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Patrick L Combettes, Jean-Christophe Pesquet. Lipschitz Certificates for Layered Network Structures Driven by Averaged Activation Operators. *SIAM Journal on Mathematics of Data Science*, 2020, 10.1137/19M1272780 . hal-02428111v1

HAL Id: hal-02428111

<https://hal.science/hal-02428111v1>

Submitted on 5 Jan 2020 (v1), last revised 9 Feb 2021 (v2)

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Lipschitz Certificates for Layered Network Structures Driven by Averaged Activation Operators*

Patrick L. Combettes[†] and Jean-Christophe Pesquet[‡]

Abstract. Obtaining sharp Lipschitz constants for feed-forward neural networks is essential to assess their robustness in the face of perturbations of their inputs. We derive such constants in the context of a general layered network model involving compositions of nonexpansive averaged operators and affine operators. By exploiting this architecture, our analysis finely captures the interactions between the layers, yielding tighter Lipschitz constants than those resulting from the product of individual bounds for groups of layers. The proposed framework is shown to cover in particular most practical instances encountered in feed-forward neural networks. Our Lipschitz constant estimates are further improved in the case of structures employing scalar nonlinear functions, which include standard convolutional networks as special cases.

1. Introduction. Artificial neural networks are becoming increasingly central tools in tasks such as learning, modeling, data processing, and decision making. As first noted in [54], neural networks are vulnerable to adversarial examples which, though close to other data inputs, lead to very different outputs. This potential lack of stability makes the networks vulnerable and unreliable in key application areas; see for instance [1, 32, 37] and the references therein. To protect networks against such instabilities various techniques have been explored [41, 45, 46, 56]. Although these defense strategies may be effective in certain scenarios, they do not provide formal guarantees of robustness for general networks and they have been shown to be breakable by new attacks; see for instance [3, 19].

It has been acknowledged for some time that the Lipschitz behavior of a network plays a key role in the analysis of its robustness [54]. Simply put, if a layered network is modeled by an operator T acting between normed spaces, with Lipschitz constant θ , given an input x and a perturbation z , we can majorize the perturbation on the output via the inequality

$$(1.1) \quad \|T(x+z) - Tx\| \leq \theta \|z\|.$$

Thus θ can be used as a certificate of robustness of the network provided that it is tightly estimated. Lipschitz regularity is also an important ingredient in the derivation of generalization bounds and approximation bounds [7, 12, 52], and of reachability conditions [49]. In [54] the estimation of θ is performed by evaluating the Lipschitz constant of the layers individually and then defining θ as the product of these constants, which typically yields pessimistic bounds. Lipschitz constants have also been computed for specific situations, e.g., [6, 35, 51, 55]. Overall, however, deriving analytically accurate constants for general contexts remains an open problem. The objective of the present paper is to address this question

*The work of P. L. Combettes was supported by the National Science Foundation under grant CCF-1715671. The work of J.-C. Pesquet was supported by Institut Universitaire de France.

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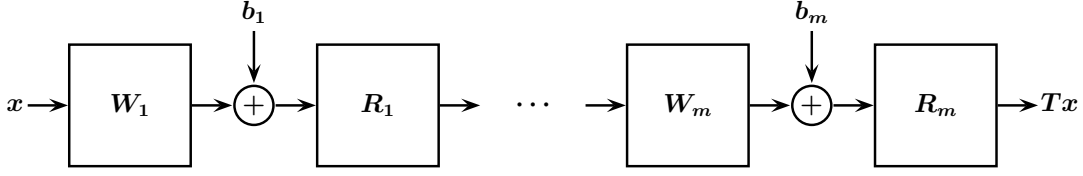


Figure 1.1. In Model 1.1, the i th layer involves a linear weight operator W_i , a bias vector b_i , and an activation operator R_i , which is assumed to be a nonlinear averaged nonexpansive operator.

for a general class of layered networks. Mathematically, our network model is described as an alternation of affine and nonlinear operators. This type of structure also arises in variational and equilibrium problems, as well as in network science, e.g., [17, 25, 29, 58]. Adopting the same terminology as in the neural network literature, where they model the activity of neurons, the nonlinear operators will be called activation operators. Our stability analysis focuses on the following m -layer model, in which the activation operators are averaged nonexpansive operators (see Fig. 1.1). Recall that an operator $R: \mathcal{H} \rightarrow \mathcal{H}$ acting on a Hilbert space \mathcal{H} is α -averaged for some $\alpha \in [0, 1]$ if there exists a nonexpansive (i.e., 1-Lipschitzian) operator $Q: \mathcal{H} \rightarrow \mathcal{H}$ such that

$$(1.2) \quad R = (1 - \alpha) \text{Id} + \alpha Q.$$

In other words, $R = \text{Id} + \alpha(Q - \text{Id})$ is an under-relaxation of a nonexpansive operator (see [9] for a detailed account). This class of operators was introduced in [5] and shown in [22] to model various problems in nonlinear analysis as it includes common operators such as projection operators, proximity operators, resolvents of monotone operators, reflection operators, gradient step operators, and various combinations thereof. Recent theoretical developments and applications to data science include [4, 10, 11, 13, 14, 16, 23, 28, 36, 43, 53, 57, 58].

Model 1.1. Let $m \geq 1$ be an integer and let $(\mathcal{H}_i)_{0 \leq i \leq m}$ be nonzero real Hilbert spaces. For every $i \in \{1, \dots, m\}$, let $W_i: \mathcal{H}_{i-1} \rightarrow \mathcal{H}_i$ be a bounded linear operator, let $b_i \in \mathcal{H}_i$, let $\alpha_i \in [0, 1]$, and let $R_i: \mathcal{H}_i \rightarrow \mathcal{H}_i$ be an α_i -averaged operator. Set

$$(1.3) \quad T = T_m \circ \dots \circ T_1, \quad \text{where } (\forall i \in \{1, \dots, m\}) \quad T_i: \mathcal{H}_{i-1} \rightarrow \mathcal{H}_i: x \mapsto R_i(W_i x + b_i).$$

Since the operators $(R_i)_{1 \leq i \leq m}$ are nonexpansive, a Lipschitz constant for T in (1.3) is

$$(1.4) \quad \theta_m = \prod_{i=1}^m \|W_i\|.$$

However, as already mentioned, this constant is usually quite loose and of limited use to assess the actual stability of the network. A novelty of our approach is to take into account the averagedness properties of the individual activation operators to capture more sharply the overall interactions between the layers, yielding tighter constants than those provided by computing bounds for groups of layers. Our specific contributions are the following:

- We show that the most common activation operators used in neural networks are averaged operators. This not only provides an a posteriori justification for Model 1.1, but also indicates that this highly structured framework should be of interest in the analysis of other properties of layered networks beyond stability.
- We derive a general expression for a Lipschitz constant of T in terms of the averaged-ness constants of the activation operators $(R_i)_{1 \leq i \leq m}$ and the norms of certain compositions of the linear operators $(W_i)_{1 \leq i \leq m}$. This Lipschitz constant is shown to lie between the simple upper bound (1.4) and the lower bound $\|W_m \circ \dots \circ W_1\|$ corresponding to a purely linear network. Our analysis applies to any type of linear operator, in particular convolutive ones, and it does not require any additional assumptions on the activation operator. In particular, differentiability is not assumed and our results therefore cover in particular networks using ReLU and max-pooling operations.
- In the common situation when the activation operators are separable, we obtain tighter Lipschitz constants for various norms.
- Under some positivity condition, we prove that the Lipschitz constant of the network reduces to that of the associated purely linear network obtained by removing the non-linear operators.

In [?], we investigated the special case of Model 1.1 in which the activation operators $(R_i)_{1 \leq i \leq m}$ are proximity operators, hence 1/2-averaged (see Section 2.1). The objective there was to study the asymptotic behavior of deep network structures rather than their stability.

The remainder of the paper is organized as follows. In Section 2.1 we provide the necessary nonlinear analysis background. In Section 2 we show that a wide array of activation operators used in neural networks are indeed nonexpansive. In Section 3 we derive general results concerning Lipschitz constants for Model 1.1. Section 4 refines this analysis in the case of separable activation operators.

2. Nonexpansive averaged activation operators.

2.1. Nonlinear analysis tools and notation. We review some key facts and definitions which will be used subsequently, see [9] for further information. Throughout \mathcal{H} is a real Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ and associated norm $\|\cdot\|$.

Let $R: \mathcal{H} \rightarrow \mathcal{H}$ be an operator and let $\alpha \in [0, 1]$. Then R is nonexpansive if it is 1-Lipschitzian, α -averaged if there exists a nonexpansive operator $Q: \mathcal{H} \rightarrow \mathcal{H}$ such that $R = (1 - \alpha)\text{Id} + \alpha Q$, and firmly nonexpansive if it is 1/2-averaged or, equivalently, if

$$(2.1) \quad (\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|Rx - Ry\|^2 + \|(\text{Id} - R)x - (\text{Id} - R)y\|^2 \leq \|x - y\|^2.$$

Let $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued operator. We denote by $\text{gra } A = \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$ the graph of A , by A^{-1} the inverse of A , i.e., the operator with graph $\{(u, x) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$, and by $J_A = (\text{Id} + A)^{-1}$ the resolvent of A . In addition, A is monotone if

$$(2.2) \quad (\forall (x, u) \in \text{gra } A)(\forall (y, v) \in \text{gra } A) \quad \langle x - y \mid u - v \rangle \geq 0,$$

and maximally monotone if there exists no monotone operator $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ such that $\text{gra } A \subset \text{gra } B \neq \text{gra } A$. We denote by $\Gamma_0(\mathcal{H})$ the class of proper lower semicontinuous convex functions

from \mathcal{H} to $]-\infty, +\infty]$. Let $f \in \Gamma_0(\mathcal{H})$. The conjugate of f is $\Gamma_0(\mathcal{H}) \ni f^*: u \mapsto \sup_{x \in \mathcal{H}} (\langle x | u \rangle - f(x))$ and the subdifferential of f is the maximally monotone operator

$$(2.3) \quad \partial f: \mathcal{H} \rightarrow 2^{\mathcal{H}}: x \mapsto \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \langle y - x | u \rangle + f(x) \leq f(y)\}.$$

For every $x \in \mathcal{H}$, the unique minimizer of $f + \|x - \cdot\|^2/2$ is denoted by $\text{prox}_f x$. We have $\text{prox}_f = J_{\partial f}$ and prox_f is therefore firmly nonexpansive.

Now let C be a nonempty convex subset of \mathcal{H} . Then ι_C is the indicator function of C (it takes values 0 on C and $+\infty$ on its complement) and $d_C: x \mapsto \min_{y \in C} \|x - y\|$ is its distance function. If C is closed, its projection operator is $\text{proj}_C = \text{prox}_{\iota_C}$.

2.2. Activators as averaged operators. We show via various illustrations that the assumption made in Model 1.1 on the activation operators covers most existing instances of feed-forward neural networks. Let us start with some key properties.

Proposition 2.1. *Let \mathcal{H} be a real Hilbert space, let $\alpha \in [0, 1]$, and let $R: \mathcal{H} \rightarrow \mathcal{H}$ be α -averaged. Then the following hold:*

- (i) *There exist a maximally monotone operator $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ and a constant $\lambda \in [0, 2]$ such that $R = \text{Id} + \lambda(J_A - \text{Id})$. Furthermore, if $\lambda \leq 1$, then R is firmly nonexpansive.*
- (ii) *Suppose that $\mathcal{H} = \mathbb{R}$. Then there exist a function $\phi \in \Gamma_0(\mathbb{R})$ and a constant $\lambda \in [0, 2]$ such that $R = \text{Id} + \lambda(\text{prox}_\phi - \text{Id})$. Furthermore, R is increasing if $\lambda \leq 1$ and R is odd if ϕ is even.*
- (iii) *Suppose that $\mathcal{H} = \mathbb{R}$ and that R is increasing. Then there exists $\phi \in \Gamma_0(\mathbb{R})$ such that $R = \text{prox}_\phi$.*

Next, we illustrate the pervasiveness of nonexpansive averaged activation operators in practice, starting with activation operators on the real line.

Example 2.2. Proposition 2.1(ii) states that activation functions on the real line can be expressed in the generic form

$$(2.4) \quad R = \text{Id} + \lambda(\text{prox}_\phi - \text{Id}), \quad \text{where } \phi \in \Gamma_0(\mathbb{R}) \quad \text{and} \quad \lambda \in [0, 2].$$

Here are a few explicit instantiations of this proximal representation.

- (i) If $\lambda = 1$, we obtain the class of proximal activation functions discussed in [?] and which was seen there to include standard instances such as the unimodal sigmoid activation function [?, Example 2.13], the saturated linear activation function [?, Example 2.5], the rectified linear unit (ReLU) activation function [?, Example 2.6], the inverse square root unit activation function [?, Example 2.9], the hyperbolic tangent activation function [?, Example 2.12], and the Elliot activation function [?, Example 2.15]. Additional examples in this category are the following. Given $\beta \in]0, +\infty[$, the capped ReLU activation function [38] is

$$(2.5) \quad (\forall x \in \mathbb{R}) \quad R(x) = \text{prox}_{\iota_{[0, \beta]}}(x) = \min\{\max\{x, 0\}, \beta\},$$

and, for $\beta \leq 1$, the exponential linear unit (ELU) function [21] is

$$(2.6) \quad (\forall x \in \mathbb{R}) \quad R(x) = \begin{cases} x, & \text{if } x \geq 0; \\ \beta(\exp(x) - 1), & \text{if } x < 0. \end{cases}$$

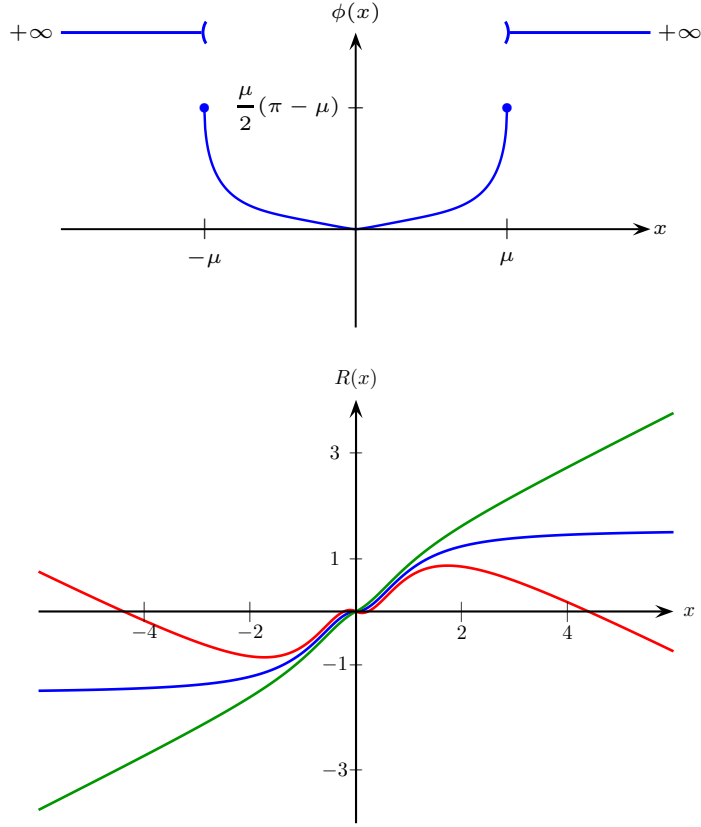


Figure 2.1. Averaged activation functions: Illustration of Example 2.2(ii). Top: The function ϕ of (2.10). Bottom: In blue, the activation operator R of (2.8) is the proximity operator of ϕ , which corresponds to $\lambda = 1$ in (2.4). The green curve corresponds to the case when $\lambda = 0.5$ in (2.4), and the red one to the case when $\lambda = 1.5$. As stated in Proposition 2.1(i), relaxations parameters $\lambda \in [0, 1]$ yield increasing activation functions. Non-monotonic averaged activation functions in (2.4) must be generated with relaxations parameters $\lambda \in]1, 2]$. As seen in Proposition 2.1(ii), since ϕ is even, R is odd.

It follows from [9, Cor. 24.5, Prop. 24.32, and Exa. 13.2(v)] that $R = \text{prox}_\phi$, where

$$(2.7) \quad (\forall x \in \mathbb{R}) \quad \phi(x) = \begin{cases} 0 & \text{if } x \geq 0; \\ (x + \beta) \ln \left(\frac{x + \beta}{\beta} \right) - x - \frac{x^2}{2}, & \text{if } -\beta < x < 0; \\ \beta - \frac{\beta^2}{2}, & \text{if } x = -\beta; \\ +\infty, & \text{if } x < -\beta. \end{cases}$$

The softplus activation function [31] $R: x \mapsto \ln((1 + e^x)/2)$ is also a proximity operator since it is nonexpansive and increasing (see Proposition 2.1(iii)).

(ii) The Geman-McClure function [30]

$$(2.8) \quad (\forall x \in \mathbb{R}) \quad R(x) = \frac{\mu \text{sign}(x)x^2}{1 + x^2}, \quad \text{where} \quad \mu = \frac{8}{3\sqrt{3}},$$

will be employed in Example 2.3. Set $\psi = |\cdot| - \arctan |\cdot| \in \Gamma_0(\mathbb{R})$. Then R is nonexpansive

and $R = \mu\psi'$. The conjugate of $\mu\psi$ is 1-strongly convex and given by $\mu\psi^*(\cdot/\mu)$, where

$$(2.9) \quad (\forall x \in \mathbb{R}) \quad \psi^*(x) = \begin{cases} \arctan \sqrt{\frac{|x|}{1-|x|}} - \sqrt{|x|(1-|x|)}, & \text{if } |x| < 1; \\ \frac{\pi}{2}, & \text{if } |x| = 1; \\ +\infty, & \text{otherwise.} \end{cases}$$

It follows from [9, Cor. 24.5] that $R = \text{prox}_\phi$ with (see Fig. 2.1)

$$(2.10) \quad \phi = \mu\psi^*\left(\frac{\cdot}{\mu}\right) - \frac{|\cdot|^2}{2}: x \mapsto \begin{cases} \mu \arctan \sqrt{\frac{|x|}{\mu-|x|}} - \sqrt{|x|(\mu-|x|)} - \frac{x^2}{2}, & \text{if } |x| < \mu; \\ \frac{\mu(\pi-\mu)}{2}, & \text{if } |x| = \mu; \\ +\infty, & \text{otherwise.} \end{cases}$$

- (iii) Take $\phi = \iota_{[0,+\infty]}$. Then we obtain the leaky ReLU activation function [40] for $0 < \lambda < 1$, the ReLU activation function for $\lambda = 1$, and the absolute value activation function [18] for $\lambda = 2$.
- (iv) The use of non-monotonic activation functions has been advocated in various papers. They turn out to be α -averaged (alternatively, in view of Proposition 2.1(ii), they are of the form (2.4) with $\lambda \in]1, 2]$). To compute the averagedness constant of a nonexpansive operator $R: \mathbb{R} \rightarrow \mathbb{R}$, one can proceed as follows. According to (1.2), we must find the smallest $\alpha \in]0, 1]$ such that $Q = \text{Id} + \alpha^{-1}(R - \text{Id})$ remains nonexpansive. This means that the supremum of the modulus of the one-sided derivatives (the derivatives if they exist) over \mathbb{R} should be one. Thus, we obtain $\alpha = 1$ for the sine activation function $R = \sin$ [44], as well as for the absolute value function $R = |\cdot|$ [18] and the mirrored ReLU activation function [60]

$$(2.11) \quad (\forall x \in \mathbb{R}) \quad R(x) = \text{proj}_{[0,1]}|x| = \begin{cases} |x|, & \text{if } |x| < 1; \\ 1, & \text{otherwise,} \end{cases}$$

$\alpha \approx 0.546$ for the swish activation function [47]

$$(2.12) \quad (\forall x \in \mathbb{R}) \quad R(x) = \frac{10x}{11(1 + \exp(-x))},$$

$\alpha \approx 0.536$ for the exponential linear squashing (ELiSH) function [8]

$$(2.13) \quad (\forall x \in \mathbb{R}) \quad R(x) = \frac{10}{11} \times \begin{cases} \frac{x}{1 + \exp(-x)}, & \text{if } x \geq 0; \\ \frac{\exp(x) - 1}{1 + \exp(-x)}, & \text{if } x < 0, \end{cases}$$

and $\alpha = (1 + \sqrt{2/e})/2$ for the Gaussian activation function $R: x \mapsto \exp(-x^2)$ [42].

Next, is a technique for lifting a proximal activation operator from \mathbb{R} to a Hilbert space \mathcal{H} .

Example 2.3. Let \mathcal{H} be a real Hilbert space, let $\lambda \in [0, 2]$, let C be a nonempty closed convex subset of \mathcal{H} , let $\phi \in \Gamma_0(\mathbb{R})$ be an even function such that ϕ^* is differentiable on $\mathbb{R} \setminus \{0\}$ with 0 as its unique minimizer. Set

$$(2.14) \quad (\forall x \in \mathcal{H}) \quad Rx = \begin{cases} (1 - \lambda)x + \frac{\lambda \text{prox}_\phi d_C(x)}{d_C(x)}(x - \text{proj}_C x), & \text{if } x \notin C; \\ (1 - \lambda)x, & \text{if } x \in C. \end{cases}$$

Then R is $\lambda/2$ -averaged. In particular, if we set $\lambda = 1$, $C = \{0\}$, $\mu = 8/(3\sqrt{3})$, and define ϕ as in (2.10), we infer that the squashing function used in capsule networks [50], that is,

$$(2.15) \quad (\forall x \in \mathcal{H}) \quad Rx = \frac{\mu \|x\|}{1 + \|x\|^2} x,$$

is a proximal activation operator.

Another construction that builds on activation functions on the real line is the following, which is reminiscent of the original multilayer perceptrons [48].

Example 2.4. Let \mathcal{H} be a separable real Hilbert space, let $\emptyset \neq \mathbb{K} \subset \mathbb{N}$, let $(e_k)_{k \in \mathbb{K}}$ be an orthonormal basis of \mathcal{H} , and let $\alpha \in [0, 1]$. For every $k \in \mathbb{K}$, let $\varrho_k: \mathbb{R} \rightarrow \mathbb{R}$ be α -averaged and such that $\varrho_k(0) = 0$. Define $R: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto \sum_{k \in \mathbb{K}} \varrho_k(\langle x | e_k \rangle) e_k$. Then R is α -averaged.

Example 2.5. Let N be a strictly positive integer, let $\omega \in [0, 1]$, and let C be a nonempty closed convex subset of \mathbb{R}^N . Set

$$(2.16) \quad R: \mathbb{R}^N \rightarrow \mathbb{R}^N: (\xi_k)_{1 \leq k \leq N} \mapsto \omega (\xi_k^\uparrow)_{1 \leq k \leq N} + (1 - \omega) \text{proj}_C (\xi_k)_{1 \leq k \leq N},$$

where $(\xi_k^\uparrow)_{1 \leq k \leq N}$ denotes the vector obtained by sorting the components of $(\xi_k)_{1 \leq k \leq N}$ in ascending order. Then R is $(1 + \omega)/2$ -averaged.

Remark 2.6. Set $C = \{(\xi_k)_{1 \leq k \leq N} \in \mathbb{R}^N \mid \xi_1 = \dots = \xi_N\}$ in Example 2.5. Then

$$(2.17) \quad R: \mathbb{R}^N \rightarrow \mathbb{R}^N: (\xi_k)_{1 \leq k \leq N} \mapsto \left(\omega \xi_k^\uparrow + \frac{1 - \omega}{N} \sum_{j=1}^N \xi_j \right)_{1 \leq k \leq N}.$$

Now set $W: \mathbb{R}^N \rightarrow \mathbb{R}: (\xi_k)_{1 \leq k \leq N} \mapsto \xi_N$. Then $W \circ R$ corresponds to the max-average pooling performed on a block of size N [39]. When $\omega = 0$, the standard average-pooling operation is obtained, which is associated with the activation operator proj_C . When $\omega = 1$, we recover the standard max-pooling operation [15], which is the main building block of maxout layers [33]. The max-pooling operator is nonexpansive.

Example 2.7. Let $2 \leq N \in \mathbb{N}$, let $\{\tau_k\}_{1 \leq k \leq N-1} \subset]-1, 1[$, and let $\theta \in \mathbb{R}$. Set

$$(2.18) \quad R: \mathbb{R}^{N-1} \rightarrow \mathbb{R}^{N-1}: (\xi_k)_{1 \leq k \leq N-1} \mapsto US \left([\tau_1 \xi_1, \dots, \tau_{N-1} \xi_{N-1}, \theta]^\top \right),$$

where $U \in \mathbb{R}^{(N-1) \times N}$ is the matrix obtained by retaining the first $(N - 1)$ rows of the identity matrix of size $N \times N$, and $S: \mathbb{R}^N \rightarrow \mathbb{R}^N: (\xi_k)_{1 \leq k \leq N} \mapsto (\xi_k^\uparrow)_{1 \leq k \leq N}$. Then R is $(1 + \max\{|\tau_1|, \dots, |\tau_{N-1}|\})/2$ -averaged.

Remark 2.8. Let $N \geq 3$ be an odd integer, let $(\tau_k)_{1 \leq k \leq N-1} \in]-1, 1[^{N-1}$, let $\theta \in \mathbb{R}$, let R be the activation operator defined in Example 2.7, and set $W: \mathbb{R}^{N-1} \rightarrow \mathbb{R}: (\xi_k)_{1 \leq k \leq N-1} \mapsto \xi_{\frac{N+1}{2}}$. Then, for every $x = (\xi_k)_{1 \leq k \leq N-1} \in \mathbb{R}^{N-1}$, $(W \circ R)x = \text{median}\{\tau_1 \xi_1, \dots, \tau_{N-1} \xi_{N-1}, \theta\}$. This corresponds to the median neuron model introduced in [2].

Remark 2.9. Multi-component averaged activation operators can be derived from the above examples. Indeed, let $(\mathcal{H}_i)_{1 \leq i \leq M}$ be real Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_M$ be their Hilbert direct sum. For every $i \in \{1, \dots, M\}$, let $\alpha_i \in [0, 1]$ and let $R_i: \mathcal{H}_i \rightarrow \mathcal{H}_i$ be α_i -averaged. Then $R: \mathcal{H} \rightarrow \mathcal{H}: (x_i)_{1 \leq i \leq M} \mapsto (R_i x_i)_{1 \leq i \leq M}$ is α -averaged with $\alpha = \max_{1 \leq i \leq M} \alpha_i$.

3. Lipschitz constants for layered networks. The objective of this section is to derive Lipschitz constants for networks conforming to Model 1.1. Note that, if $m = 1$, a Lipschitz constant of T is clearly $\theta_1 = \|W_1\|$ since R_1 is nonexpansive. We shall therefore focus henceforth on the case $m \geq 2$. Throughout, the following notation is employed.

Notation 3.1. Let $2 \leq m \in \mathbb{N}$ and $k \in \{1, \dots, m-1\}$. Then

$$(3.1) \quad \mathbb{J}_{m,k} = \begin{cases} \{(j_1, \dots, j_k) \in \mathbb{N}^k \mid 1 \leq j_1 < \dots < j_k \leq m-1\}, & \text{if } k > 1; \\ \{1, \dots, m-1\}, & \text{if } k = 1 \end{cases}$$

and, for every $(j_1, \dots, j_k) \in \mathbb{J}_{m,k}$,

$$(3.2) \quad \sigma_{m;\{j_1, \dots, j_k\}} = \|W_m \circ \dots \circ W_{j_k+1}\| \|W_{j_k} \circ \dots \circ W_{j_{k-1}+1}\| \dots \|W_{j_1} \circ \dots \circ W_1\|.$$

Theorem 3.2. Consider the setting of Model 1.1 with $m \geq 2$. Set

$$(3.3) \quad (\forall \mathbb{J} \subset \{1, \dots, m-1\}) \quad \beta_{m;\mathbb{J}} = \left(\prod_{j \in \mathbb{J}} \alpha_j \right) \prod_{j \in \{1, \dots, m-1\} \setminus \mathbb{J}} (1 - \alpha_j)$$

and

$$(3.4) \quad \theta_m = \beta_{m;\emptyset} \|W_m \circ \dots \circ W_1\| + \sum_{k=1}^{m-1} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m,k}} \beta_{m;\{j_1, \dots, j_k\}} \sigma_{m;\{j_1, \dots, j_k\}}.$$

Then θ_m is a Lipschitz constant of T .

The following proposition features some important special cases.

Proposition 3.3. Consider the setting of Model 1.1 with $m \geq 2$, and let θ_m be defined as in (3.4). Then the following hold:

- (i) $\|W_m \circ \dots \circ W_1\| \leq \theta_m \leq \prod_{i=1}^m \|W_i\|$.
- (ii) Suppose that, for every $i \in \{1, \dots, m-1\}$, $R_i = \text{Id}$. Then $\theta_m = \|W_m \circ \dots \circ W_1\|$.
- (iii) Suppose that, for every $i \in \{1, \dots, m-1\}$, R_i is purely nonexpansive in the sense that $\alpha_i = 1$ is its smallest averaging constant. Then $\theta_m = \prod_{i=1}^m \|W_i\|$.
- (iv) Suppose that, for every $i \in \{1, \dots, m-1\}$, R_i is firmly nonexpansive. Then

$$(3.5) \quad \theta_m = \frac{1}{2^{m-1}} \left(\|W_m \circ \dots \circ W_1\| + \sum_{k=1}^{m-1} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m,k}} \sigma_{m;\{j_1, \dots, j_k\}} \right).$$

(v) Set $\alpha_0 = \theta_0 = 1$. Then

$$(3.6) \quad \theta_m = \sum_{i=0}^{m-1} \alpha_i \theta_i \left(\prod_{q=i+1}^{m-1} (1 - \alpha_q) \right) \|W_m \circ \dots \circ W_{i+1}\|.$$

Remark 3.4. Proposition 3.3(i)–3.3(iii) show that the tightest bound in terms of stability corresponds to a linear network, while the loosest corresponds to a network with nonlinearities having no stronger property than nonexpansiveness.

Example 3.5. To illustrate the improvement of the proposed bound over the classical product norm estimate, we consider a fully connected network using proximal activation operators with $m = 3$, $\mathcal{H}_0 = \mathbb{R}^8$, $\mathcal{H}_1 = \mathbb{R}^{10}$, $\mathcal{H}_2 = \mathbb{R}^6$, and $\mathcal{H}_3 = \mathbb{R}^3$. The entries of the matrices $(W_i)_{1 \leq i \leq 3}$ are generated randomly and independently according to a normal distribution. We evaluate the Lipschitz constant estimate θ_3 provided by (3.5) and its lower bound in Proposition 3.3(i). The average (resp. minimal) value of $\theta_3 / (\|W_1\| \|W_2\| \|W_3\|)$ computed over 1000 realizations is approximately equal to 0.6699 (resp. 0.5112), while the average (resp. minimal) value of $\|W_3 W_2 W_1\| / (\|W_1\| \|W_2\| \|W_3\|)$ is approximately equal to 0.3747 (resp. 0.1208).

We close this section by observing that the Lipschitz constant exhibited in Theorem 3.2 is a componentwise increasing function of the averagedness constants of the activation operators.

Proposition 3.6. Consider the setting of Model 1.1 with $m \geq 2$. Make the Lipschitz constant θ_m in Theorem 3.2 a function of $(\alpha_1, \dots, \alpha_{m-1}) \in [0, 1]^{m-1}$. Let $(\alpha_i)_{1 \leq i \leq m-1} \in [0, 1]^{m-1}$ and $(\alpha'_i)_{1 \leq i \leq m-1} \in [0, 1]^{m-1}$ be such that $(\forall i \in \{1, \dots, m-1\}) \alpha_i \leq \alpha'_i$. Then $\theta_m(\alpha_1, \dots, \alpha_{m-1}) \leq \theta_m(\alpha'_1, \dots, \alpha'_{m-1})$.

Remark 3.7. Proposition 3.6 suggests that, in terms of stability, it is better to use proximal activation operators, such as those listed in Example 2.2(i)–(ii), than α -averaged activation operators for which $\alpha > 1/2$, such as those mentioned in Example 2.2(iv).

4. Networks using separable activation operators. We show that sharper Lipschitz constants can be derived in the case of networks featuring the type of separable structure described in Example 2.4. Note that this class of networks is the most commonly used, standard convnets being special cases. The following notation will be used.

Notation 4.1. Let \mathcal{H} be a separable real Hilbert space, let $\emptyset \neq \mathbb{K} \subset \mathbb{N}$, let $E = (e_k)_{k \in \mathbb{K}}$ be an orthonormal basis of \mathcal{H} , and let I be a nonempty bounded subset of \mathbb{R} . Then

$$(4.1) \quad \mathcal{D}_I(E) = \left\{ \Lambda: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto \sum_{k \in \mathbb{K}} \lambda_k \langle x | e_k \rangle e_k \mid \{\lambda_k\}_{k \in \mathbb{K}} \subset I \right\}.$$

4.1. General results.

Theorem 4.2. Consider the setting of Model 1.1 with $m \geq 2$. For every $i \in \{1, \dots, m-1\}$, suppose that \mathcal{H}_i is separable, let $\emptyset \neq \mathbb{K}_i \subset \mathbb{N}$, let $E_i = (e_{i,k})_{k \in \mathbb{K}_i}$ be an orthonormal basis of \mathcal{H}_i , and, for every $k \in \mathbb{K}_i$, let $\varrho_{i,k}: \mathbb{R} \rightarrow \mathbb{R}$ be α_i -averaged and such that $\varrho_{i,k}(0) = 0$. Assume that

$$(4.2) \quad (\forall i \in \{1, \dots, m-1\}) \quad R_i: \mathcal{H}_i \rightarrow \mathcal{H}_i: x \mapsto \sum_{k \in \mathbb{K}_i} \varrho_{i,k}(\langle x | e_{i,k} \rangle) e_{i,k}$$

and define

$$(4.3) \quad \vartheta_m = \sup_{\Lambda_1 \in \mathcal{D}_{\{1-2\alpha_1, 1\}}(E_1)} \|W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|.$$

$$\vdots$$

$$\Lambda_{m-1} \in \mathcal{D}_{\{1-2\alpha_{m-1}, 1\}}(E_{m-1})$$

Then the following hold:

- (i) ϑ_m is a Lipschitz constant of the operator T of (1.3).
- (ii) Define θ_m as in (3.4). Then $\|W_m \circ \cdots \circ W_1\| \leq \vartheta_m \leq \theta_m$.

Example 4.3. Consider the same setting as in Example 3.5 and assume now that the activation operators R_1 and R_2 are separable (for example, a ReLU layer, followed by a sigmoid layer, and finally a softmax operation). The average (resp. minimal) value of $\vartheta_3 / (\|W_1\| \|W_2\| \|W_3\|)$ computed over 1000 realizations is approximately equal to 0.4528 (resp. 0.2424). In agreement with Theorem 4.2(ii), this estimation of the Lipschitz constant is better than θ_3 and significantly sharper than $\|W_1\| \|W_2\| \|W_3\|$.

Remark 4.4. An expression similar to (4.3) was proposed empirically in [51] for a multi-layer perceptron operating on finite-dimensional spaces under the additional assumption that the activation operators are continuously differentiable and firmly nonexpansive.

Remark 4.5. In Theorem 4.2, make the additional assumption that, for some $i \in \{1, \dots, m-1\}$, the functions $(\varrho_{i,k})_{k \in \mathbb{K}_i}$ are increasing. Then it follows from Proposition 2.1(iii) that there exist functions $(\phi_{i,k})_{k \in \mathbb{K}_i}$ in $\Gamma_0(\mathbb{R})$ such that $(\forall k \in \mathbb{K}_i) \varrho_{i,k} = \text{prox}_{\phi_{i,k}}$. In addition, for every $k \in \mathbb{K}_i$, since $\varrho_{i,k}(0) = 0$ and since the set of minimizers of $\phi_{i,k}$ coincides with the set of fixed points of $\text{prox}_{\phi_{i,k}}$ [9, Proposition 12.29], we deduce that $\phi_{i,k}$ is minimized at 0. Furthermore, $\alpha_i = 1/2$ and $R_i = \text{prox}_{\varphi_i}$, where $(\forall x \in \mathcal{H}_i) \varphi_i(x) = \sum_{k \in \mathbb{K}_i} \phi_{i,k}(\langle x | e_{i,k} \rangle)$. Such a construction is used in [24, 26, 27].

As in Proposition 3.6, the Lipschitz constant exhibited in Theorem 4.2 turns out to be a componentwise increasing function of the averagedness constants of the activation operators.

Proposition 4.6. Consider the setting of Model 1.1 with $m \geq 2$. For every $i \in \{1, \dots, m-1\}$, suppose that \mathcal{H}_i is separable, let $\emptyset \neq \mathbb{K}_i \subset \mathbb{N}$, and let $E_i = (e_{i,k})_{k \in \mathbb{K}_i}$ be an orthonormal basis of \mathcal{H}_i . Define $\vartheta_m: [0, 1]^{m-1} \rightarrow [0, +\infty[$ by

$$(4.4) \quad \vartheta_m: (\alpha_1, \dots, \alpha_{m-1}) \mapsto \sup_{\Lambda_1 \in \mathcal{D}_{\{1-2\alpha_1, 1\}}(E_1)} \|W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|.$$

$$\vdots$$

$$\Lambda_{m-1} \in \mathcal{D}_{\{1-2\alpha_{m-1}, 1\}}(E_{m-1})$$

Let $(\alpha_i)_{1 \leq i \leq m-1} \in [0, 1]^{m-1}$ and $(\alpha'_i)_{1 \leq i \leq m-1} \in [0, 1]^{m-1}$ be such that $(\forall i \in \{1, \dots, m-1\}) \alpha_i \leq \alpha'_i$. Then $\vartheta_m(\alpha_1, \dots, \alpha_{m-1}) \leq \vartheta_m(\alpha'_1, \dots, \alpha'_{m-1})$.

4.2. Extension to non-Hilbertian norms. In certain applications, Hilbertian norms may not be the most relevant measures to quantify errors. We now state a variant of Theorem 4.2 which holds for alternative norms. It involves embeddings of Hilbert spaces; standard examples can be found in [59]. Let us also point out that these embedding conditions are automatically satisfied if the spaces are finite-dimensional.

Proposition 4.7. Consider the setting of Model 1.1 with $m \geq 2$. For every $i \in \{1, \dots, m\}$, suppose that \mathcal{H}_i is separable, let $\emptyset \neq \mathbb{K}_i \subset \mathbb{N}$, let $E_i = (e_{i,k})_{k \in \mathbb{K}_i}$ be an orthonormal basis of \mathcal{H}_i , and, for every $k \in \mathbb{K}_i$, let $\varrho_{i,k}: \mathbb{R} \rightarrow \mathbb{R}$ be α_i -averaged and such that $\varrho_{i,k}(0) = 0$. Let \mathcal{G}_0 be the normed space obtained by equipping the vector space underlying \mathcal{H}_0 with a norm for which \mathcal{G}_0 is continuously embedded in \mathcal{H}_0 , and let \mathcal{G}_m be the normed space obtained by equipping the vector space underlying \mathcal{H}_m with a norm for which \mathcal{H}_m is continuously embedded in \mathcal{G}_m . Assume that

$$(4.5) \quad (\forall i \in \{1, \dots, m\}) \quad R_i: \mathcal{H}_i \rightarrow \mathcal{H}_i: x \mapsto \sum_{k \in \mathbb{K}_i} \varrho_{i,k}(\langle x | e_{i,k} \rangle) e_{i,k}.$$

Then

$$(4.6) \quad \vartheta_m = \sup_{\substack{\Lambda_1 \in \mathcal{D}_{\{1-2\alpha_1, 1\}}(E_1) \\ \vdots \\ \Lambda_m \in \mathcal{D}_{\{1-2\alpha_m, 1\}}(E_m)}} \|\Lambda_m \circ W_m \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}$$

is a Lipschitz constant of $T: \mathcal{G}_0 \rightarrow \mathcal{G}_m$.

Corollary 4.8. Consider the setting of Model 1.1 with $m \geq 2$. Define \mathcal{G}_0 and $(R_i)_{1 \leq i \leq m}$ as in Proposition 4.7, let $p \in [1, +\infty]$, and let $\{\omega_k\}_{k \in \mathbb{K}_m} \subset]0, +\infty[$ be such that one of the following holds:

- (i) $p \in [1, 2[$ and $\sum_{k \in \mathbb{K}_m} \omega_k^{2/(2-p)} < +\infty$.
- (ii) $p \in [2, +\infty]$ and $\sup_{k \in \mathbb{K}_m} \omega_k < +\infty$.

Let \mathcal{G}_m be the normed space obtained by equipping the vector space underlying \mathcal{H}_m with the norm

$$(4.7) \quad (\forall x \in \mathcal{H}_m) \quad \|x\|_{\mathcal{G}_m} = \begin{cases} \left| \sum_{k \in \mathbb{K}_m} \omega_k |\langle x | e_{m,k} \rangle|^p \right|^{1/p}, & \text{if } p < +\infty; \\ \sup_{k \in \mathbb{K}_m} \omega_k |\langle x | e_{m,k} \rangle|, & \text{if } p = +\infty. \end{cases}$$

Then

$$(4.8) \quad \vartheta_m = \sup_{\substack{\Lambda_1 \in \mathcal{D}_{\{1-2\alpha_1, 1\}}(E_1) \\ \vdots \\ \Lambda_{m-1} \in \mathcal{D}_{\{1-2\alpha_{m-1}, 1\}}(E_{m-1})}} \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}$$

is a Lipschitz constant of $T: \mathcal{G}_0 \rightarrow \mathcal{G}_m$.

4.3. Networks with positive weights. Under certain positivity assumptions, the constant ϑ_m of (4.3) and (4.8) can be simplified.

Assumption 4.9. Consider the setting of Model 1.1 with $m \geq 2$. For every $i \in \{0, \dots, m\}$, suppose that \mathcal{H}_i is separable, let $\emptyset \neq \mathbb{K}_i \subset \mathbb{N}$, and let $E_i = (e_{i,k})_{k \in \mathbb{K}_i}$ be an orthonormal basis of \mathcal{H}_i . For every $(k_0, \dots, k_m) \in \mathbb{K}_0 \times \dots \times \mathbb{K}_m$, set

$$(4.9) \quad \mu_{k_0, \dots, k_m} = \langle W_1 e_{0, k_0} | e_{1, k_1} \rangle \cdots \langle W_m e_{m-1, k_{m-1}} | e_{m, k_m} \rangle.$$

We suppose that

$$(4.10) \quad (\forall (k_0, \dots, k_m) \in \mathbb{K}_0 \times \dots \times \mathbb{K}_m) (\forall (l_0, \dots, l_{m-1}) \in \mathbb{K}_0 \times \dots \times \mathbb{K}_{m-1}) \\ \mu_{k_0, \dots, k_{m-1}, k_m} \mu_{l_0, \dots, l_{m-1}, k_m} \geq 0.$$

Example 4.10. Consider the particular case of Model 1.1 in which, for every $i \in \{0, \dots, m\}$, $N_i \in \mathbb{N} \setminus \{0\}$, $\mathcal{H}_i = \mathbb{R}^{N_i}$, E_i is the canonical basis of \mathbb{R}^{N_i} and, for every $k \in \{1, \dots, N_i\}$, $\chi_{i,k} \in \{-1, 1\}$ with the additional condition that, for every $l \in \{1, \dots, N_0\}$, $\chi_{0,k} = \chi_{0,l}$. Further, for every $i \in \{1, \dots, m\}$, the matrix $W_i = [w_{i,k,l}]_{1 \leq k \leq N_i, 1 \leq l \leq N_{i-1}} \in \mathbb{R}^{N_i \times N_{i-1}}$ satisfies

$$(4.11) \quad (\forall k \in \{1, \dots, N_i\}) (\forall l \in \{1, \dots, N_{i-1}\}) \quad w_{i,k,l} = \chi_{i,k} \chi_{i-1,l} |w_{i,k,l}|.$$

Then Assumption 4.9 holds. This is true in particular if, for every $i \in \{1, \dots, m\}$, $\{w_{i,k,l}\}_{1 \leq k \leq N_i, 1 \leq l \leq N_{i-1}} \subset [0, +\infty[$, which corresponds to positively weighted networks. See [20] for the design of such networks.

In the following result, the Lipschitz constant of the network (1.3) coincides with that of the linear network $W_m \circ \dots \circ W_1$ for standard choices of norms.

Proposition 4.11. Suppose that the assumptions of Corollary 4.8 are satisfied, that

$$(4.12) \quad (\forall (\xi_k)_{k \in \mathbb{K}_0} \in \ell^2(\mathbb{K}_0)) \quad \left\| \sum_{k \in \mathbb{K}_0} \xi_k e_{0,k} \right\|_{\mathcal{G}_0} = \left\| \sum_{k \in \mathbb{K}_0} |\xi_k| e_{0,k} \right\|_{\mathcal{G}_0},$$

and that Assumption 4.9 holds. Then the Lipschitz constant ϑ_m of $T: \mathcal{G}_0 \rightarrow \mathcal{G}_m$ in (4.8) reduces to $\vartheta_m = \|W_m \circ \dots \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}$.

We show below that the Lipschitz constant of a positively weighted network associated with weight operators $(W_i)_{1 \leq i \leq m}$ and nonseparable activation operators is not necessarily $\|W_m \circ \dots \circ W_1\|$.

Example 4.12. Consider the toy version of Model 1.1 in which $m = 2$, $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^2$. Set $\varphi: x = (\xi_1, \xi_2) \mapsto \phi(\xi_1) + \phi(\xi_2)$, where

$$(4.13) \quad \phi: \mathbb{R} \rightarrow]-\infty, +\infty]: \xi \mapsto \begin{cases} \frac{(1+\xi) \ln(1+\xi) + (1-\xi) \ln(1-\xi) - \xi^2}{2} & \text{if } |\xi| < 1; \\ \ln(2) - 1/2 & \text{if } |\xi| = 1; \\ +\infty, & \text{if } |\xi| > 1. \end{cases}$$

Let $\xi \in]-1, 1[= \text{dom } \phi' = \text{dom } (\text{Id} + \phi') = \text{ran } \text{prox}_\phi$. Then $\xi + \phi'(\xi) = \text{arctanh}(\xi)$ and therefore $\varrho = (\text{Id} + \phi')^{-1} = \tanh$. Consequently, we derive from [?, Example 2.13] that $(\forall x = (\xi_1, \xi_2) \in \mathbb{R}^2) \text{prox}_\varphi x = (\tanh(\xi_1), \tanh(\xi_2))$. Now set

$$(4.14) \quad b_1 = b_2 = 0, \quad U = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 10 & 2 \\ 7 & 4 \end{bmatrix},$$

$R_1 = \text{prox}_{\varphi \circ U} = U \circ \text{prox}_\varphi \circ U$ [27, Lemma 2.8], and $R_2 = \text{Id}$. Then $\|W_2 W_1\| \approx 54.72$. If the input $x = (-3.4, 2)$ is perturbed by $z = 10^{-4} \times (1, \sqrt{3})$, we get $\|T(x+z) - Tx\|/\|z\| \approx 58.18$,

which shows that, although W_1 and W_2 have strictly positive entries, the Lipschitz constant is larger than $\|W_2W_1\|$. Note that, in this scenario, the constant of (3.4) is $\theta_2 = (\|W_2W_1\| + \|W_2\|\|W_1\|)/2 \approx 60.50$. A sharper Lipschitz constant can be obtained by noticing that this network is equivalent to a network in which W_1 , W_2 , and R_1 are replaced by $W'_1 = UW_1$, $W'_2 = W_2U$, and $R'_1 = \text{prox}_\varphi$. Since R'_1 is separable, the constant of (4.4) is $\vartheta_2 \approx 59.54$. In contrast, the naive bound of (1.4) is about 66.29.

For separable activators in finite-dimensional spaces, we have the following result, which does not require Assumption 4.9.

Proposition 4.13. *Consider the setting of Model 1.1 with $m \geq 2$. Suppose that the assumptions of Corollary 4.8 hold and that $\|\cdot\|_{\mathcal{G}_0}$ satisfies (4.12). In addition, assume that, for every $i \in \{0, \dots, m\}$, $\mathcal{H}_i = \mathbb{R}^{N_i}$ and E_i is the canonical basis of \mathbb{R}^{N_i} . For every $i \in \{1, \dots, m\}$, let A_i denote the matrix obtained by taking the absolute values of the entries of the matrix W_i . Then the Lipschitz constant ϑ_m of $T: \mathcal{G}_0 \rightarrow \mathcal{G}_m$ in (4.8) satisfies $\vartheta_m \leq \|A_m \cdots A_1\|_{\mathcal{G}_0, \mathcal{G}_m}$.*

5. Conclusion. Using advanced tools from nonlinear analysis, we have derived sharp Lipschitz constants for layered network structures involving compositions of nonexpansive averaged operators and affine operators. This framework has been shown to model feed-forward neural networks. Among the many avenues of future research that this work suggests, it would be interesting to exploit it to devise training strategies to achieve better robustness. The proposed nonexpansive operator machinery could also be used to design network architectures with smaller Lipschitz constants.

Appendix A. Technical lemmas.

Lemma A.1. [24, Proposition 2.4] *Let R be a function defined from \mathbb{R} to \mathbb{R} . Then R is the proximity operator of a function in $\Gamma_0(\mathbb{R})$ if and only if it is nonexpansive and increasing.*

Lemma A.2. *Let $q \in \mathbb{N} \setminus \{0\}$ and, for every $i \in \{1, \dots, q\}$, let S_i be a nonempty subset of a real vector space \mathcal{X}_i . Let $\psi: \mathcal{X}_1 \times \cdots \times \mathcal{X}_q \rightarrow \mathbb{R}$ be a function which is convex with respect to each of its q coordinates. Set $\mathcal{S} = S_1 \times \cdots \times S_q$ and let $\text{conv } \mathcal{S}$ be its convex envelope. Then $\sup \psi(\mathcal{S}) = \sup \psi(\text{conv } \mathcal{S})$.*

Proof. Set $\mu = \sup \psi(\mathcal{S})$. Clearly, $\mu \leq \sup \psi(\text{conv } \mathcal{S})$. Now take $\mathbf{x} \in \text{conv } \mathcal{S}$. Then $\mathbf{x} = \sum_{j \in I} \alpha_j \mathbf{x}_j$, where $(\alpha_j)_{j \in I}$ is a finite family in $]0, 1]$ such that $\sum_{j \in I} \alpha_j = 1$ and, for every $j \in I$, $\mathbf{x}_j = (x_{j,i})_{1 \leq i \leq q}$, with $(\forall i \in \{1, \dots, q\}) x_{j,i} \in S_i$. Note that $(\forall (j_1, \dots, j_q) \in I^q) (x_{j_1,1}, \dots, x_{j_q,q}) \in \mathcal{S}$. Therefore,

$$\begin{aligned}
\psi(\mathbf{x}) &= \psi\left(\sum_{j_1 \in I} \alpha_{j_1} x_{j_1,1}, \dots, \sum_{j_q \in I} \alpha_{j_q} x_{j_q,q}\right) \\
&\leq \sum_{j_1 \in I} \alpha_{j_1} \psi\left(x_{j_1,1}, \sum_{j_2 \in I} \alpha_{j_2} x_{j_2,2}, \dots, \sum_{j_q \in I} \alpha_{j_q} x_{j_q,q}\right) \\
&\quad \vdots \\
&\leq \sum_{j_1 \in I, \dots, j_q \in I} \left(\prod_{i=1}^q \alpha_{j_i}\right) \psi(x_{j_1,1}, \dots, x_{j_q,q}) \leq \mu.
\end{aligned}
\tag{A.1}$$

Hence, $\sup \psi(\text{conv } \mathbf{S}) = \sup_{\mathbf{x} \in \text{conv } \mathbf{S}} \psi(\mathbf{x}) \leq \mu$. ■

Lemma A.3. *Let \mathcal{H} be a separable real Hilbert space, let $\emptyset \neq \mathbb{K} \subset \mathbb{N}$, let $E = (e_k)_{k \in \mathbb{K}}$ be an orthonormal basis of \mathcal{H} , and let $\alpha \in [0, 1]$. For every $k \in \mathbb{K}$, let $\varrho_k: \mathbb{R} \rightarrow \mathbb{R}$ be α -averaged and such that $\varrho_k(0) = 0$. Define $R: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto \sum_{k \in \mathbb{K}} \varrho_k(\langle x | e_k \rangle) e_k$, and fix x and y in \mathcal{H} . Then there exists $\Lambda \in \mathcal{D}_{[1-2\alpha, 1]}(E)$ such that $Rx - Ry = \Lambda(x - y)$.*

Proof. We saw in Example 2.4 that R is well defined. We have

$$(A.2) \quad Rx - Ry = \sum_{k \in \mathbb{K}} (\varrho_k(\langle x | e_k \rangle) - \varrho_k(\langle y | e_k \rangle)) e_k.$$

For every $k \in \mathbb{K}$, there exists a nonexpansive $\theta_k: \mathbb{R} \rightarrow \mathbb{R}$ such that $\varrho_k = (1 - \alpha) \text{Id} + \alpha \theta_k$ and, therefore,

$$(A.3) \quad \varrho_k(\langle x | e_k \rangle) - \varrho_k(\langle y | e_k \rangle) = (1 - \alpha)(\langle x | e_k \rangle - \langle y | e_k \rangle) + \alpha(\theta_k(\langle x | e_k \rangle) - \theta_k(\langle y | e_k \rangle)).$$

Consequently, for every $k \in \mathbb{K}$, there exists $\lambda_k \in [1 - 2\alpha, 1]$ such that

$$(A.4) \quad (1 - \alpha)(\langle x | e_k \rangle - \langle y | e_k \rangle) + \alpha(\theta_k(\langle x | e_k \rangle) - \theta_k(\langle y | e_k \rangle)) = \lambda_k(\langle x | e_k \rangle - \langle y | e_k \rangle).$$

We deduce from (A.2) that $Rx - Ry = \sum_{k \in \mathbb{K}} \lambda_k(\langle x | e_k \rangle - \langle y | e_k \rangle) e_k$, as claimed. ■

Appendix B. Proofs of main results.

B.1. Proof of Proposition 2.1. (i): As seen in (1.2), there exists a nonexpansive operator $Q: \mathcal{H} \rightarrow \mathcal{H}$ such that $R = (1 - \alpha) \text{Id} + \alpha Q$. However, by [9, Prop. 4.4 and Cor. 23.9], there exists a maximally monotone operator $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ such that $Q = 2J_A - \text{Id}$. Hence, $R = \text{Id} + \lambda(J_A - \text{Id})$, where $\lambda = 2\alpha \in [0, 2]$. For the last claim, notice that, since J_A is firmly nonexpansive [9, Cor. 23.9], so is $R = (1 - \lambda) \text{Id} + \lambda J_A$ as a convex combination of two firmly nonexpansive operators [9, Exa. 4.7].

(ii) \Rightarrow (i): It follows from [9, Cor. 22.23] that there exists $\phi \in \Gamma_0(\mathbb{R})$ such that $A = \partial\phi$, which provides the expression for R . The increasingness claim follows from Lemma A.1. Finally, if ϕ is even, then prox_ϕ is odd [9, Prop. 24.10] and so is R .

(iii): This follows from Lemma A.1.

B.2. Proof of Example 2.3. Let σ_C be the support function of C and set $f = \sigma_C + \phi \circ \|\cdot\| \in \Gamma_0(\mathcal{H})$. Then it follows from [9, Prop. 24.30] and (2.14) that $R = \text{Id} + \lambda(\text{prox}_f - \text{Id})$. However, since prox_f is firmly nonexpansive, it is $1/2$ -averaged, which makes R a $\lambda/2$ -averaged operator. Now consider the function ϕ of (2.10). Then it is an even function in $\Gamma_0(\mathbb{R})$ with 0 as its unique minimizer. Next, set $\psi = |\cdot| - \arctan |\cdot|$. As seen in Example 2.2(ii), $\phi = \mu\psi^*(\cdot/\mu) - |\cdot|^2/2$ and $\text{dom } \psi^*$ is bounded. Therefore $\text{dom } \phi = \mu \text{dom } \psi^*$ is bounded. In turn, ϕ is supercoercive and we derive from [9, Prop. 14.15] that $\text{dom } \phi^* = \mathbb{R}$. Hence, since $\phi = \phi^{**}$ is strictly convex, it follows derive from [9, Prop. 18.9] that ϕ^* is differentiable on \mathbb{R} . In addition, $d_C = \|\cdot\|$. Altogether, (2.14) reduces to

$$(B.1) \quad (\forall x \in \mathcal{H}) \quad Rx = \begin{cases} \frac{\text{prox}_\phi \|x\|}{\|x\|} x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$$

and hence, in view of Example 2.2(ii), to (2.15).

B.3. Proof of Example 2.4. Let $x \in \mathcal{H}$ and $y \in \mathcal{H}$. It follows from the nonexpansiveness of the functions $(\varrho_k)_{k \in \mathbb{K}}$ that

$$(B.2) \quad \sum_{k \in \mathbb{K}} |\varrho_k(\langle x | e_k \rangle)|^2 = \sum_{k \in \mathbb{K}} |\varrho_k(\langle x | e_k \rangle) - \varrho_k(0)|^2 \leq \sum_{k \in \mathbb{K}} |\langle x | e_k \rangle - 0|^2 = \|x\|^2.$$

Hence, R is well defined. For every $k \in \mathbb{K}$, (1.2) there exists a nonexpansive function $\theta_k: \mathbb{R} \rightarrow \mathbb{R}$ such that $\varrho_k = (1 - \alpha)\text{Id} + \alpha\theta_k$. Hence, $Rx = (1 - \alpha)x + \alpha Qx$, where $Qx = \sum_{k \in \mathbb{K}} \theta_k(\langle x | e_k \rangle)e_k$. Therefore,

$$(B.3) \quad \|Qx - Qy\|^2 = \sum_{k \in \mathbb{K}} |\theta_k(\langle x | e_k \rangle) - \theta_k(\langle y | e_k \rangle)|^2 \leq \sum_{k \in \mathbb{K}} |\langle x | e_k \rangle - \langle y | e_k \rangle|^2 = \|x - y\|^2.$$

This shows that Q is nonexpansive and hence that R is α -averaged.

B.4. Proof of Example 2.5. Let S be the sorting operator of Example 2.7. Then

$$(B.4) \quad \begin{aligned} (\forall x \in \mathbb{R}^N)(\forall y \in \mathbb{R}^N) \quad \|Sx - Sy\|^2 &= \|Sx\|^2 - 2\langle Sx | Sy \rangle + \|Sy\|^2 \\ &= \|x\|^2 - 2\langle x | y \rangle + \|y\|^2 \\ (B.5) \quad &\leq \|x\|^2 - 2\langle x | y \rangle + \|y\|^2 \\ &= \|x - y\|^2, \end{aligned}$$

where (B.4) follows from [34, Thm. 368]. This shows that S is nonexpansive. Furthermore, $Q = 2\text{proj}_C - \text{Id}$ is nonexpansive [9, Cor. 4.18]. Note that

$$(B.6) \quad (1 - \omega)\text{proj}_C + \omega S = \left(1 - \frac{1 + \omega}{2}\right) \text{Id} + \frac{1 + \omega}{2} \left(\frac{1 - \omega}{1 + \omega} Q + \frac{2\omega}{1 + \omega} S\right).$$

Since $((1 - \omega)Q + 2\omega S)/(1 + \omega)$ is nonexpansive as a convex combination of nonexpansive operators, the operator $(1 - \omega)\text{proj}_C + \omega S$ is $(1 + \omega)/2$ -averaged.

B.5. Proof of Example 2.7. Set $A = \text{Diag}(\tau_1, \dots, \tau_{N-1})$. Let x and y be in \mathbb{R}^{N-1} , and define $\tilde{x} = [(Ax)^\top, \theta]^\top$ and $\tilde{y} = [(Ay)^\top, \theta]^\top$. As seen in (B.5), S is nonexpansive. Consequently,

$$(B.7) \quad \begin{aligned} \|Rx - Ry\| &= \|US\tilde{x} - US\tilde{y}\| \leq \|U\| \|S\tilde{x} - S\tilde{y}\| = \|S\tilde{x} - S\tilde{y}\| \\ &\leq \|\tilde{x} - \tilde{y}\| = \|Ax - Ay\| \leq \max\{|\tau_1|, \dots, |\tau_{N-1}|\} \|x - y\|. \end{aligned}$$

This shows that R is Lipschitzian with constant $\max\{|\tau_1|, \dots, |\tau_{N-1}|\} < 1$. It is thus α -averaged with $\alpha = (1 + \max\{|\tau_1|, \dots, |\tau_{N-1}|\})/2$ [9, Prop. 4.38].

B.6. Proof of Theorem 3.2. Note that, for every $i \in \{1, \dots, m\}$, $P_i = R_i(\cdot + b_i)$ is α_i -averaged and, therefore, there exists a nonexpansive operator $Q_i: \mathcal{H}_i \rightarrow \mathcal{H}_i$ such that $P_i = (1 - \alpha_i)\text{Id} + \alpha_i Q_i$. Since $T = P_m \circ W_m \circ \dots \circ P_1 \circ W_1$ and P_m is nonexpansive, it suffices to show that

$$(B.8) \quad \theta_m \text{ is a Lipschitz constant of } W_m \circ \dots \circ P_1 \circ W_1.$$

Let us prove this result by induction. Let $x \in \mathcal{H}_0$ and $y \in \mathcal{H}_0$. If $m = 2$, we derive from the nonexpansiveness of Q_1 that

$$\begin{aligned}
& \|(W_2 \circ P_1 \circ W_1)x - (W_2 \circ P_1 \circ W_1)y\| \\
&= \|(W_2 \circ ((1 - \alpha_1) \text{Id} + \alpha_1 Q_1) \circ W_1)x - (W_2 \circ ((1 - \alpha_1) \text{Id} + \alpha_1 Q_1) \circ W_1)y\| \\
&\leq (1 - \alpha_1)\|(W_2 \circ W_1)(x - y)\| + \alpha_1\|(W_2 \circ Q_1 \circ W_1)x - (W_2 \circ Q_1 \circ W_1)y\| \\
&\leq (1 - \alpha_1)\|W_2 \circ W_1\| \|x - y\| + \alpha_1\|W_2\| \|Q_1(W_1x) - Q_1(W_1y)\| \\
&\leq (1 - \alpha_1)\|W_2 \circ W_1\| \|x - y\| + \alpha_1\|W_2\| \|W_1(x - y)\| \\
\text{(B.9)} \quad &\leq ((1 - \alpha_1)\|W_2 \circ W_1\| + \alpha_1\|W_2\| \|W_1\|)\|x - y\|.
\end{aligned}$$

Hence, T is Lipschitzian with constant

$$\text{(B.10)} \quad (1 - \alpha_1)\|W_2 \circ W_1\| + \alpha_1\|W_2\| \|W_1\| = \beta_{2;\emptyset}\|W_2 \circ W_1\| + \beta_{2;\{1\}}\|W_2\| \|W_1\| = \theta_2.$$

Now assume that $m > 2$ and that (B.8) holds at order $m - 1$. Then

$$\begin{aligned}
& \|(W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)y\| \\
&= \|(W_m \circ ((1 - \alpha_{m-1}) \text{Id} + \alpha_{m-1} Q_{m-1}) \circ \cdots \circ P_1 \circ W_1)x \\
&\quad - (W_m \circ ((1 - \alpha_{m-1}) \text{Id} + \alpha_{m-1} Q_{m-1}) \circ \cdots \circ P_1 \circ W_1)y\| \\
&\leq (1 - \alpha_{m-1})\|(W_m \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)y\| \\
&\quad + \alpha_{m-1}\|(W_m \circ Q_{m-1} \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ Q_{m-1} \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)y\| \\
&\leq (1 - \alpha_{m-1})\|(W_m \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)y\| \\
&\quad + \alpha_{m-1}\|W_m\| \|(Q_{m-1} \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (Q_{m-1} \circ W_{m-1} \circ \cdots \circ P_1 \circ W_1)y\|.
\end{aligned}$$

Hence, the nonexpansiveness of Q_{m-1} yields

$$\begin{aligned}
& \|(W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)y\| \\
&\leq (1 - \alpha_{m-1})\|(W_m \circ W_{m-1} \circ P_{m-2} \circ \cdots \circ W_1)x - (W_m \circ W_{m-1} \circ P_{m-2} \circ \cdots \circ W_1)y\| \\
&\quad + \alpha_{m-1}\|W_m\| \|(W_{m-1} \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)x - (W_{m-1} \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)y\|. \\
\text{(B.11)} \quad &
\end{aligned}$$

On the other hand, the induction hypothesis yields

$$\begin{aligned}
& \|(W_{m-1} \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)x - (W_{m-1} \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)y\| \\
&\leq \theta_{m-1}\|x - y\| \\
&= \left(\beta_{m-1;\emptyset}\|W_{m-1} \circ \cdots \circ W_1\| + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} \beta_{m-1;\{j_1, \dots, j_k\}} \sigma_{m-1;\{j_1, \dots, j_k\}} \right) \|x - y\|. \\
\text{(B.12)} \quad &
\end{aligned}$$

Similarly, replacing W_{m-1} by $W_m \circ W_{m-1}$ above, we get

$$\begin{aligned} & \|((W_m \circ W_{m-1}) \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)x - ((W_m \circ W_{m-1}) \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)y\| \\ & \leq \left(\beta_{m-1; \emptyset} \|W_m \circ W_{m-1} \circ \cdots \circ W_1\| + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} \beta_{m-1; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \right) \|x - y\|. \end{aligned}$$

(B.13)

Using (B.11), and then inserting (B.13) and (B.12), we obtain

$$\begin{aligned} & \|(W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)y\| \\ & \leq (1 - \alpha_{m-1}) \|(W_m \circ W_{m-1} \circ P_{m-2} \circ \cdots \circ W_1)x - (W_m \circ W_{m-1} \circ P_{m-2} \circ \cdots \circ W_1)y\| \\ & \quad + \alpha_{m-1} \|W_m\| \|(W_{m-1} \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)x - (W_{m-1} \circ P_{m-2} \circ \cdots \circ P_1 \circ W_1)y\| \\ & \leq (1 - \alpha_{m-1}) \times \\ & \quad \left(\beta_{m-1; \emptyset} \|W_m \circ W_{m-1} \circ \cdots \circ W_1\| + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} \beta_{m-1; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \right) \|x - y\| \\ & \quad + \alpha_{m-1} \|W_m\| \times \\ & \quad \left(\beta_{m-1; \emptyset} \|W_{m-1} \circ \cdots \circ W_1\| + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} \beta_{m-1; \{j_1, \dots, j_k\}} \sigma_{m-1; \{j_1, \dots, j_k\}} \right) \|x - y\|, \end{aligned}$$

(B.14)

Furthermore, we deduce from (3.3) that

$$(B.15) \quad (\forall \mathbb{J} \subset \{1, \dots, m-1\}) \quad \beta_{m; \mathbb{J}} = \begin{cases} (1 - \alpha_{m-1}) \beta_{m-1; \mathbb{J}}, & \text{if } m-1 \notin \mathbb{J}; \\ \alpha_{m-1} \beta_{m-1; \mathbb{J} \setminus \{m-1\}}, & \text{if } m-1 \in \mathbb{J}. \end{cases}$$

Therefore

$$(B.16) \quad \begin{cases} \beta_{m; \emptyset} = (1 - \alpha_{m-1}) \beta_{m-1; \emptyset} \\ \beta_{m; \{j_1, \dots, j_k\}} = (1 - \alpha_{m-1}) \beta_{m-1; \{j_1, \dots, j_k\}} & \text{if } m-1 \notin \{j_1, \dots, j_k\} \\ \beta_{m; \{j_1, \dots, j_k\}} = \alpha_{m-1} \beta_{m-1; \{j_1, \dots, j_k\} \setminus \{m-1\}} & \text{if } m-1 \in \{j_1, \dots, j_k\}, \end{cases}$$

which implies that, if $m-1 \notin \{j_1, \dots, j_k\}$, then $\beta_{m; \{j_1, \dots, j_k, m-1\}} = \alpha_{m-1} \beta_{m-1; \{j_1, \dots, j_k\}}$. Hence,

(B.14) yields

$$\begin{aligned}
& \| (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)y \| / \|x - y\| \\
& \leq \beta_{m;\emptyset} \|W_m \circ W_{m-1} \circ \cdots \circ W_1\| \\
& \quad + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} (1 - \alpha_{m-1}) \beta_{m-1; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\
& \quad + \alpha_{m-1} \beta_{m-1; \emptyset} \|W_m\| \|W_{m-1} \circ \cdots \circ W_1\| \\
& \quad + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} \alpha_{m-1} \beta_{m-1; \{j_1, \dots, j_k\}} \|W_m\| \sigma_{m-1; \{j_1, \dots, j_k\}} \\
& = \beta_{m;\emptyset} \|W_m \circ W_{m-1} \circ \cdots \circ W_1\| + \beta_{m; m-1} \sigma_{m; \{m-1\}} \\
& \quad + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m, k} \setminus \{m-1\}} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\
& \quad + \sum_{k=1}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m-1, k}} \beta_{m; \{j_1, \dots, j_k, m-1\}} \sigma_{m; \{j_1, \dots, j_k, m-1\}} \\
& = \beta_{m;\emptyset} \|W_m \circ \cdots \circ W_1\| + \sum_{j=1}^m \beta_{m; \{j\}} \sigma_{m; \{j\}} \\
& \quad + \sum_{k=2}^{m-2} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m, k} \setminus \{m-1\}} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\
& \quad + \sum_{k=2}^{m-1} \sum_{\substack{(j_1, \dots, j_k) \in \mathbb{J}_{m, k} \\ j_k = m-1}} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\
& = \beta_{m;\emptyset} \|W_m \circ \cdots \circ W_1\| + \sum_{k=1}^{m-1} \sum_{(j_1, \dots, j_k) \in \mathbb{J}_{m, k}} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\
& = \theta_m.
\end{aligned}
\tag{B.17}$$

Thus, we obtain

$$\| (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)x - (W_m \circ P_{m-1} \circ \cdots \circ P_1 \circ W_1)y \| \leq \theta_m \|x - y\|,
\tag{B.18}$$

which establishes (B.8).

B.7. Proof of Proposition 3.3. Define $(\beta_{m; \mathbb{J}})_{\mathbb{J} \subset \{1, \dots, m-1\}}$ as in (3.3).

(i): For every $k \in \{1, \dots, m-1\}$ and every $(j_1, \dots, j_k) \in \mathbb{J}_{m, k}$, (3.2) yields

$$\|W_m \circ \cdots \circ W_1\| \leq \sigma_{m; \{j_1, \dots, j_k\}} \leq \prod_{i=1}^m \|W_i\|.
\tag{B.19}$$

Consequently, it follows from (3.4) that

$$(B.20) \quad \|W_m \circ \dots \circ W_1\| \sum_{\mathbb{J} \subset \{1, \dots, m-1\}} \beta_{m; \mathbb{J}} \leq \theta_m \leq \left(\prod_{i=1}^m \|W_i\| \right) \sum_{\mathbb{J} \subset \{1, \dots, m-1\}} \beta_{m; \mathbb{J}}.$$

In view of (3.3), $(\beta_{m; \mathbb{J}})_{\mathbb{J} \subset \{1, \dots, m-1\}}$ is the discrete probability distribution of a vector of $m-1$ independent Bernoulli random variables. Hence, $\sum_{\mathbb{J} \subset \{1, \dots, m-1\}} \beta_{m; \mathbb{J}} = 1$ in (B.20).

(ii): For every $i \in \{1, \dots, m-1\}$, $\alpha_i = 0$. Therefore, in view of (3.3),

$$(B.21) \quad (\forall \mathbb{J} \subset \{1, \dots, m-1\}) \quad \beta_{m; \mathbb{J}} = \begin{cases} 1, & \text{if } \mathbb{J} = \emptyset; \\ 0, & \text{if } \mathbb{J} \neq \emptyset. \end{cases}$$

Hence, the result follows from (3.4).

(iii): For every $i \in \{1, \dots, m-1\}$, $\alpha_i = 1$. Therefore, in view of (3.3),

$$(B.22) \quad (\forall \mathbb{J} \subset \{1, \dots, m-1\}) \quad \beta_{m; \mathbb{J}} = \begin{cases} 1, & \text{if } \mathbb{J} = \{1, \dots, m-1\}; \\ 0, & \text{if } \mathbb{J} \neq \{1, \dots, m-1\}. \end{cases}$$

Invoking (3.4) allows us to conclude.

(iv): For every $i \in \{1, \dots, m-1\}$ $\alpha_i = 1/2$. Hence, (3.3) yields $(\forall \mathbb{J} \subset \{1, \dots, m-1\}) \beta_{m; \mathbb{J}} = 2^{1-m}$. Invoking once again (3.4) yields the result.

(v): It follows from (3.2) that

$$(B.23) \quad \begin{aligned} & \sum_{k=1}^{m-1} \sum_{1 \leq j_1 < \dots < j_k \leq m-1} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\ &= \sum_{k=1}^{m-1} \sum_{1 \leq j_1 < \dots < j_k \leq m-1} \beta_{m; \{j_1, \dots, j_k\}} \|W_m \circ \dots \circ W_{j_{k+1}}\| \|W_{j_k} \circ \dots \circ W_{j_{k-1}+1}\| \\ & \quad \dots \|W_{j_1} \circ \dots \circ W_1\|. \end{aligned}$$

Next, we decompose this expression in a sum of terms depending on the value i taken by j_k , namely

$$(B.24) \quad \begin{aligned} & \sum_{k=1}^{m-1} \sum_{1 \leq j_1 < \dots < j_k \leq m-1} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\ &= \sum_{i=1}^{m-1} \beta_{m; \{i\}} \|W_m \circ \dots \circ W_{i+1}\| \|W_i \circ \dots \circ W_1\| \\ & \quad + \sum_{k=2}^{i-1} \sum_{1 \leq j_1 < \dots < j_{k-1} \leq i-1} \beta_{m; \{j_1, \dots, j_{k-1}, i\}} \|W_m \circ \dots \circ W_{i+1}\| \|W_i \circ \dots \circ W_{j_{k-1}+1}\| \\ & \quad \dots \|W_{j_1} \circ \dots \circ W_1\|. \end{aligned}$$

In addition, for every $(j_1, \dots, j_{k-1}) \in \mathbb{J}_{i, k-1}$, we derive from (3.3) that

$$\begin{aligned}
\beta_{m; \{j_1, \dots, j_{k-1}, i\}} &= \left(\prod_{j \in \{j_1, \dots, j_{k-1}, i\}} \alpha_j \right) \prod_{j \in \{1, \dots, m-1\} \setminus \{j_1, \dots, j_{k-1}, i\}} (1 - \alpha_j) \\
&= \alpha_i \left(\prod_{j \in \{j_1, \dots, j_{k-1}\}} \alpha_j \right) \left(\prod_{q=i+1}^{m-1} (1 - \alpha_q) \right) \prod_{j \in \{1, \dots, i-1\} \setminus \{j_1, \dots, j_{k-1}\}} (1 - \alpha_j) \\
\text{(B.25)} \quad &= \alpha_i \left(\prod_{q=i+1}^{m-1} (1 - \alpha_q) \right) \beta_{i; \{j_1, \dots, j_{k-1}\}}.
\end{aligned}$$

Using the above equality in (B.24), factorizing common factors, and using (3.4) yields

$$\begin{aligned}
&\sum_{k=1}^{m-1} \sum_{1 \leq j_1 < \dots < j_k \leq m-1} \beta_{m; \{j_1, \dots, j_k\}} \sigma_{m; \{j_1, \dots, j_k\}} \\
&= \sum_{i=1}^{m-1} \alpha_i \left(\prod_{q=i+1}^{m-1} (1 - \alpha_q) \right) \|W_m \circ \dots \circ W_{i+1}\| \left(\beta_{i; \emptyset} \|W_i \circ \dots \circ W_1\| \right. \\
&\quad \left. + \sum_{k=2}^i \sum_{1 \leq j_1 < \dots < j_{k-1} \leq i-1} \beta_{i; \{j_1, \dots, j_{k-1}\}} \|W_i \circ \dots \circ W_{j_{k-1}+1}\| \dots \|W_{j_1} \circ \dots \circ W_1\| \right) \\
\text{(B.26)} \quad &= \sum_{i=1}^{m-1} \alpha_i \theta_i \left(\prod_{q=i+1}^{m-1} (1 - \alpha_q) \right) \|W_m \circ \dots \circ W_{i+1}\|,
\end{aligned}$$

and we obtain (3.6).

B.8. Proof of Proposition 3.6. Let $l \in \{1, \dots, m-1\}$ and set

$$\text{(B.27)} \quad (\forall \mathbb{J} \subset \{1, \dots, m-1\} \setminus \{l\}) \quad \beta_{m, l; \mathbb{J}} = \left(\prod_{j \in \mathbb{J}} \alpha_j \right) \prod_{j \in \{1, \dots, m-1\} \setminus (\mathbb{J} \cup \{l\})} (1 - \alpha_j).$$

For every $k \in \{1, \dots, m-1\}$ and every $(j_1, \dots, j_k) \in \mathbb{J}_{m, k}$, (3.2) yields

$$\text{(B.28)} \quad \sigma_{m; \{j_1, \dots, j_k\}} \leq \sigma_{m; \{j_1, \dots, j_k\} \cup \{l\}}.$$

We infer from (3.4) that

$$\begin{aligned}
& \theta_m(\alpha_1, \dots, \alpha_{m-1}) \\
&= (1 - \alpha_l)\beta_{m,l;\emptyset}\|W_m \circ \dots \circ W_1\| + \sum_{k=1}^{m-1} \sum_{\substack{(j_1, \dots, j_k) \in \mathbb{J}_{m,k} \\ l \in \{j_1, \dots, j_k\}}} \alpha_l \beta_{m,l;\{j_1, \dots, j_k\} \setminus \{l\}} \sigma_{m;\{j_1, \dots, j_k\}} \\
&+ \sum_{k=1}^{m-2} \sum_{\substack{(j_1, \dots, j_k) \in \mathbb{J}_{m,k} \\ l \notin \{j_1, \dots, j_k\}}} (1 - \alpha_l) \beta_{m,l;\{j_1, \dots, j_k\}} \sigma_{m;\{j_1, \dots, j_k\}} \\
&= \beta_{m,l;\emptyset} \left((1 - \alpha_l) \|W_m \circ \dots \circ W_1\| + \alpha_l \|W_m \circ \dots \circ W_{l+1}\| \|W_l \circ \dots \circ W_1\| \right) \\
\text{(B.29)} \quad &+ \sum_{k=1}^{m-2} \sum_{\substack{(j_1, \dots, j_k) \in \mathbb{J}_{m,k} \\ l \notin \{j_1, \dots, j_k\}}} \beta_{m,l;\{j_1, \dots, j_k\}} \left((1 - \alpha_l) \sigma_{m;\{j_1, \dots, j_k\}} + \alpha_l \sigma_{m;\{j_1, \dots, j_k\} \cup \{l\}} \right).
\end{aligned}$$

It follows from (B.28) that

$$\begin{aligned}
\frac{\partial \theta_m}{\partial \alpha_l}(\alpha_1, \dots, \alpha_{m-1}) &= \beta_{m,l;\emptyset} \left(\|W_m \circ \dots \circ W_{l+1}\| \|W_l \circ \dots \circ W_1\| - \|W_m \circ \dots \circ W_1\| \right) \\
\text{(B.30)} \quad &+ \sum_{k=1}^{m-2} \sum_{\substack{(j_1, \dots, j_k) \in \mathbb{J}_{m,k} \\ l \notin \{j_1, \dots, j_k\}}} \beta_{m,l;\{j_1, \dots, j_k\}} \left(\sigma_{m;\{j_1, \dots, j_k\} \cup \{l\}} - \sigma_{m;\{j_1, \dots, j_k\}} \right) \geq 0,
\end{aligned}$$

which concludes the proof.

B.9. Proof of Theorem 4.2. (i): For every $i \in \{1, \dots, m\}$, set $P_i = R_i(\cdot + b_i)$ and $(\forall k \in \mathbb{K}_i)$ $\pi_{i,k} = \varrho_{i,k}(\cdot + \langle b_i | e_{i,k} \rangle)$. Note that, for every $i \in \{1, \dots, m\}$ and every $k \in \mathbb{K}_i$, $\pi_{i,k}$ is α_i -averaged. Furthermore,

$$\text{(B.31)} \quad (\forall i \in \{1, \dots, m-1\})(\forall x \in \mathcal{H}_i) \quad P_i x = \sum_{k \in \mathbb{K}_i} \pi_{i,k}(\langle x | e_{i,k} \rangle) e_{i,k}.$$

Now fix x and y in \mathcal{H}_0 . It follows from (1.3) and the nonexpansiveness of P_m that

$$\text{(B.32)} \quad \|Tx - Ty\| \leq \|(W_m \circ P_{m-1} \circ W_{m-1} \circ \dots \circ P_1 \circ W_1)x - (W_m \circ P_{m-1} \circ W_{m-1} \circ \dots \circ P_1 \circ W_1)y\|.$$

In view of Lemma A.3, for every $i \in \{1, \dots, m-1\}$, there exists $\Lambda_i \in \mathcal{D}_{[1-2\alpha_i, 1]}(E_i)$ such that

$$\begin{aligned}
\text{(B.33)} \quad & (P_i \circ W_i \circ \dots \circ P_1 \circ W_1)x - (P_i \circ W_i \circ \dots \circ P_1 \circ W_1)y \\
&= \Lambda_i \left((W_i \circ P_{i-1} \circ \dots \circ P_1 \circ W_1)x - (W_i \circ P_{i-1} \circ \dots \circ P_1 \circ W_1)y \right).
\end{aligned}$$

Recursive application of this identity yields

$$\begin{aligned}
\text{(B.34)} \quad & (P_{m-1} \circ W_{m-1} \circ \dots \circ P_1 \circ W_1)x - (P_{m-1} \circ W_{m-1} \circ \dots \circ P_1 \circ W_1)y \\
&= (\Lambda_{m-1} \circ W_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)(x - y).
\end{aligned}$$

This implies that $\|Tx - Ty\| \leq \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\| \|x - y\|$. Thus, a Lipschitz constant of T is

$$(B.35) \quad \vartheta_m = \sup_{\Lambda_1 \in \mathcal{D}_{[1-2\alpha_1, 1]}(E_1)} \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\| \\ \vdots \\ \sup_{\Lambda_{m-1} \in \mathcal{D}_{[1-2\alpha_{m-1}, 1]}(E_{m-1})}$$

Set $S = \{1 - 2\alpha_1, 1\}^{\mathbb{K}_1} \times \dots \times \{1 - 2\alpha_{m-1}, 1\}^{\mathbb{K}_{m-1}}$ and $C = [1 - 2\alpha_1, 1]^{\mathbb{K}_1} \times \dots \times [1 - 2\alpha_{m-1}, 1]^{\mathbb{K}_{m-1}}$. For every $i \in \{1, \dots, m-1\}$, $\Lambda_i: \mathcal{H}_i \rightarrow \mathcal{H}_i$ is generated from a sequence $(\lambda_{i,k})_{k \in \mathbb{K}_i}$ in $[1 - 2\alpha_i, 1]$ via the construction described in (4.1). The function

$$(B.36) \quad \psi: \begin{array}{ccc} C & & \rightarrow \mathbb{R} \\ ((\lambda_{1,k})_{k \in \mathbb{K}_1}, \dots, (\lambda_{m-1,k})_{k \in \mathbb{K}_{m-1}}) & \mapsto & \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\| \end{array}$$

is convex with respect to each of its coordinates. Hence, we deduce from Lemma A.2 that $\sup \psi(C) = \sup \psi(\text{conv } S) = \sup \psi(S)$, as claimed.

(ii): For every $i \in \{1, \dots, m-1\}$, the identity operator Id_i of \mathcal{H}_i lies in $\mathcal{D}_{\{1-2\alpha_i, 1\}}(E_i)$. Hence, $\vartheta_m \geq \|W_m \circ \text{Id}_{m-1} \circ \dots \circ \text{Id}_1 \circ W_1\| = \|W_m \circ \dots \circ W_1\|$. For every $i \in \{1, \dots, m-1\}$, let $\Lambda_i \in \mathcal{D}_{\{1-2\alpha_i, 1\}}(E_i)$ and note that the linear operator

$$(B.37) \quad \Theta_i = \begin{cases} \frac{\Lambda_i - (1 - \alpha_i) \text{Id}_i}{\alpha_i}, & \text{if } \alpha_i \neq 0; \\ 0, & \text{otherwise} \end{cases}$$

is nonexpansive. Performing the same kind of decomposition as in the proof of Theorem 3.2 yields

$$(B.38) \quad \begin{aligned} & \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\| \\ &= \|W_m \circ ((1 - \alpha_{m-1}) \text{Id}_{m-1} + \alpha_{m-1} \Theta_{m-1}) \circ \dots \circ ((1 - \alpha_1) \text{Id}_1 + \alpha_1 \Theta_1) \circ W_1\| \\ &\leq \theta_m, \end{aligned}$$

and allows us to conclude that $\vartheta_m \leq \theta_m$.

B.10. Proof of Proposition 4.6. It follows from (B.35) that

$$(B.39) \quad \begin{aligned} \vartheta_m(\alpha_1, \dots, \alpha_{m-1}) &= \sup_{\Lambda_1 \in \mathcal{D}_{[1-2\alpha_1, 1]}(E_1)} \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\| \\ &\quad \vdots \\ &\quad \sup_{\Lambda_{m-1} \in \mathcal{D}_{[1-2\alpha_{m-1}, 1]}(E_{m-1})} \\ &\leq \sup_{\Lambda_1 \in \mathcal{D}_{[1-2\alpha'_1, 1]}(E_1)} \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\| \\ &\quad \vdots \\ &\quad \sup_{\Lambda_{m-1} \in \mathcal{D}_{[1-2\alpha'_{m-1}, 1]}(E_{m-1})} \\ &= \vartheta_m(\alpha'_1, \dots, \alpha'_{m-1}), \end{aligned}$$

as claimed.

B.11. Proof of Proposition 4.7. Let us first note that, because of the embeddings, $W_1: \mathcal{G}_0 \rightarrow \mathcal{H}_1$ is continuous and, likewise, every $\Lambda_m \in \mathcal{D}_{[1-2\alpha_m, 1]}(E_m)$ is continuous from \mathcal{H}_m to \mathcal{G}_m . Hence, for every $(\Lambda_i)_{1 \leq i \leq m} \in \mathcal{D}_{[1-2\alpha_1, 1]}(E_1) \times \cdots \times \mathcal{D}_{[1-2\alpha_m, 1]}(E_{m-1})$, $\Lambda_m \circ W_m \circ \cdots \circ \Lambda_1 \circ W_1: \mathcal{G}_0 \rightarrow \mathcal{G}_m$ is continuous. We now follow the same argument as in the proof of Theorem 4.2. Let x and y be in \mathcal{G}_0 . For every $i \in \{1, \dots, m\}$, there exists $\Lambda_i \in \mathcal{D}_{[1-2\alpha_i, 1]}(E_i)$ such that

$$(B.40) \quad Tx - Ty = (\Lambda_m \circ W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1)(x - y).$$

Thus, $\|Tx - Ty\|_{\mathcal{G}_m} \leq \|\Lambda_m \circ W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \|x - y\|_{\mathcal{G}_0}$, which leads to (4.6).

B.12. Proof of Corollary 4.8. Since, for every $x \in \mathcal{H}_m$, $(\langle x | e_{m,k} \rangle)_{k \in \mathbb{K}_m} \in \ell^2(\mathbb{K}_m)$, it follows from Hölder's inequality that $\|\cdot\|_{\mathcal{G}_m}$ in (4.7) is well defined and does provide a continuous embedding of \mathcal{H}_m in \mathcal{G}_m . As in the proof of Theorem 4.2, it is enough to take the supremum in (4.8) over $\mathbf{D} = \mathcal{D}_{[1-2\alpha_1, 1]}(E_1) \times \cdots \times \mathcal{D}_{[1-2\alpha_{m-1}, 1]}(E_{m-1})$. For every $i \in \{1, \dots, m\}$, let $\Lambda_i \in \mathcal{D}_{[1-2\alpha_i, 1]}(E_i)$. Then

$$(B.41) \quad \|\Lambda_m \circ W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \leq \|\Lambda_m\|_{\mathcal{G}_m, \mathcal{G}_m} \|W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}.$$

Let us designate by $(\lambda_{m,k})_{k \in \mathbb{K}_m}$ the sequence in $[1 - 2\alpha_m, 1]$ involved in the construction of Λ_m in (4.1). If $p < +\infty$, then

$$(B.42) \quad \begin{aligned} (\forall x \in \mathcal{H}_m) \quad \|\Lambda_m x\|_{\mathcal{G}_m} &= \left\| \sum_{k \in \mathbb{K}_m} \lambda_{m,k} \langle x | e_{m,k} \rangle e_{m,k} \right\|_{\mathcal{G}_m} = \left| \sum_{k \in \mathbb{K}_m} \omega_k |\lambda_{m,k} \langle x | e_{m,k} \rangle|^p \right|^{1/p} \\ &\leq \left| \sum_{k \in \mathbb{K}_m} \omega_k |\langle x | e_{m,k} \rangle|^p \right|^{1/p} = \|x\|_{\mathcal{G}_m}, \end{aligned}$$

which shows that $\|\Lambda_m\|_{\mathcal{G}_m, \mathcal{G}_m} \leq 1$. This inequality holds analogously if $p = +\infty$. We then deduce from (B.41) that

$$(B.43) \quad \vartheta_m \leq \sup_{(\Lambda_1, \dots, \Lambda_{m-1}) \in \mathbf{D}} \|W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}.$$

On the other hand, it follows from (4.6) that

$$(B.44) \quad \vartheta_m \geq \sup_{(\Lambda_1, \dots, \Lambda_{m-1}) \in \mathbf{D}} \|\text{Id}_m \circ W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m},$$

which concludes the proof.

B.13. Proof of Proposition 4.11. For every $i \in \{1, \dots, m-1\}$, let $\Lambda_i \in \mathcal{D}_{[1-2\alpha_i, 1]}(E_i)$ and let $(\lambda_{i,k})_{k \in \mathbb{K}_i}$ be the associated sequence in (4.1). Define

$$(B.45) \quad (\forall k \in \mathbb{K}_m) \quad \lambda_{m,k} = \begin{cases} -1, & \text{if } (\exists (k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}) \mu_{k_0, \dots, k_{m-1}, k} < 0; \\ 1, & \text{otherwise,} \end{cases} \quad 23$$

and set

$$(B.46) \quad \Lambda_m: \mathcal{H}_m \rightarrow \mathcal{H}_m: x \mapsto \sum_{k \in \mathbb{K}_m} \lambda_{m,k} \langle x | e_{m,k} \rangle e_{m,k}, \quad \text{and} \quad V_m = \Lambda_m W_m.$$

Then, because of (4.10),

$$(B.47) \quad (\forall (k_0, \dots, k_m) \in \mathbb{K}_0 \times \dots \times \mathbb{K}_m) \\ \langle W_1 e_{0,k_0} | e_{1,k_1} \rangle \cdots \langle W_{m-1} e_{m-2,k_{m-2}} | e_{m-1,k_{m-1}} \rangle \langle V_m e_{m-1,k_{m-1}} | e_{m,k_m} \rangle \geq 0.$$

In addition, it follows from (4.7) and (B.45) that

$$(B.48) \quad \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} = \|\Lambda_m \circ V_m \circ \Lambda_{m-1} \circ W_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \\ = \|V_m \circ \Lambda_{m-1} \circ W_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}.$$

Therefore, without loss of generality, we assume that

$$(B.49) \quad (\forall (k_0, \dots, k_m) \in \mathbb{K}_0 \times \dots \times \mathbb{K}_m) \quad \mu_{k_0, \dots, k_m} \geq 0.$$

Let us now show that

$$(B.50) \quad \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \leq \|W_m \circ \dots \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}.$$

Let $\varepsilon \in]0, +\infty[$. Then there exists $x \in \mathcal{H}_0$ such that $\|x\|_{\mathcal{G}_0} = 1$ and

$$(B.51) \quad \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \leq \|(W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x\|_{\mathcal{G}_m} + \varepsilon.$$

If $p < +\infty$ in (4.7), this yields

$$(B.52) \quad \|W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \\ \leq \left| \sum_{k_m \in \mathbb{K}_m} \omega_{k_m} |\langle (W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x | e_{m,k_m} \rangle|^p \right|^{1/p} + \varepsilon.$$

On the other hand,

$$(B.53) \quad (W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x \\ = \sum_{k_{m-1} \in \mathbb{K}_{m-1}} \langle (\Lambda_{m-1} \circ W_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x | e_{m-1,k_{m-1}} \rangle W_m e_{m-1,k_{m-1}}$$

which, in view of (4.1), implies that

$$(\forall k_m \in \mathbb{K}_m) \langle (W_m \circ \Lambda_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x | e_{m,k_m} \rangle \\ = \sum_{k_{m-1} \in \mathbb{K}_{m-1}} \langle (\Lambda_{m-1} \circ W_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x | e_{m-1,k_{m-1}} \rangle \langle W_m e_{m-1,k_{m-1}} | e_{m,k_m} \rangle \\ = \sum_{k_{m-1} \in \mathbb{K}_{m-1}} \lambda_{m-1,k_{m-1}} \langle W_m e_{m-1,k_{m-1}} | e_{m,k_m} \rangle \langle (\Lambda_{m-1} \circ W_{m-1} \circ \dots \circ \Lambda_1 \circ W_1)x | e_{m-1,k_{m-1}} \rangle.$$

Using (4.9) recursively yields

$$(B.54) \quad (\forall k_m \in \mathbb{K}_m) \quad \langle (W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1)x \mid e_{m,k_m} \rangle = \sum_{(k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}} \mu_{k_0, \dots, k_m} \lambda_{m-1, k_{m-1}} \cdots \lambda_{1, k_1} \langle x \mid e_{0, k_0} \rangle.$$

We then deduce from (B.49) that

$$(B.55) \quad (\forall k_m \in \mathbb{K}_m) \quad \left| \langle (W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1)x \mid e_{m,k_m} \rangle \right| = \left| \sum_{(k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}} \mu_{k_0, \dots, k_m} \lambda_{m-1, k_{m-1}} \cdots \lambda_{1, k_1} \langle x \mid e_{0, k_0} \rangle \right| \leq \sum_{(k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}} \mu_{k_0, \dots, k_m} |\lambda_{m-1, k_{m-1}}| \cdots |\lambda_{1, k_1}| |\langle x \mid e_{0, k_0} \rangle| \leq \sum_{(k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}} \mu_{k_0, \dots, k_m} |\langle x \mid e_{0, k_0} \rangle|.$$

Set $y = \sum_{k_0 \in \mathbb{K}_0} |\langle x \mid e_{0, k_0} \rangle| e_{0, k_0}$. In view of (4.12), $\|y\|_{\mathcal{G}_0} = \|x\|_{\mathcal{G}_0} = 1$. Thus, (B.55) yields

$$(B.56) \quad \left| \langle (W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1)x \mid e_{m,k_m} \rangle \right| \leq \sum_{(k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}} \mu_{k_0, \dots, k_m} \langle y \mid e_{0, k_0} \rangle = \langle (W_m \circ \cdots \circ W_1)y \mid e_{m,k_m} \rangle.$$

It then follows from (B.51) and the fact that $\|y\|_{\mathcal{G}_0} = 1$ that

$$(B.57) \quad \|W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \leq \left| \sum_{k_m \in \mathbb{K}_m} \omega_{k_m} |\langle (W_m \circ \cdots \circ W_1)y \mid e_{m,k_m} \rangle|^p \right|^{1/p} + \varepsilon \leq \|(W_m \circ \cdots \circ W_1)y\|_{\mathcal{G}_m} + \varepsilon \leq \|W_m \circ \cdots \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} + \varepsilon.$$

The same inequality is obtained similarly for $p = +\infty$. This establishes (B.50), which leads to

$$(B.58) \quad \sup_{\substack{\Lambda_1 \in \mathcal{D}_{\{1-2\alpha_1, 1\}}(E_1) \\ \vdots \\ \Lambda_{m-1} \in \mathcal{D}_{\{1-2\alpha_{m-1}, 1\}}(E_{m-1})}} \|W_m \circ \Lambda_{m-1} \circ \cdots \circ \Lambda_1 \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \leq \|W_m \circ \cdots \circ W_1\|_{\mathcal{G}_0, \mathcal{G}_m}.$$

Since the converse inequality holds straightforwardly (see (B.44)), the proof is complete.

B.14. Proof of Proposition 4.13. We use arguments similar to those of the proof of Proposition 4.11. For every $i \in \{1, \dots, m-1\}$, let $\Lambda_i \in \mathcal{D}_{\{1-2\alpha_i, 1\}}(E_i)$. There exists $x \in \mathcal{H}_0$ such that $\|x\|_{\mathcal{G}_0} = 1$ and

$$(B.59) \quad \|W_m \Lambda_{m-1} \cdots \Lambda_1 W_1\|_{\mathcal{G}_0, \mathcal{G}_m} = \|(W_m \Lambda_{m-1} \cdots \Lambda_1 W_1)x\|_{\mathcal{G}_m}.$$

On the other hand, for every $k_m \in \mathbb{K}_m$,

$$(B.60) \quad |\langle W_m \Lambda_{m-1} \cdots \Lambda_1 W_1 x \mid e_{m,k_m} \rangle| \leq \sum_{(k_0, \dots, k_{m-1}) \in \mathbb{K}_0 \times \cdots \times \mathbb{K}_{m-1}} |\mu_{k_0, \dots, k_m}| |\langle x \mid e_{0,k_0} \rangle|.$$

Setting $y = \sum_{k_0 \in \mathbb{K}_0} |\langle x \mid e_{0,k_0} \rangle| e_{0,k_0}$ yields $|\langle W_m \Lambda_{m-1} \cdots \Lambda_1 W_1 x \mid e_{m,k_m} \rangle| \leq \langle (A_m \cdots A_1) y \mid e_{m,k_m} \rangle$, and (B.59) implies that $\|W_m \Lambda_{m-1} \cdots \Lambda_1 W_1\|_{\mathcal{G}_0, \mathcal{G}_m} \leq \|A_m \cdots A_1\|_{\mathcal{G}_m} \leq \|A_m \cdots A_1\|_{\mathcal{G}_0, \mathcal{G}_m}$, which concludes the proof.

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