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Exploring with digital media to understand trigonometric functions through periodicity

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Trigonometry consists of a multifaceted mathematical field whose fundamental concepts, sine and cosine, have many different representations, some of them approached throughout secondary education. Unfortunately, many of its aspects are taught individually, leading students to make isolated, unconnected meanings on them. This empirical study discusses an alternative view towards trigonometry, in an attempt to create connections among the different aspects, under the scope of one meaningful context; that of periodicity. The use of digital media enables the application of this unconventional proposal, through a set of specially designed activities. In this paper we present a brief description of the main study-in-progress, as well as the results gained by its first implementation to 9th- Grade students, in terms of their meaning making on trigonometric concepts.

Keywords: learning trigonometry, periodicity, digital tools, meaning making

Periodicity for meaning making in Trigonometry

Research on teaching and learning trigonometry has been given surprisingly little attention in relation to, say, algebra or calculus. Studies regarding this field show that students develop weak and narrow understandings on the fundamental concepts of sine and cosine. They also develop a fragmented, disconnected view of the various related representations, such as the triangle model, the unit circle model and Cartesian graphs (Weber, 2008; Gür, 2009; Moore, 2010; Demir & Heck, 2013). The problem is not unrelated to epistemological debate on the nature and functionality of trigonometry in mathematics. Newson and Randolph (1946) argue that the problem has its origins in the predominant definition of sine and cosine in terms of angles, something which they perceived as related to a narrow application of trigonometry. They compare this to the attempt to define arithmetic as the “science of money”, as it is based on one of its limited applications. They instead proposed defining trigonometry as the mathematical science concerned with the trigonometric functions, whose arguments may denote time, or any other magnitude, or just a real number without any connotation. Hirsch, Winhold and Nichols (1991) characterize traditional trigonometry instruction as “memorization of isolated facts and procedures” that is unable to support a robust understanding of the subject. They also suggest a shift in emphasis towards trigonometric functions themselves and their applications at modeling periodic phenomena. Weber (2008) stresses two additional important obstacles that students deal with when learning trigonometric functions: they are initially familiar to sine and cosine as algorithms of ratios within the right-triangle context rather than as procedures regarding any given angle. The right-triangle model consists of a restriction on perceiving any other representation of trigonometric functions, as no links are ever made to this initial approach. Further, trigonometric functions are typically among the first functions that students cannot evaluate directly by performing arithmetic operations; that makes them even more complicated in order to be deeply understood. The literature focus is thus two-fold. Firstly, on posing

the problems that reside in learning trigonometry as currently portrayed from an epistemology perspective mainly emphasizing the right-triangle sine and cosine constructs (Blackett & Tall, 1991; Breidenbach et al., 1992). Secondly on fostering one representation of trigonometric functions as more important over another, while proposing a certain type of exercises (Kendal and Stacey, 1997; Weber, 2008). A fresh alternative look towards this field is presented by Demir and Heck (2013) and focuses on promoting integrated understanding of trigonometric functions by connecting their three representations within a dynamic geometry environment. In this paper we address trigonometry as a field for generating meanings around periodic covariation placing it at the centre of pedagogical focus, in line with with Demir and Heck's idea and Newson and Randolph's epistemological arguments. We follow on from prior research on the use of digital resources for students to develop understandings of periodicity through trigonometric functions (Gavrilis and Kynigos, 2006), but expand the idea of periodicity beyond the right-triangle context.

Design considerations

In our Lab, we have a history of employing two constructs which for us have been fundamental to the generation of new ideas to exploit various affordances of expressive digital media for mathematical meaning making (Kynigos, 2015). The first is that of 'restructuring' coined by Willensky and Papert in 2010, where the designer questions the existing structure and emphasis of a mathematics curriculum allowing for a fresh look for meaning making opportunities in new structures and perspectives of mathematical ideas, given the new expressivity affordances of digital media. The second is that of 'conceptual field' (Vergnaud, 2009), where these new structures are perceived as an integral part of re-configuring sets of closely related concepts, using a diverse but connected set of meaningful representations for these and creating a set of problem situations where these play a central role for their resolution. So, for us, periodicity was a central characteristic of a special kind of mathematical function such as sine-cosine and tangent connected to diverse representations and part of physical phenomena such as tide. Adopting this perspective, we perceived the design of the tasks we prepared for students under the scope of the conceptual field of periodicity. Into this field, notions of trigonometry, geometry, algebra and physics are linked together so as to strengthen their embedded meanings. Trigonometric functions can be approached through different situations within the meaningful context of periodic phenomena, where students have the opportunity to derive their properties and their actual value. Our assumption is that in that way, we enhance the meaning-making process on these concepts and their various representations within meaningful situations (Noss and Hoyles, 1996) and provide flexible connections between them, as they all share the feature of periodic nature. We saw our perspective and design principle as uniquely enabled by special digital simulations and microworlds which we developed for students to use as media for expressing mathematical ideas. Firstly, we used an available Dynamic Geometry System, Geogebra, to create a simulation of the periodic phenomenon of tide connected to the representation of both the graph of the trigonometric functions and the one of the unit circle that model the phenomenon. Then, we developed a programmable microworld with the MaLT2 tool, which integrates programming and dynamic manipulation of variable values (Kynigos and Zantzos, 2017), to provide opportunities for students to make connections between periodicity and trigonometry of angles through a code-editing task. Our aim was to shed light on what meanings of

the trigonometric field can be produced by students, who engage with the designed tasks around periodicity.

Research setting and tasks

The tasks designed for this study reflected the ideas described above. They were separated into two phases, each of which corresponded to different but complementary intentions by the designers. The first phase was related to the phenomenon of tide (in a non-realistic yet close to students' perception way) simulated in the dynamic environment of Geogebra in order to be modeled in any way possible by students. The second one was dealing with the periodic change of the vertical lines of a right triangle in the programmable media of MalT2, requiring correlation of periodic functions with the trigonometric ones and leading to formalization.

Modeling Periodic Phenomenon

We designed a model in Geogebra that visualizes the periodic rise and fall of sea levels according to the sinusoidal function $f(t)=\sin t$, where the variable t stands for the time passing in hours (Figure 1a). This phenomenon was simulated so that the sea would cover and uncover periodically the surface of an island. It could be observed by moving the cursor of the variable t till it reached the value of 30 hours and then it would start over. The initial task students would be challenged with was to make a schedule for future most suitable time for visits to the island; ones that would last the longest in order to collect its valuable shells. The final task was the construction of a math formula that would receive a value of any future time and export the exact height of the sea levels. The available digital tools were split into two representations which were given to students during the first phase along with a worksheet directing the above exploration process. The first one involved the "trace-leaving" point $M(t,y)$ available for dragging into a Cartesian system (Figure 1b). By following the height that corresponded to every time value, starting from 0 and moving forwards, the trace would form the sinusoidal graph. The second representation was that of the unit circle. Adopting the "wrapping function" model (Podbelsek, 1972), we made a cursor that enables the wrapping around the unit circle of a segment whose starting point is the $(1,0)$ and has the length of the exact same value as the time paused (Figure 1c). When the segment was fully wrapped, the stopping point's coordinates revealed the sine and cosine of that value as well as a corresponding arc/angle. The tasks required the specification of the relationship between time and the height of sea levels in order to find a convenient way to predict the island's uncovered surface at any future time. The particularity of this phase is the fact that the words sine or trigonometry were not mentioned anywhere so as for students to proceed to formalization.

Formalization of the Trigonometric Functions

The second phase interfered in order to "fill" the gaps that the first phase left uncovered; those are the formalization that the functions responsible for the periodic change are the trigonometric ones and their possible reference to angles or any other magnitude. For this matter, we designed an artefact in the programmable environment of MalT2. The code that produced the turtle's trace for this artefact was hidden from the students, whose task was to discover it and reproduce it. The only accessible source was the product of the code: the artefact that represented a right triangle (Figure 2) and the periodic changes of its shape while manipulating the value of its unique variable through

the uni-dimensional variation tool. An additional output that students had access to was the values of the length of every triangle's side. The relations that needed to be "unmasked" were the sinusoidal and cosinusoidal covariation between the visible variable t -that corresponded to a triangle's angle- and a triangle's vertical side.

Findings

The study described above was an educational intervention designed according to the methodology of "design experiments" (Collins et al., 2004). The focus was both on the meaning-making process by students and their interaction with the digital tools. Three small groups of 9th Grade students from a public Experimental School in Athens participated in a first implementation of this study. It took place in the pc-lab of the school during after-class mathematics courses for totally six teaching hours within two weeks.

Phase 1

All three groups engaged with the tasks described above generated rich meanings on the embedded concepts. We analyze below some representative dialogues that help us perceive aspects of this meaning making process.

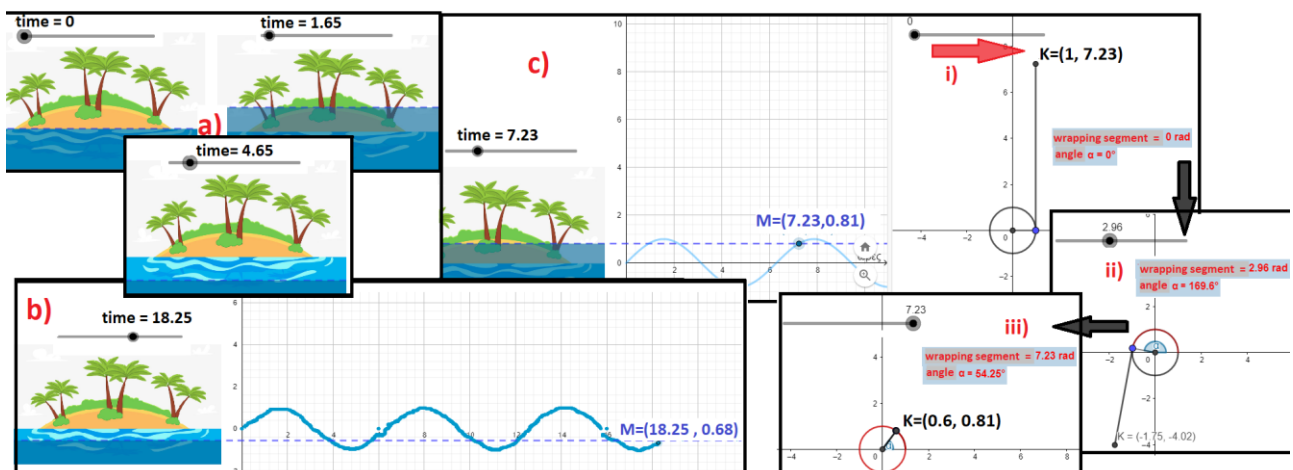


Figure 1: The phenomenon of tide and its two representations at Geogebra

At first, students studied the phenomenon of tide and the way it was affecting the represented island. While studying the simulation and the questions of the worksheet, they all followed a similar thinking flow, presented in this discussion:

Student 1: Well, we could say that the sea levels are going up and down in a steady rhythm. There is a highest and a lowest point the levels reach within a standard period of time; almost 6 hours? In order to predict it, we need to know the relation between time and height, but we don't know the function formula.

Student 2: The best time for visiting the island is when the height is at level 0 and goes downwards. This level is reached many times, maybe even infinite ones. But I can't say their exact value after the 30th hour. But I am sure there is a way to find out because there is a standard pattern. It's like a circle repeated endlessly.

In our understanding, these lines indicate a first approach of the trigonometric functions based on their periodic nature. Dragging the point M helped them express the relation between time and height until the 30th hour. That way they constructed the sinusoidal graph, which inspired them with ideas for predicting accurately the future evolving of the phenomenon:

- S1: The most suitable moments for visiting are at 3.14, 9.42 and 15.7 hours.
- S2: These numbers are odd multiples of 3.14. Therefore we can determine the most suitable for a visit future moments by computing as many multiples as we can. But I can't think of a way to determine the exact height level at any future time.

The graph provided them with the conception that the phenomenon follows a specific pattern defined by its period. What concerned them is the fact that they were unable to find a way to predict the exact correspondence between sea levels and any future time. This enlightenment came during their interaction with the second representation; that of the unit circle. By wrapping the segment - that equals with the time value- around the circle, students realized that the stopping point reveals the height of the sea levels by its ordinate. This procedure improved the way they perceived the periodic nature and led them to discover a convenient way for predicting the phenomenon at future time. They counted the times the segment was wrapped to a full circle and determined the height according to the wrapping of the remaining segment. A possible interpretation of this achievement is the fact that the circle model reminded them of a familiar concept; that of division.

- S4: We must find how many times this whole segment will be wrapping a whole circle. Let's divide it with 6.28.¹ The remainder gives the length of the segment that will determine the height.
- S5: Yeah, all we have to do is wrap the remainder value segment and see the ordinate of K where it stops. The remainder is always lower than 6.28. We can use that as a formula to predict the exact height of sea levels at any future time.

This realization came when they were asked to predict the height of the sea levels at the 300th hour since the beginning of the observation. That led them to the modeling of the phenomenon and the conquest of the wider challenge of the study. A strong link was thus established between the trigonometric function and the notions of periodicity and predictability. However the study is incomplete; no student made the connection between the trigonometric representations and trigonometry itself. As a result, the meanings they produced were rich in trigonometric features but, not surprisingly, they did not correspond to formal trigonometric labels.

Phase 2

As mentioned above, this phase aimed to establish links between periodic covariation and formalized trigonometry. Students started their exploration by manipulating the cursor and observing the changes happening to the triangle. The following parts of their conversation represent their initial thoughts and their completion:

¹ They had already pointed out that an exact circle is fully wrapped when $t=6.28$; thus its perimeter equals to 6.28.

S1: The things here that remain stable are a right angle and the length of the hypotenuse, which is always 100, no matter what value the (variable) t takes.

S2: Yes, except from 0. Oh, and 90. And 180, too. Bet it's 0 at every multiple of 90.

Students S1 and S2 were seeking for regularities in order to find the “unaffected” from the cursor’s movement magnitudes and start the code writing from these constants. Then, while trying to figure out the nature of the observing variation and decode it, many trigonometrically interesting things emerged:

S3: It’s weird because while one vertical side grows, the other one shortens. I think they grow with a steady rhythm. (...) It repeats itself after a period of time. Just like the sea levels did.

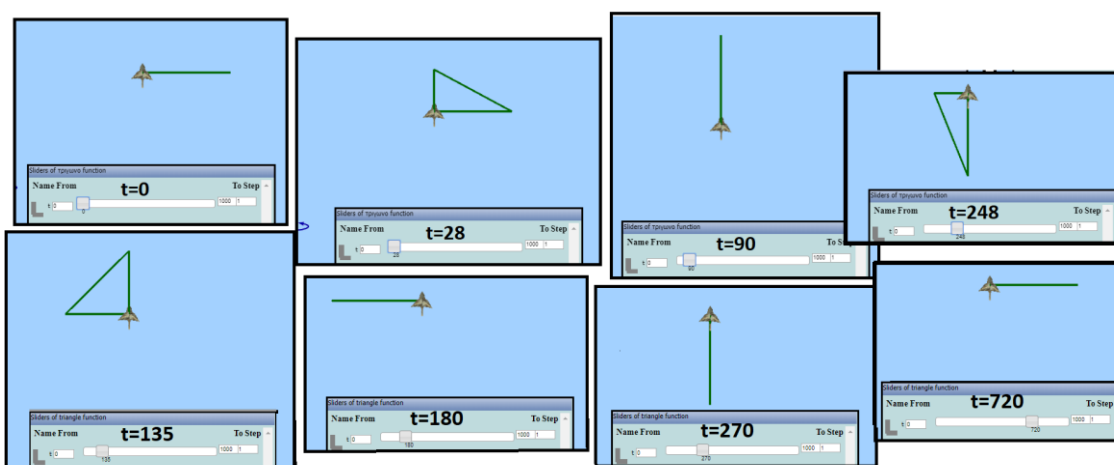


Figure 2: Manipulating the value of the variable t causes changes to the artefact

S1: In order to make a right triangle that remains right, we know that this angle is 90 degrees, so the other two must sum up to 180. Now, can we find the vertical sides based on these constant elements?

S2: Guys! What if we use the variable t as angle and not as length? That way we can find that side by the relation “sine t equals the ratio of the opposite side to the hypotenuse”. That way we find the opposite side!

S1: YEAH! Genius! We can do the same thing with the last side, but using cosine instead.

```
TO triangle :t
  fd 100*sin(:t) rt 90+:t fd 100 rt 180-:t
  fd 100*cos(:t) rt 90
END
triangle 30 |
```

Figure 3: The code producing the artefact made by the students

These students followed the right conceptual path in order to conceive the idea that led them to the interpretation of the artefact. Being familiar with the right triangle model, they easily discovered the

trigonometric relation by thinking of the variable t as an angle. Based on that realization, they managed to correspond each unknown element of the triangle to a functional relation with the known ones. Thus, considering sine and cosine as functions, they got to the discovery of the right code. (Figure 3)

Student S3 made a crucial comment on the completion of the meaning-making process; he realized the resemblance to the phenomenon of tide examined at the previous phase. After the discovery of the functional code, he completed his thought with an even more remarkable comment:

S3: Therefore sine and cosine are responsible for the growing and shortening of the segments. It depends on the angle; when an angle grows the one side, it shortens the other one with the same period. I think it's the first time I realize that.

He expressed the connection between trigonometry and periodicity, building a strong conceptual link between them. We found it quite interesting that he intuitively he connected variability in sine and cosine with segment length rather than angle. This signified the mental abstraction that trigonometric functions can be used in order to express periodic change of the value of a segment. As a result, the completion of this phase came along with the awareness that the value of trigonometric functions goes far beyond the right triangle model.

Conclusion

The students seemed to find periodicity as a familiar concept and to be able to think about functional relations and trigonometric functions to gradually see the mathematical underpinning of periodicity. The tasks allowed them to observe and to express trigonometric functions as elements representing periodicity. They first produced meanings on various trigonometric notions and afterwards they established links between them. They approached sine and cosine as functions and not as just numbers and they listed many of their properties based on their periodic nature. They finally realized that trigonometry is strongly connected with periodicity and thus it can be exploited for modeling periodic phenomena which are of great scientific importance. The role of the digital media was integral during the students' meaning making progress. They managed to interpret periodicity using trigonometric terms, thanks to their exploration with these digital tools. They provided an alternative dynamic and constructionist way to visualize, model and manipulate dynamically periodic situations. It is very difficult to represent periodicity in an alternative way open to exploration. So, this study indicates that it might be interesting to further research the potential of using digital representations of trigonometry so that some constituent elements can be understood through the fertile conceptual field of periodicity rather than the curriculum based focus on angles within a triangle. Maybe this could provide a solution to the constant problem of connectivity among trigonometric concepts and representations and, as a consequence, a promising proposal for learning trigonometry.

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