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# Issues in modelling terms involving subtraction in a manipulative environment for linear equations—and a possible solution

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*In this paper, we present and utilize theories about interface design and didactical models to evaluate how existing manipulatives models for linear equations represent terms involving subtraction. We then outline a redesign of subtraction zones for algebra tiles that can be implemented on a screen. The presented work is intended to serve as an example case for theory-guided design of digital or digitally enhanced didactical models, anticipating the all-important learning activities with the resulting models.*

*Keywords: Equations (mathematics), educational technology, instructional systems, instructional design, design requirements.*

## **Introduction: Manipulatives environments for linear equations**

To help students understand equations and how to solve them, different tangible models are in use. Each consists of material objects or manipulatives (standing for numbers and variables) and an environment that specifies the relationship between the variables and numbers represented by the manipulatives placed in the environment. Probably the best-known model is the balance scale, where known and unknown weights on both sides of a scale are supposed to be kept in balance. Another model uses two sets of objects, some hidden in containers. The two sets are stated to have equal numbers of objects and the equation is framed as a puzzle, to find the number of hidden objects. Affolter et al. (2003) have used matches in matchboxes, Radford and Grenier (1996, p. 184) cards in envelopes. In this model the equality, which had a physical parallel in the balance model, is an abstract social convention. A further abstraction is made when *algebra tiles* (or blocks) are used. Known quantities are represented by squares (or cubes) and unknowns by oblong rectangles (cuboids). They are placed on a mat divided into two sides and students are informed that the amount represented by the tiles on the left side is the same as that on the right side. This model was developed because it is applicable to manipulating quadratic expressions, but has since also been used for linear equations.

We are involved in a research project with the goal of developing an “intelligent” system of algebra tiles with the ability to give feedback to its users (Reinschlüssel et al., 2018). The algebra tiles model was chosen because it can be used as the concepts of algebra are developed further; avoiding the need to introduce a new model for manipulating quadratic expressions, etc. Furthermore, the model allows the representation of terms involving subtraction (such as “ $x - 3$ ”) differently from additive terms involving negative values (“ $x + (-3)$ ”). In the design process, however, shortcomings in the representation of subtraction in the algebra tiles environment became apparent, offering a chance to reconsider its design. The considerations guiding this redesign, specifically describing how a concrete model can sensibly represent equations involving subtraction, are the focus of this theoretical paper.

Theories on (educational) interface design as well as experiences from early design cycles are used to answer the following research questions:

- What are the specific disadvantages of the ways different models represent subtraction?
- How can subtraction be represented in a way that preserves advantages of existing designs but avoids the identified problems?

### **Line of argumentation**

This paper aims to advance the understanding of the design of technology for teaching and learning mathematics. For a theoretically informed discussion of the two questions, approaches from human-computer interaction studies and mathematics education are introduced and set into relation with each other. The subsequent section probes existing models' representation of subtraction. We argue that there are advantages in representing subtraction as in the algebra tiles model but that there are certain specific shortcomings. We show how these shortcomings can be addressed in a screen-based redesign of the algebra tiles model. The discussion will be used to highlight the relevance of the presented work for the design of digital or digitally enhanced didactical models and to point out possibilities how activities can be designed around the layout that we claim solves the identified problems.

### **State of the art and existing theoretical approaches**

There has been little rigorous research regarding the representation of subtraction in manipulative models for equations (see Vlassis, 2002). Thus, it is all the more advisable to take notice of the rich theory around (educational) interface design in general.

Goguen (1999) builds his theory for user interface design on the claim that “user interfaces are representations, that their quality is determined by what they preserve, and that this can be an effective basis for design” (p. 243). To investigate the quality of user interfaces, Goguen defines and investigates *mappings* between *sign systems*, indicating the possibility for mathematics educators to do so for the representation of a given mathematical sign system (such as linear equations) through a certain manipulative environment (such as algebra tiles).

The focus on mappings has been elaborated with regard to teaching models in mathematics education by English and Halford (1995), who name four principles of learning by analogy that can help to analyze (and to design) didactical models (pp. 100–102):

- Learners need to be able to understand the structural properties of the model.
- The mapping between model and concept to be learnt shall be unambiguous to the learners.
- The attributes of objects of the model that are relevant to the concept to be learnt form a cohesive conceptual structure—other attributes shall be designed to not disturb this structure.
- A model shall be applicable to a range of instances.

Taking up the language of mappings, Vlassis (2002), with regard to the balance scale model, goes so far as to claim that “the isomorphism between the object itself and the mathematical notions implied allows students to form a mental image of the operations that they have to apply” (p. 355). However, investigating and optimizing the characteristics of a model will not suffice, and claims

that ascribe the success of material models to their alleged relationship to the mathematics to be learnt may be premature. English and Halford (1995) are very clear that a model “will fail miserably if the accompanying explanation is unclear” (p. 99). Gravemeijer (2011) criticizes denoting models as transparent on the grounds that it may suggest transparency to be a feature of the model, not taking into account that instruction may nevertheless fail to offer students a help in seeing mathematical meaning beyond the mathematics they have already learnt. Meira (1998) argues that transparency should not be seen as an inherent feature of objects used in a model but as a *process* “mediated by unfolding activities and users’ participation in ongoing sociocultural practices” (p. 121). A model that properly illustrates the mathematics to be taught is nothing but a necessary first step.

These considerations let us return to Goguen’s work. With its reliance on semiotics it avoids the epistemological problem of what form of existence is ascribed a) to the representation and b) to the mathematics to be represented. Regarding the former, he writes: “what ‘really exists’ (in the sense of physics) are the photons coming off the screen; the structure that we see is our own construction” (Goguen, 1999, p. 271). Therefore, the models created “are definitely not adequate in certain respects, some known and some unknown” (p. 271). What is then needed in order to learn with such necessarily imperfect models is illuminated by a range of approaches in mathematics education that take learning processes with technology to be thought of—and therefore designed—as processes of semiotic mediation (Mariotti, 2002; cf. Meira, 2008; Duval, 2017).

In the subsequent discussion of existing models, we will follow Goguen’s view on interface design, focusing on the mappings between the interface and the mathematics to be represented, which both are taken as sign systems. This view makes our approach compatible with semiotic theories of learning that may later be helpful regarding learning processes around the designed model. Where appropriate, we enrich this approach by referring to the principles devised by English and Halford.

## **Discussion of existing manipulatives models representing subtraction**

We are particularly interested here in the mapping between negative integers and the operation of subtraction in algebra, and negative tiles and various representations of subtraction in the algebra tile model. In algebra, numbers are distinct from operations, it is possible to subtract from an unknown quantity, it is possible to subtract negative integers, and it is possible to divide a term involving subtraction (i.e.,  $3x - 9$ ). Here we discuss ways in which these aspects of the sign system of algebra map onto aspects of the sign system of algebra tiles.

### **Representing subtraction only by action**

The basic implementations of the balance scale and container models do not allow for terms involving subtraction as part of an equation. The only way students experience subtraction with these models is as an action carried out on one representation to produce an equivalent representation. One cannot represent  $x - 3 = 6$ , but one can represent  $x + 3 = 6$ , and then subtract three on both sides (i.e.,  $x + 3 - 3 = 6 - 3$ ). The subtraction on each side would not be represented by material objects, but by actions of taking away. Only the result of the subtraction, i.e.  $x = 3$ , is materially represented. Thus, the distinction between an operation (represented as an action) and a number (represented by objects) is preserved, but at the cost of not being able to represent

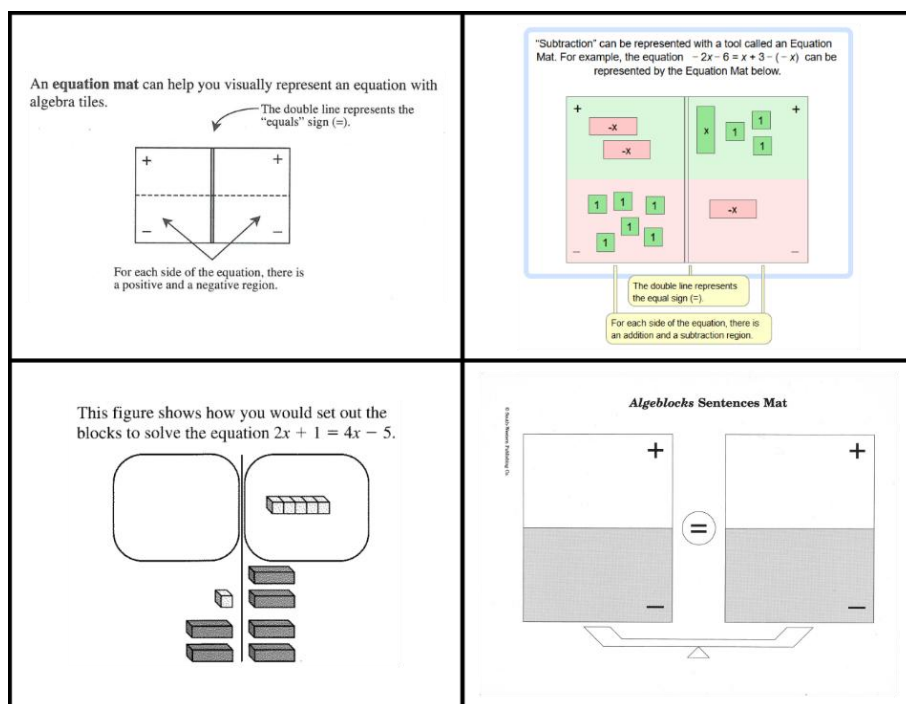
subtraction from an unknown quantity. When progressing to equations that cannot be represented in this way, students are expected to accept that the rules that applied to positive numbers and variables connected by addition and multiplication similarly apply to terms including negative numbers and fractions, subtraction and division. Some students, however, may find this difficult. Vlassis (2002) found that students instructed with the balance scale model had problems as soon as subtraction and negative integers were present in symbolic equations to be solved. She reasoned that “the negatives place the equation [...] on an abstract level. It is no longer possible to refer back to a concrete model or to arithmetic“ (p. 350).

### **Representing subtraction with objects that stand for negative integers**

An alternative is to introduce objects that stand for negative integers and to define subtraction of positive integers as addition of negative integers. In some virtual implementations, the balance model is amended with “negative weights”, e.g. represented by helium balloons. Unbound by problems of physics, the algebra tiles model allows the introduction of opposite tiles for both the unit tiles and the variable tiles. However, what is then represented are *negative numbers* and *variables with an opposite sign*. All terms in the equations are still *added*. Subtraction from an unknown quantity like  $x - 3 = 6$  can be represented (e.g., Balka & Boswell, 2006) but the distinction between operations and numbers is blurred. However, subtraction of negative integers is not possible in such a model, except by accepting a priori that this would mean addition. Hence it is clear that these models, as sign systems, do not involve elements that directly represent terms involving subtraction in linear equations.

### **Representing subtraction with a fixed subtraction zone above or below an addition zone**

Many algebra tiles models introduce a “subtraction zone” to allow terms involving a subtraction (such as “ $x - 3$ ”) to be distinguished from additive terms involving negative values (“ $x + (-3)$ ”). Tiles representing subtracted terms are placed in the subtraction zone. In all models that we are aware of, the subtraction zone is placed below or above the addition zone (see examples in Fig. 1).



**Figure 1: Layout of subtraction zones in different algebra tiles environments, clockwise from top left (disregard the texts): textbook *Algebra Connections* (Dietiker & Baldinger, 2006, p. 69), implemented in an applet under <https://technology.cpm.org/general/tiles/>; applet developed at the Freudenthal Institute (<https://app.dwo.nl/dwo/apps/player.html#202852>, step 8); Algeblocks® mat with an iconic scale; textbook *Algebra* by Wah & Picciotto (1994, p. 212) with so-called minus areas above.**

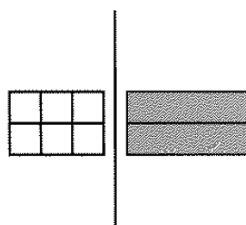
The use of subtraction zones has the potential to provoke fruitful thoughts and discussions. For example, they can help illuminate the relationship between subtraction and negative integers: by placing representations of  $a - b$  on both sides (i.e.,  $a$  tiles in the addition zone and  $b$  tiles the subtraction zone), students can add  $b$  negative tiles to both zones on one side to make sense of the identity  $a - b = a + (-b)$ . The spatial juxtaposition of a zone for addition and one for subtraction highlights the basic concept of subtraction as comparison and may already be familiar to students from arithmetic (e.g., English & Halford, 1995, p. 157–159). According to Stacey, Helme, Archer, and Condon (2001), such familiarity with an idea from earlier studies can promote engagement with a model. However, there are still some shortcomings that shall be discussed in the following paragraphs.

First, subtracting a term that itself involves subtraction (e.g.  $3 - (4 - x)$ ) and therefore discussing the rules of dealing with bracketed terms is not possible.

Second, the traditional design of the subtraction zone may lead to misunderstandings about division as an equivalent transformation. Dividing the expression on each side by the same number can be represented with algebra tiles in the last step by evenly assigning the number of unit tiles to the remaining unknown tiles (see Fig. 2). This suggests that as long as some tiles are in the subtraction zones, division is not possible. Unlike in the algebra sign system, it is not possible to divide a term like  $2x - 6$  involving subtraction. As Goguen points out, not all aspects of the original sign system can and should always be represented in the sign system of the model. However, the limitation that

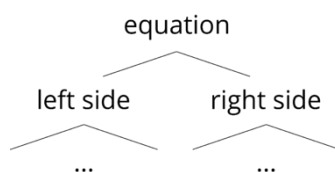
an equation like  $2x - 6 = 0$  can only be divided by 2 after it has been transformed into  $2x = 6$  seems arbitrary—and can be avoided, as we will show later.

The remaining blocks (those that cannot be matched with blocks on the other side) can then be rearranged to make it easy to see the solution to the equation.



**Figure 2: The equivalent transformation of division as even assignment of unit tiles to unknown tiles (Wah & Piciotto, 1994, p. 212).**

Third, the primary distinction between left and right can become submerged. The designs of the environments shown in Figure 1 all visually distinguish between the vertical boundary (between the left and right side of the equation) and the horizontal boundary (between the addition zone and the subtraction zone on each side). However, an analysis of how well the first principle of English and Halford—clarity of the model—is met, shows some differences: In two of the models the vertical boundary is more marked (top left: double solid line vs. single dashed line, bottom right: gap vs. no gap, balance scale as framing image), corresponding to the primacy of separation of the sides of the equation: However we parse the expressions on each side, it is their relation to each other as a whole that is most important (see Fig. 3). The bottom left model in Figure 1 highlights this by having only one real boundary that suggests two sides; the boundary of the subtraction zones seem to lie within their respective side. Only in the top right model some observers may find the continuously colored fields to suggest a stronger connection between the zones on both sides than within each side.



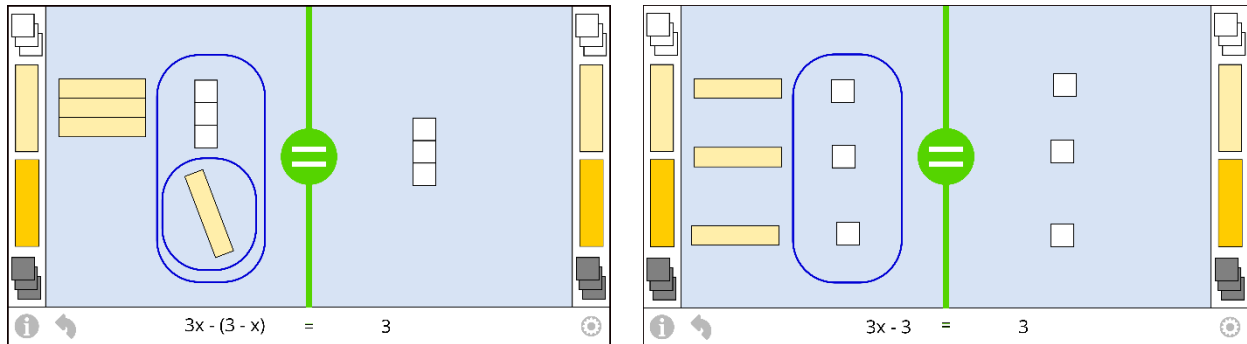
**Figure 3: Structure tree for linear equations highlighting that any operation on each side of the equation should be less prominent than the distinction between the two sides**

### Addressing the shortcomings of existing models

In the design of Wah and Piciotto (1994, see Fig. 1, bottom left), the subtraction zone can be interpreted as lying within the addition zone on each side. When the environment is shown on a screen, one can start with the addition zone only and allow students to draw the subtraction zone whenever they need it, and to change its size and position as circumstances require. Furthermore, nested subtraction zones are possible, representing nested expressions (see Fig. 4, left). This can be

helpful in modelling realistic situations without referring to learnt rules for subtracting negative integers at the outset. So for example, an equation like  $3x - (3 - x) = 3$  can be modelled without using negative integers, which is desirable if the equation models a situation where all quantities are positive. The subtracted terms are eliminated by adding their equivalents to both sides, reinforcing the concept that subtraction and addition are inverse operations.

The possibility of drawing the subtraction zone anywhere also solves the problem of division. It is now possible by drawing the subtraction zone(s) in a way that allows setting up groups horizontally across addition and subtraction zones (see Fig. 4, right).



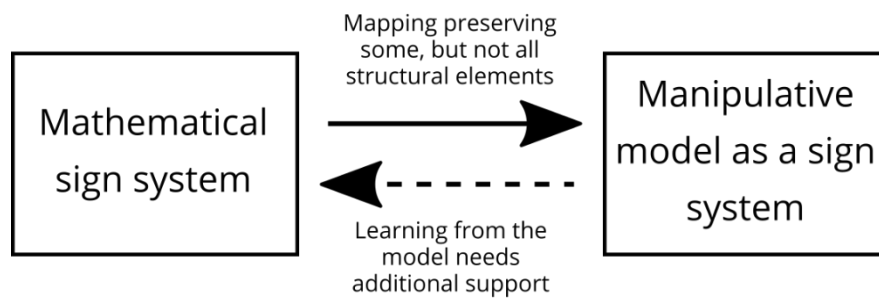
**Figure 4:** A flexible subtraction zone allows representing nested expressions (see left) and to arrange tiles so that the result of a division becomes perceptible ( $x - 1 = 1$  is equivalent to  $3x - 3 = 3$ , see right).

## Discussion and outlook

Quite regularly, instructional designs have such a long use history that it is hard to find their origins, let alone the thinking behind them. Their popularity and success may well be a result of the intuition of experienced teachers. However, as this paper shows, even models used in textbooks may have significant shortcomings. The redesign of familiar models for new technologies offers a chance to reconsider their foundations: Are the models the best way to represent the mathematics we want students to become aware of? This paper presents a set of theoretical ideas that have helped us to identify problems with how subtraction is modelled in different manipulative environments. Moreover, we do not only highlight shortcomings; the theories presented here positively offer a methodology to think of better designs. This theoretical analysis complements the experience and intuition that have guided the development of models hitherto. We understand mathematical knowledge to be learnt, and models to be designed, as sign systems that need to be linked by a mapping with certain qualities that preserve important aspects of the mathematics. Considering these qualities gives us a way to argue why a certain design should be preferred over a traditional approach.

A well-designed model is nothing but a necessary first step. As noted above, no model is transparent; like any sign system it must be interpreted. Moreover, as Goguen notes, the mapping from one sign system to another cannot be one-to-one, something must be lost. Both the learning *of* the model and learning *from* the model must be facilitated. We diagram this situation in Fig. 5.





**Figure 5: The mathematical sign system (here: linear equations) is mapped to the model sign system (here: the algebra tiles model) in an incomplete manner, necessitating additional support.**

To support learning with a model sensible instructional approaches need also to be designed. The focus of this paper does not allow us to go beyond some suggestions: The algebra tiles model can and should first be used without negative tiles or subtraction zones to introduce the initial equivalent transformations (subtracting and dividing). In this way, confusion about the meaning of subtraction zones is avoided as long as they are not needed (see third principle of learning by analogy quoted above). The need to represent equations including terms involving subtraction may then come from a problem situation that is modelled, in which something must be taken away from an unknown quantity. Representing the not-yet-possible subtraction as a set of tiles in a subtraction zone can make a link between the well-established concept of subtraction as taking away and the concept of subtraction as comparison. As can be seen from these suggestions for teaching practice, the application of theories in this paper has the potential to be followed upon by research that explores “how technology mediates knowledge and the consequences of mediation on the knowledge itself” (Trgalová, Clark-Wilson, & Weigand, 2018, p. 152).

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