp_{\cdot x})(y)(\Delta_i g)(y)dxdy.$$

For the case *(a)*, there exists $b \in \mathbb{N}$ such that $|\Delta_i g(y)| \lesssim 2^{-i\beta}|y|^b_*$. Since $f^\sharp \in \mathcal{C}^\alpha_{a+b}(\mathbb{R}^d \times \mathbb{R}^d)$, for any $a \in \mathbb{N}$ one has

$$\begin{aligned}
\int |P_i(y-u)||f^\sharp_{ux}|du &\lesssim \int |P_i(y-u)||u-x|^\alpha\big(|u|^{-a-b}_* + |x|^{-a-b}_*\big)\, du \\
&\lesssim \int |P_i(y-u)|\big(|u-y|^\alpha + |y-x|^\alpha\big)\big(|u|^{-a-b}_* + |x|^{-a-b}_*\big)\, du \\
&\lesssim \big(|x|^{-a}_* + |y|^{-a}_*\big)\big(2^{-i\alpha} + |y-x|^\alpha\big)
\end{aligned}$$

by Lemma 22. Hence we have

$$\begin{aligned}
&\Big|\mathbf{Q}_j\big((\mathsf{P}_{f^\sharp_{\cdot x}} g)(y)\big)(z)\Big| \\
&\lesssim \sum_{|i-j|\leqslant 4} \int |P_j(z-x)||Q_j(z-y)||\big(S_i f^\sharp_{\cdot x}\big)(y)||(\Delta_i g)(y)|dxdy \\
&\lesssim \sum_{|i-j|\leqslant 4} \int |P_j(z-x)||Q_j(z-y)|\big(|x|^{-a-b}_* + |y|^{-a-b}_*\big)|y|^b_*\big(2^{-i\alpha} + |y-x|^\alpha\big)\, 2^{-i\beta}\, dxdy \\
&\lesssim \sum_{|i-j|\leqslant 4} \int |P_j(z-x)||Q_j(z-y)|\big(|x|^{-a-b}_*|y|^b_* + |y|^{-a}_*\big)\big(2^{-i\alpha} + |z-x|^\alpha + |z-y|^\alpha\big)\, 2^{-i\beta}\, dxdy \\
&\lesssim \sum_{|i-j|\leqslant 4} |z|^{-a}_*\big(2^{-i\alpha} + 2^{-j\alpha}\big)\, 2^{-i\beta} \lesssim |z|^{-a}_* 2^{-j(\alpha+\beta)}.
\end{aligned}$$

For the case *(b)*, since $|\Delta_i g(y)| \lesssim 2^{-i\beta}|y|^{-a}_*$ for any $a \in \mathbb{N}$ and

$$\int |P_i(y-u)||f^\sharp_{ux}|du \lesssim \int |P_i(y-u)||u-x|^\alpha du \lesssim 2^{-i\alpha} + |y-x|^\alpha,$$

we have

$$\begin{aligned}
&\Big|\mathbf{Q}_j\big((\mathsf{P}_{f^\sharp_{\cdot x}} g)(y)\big)(z)\Big| \\
&\lesssim \sum_{|i-j|\leqslant 4} \int |P_j(z-x)||Q_j(z-y)||(S_i f^\sharp_{\cdot x})(y)||(\Delta_i g)(y)|\, dxdy \\
&\lesssim \sum_{|i-j|\leqslant 4} \int |P_j(z-x)||Q_j(z-y)||y|^{-a}_*\big(2^{-i\alpha} + |y-x|^\alpha\big)\, 2^{-i\beta}\, dxdy
\end{aligned}$$

$$\lesssim \sum_{|i-j|\leqslant 4} \int |P_j(z-x)||Q_j(z-y)||y|_*^{-a}\big(2^{-i\alpha}+|z-x|^\alpha+|z-y|^\alpha\big)\,2^{-i\beta}\,dxdy$$
$$\lesssim \sum_{|i-j|\leqslant 4} |z|_*^{-a}\big(2^{-i\alpha}+2^{-j\alpha}\big)2^{-i\beta} \lesssim |z|_*^{-a}2^{-j(\alpha+\beta)}.$$

By Proposition 23, we are done. ▷

Proposition 25. [4, Proposition 9] *Let $\gamma \in \mathbb{R}$ and $\beta_0 \in \mathbb{R}$ be given together with a family Λ_x of distributions on $\mathbb{R}^d$, indexed by $x \in \mathbb{R}^d$. Assume one has*

$$\sup_{x\in\mathbb{R}^d} |x|_*^a \|\Lambda_x\|_{C^{\beta_0}} < \infty$$

for any $a > 0$ and one can decompose $(\Lambda_y - \Lambda_x)$ under the form

$$\Lambda_y - \Lambda_x = \sum_{\ell=1}^{L} c_{yx}^\ell\, \Theta_x^\ell$$

for L finite, $\mathbb{R}^d$-indexed distributions Θ_x^ℓ, and real-valued coefficients c_{yx}^ℓ depending measurably on x and y. Assume that for each ℓ there exists $\beta_\ell < \gamma$ such that either of the following conditions holds.

(a) $\Theta^\ell \in D_{\mathrm{rap}}^{\beta_\ell}$ and $c^\ell \in \mathcal{C}^{\gamma-\beta_\ell}(\mathbb{R}^d \times \mathbb{R}^d)$.
(b) $\Theta^\ell \in D^{\beta_\ell}$ and $c^\ell \in \mathcal{C}_{\mathrm{rap}}^{\gamma-\beta_\ell}(\mathbb{R}^d \times \mathbb{R}^d)$.

Moreover, assume that one can decompose $(\Theta_x^\ell - \Theta_z^\ell)$ again under the form

$$\Theta_x^\ell - \Theta_z^\ell = \sum_{m=1}^{M^\ell} d_{xz}^{\ell m}\Omega_z^{\ell m}$$

for M^ℓ finite, $\mathbb{R}^d$-indexed distributions $\Omega_z^{\ell m}$, and a real-valued coefficients $d_{xz}^{\ell m}$ depending measurably on x and z. Assume that for each ℓ there exists $\beta_{\ell m} < \beta_\ell$ such that any one of the following conditions holds.

(c) Under (a), one has $\Omega^{\ell m} \in D_{\mathrm{rap}}^{\beta_{\ell m}}$ and $d^{\ell m} \in \mathcal{C}^{\beta_\ell-\beta_{\ell m}}(\mathbb{R}^d \times \mathbb{R}^d)$.
(d) Under (a), one has $\Omega^{\ell m} \in D^{\beta_{\ell m}}$ and $d^{\ell m} \in \mathcal{C}_{\mathrm{rap}}^{\beta_\ell-\beta_{\ell m}}(\mathbb{R}^d \times \mathbb{R}^d)$.
(e) Under (b), one has $\Omega^{\ell m} \in D^{\beta_{\ell m}}$ and $d^{\ell m} \in \mathcal{C}^{\beta_\ell-\beta_{\ell m}}(\mathbb{R}^d \times \mathbb{R}^d)$.

Write $\mathbf{P}(\Lambda)$ for $\mathbf{P}_{y,z}\big(\Lambda_y(z)\big)$ below.

- *If $\gamma > 0$, then there exists a unique function $\lambda \in C_a^\gamma(\mathbb{R}^d)$ such that*
$$\Big\{\big(\mathbf{P}(\Lambda)+\lambda\big) - \Lambda_x\Big\}_{x\in\mathbb{R}^d} \in D_{\mathrm{rap}}^\gamma.$$
- *If $\gamma < 0$, then*
$$\big\{\mathbf{P}(\Lambda) - \Lambda_x\big\}_{x\in\mathbb{R}^d} \in D_{\mathrm{rap}}^\gamma.$$

Consequently, $\mathbf{P}(\Lambda) \in C_{\mathrm{rap}}^{\beta_0}(\mathbb{R}^d)$. If furthermore $\Lambda \in D_{\mathrm{rap}}^\gamma$, then $\mathbf{P}(\Lambda) \in C_{\mathrm{rap}}^\gamma(\mathbb{R}^d)$.

Proof – In view of [4, Proposition 9], it is sufficient to show that

$$\sup_{x\in\mathbb{R}^d} |x|_*^a\big|\Delta_j\big(\mathbf{P}(\Lambda)-\Lambda_x\big)(x)\big| \lesssim 2^{-j\gamma} \tag{B.1}$$

We write for that purpose

$$\mathbf{P}(\Lambda)(y) - \Lambda_x(y) = \sum_{j\geqslant -1}\sum_{\ell=1}^{L} \iint P_j(y-u)Q_j(y-v)c_{ux}^\ell\Theta_x^\ell(v)dudv - \mathcal{S}(\Lambda_x).$$

For the second term,

$$\sup_x |x|_*^a \big|\Delta_j \mathcal{S}(\Lambda_x)(x)\big| \lesssim 2^{-jr} \sup_x |x|_*^a \big\|\mathcal{S}(\Lambda_x)\big\|_{C^r}$$
$$\lesssim 2^{-jr} \sup_x |x|_*^a \|\Lambda_x\|_{C^{\beta_0}} \lesssim 2^{-jr}$$

for any $r > 0$. Note that

$$\left|\int Q_j(y-v)\Theta_x^\ell(v)dv\right| = \sum_{m=1}^{M^\ell} \left|d_{xy}^{\ell m} \int Q_j(y-v)\Omega_y^{\ell m}(v)dv\right|$$
$$\lesssim (|x|_*^{-a} + |y|_*^{-a}) \sum_{m=1}^{M^\ell} |x-y|^{\beta_\ell - \beta_{\ell m}} 2^{-j\beta_{\ell m}}$$

for (c) and (d), or

$$\left|\int Q_j(y-v)\Theta_x^\ell(v)dv\right| \lesssim \sum_{m=1}^{M^\ell} |x-y|^{\beta_\ell - \beta_{\ell m}} 2^{-j\beta_{\ell m}}$$

for (e). Hence we can conclude (B.1) by using Lemma 22. ▷

Corollary 26. *Given a concrete regularity structure $\mathscr{T}$ satisfying assumptions* **(A)** *and* **(B)** *and given a rapidly decreasing model $\mathsf{M} = (\mathsf{g}, \Pi)$, we define the map $\mathbf{R} : \mathcal{D}^\gamma_{\mathrm{rap}}(T, \mathsf{g}) \to C^{\beta_0}_{\mathrm{rap}}$ by*

$$\mathbf{R}\boldsymbol{f} = \mathbf{P}_{x,y}\big((\Pi_x^{\mathsf{g}} \boldsymbol{f}(x))(y)\big).$$

Then one has

$$\big(\mathbf{R}\boldsymbol{f} - \Pi_x^{\mathsf{g}} \boldsymbol{f}(x)\big)_{x \in \mathbb{R}^d} \in D^\gamma_{\mathrm{rap}}.$$

Proof – Let $\Lambda_x = \Pi_x^{\mathsf{g}} \boldsymbol{f}(x)$. Since

$$\Lambda_y - \Lambda_x = \sum_{\tau \in \mathcal{B}} \langle \tau', \widehat{\mathsf{g}_{xy}} \boldsymbol{f}(y) - \boldsymbol{f}(x)\rangle \Pi_x^{\mathsf{g}} \tau$$

and

$$\Pi_x^{\mathsf{g}} \tau - \Pi_z^{\mathsf{g}} \tau = \sum_{\sigma < \tau} \mathsf{g}_{zx}(\tau/\sigma) \Pi_z^{\mathsf{g}} \sigma,$$

we can check (a)-(e) by definitions on the regularity structure $\mathscr{T}$. ▷

Proof of Theorem 7 – Consider the first formula (3.2). First we show that, for each $\tau \in \mathcal{B}^+$ we have

$$\mathsf{g}(\tau) = \sum_{\mathbf{1} <^+ \nu <^+ \tau,\, \nu \in \mathcal{B}^+} \mathsf{P}_{\mathsf{g}(\tau/^+\nu)}[\nu]^{\mathsf{g}} + [\tau]^{\mathsf{g}}, \tag{B.2}$$

where

- $[\nu]^{\mathsf{g}} \in C^{|\nu|}_{\mathrm{rap}}(\mathbb{R}^d)$, if $\nu \in \mathcal{B}^+ \backslash \mathcal{B}^+_X$,
- $[\nu]^{\mathsf{g}} \in C^\infty_{\mathrm{poly}}(\mathbb{R}^d)$, if $\nu \in \mathcal{B}^+_X$.

If $\tau = X^k$, then since $\Delta^+ X^k = \sum_\ell \binom{k}{\ell} X^\ell \otimes X^{k-\ell}$ we have

$$\mathsf{g}(X^k) = \sum_{0<\ell<k} \binom{k}{\ell} \mathsf{P}_{\mathsf{g}(X^\ell)}[X^{k-\ell}]^{\mathsf{g}} + [X^k]^{\mathsf{g}}.$$

Since $\mathsf{g}_x(X^k) = x^k$ is a function belonging to $C^\infty_{\mathrm{poly}}(\mathbb{R}^d)$, by an induction we have $[X^k]^{\mathsf{g}} \in C^\infty_{\mathrm{poly}}(\mathbb{R}^d)$. Now let $\tau \in \mathcal{B}^+ \backslash \mathcal{B}^+_X$. Recall the formula obtained in [4]

$$[\tau]^{\mathsf{g}} = \mathscr{S}\mathsf{g}(\tau) + \mathbf{P}_{x,y}\big(\mathsf{g}_{yx}(\tau)\big)$$
$$+ \sum_{n=1}^\infty (-1)^{n-1} \sum_{\mathbf{1} <^+ \sigma_{n+1} <^+ \cdots <^+ \sigma_1 <^+ \tau} \mathsf{R}^\circ\Big(\mathsf{g}(\tau/^+\sigma_1)\cdots \mathsf{g}(\sigma_{n-1}/^+\sigma_n), \mathsf{g}(\sigma_n/^+\sigma_{n+1}), [\sigma_{n+1}]^{\mathsf{g}}\Big).$$

Since $\tau \in \mathcal{B}^+\backslash\mathcal{B}_X^+$, we have $\mathscr{S}\mathsf{g}(\tau) \in C_{\mathrm{rap}}^\infty(\mathbb{R}^d)$ and $\mathbf{P}_{x,y}(\mathsf{g}_{yx}(\tau)) \in C_{\mathrm{rap}}^{|\tau|}(\mathbb{R}^d)$. For the R° terms, we apply Proposition 24. Recall the expansion formula obtained in [4];

$$\begin{aligned}&\mathsf{g}_y(\tau/^+\sigma) - \mathsf{g}_x(\tau/^+\sigma)\\ &= \sum_{n=1}^\infty (-1)^{n-1} \sum_{\sigma<^+\sigma_n<^+\cdots<^+\sigma_1<^+\tau} \mathsf{g}_x(\tau/^+\sigma_1)\cdots \mathsf{g}_x(\sigma_{n-1}/^+\sigma_n)\Big(\mathsf{g}_y(\sigma_n/^+\sigma) - \mathsf{g}_x(\sigma_n/^+\sigma)\Big)\\ &\quad + \mathsf{g}_{yx}(\tau/^+\sigma).\end{aligned}$$

If $\sigma \in \mathcal{B}_X^+$, since $\tau/^+\sigma \in \mathrm{span}(\mathcal{B}^+\backslash\mathcal{B}_X^+)$, we have $\mathsf{g}_{yx}(\tau/^+\sigma) \in \mathcal{C}_{\mathrm{rap}}^{|\tau|-|\sigma|}(\mathbb{R}^d\times\mathbb{R}^d)$. For the sum over $\sigma <^+ \sigma_n <^+ \cdots <^+ \sigma_1 <^+ \tau$, we can see that at least one element among

$$\mathsf{g}(\tau/^+\sigma_1), \quad \dots, \quad \mathsf{g}(\sigma_{n-1}/^+\sigma_n), \quad \mathsf{g}(\sigma_n/^+\sigma)$$

belongs to $L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$. Indeed, if $\sigma_n \notin \mathcal{B}_X^+$ then $\mathsf{g}(\sigma_n/^+\sigma) \in L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$. Otherwise, if $\sigma_{n-1} \notin \mathcal{B}_X^+$ then $\mathsf{g}(\sigma_{n-1}/^+\sigma_n) \in L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$. Since $\tau \notin \mathcal{B}_X^+$, for at least one i we have $\mathsf{g}(\sigma_i/^+\sigma_{i+1}) \in L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$. Since $L_{\mathrm{poly}}^\infty(\mathbb{R}^d)\cdot L_{\mathrm{rap}}^\infty(\mathbb{R}^d) \subset L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$, we can apply Proposition 24-(1) to get

$$\sum_{n=1}^\infty (-1)^{n-1} \sum_{\sigma<^+\sigma_n<^+\cdots<^+\sigma_1<^+\tau} \mathsf{R}^\circ\Big(\mathsf{g}(\tau/^+\sigma_1)\cdots\mathsf{g}(\sigma_{n-1}/^+\sigma_n), \mathsf{g}(\sigma_n/^+\sigma), [\sigma]^{\mathsf{g}}\Big) \in C_{\mathrm{rap}}^{|\tau|}(\mathbb{R}^d).$$

If $\sigma \notin \mathcal{B}_X^+$, since $[\sigma]^{\mathsf{g}} \in C_{\mathrm{rap}}^{|\sigma|}(\mathbb{R}^d)$ we can apply Proposition 24-(2) to get the same estimate. Hence we obtain the required estimates in the formula (B.2).

To get (3.2) from (B.2), it is sufficient to show

$$[\![\tau]\!]^{\mathsf{g}} - [\tau]^{\mathsf{g}} \in C_{\mathrm{rap}}^\infty(\mathbb{R}^d) \tag{B.3}$$

for any $\tau \in \mathcal{B}^+\backslash\mathcal{B}_X^+$. Assume that all $\nu \in \mathcal{B}^+\backslash\mathcal{B}_X^+$ with $|\nu| < |\tau|$ satisfy (B.3). Then we have

$$\begin{aligned}[\![\tau]\!]^{\mathsf{g}} - [\tau]^{\mathsf{g}} &= \sum_{\mathbf{1}<^+\nu<^+\tau} \mathsf{P}_{\mathsf{g}(\tau/^+\nu)}[\nu]^{\mathsf{g}} - \sum_{\mathbf{1}<^+\nu<^+\tau,\,\nu\notin\mathcal{B}_X^+} \mathsf{P}_{\mathsf{g}(\tau/^+\nu)}[\![\nu]\!]^{\mathsf{g}}\\ &= \sum_{\mathbf{1}<^+\nu<^+\tau,\,\nu\notin\mathcal{B}_X^+} \mathsf{P}_{\mathsf{g}(\tau/^+\nu)}\big([\nu]^{\mathsf{g}} - [\![\nu]\!]^{\mathsf{g}}\big) + \sum_{k\neq0} \mathsf{P}_{\mathsf{g}(\tau/^+X^k)}[X^k]^{\mathsf{g}}.\end{aligned}$$

The first term belongs to $C_{\mathrm{rap}}^\infty(\mathbb{R}^d)$ by assumption. For the second term, since $[X^k]^{\mathsf{g}} \in C_{\mathrm{poly}}^\infty(\mathbb{R}^d)$ and $\mathsf{g}(\tau/^+X^k) \in L_{\mathrm{rap}}^\infty(\mathbb{R}^d)$, we can complete the proof.

One can obtain formula (3.3) in the similar way. The only difference is that we use Proposition 25 to get $\mathbf{P}_{x,y}\big((\Pi_x\sigma)(y)\big) \in C_{\mathrm{rap}}^{|\sigma|}(\mathbb{R}^d)$, for any $\sigma \in \mathcal{B}\backslash\mathcal{B}_X$. ▷

We define here the two-parameter extension $\mathbf{P}^m$ of the modified paraproduct P^m. Note that, there is an annulus $\mathcal{A} \subset \mathbb{R}^d$ such that the Fourier transform of the function

$$x \mapsto P_j(x-y)Q_j(x-z)$$

is contained in $2^j\mathcal{A}$ (independently to y, z). Let χ be a smooth function on $\mathbb{R}^d$ supported in a larger annulus $\mathcal{A}'$ and such that $\chi \equiv 1$ on $\mathcal{A}$. Letting $R_j = \mathcal{F}^{-1}\big(\chi(2^{-j}\cdot)\big)$, we have

$$(\mathbf{Q}_j\Lambda)(x) = \iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} R_j(x-w)P_j(w-y)Q_j(w-z)\Lambda(y,z)dydzdw.$$

For $m \in \mathbb{Z}$, set

$$\begin{aligned}Q_j^{-m} &:= \mathcal{F}^{-1}\big(|\cdot|^{-m}\rho_j\big),\\ R_j^m &:= \mathcal{F}^{-1}\big(|\cdot|^m\chi(2^{-j}\cdot)\big);\end{aligned}$$

then they are smooth functions such that $Q_j^{-m} = |\nabla|^{-m}Q_j$ and $R_j^m = |\nabla|^m R_j$.

Definition 27. *For any $m \in \mathbb{N}$ and any two-variable distribution Λ on $\mathbb{R}^d \times \mathbb{R}^d$, define*

$$(\mathbf{Q}_j^m \Lambda)(x) := \iiint_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} R_j^m(x-w) P_j(w-y) Q_j^{-m}(w-z) \Lambda(y,z) dydzdw,$$

$$(\mathbf{P}^m \Lambda)(x) := \sum_{j \geqslant 1} (\mathbf{Q}_j^m \Lambda)(x).$$

If necessary, we emphasize the integrated variables by writing

$$\mathbf{P}^m \Lambda = \mathbf{P}^m_{y,z}\big(\Lambda(y,z)\big).$$

For the special case $\Lambda(y,z) = f(y)g(z)$, we have the consistency relation

$$\mathbf{P}^m \Lambda = \mathsf{P}^m_f g.$$

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