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Backwards mixing

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In three-dimensional time-periodic advection of a passive scalar, there are three non-zero Lyapunov exponents; because of incompressibility, the sum of those exponents is zero. Using direct numerical simulation of the advection-diffusion equation, we show that when one is positive and the two others are negative, spatial structures of the scalar field in the form of filaments are generated by the flow. When reversing the flow in time, we change the sign of all Lyapunov exponents, and obtain sheets. While dimensional analysis suggests that diffusion by thinner structures is larger, so that the sheets, associated to the most negative Lyapunov exponent, should dissipate more rapidly the scalar energy, we find numerically that the scalar energy decay is the same for both types of structures. We prove this result using a symmetry argument on the advection-diffusion operator. This evidences that the decay of the variance of a scalar field is not linked to the sign of the intermediate exponent.

I. INTRODUCTION

Mixing plays an important role in many industrial processes and natural phenomena; more importantly, the decay rate of the variance of a passive scalar field, advected by turbulent or laminar flows, through the combined effects of mechanical stirring and molecular diffusion [1], has received considerable interest in the last decades. It is governed by the advection-diffusion equation,

$$\frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = D \nabla^2 c, \quad (1)$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field, $c(\mathbf{x}, t)$ is the concentration field of the species to be mixed, and D is the molecular diffusion. The velocity field is spatially smooth and satisfies the incompressibility condition $\nabla \cdot \mathbf{v} = 0$, and we denote $c_0(\mathbf{x}) = c(\mathbf{x}, 0)$ the initial condition for the scalar. The Péclet number is defined as $Pe = UL/D$, where U and L are the characteristic velocity and length scale. In closed flows (or flows with periodic boundary conditions), the spatial mean concentration is constant in time, so that for any time t , $\langle c(\mathbf{x}, t) \rangle = \langle c_0(\mathbf{x}) \rangle$, where $\langle \cdot \rangle$ denotes the spatial mean; because equation 1 is linear, we furthermore assume without loss of generality that $\langle c_0(\mathbf{x}) \rangle = 0$, and consider $c(\mathbf{x}, t)$ as the deviation from the mean. In the case of chaotic advection, or for turbulent flows in the Batchelor regime, R. Pierrehumbert showed the existence of strange eigenmodes $\hat{c}(\mathbf{x}, t)$, such that after a transient time the concentration field is such that [2, 3]

$$c(\mathbf{x}, t) = \hat{c}(\mathbf{x}, t) \exp^{-\hat{\lambda}t}, \quad (2)$$

see also Giona *et al.* [4, 5], or Haller and Yuan [6]. The strange eigenmodes $\hat{c}(\mathbf{x}, t)$ may be periodic in time or stationary, so that concentration spectra are periodic in time (or identically the same) after normalization by the exponential decay. The rate of decay $\hat{\lambda}$ is the eigenvalue of the linear equation 1.

Note that in the case of a flow with non-slip walls, the decay is not exponential but rather algebraic [7–11].

In order to predict $\hat{\lambda}$, two theories have been proposed: (i) the Global Transport theory (GTT) estimates the decay rate in a domain of size L as the slowest mode of equation 1, which is $\langle c \rangle = \cos(k_D x) \exp(-D k_D^2 t)$, with $k_D = 2\pi/L$ so that $\hat{\lambda} = D k_D^2$; (ii) the Local Lagrangian Stretching (LLS) [12–15] is based on Kraichnan theory [16]: the rate of decay is related to the statistics of the stretching factors of a line element; because those statistics do not involve diffusion, this theory is only valid in the limit of very large Péclet numbers. Haynes and Vanneste [17] have unified the problem by explaining that the two solutions are correct under certain conditions: when the spatial scalar scale is much larger than that of the velocity field (for instance with turbulent flows), the Global Transport gives the correct solution, while when the scales are comparable in size, the local theory gives a reasonable answer. K. Ngan and J. Vanneste [18] have studied both cases numerically, in a 3D implementation of the sine-flow [2, 3]. They chose velocity-fields of scale either small compared to the size of the domain, or of comparable size, and checked the theory.

In advection by chaotic streamlines of a 3D stationary flow [19–22], there are 3 Lyapunov exponents, one of which is zero (two points extremely close on the same streamline never separate exponentially). Furthermore, if the flow is incompressible, the sum of those exponents is zero: hence, if the stationary flow has chaotic streamlines, it has two non-zero Lyapunov exponents, of equal moduli and opposite signs, similarly as for 2D, time-periodic chaotic flows. When the flow is both 3D and time-periodic [18, 23], there can be another non-zero Lyapunov exponent, the sum of all

three being zero because of incompressibility. Therefore, there are two possibilities: either two Lyapunov exponents are positive, the third one being negative, and larger in amplitude, or only one exponent is positive, the two others being negative. When a flow is reversed in time, the Lyapunov exponents change sign, and so does the intermediate Lyapunov exponent. In turbulence, there are also three non zero Lyapunov exponents. While the intermediate exponent could have any sign [14], it has always been found to be positive [24–26]: this implies two directions of expansion, corresponding to the two positive exponents, and a “strongly contracting” direction, associated to the negative Lyapunov exponent. In particular, A. Pumir showed that when the concentration gradient $\mathbf{G} = \nabla c$ aligns with the most contracting direction (corresponding to the negative exponent), the gradients grow, and the scalar energy $\langle c^2 \rangle$ decreases [25]. This can be explained as follows: the equation of concentration gradients is obtained by taking the gradient of equation 1,

$$\frac{dG_i}{dt} = D \partial_j^2 G_i - G_j \partial_i v_j, \quad (3)$$

where $d/dt = \partial/\partial t + v_j \partial_j$ stands for the material derivative; it is clear that the concentration gradients grow because of negative velocity gradients (contractions). Concomitantly, the decay of scalar energy obeys the following equation (for a closed flow or with periodic boundary conditions):

$$\frac{d\langle c^2 \rangle}{dt} = -2D \int_{\Omega} (\nabla c)^2 d^3x; \quad (4)$$

therefore increase of scalar gradients implies decrease of scalar energy, as observed by Pumir [25].

Following Balkovsky and Fouxon [14], the rate of decay should depend on the sign of the intermediate Lyapunov exponent. However, the LLS theory predicts that, *in the limit of large Péclet numbers*, a flow and its inverse (by reversing time) lead to the same decay rate of scalar energy. In their numerical study, Ngan and Vanneste [18] found the rate of decay for their flow and its inverse to be identical (therefore independent on the sign of the intermediate eigenvalue). However, in their case, mixing was performed using a 3D Poincaré map (purely advecting stage) followed by a purely diffusive stage; therefore they do not solve equation 1, but rather use a simpler scheme.

We propose here to answer the question of the importance of the sign of the intermediate Lyapunov exponent, using a 3D time-periodic flow-field that exhibits chaotic advection, by solving numerically equation 1. The result is then proved in the general case.

II. LAGRANGIAN PROPERTIES

A. Flow-fields

In order to obtain three non-zero Lyapunov exponents, the flow-field has to be time-dependent. We chose a periodic flow-field of period T , composed of three different stationary flows acting during a lapse of time $T/3$; the “basic brick”, whose velocity-field is sketched in figure 1, is that of a stationary 3-D (forced) Stokes flow in a cube with slipping boundaries: it is the sum of a steady main vortex, \vec{U}_1 , and of two counter-rotating steady plane vortices (\vec{U}_2):

$$v_x = -U_1 \sin \pi x \cos \pi z \quad (5)$$

$$v_y = -2U_2 \sin \pi y \cos 2\pi z \quad (6)$$

$$v_z = U_1 \cos \pi x \sin \pi z + U_2 \cos \pi y \sin 2\pi z, \quad (7)$$

where the constants U_1 and U_2 satisfy the normalization condition (constant dissipation) $U_1^2 + 5U_2^2/2 = 1$ [27]. in contrast to flows in complex geometries where hybrid numerical approaches are necessary [28–30], here the cubic geometry, associated with the analytical expression of the flow-field, allows direct numerical simulation of the advection-diffusion equation at high Péclet number [27]. In the following we will consider the case $U_1 = 0.25$, flow for which both chaos is global and mixing is the most efficient [27]. In order to make the flow time-periodic, the basic brick is turned twice, every $T/3$ where T is the period, as shown in figure 1. This pattern of duration T is then repeated thereafter. Flow B is obtained by reversing flow A in time, *i.e.* $\mathbf{v}_B(\mathbf{x}, t) = -\mathbf{v}_A(\mathbf{x}, -t)$.

B. Lyapunov exponents

As expected for 3D time-periodic flows, the two situations depicted before for the Lyapunov exponents are encountered, see figure 2: Flow A has two positive Lyapunov exponents, the third one being negative and larger in

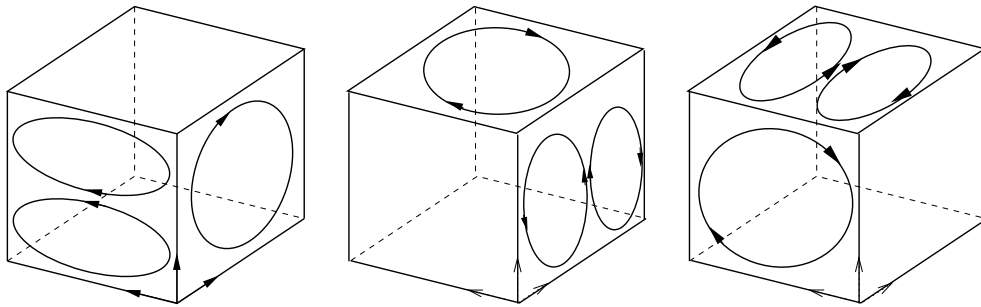


FIG. 1: Flow-field A: the basic brick is turned three times so as to obtain three non-zero Lyapunov exponents. Each configuration lasts a lapse of time $T/3$. The flow is made periodic in time by repeating those three steps. Flow B is obtained by reversing flow A in time, *i.e.* $\mathbf{v}_B(\mathbf{x}, t) = -\mathbf{v}_A(\mathbf{x}, -t)$.

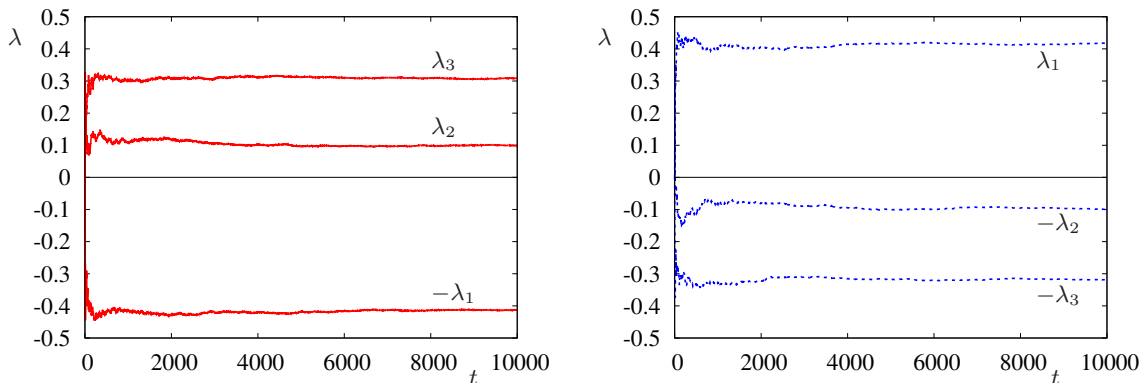


FIG. 2: Lyapunov exponents for flow A (—), and flow B (---), with $T = 3$. In both cases, the sum of the Lyapunov exponents is zero. For flow A, one has two positive Lyapunov exponents, and a negative one; the situation is reversed for flow B, where the Lyapunov exponents have same amplitude and opposite signs.

amplitude, whereas flow B has only one positive exponent, the two others being negative. Moreover, since flow B is the time-reversed of flow A, the Lyapunov exponents of flow B are exactly the opposite of those obtained with flow A. We name the Lyapunov exponents of flow A after $\lambda_3 > \lambda_2$ and $-\lambda_1$, with all $\lambda_i > 0$ ($i = 1, 2, 3$); therefore flow B has Lyapunov exponents $\lambda_1, -\lambda_2$ and $-\lambda_3$ (figure 2). The period T is chosen to maximize λ_2 compared to λ_3 : λ_2/λ_3 is maximum for $T = 3$.

III. MIXING

Equation 1 is solved for flow A and B using direct numerical simulation with a spectral method (see details in [27]). Because Flow A is associated with two positive Lyapunov exponents, stretching occurs in two dimensions and one should expect planar scalar structures, whereas flow B, with only one positive Lyapunov exponent, should lead to unidirectionally stretched structures. This is exactly what is observed in figure 3 (see also the two movies joined as supplemental material [31, 32]).

Scalar energy spectra are classical quantities in mixing [2, 33–36]. As in the case of a stationary flow-field, after a transient phase, the normalized energy spectra become stationary and independent on the initial condition, in agreement with equation 2. The non dimensional spectrum $E_\theta(k)$, and the scalar dissipation $k^2 E_\theta(k)$ are shown in figure 4: it is observed that there is more scalar energy at small scales (large k) for flow A (associated with the most negative Lyapunov exponent), and that the maximum of dissipation occurs at a larger wavenumber (hence a smaller lengthscale) for flow A compared to flow B. This is in agreement with the physics contained in equation 3, where larger scalar gradients are produced when strongly negative velocity gradients are encountered, associated to small length scales. All this suggests that flow B mixes more: with a strongly negative Lyapunov exponent, the scalar structures are very thin, so that molecular diffusion is more efficient.

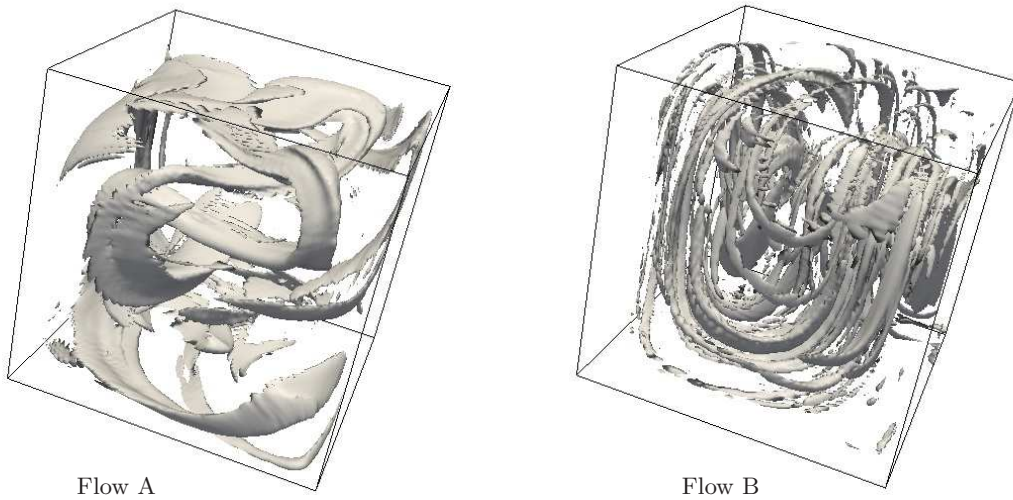


FIG. 3: Iso-scalar structures, at the same time in the exponential decay phase and from the same view point, for the same range of scalar values, with $T = 3$. For flow A, with two positive Lyapunov exponents, planar scalar structures are clearly visible; for flow B, with only one positive Lyapunov exponent, the structures are unidirectionally stretched. In both cases, the Péclet number is $Pe = 10^5$.

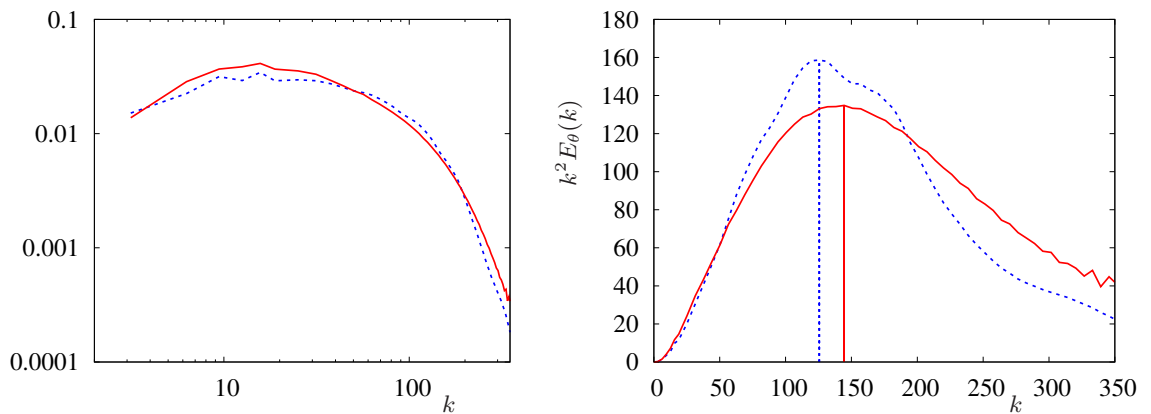


FIG. 4: left: non dimensional spectra $E_\theta(k)$ (log-log scale); right: scalar dissipation $k^2 E_\theta(k)$ (lin-lin scale). —: flow A (one single strongly negative exponent); - - -: flow B (two negative exponents). the maximum of dissipation corresponds to a higher wavenumber (therefore to a smaller length scale) for flow A. In both cases, the Péclet number is $Pe = 10^5$.

In order to check this, we quantify mixing using the standard mean deviation

$$\tilde{c}(t) = \sqrt{\langle c^2 \rangle}, \quad (8)$$

which decay is related to equation 4. Figure 5 shows the typical behavior of \tilde{c} as a function of time for a Péclet number $Pe = 10^5$. After a transient stage (that depends mostly on the initial shape of the blob to be mixed), the decay is exponential, with a slope independent on the initial condition. Surprisingly, the decay rate is exactly the same for flows A and B!

This can be explained as follows: Equation 1 can be rewritten:

$$\frac{\partial c}{\partial t} = \mathcal{L}c, \quad (9)$$

where \mathcal{L} is a linear operator defined as

$$\mathcal{L}c = -\mathbf{v} \cdot \nabla c + D \nabla^2 c. \quad (10)$$

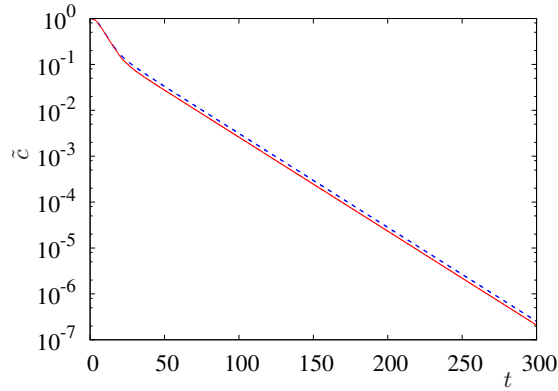


FIG. 5: Decay of the standard mean deviation of c , $\tilde{c} = \sqrt{\langle c^2 \rangle}$, as a function of time ($Pe = 10^5$). —: flow A; - - -: flow B. After a transient stage the decay becomes exponential, with an identical decay rate for the two flows.

As proved in the appendix, changing \mathbf{v} into $-\mathbf{v}$ turns \mathcal{L} into its adjoint \mathcal{L}^* . Because flow B is such that $\mathbf{v}_B(\mathbf{x}, t) = -\mathbf{v}_A(\mathbf{x}, -t)$, then the two operators acting on flow A and B are adjoint. More particularly in our flow composed of three successive stationary flows, let $\mathbf{v}_i^{T/3}$ ($i \in \{1, 2, 3\}$) be the velocity-field associated with the i th stage of flow A, and \mathcal{L}_i the associated linear operator. We denote by $L_i^{T/3}$ the global action of \mathcal{L}_i on c for the duration $T/3$. Then $L^T = L_3^{T/3} \circ L_2^{T/3} \circ L_1^{T/3}$ is the global operator for one period of flow A. Thus the global operator acting on flow B is $(L_1^{T/3})^* \circ (L_2^{T/3})^* \circ (L_3^{T/3})^* = (L^T)^*$. The operators are adjoint, and two adjoint operators have conjugate eigenvalues, with same real part. Because decay is associated with the real part of the eigenvalue, the decay rate is asymptotically identical for both flows.

A similar reasoning had been used by Favier & Proctor [37], who considered the growth rate of the dynamo in steady (reversible) flows: they showed that, due to the adjointness of the induction operator, the growth rate of the dynamo was identical for two different types of boundary conditions (magnetic field whether tangent or normal to the boundaries); the corresponding magnetic eigenmodes, had, like in our case, a completely different topology (see also references [38, 39]). Ngan and Vanneste also proved this property for their flow, considering the adjointness of each phase (reversible Poincaré map, followed by a purely diffusing stage). The proof is even more general here, since it can be applied to any time-periodic flow (real or modeled) where advection and diffusion act simultaneously.

IV. CONCLUSION

Using a counterexample based on a flow exhibiting chaotic advection, we have shown that the decay of scalar energy is independent on the sign of the intermediate Lyapunov exponent, and next proved it mathematically.

This result is however physically counterintuitive: since flow A creates larger scalar gradients, one could expect it to dissipate more than flow B. Let us come back to equation 3, governing the production of scalar gradients; it can be rewritten:

$$\underbrace{\frac{1}{|\mathbf{G}|} \frac{d|\mathbf{G}|}{dt}}_{(0)} = D \underbrace{\frac{G_i}{G^2} \partial_j^2 G_i}_{(a)} - \underbrace{\frac{G_i G_j}{G^2} \partial_i v_j}_{(b)}. \quad (11)$$

Because the non dimensional concentration spectra are unchanged after a transient stage, the Batchelor scale of the flow, corresponding to the maximum of dissipation, is constant in time. Therefore, the maximum concentration gradient $|\mathbf{G}|$ only varies by dilution of scalar, *i.e.* $(0) \sim 1/\tilde{c} d\tilde{c}/dt \sim -0,046$. The order of magnitude of term (b) , describing creation of gradients, is $(b) \sim |\lambda|$, where λ is the most negative Lyapunov exponent (most negative velocity gradients). Hence $(b) \sim 0.4$ for flow A, and 0.3 for flow B: in both cases, $|(0)| \ll (b)$, so that the net creation of scalar gradients is negligible: although there is no forcing here (hence scalar energy decays and the situation is not stationary a priori), the state is quasi stationary. This implies that $(b) \sim -(a)$: in this case of good mixing, the scalar gradients are dissipated at the same rate as they are created by the flow [40]. Using the orders of magnitude of both terms, we obtain $k^2 \sim |\lambda|/D$, where λ is the most negative Lyapunov exponent, so as to have the most positive term

(b) possible. Indeed here, the maximum in figure 4 corresponds to a wavenumber $k = \sqrt{|\lambda|/(2D)}$ with $D = 10^{-5}$ ($k = 141$ for flow A with $|\lambda| \approx 0.4$, and $k = 122$ for flow B with $|\lambda| \approx 0.3$): this would not hold if the term (0) were not negligible. Actually, the exponential decay of variance is not physically controlled by regions of large stretching, but rather by a few small fluid blobs that remain unstretched for long times [11, 17, 18]. Note finally that Probability Density Functions of $\tilde{c}/\langle\tilde{c}\rangle$ (not shown here) are identical for both flows, in agreement with the idea that the rate of decay is identical (see [41]).

V. ACKNOWLEDGMENTS:

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Appendix A: Adjoint operator of \mathcal{L}

Using the definition of \mathcal{L} , one has:

$$\int c_2 \mathcal{L} c_1 d^3v = \int (c_2 \nabla \cdot (-\mathbf{v}c_1) + Dc_2 \nabla^2 c_1) d^3v. \quad (\text{A1})$$

- The diffusion term writes:

$$\int \nabla \cdot (Dc_2 \nabla c_1) d^3v - \int (D\nabla c_2 \cdot \nabla c_1) d^3v \quad (\text{A2})$$

The term with the divergence disappears when dealing with non permeable walls, or when periodic boundary conditions are considered; the other term is symmetric in c_1 and c_2 , so that their role can be exchanged.

- The advecting term writes:

$$\int \nabla \cdot (-\mathbf{v}c_1c_2) d^3v + \int \mathbf{v}c_1 \cdot \nabla c_2 d^3v. \quad (\text{A3})$$

Here again the divergence term disappears with boundary conditions, and the advecting term writes

$$\int c_1 \nabla \cdot (\mathbf{v}c_2) d^3v. \quad (\text{A4})$$

Finally:

$$\int c_2 \mathcal{L} c_1 d^3v = \int c_1 \nabla \cdot (\mathbf{v}c_2 + D\nabla c_2) d^3v, \quad (\text{A5})$$

and

$$\mathcal{L}^* c = \nabla \cdot (\mathbf{v}c + D\nabla c). \quad (\text{A6})$$

By comparing equations 10 and A6, one finds that the adjoint \mathcal{L}^* is obtained from \mathcal{L} by changing \mathbf{v} into $-\mathbf{v}$.

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- [1] E. Villermaux. Mixing versus stirring. *Annual Review of Fluid Mechanics*, 51(1):245–273, 2019.
[2] R. Pierrehumbert. On tracer microstructure in the large-eddy dominated regime. *Chaos, Solitons fractals*, 4:1091–1110, 1994.
[3] R. T. Pierrehumbert. Lattice modes of advection-diffusion. *Chaos*, 10(1):61–73, 2000.
[4] M. Giona, A. Adrover, S. Cerbelli, and V. Vitacolonna. Spectral properties and transport mechanisms of partially chaotic bounded flows in the presence of diffusion. *Phys. Rev. Lett.*, 92(11):114101–1–4, 2004.
[5] M. Giona, S. Cerbelli, and V. Vitacolonna. Universality and imaginary potentials in advection–diffusion equations in closed flows. *J. Fluid Mech.*, 513:221–237, 2004.

- [6] G. Haller and G. Yuan. Lagrangian coherent structures and mixing in two-dimensional turbulence. *Physica D: Nonlinear Phenomena*, 147(3-4):352–370, 2000.
- [7] S.W. Jones and W.R. Young. Shear dispersion and anomalous diffusion by chaotic advection. *J. Fluid Mech.*, 1994.
- [8] M. Chertkov and V. Lebedev. Decay of scalar turbulence revisited. *Phys. Rev. Lett.*, 90(3):034501, 2003.
- [9] H. Salman and P. H. Haynes. A numerical study of passive scalar evolution in peripheral regions. *Phys. Fluids*, 19(6):067101, 2007.
- [10] E. Gouillart, N. Kuncio, O. Dauchot, B. Dubrulle, S. Roux, and J.-L. Thiffeault. Walls Inhibit Chaotic Mixing. *Phys. Rev. Lett.*, 99(11):114501, 2007.
- [11] F. Raynal and Ph. Carrière. The distribution of time of flight in three dimensional stationary chaotic advection. *Phys. Fluids*, 27(4):043601, 2015.
- [12] T. M. Antonsen, Z. Fan, E. Ott, and E. Garcia-Lopez. The role of chaotic orbits in the determination of power spectra of passive scalars. *Phys. Fluids*, 8(11):3094–3104, 1996.
- [13] DT Son. Turbulent decay of a passive scalar in the batchelor limit: Exact results from a quantum-mechanical approach. *Physical Review E*, 59(4):R3811, 1999.
- [14] E. Balkovsky and A. Fouxon. Universal long-time properties of lagrangian statistics in the batchelor regime and their application to the passive scalar problem. *Phys. Rev. E*, 60:4164–4174, 1999.
- [15] G. Falkovich, K. Gawędzki, and M. Vergassola. Particles and fields in fluid turbulence. *Rev. Mod. Phys.*, 73:913–975, 2001.
- [16] R. H. Kraichnan. Convection of a passive scalar by a quasi-uniform random straining field. *J. Fluid Mech.*, 64:737–762, 1974.
- [17] P. H. Haynes and J. Vanneste. What controls the decay of passive scalars in smooth flows? *Physics of Fluids*, 17(9):097103, 2005.
- [18] K. Ngan and J. Vanneste. Scalar decay in a three-dimensional chaotic flow. *Phys. Rev. E*, 83:056306, 2011.
- [19] H. Aref. Stirring by chaotic advection. *J. Fluid Mech.*, 143:1–21, 1984.
- [20] V. Rom-Kedar, A. Leonard, and S. Wiggins. An analytical study of the transport, mixing and chaos in an unsteady vortical flow. *J. Fluid Mech.*, 214:347–394, 1990.
- [21] H. Aref, J. R. Blake, M. Budišić, S. S. S. Cardoso, J. H. E. Cartwright, H. J. H. Clercx, K. El Omari, U. Feudel, R. Golestanian, E. Gouillart, G. F. van Heijst, T. S. Krasnopolskaya, Y. Le Guer, R. S. MacKay, V. V. Meleshko, G. Metcalfe, I. Mezić, A. P. S. de Moura, O. Piro, M. F. M. Speetjens, R. Sturman, J.-L. Thiffeault, and I. Tuval. Frontiers of chaotic advection. *Rev. Mod. Phys.*, 89:025007, 2017.
- [22] L. D. Smith, P. B. Umbanhowar, R. M. Lueptow, and J. M. Ottino. The geometry of cutting and shuffling: An outline of possibilities for piecewise isometries. *Physics Reports*, 2019.
- [23] N. R. Moharana, M. F. M. Speetjens, R. R. Trieling, and H. J. H. Clercx. Three-dimensional lagrangian transport phenomena in unsteady laminar flows driven by a rotating sphere. *Physics of Fluids*, 25(9):093602, 2013.
- [24] S. Girimaji and S. Pope. Material-element deformation in isotropic turbulence. *Journal of fluid mechanics*, 220:427–458, 1990.
- [25] A. Pumir. A numerical study of the mixing of a passive scalar in three dimensions in the presence of a mean gradient. *Physics of Fluids*, 6(6):2118–2132, 1994.
- [26] J. Bec, L. Biferale, G. Boffetta, M. Cencini, S. Musacchio, and F. Toschi. Lyapunov exponents of heavy particles in turbulence. *Physics of Fluids*, 18(9):091702, 2006.
- [27] V. Toussaint, Ph. Carrière, and F. Raynal. A numerical Eulerian approach to mixing by chaotic advection. *Phys. Fluids*, 7:2587–2600, 1995.
- [28] O. Gorodetskyi, M. Giona, and P. D. Anderson. Exploiting numerical diffusion to study transport and chaotic mixing for extremely large péclet values. *EPL (Europhysics Letters)*, 97(1):14002, jan 2012.
- [29] O. Gorodetskyi, M.F.M. Speetjens, and P.D. Anderson. Eigenmode analysis of advective–diffusive transport by the compact mapping method. *European Journal of Mechanics-B/Fluids*, 49:1–11, 2015.
- [30] A. Borgogna, M.A. Murmura, M.C. Annesini, M. Giona, and S. Cerbelli. A hybrid numerical approach for predicting mixing length and mixing time in microfluidic junctions from moderate to arbitrarily large values of the Péclet number. *Chemical Engineering Science*, 196:247 – 264, 2019.
- [31] See Supplemental Material, video iso1.avi, representing a blob of dye that is mixed by flow A: the scalar structures obtained are clearly sheets of scalar.
- [32] See Supplemental Material, video inv_iso1.avi, representing a blob of dye that is mixed by flow B, the inverse in time of flow A: the scalar structures obtained are clearly elongated.
- [33] B. S. Williams, D. Marteau, and J. P. Gollub. Mixing of a passive scalar in magnetically forced two-dimensional turbulence. *Physics of Fluids (1994-present)*, 9(7):2061–2080, 1997.
- [34] V. Toussaint, Ph. Carrière, J. Scott, and J.N. Gence. Spectral decay of a passive scalar in chaotic mixing. *Phys. Fluids*, 12(11):2834–2844, 2000.
- [35] M.-C. Jullien, P. Castiglione, and P. Tabeling. Experimental observation of batchelor dispersion of passive tracers. *Phys. Rev. Lett.*, 85:3636–3639, Oct 2000.
- [36] P. Meunier and E. Villermaux. The diffusive strip method for scalar mixing in two dimensions. *Journal of Fluid Mechanics*, 662:134–172, 11 2010.
- [37] B. Favier and M. R. E. Proctor. Growth rate degeneracies in kinematic dynamos. *Phys. Rev. E*, 88:031001, 2013.
- [38] M. R. E. Proctor. On the eigenvalues of kinematic α -effect dynamos. *Astronomische Nachrichten*, 298(1):19–25, 1977.
- [39] M. R. E. Proctor. The role of mean circulation in parity selection by planetary magnetic fields. *Geophysical & Astrophysical Fluid Dynamics*, 8(1):311–324, 1977.

- [40] F. Raynal and J.N. Gence. Energy saving in chaotic laminar mixing. *Int. J. Heat Mass Transfer*, 40(14):3267–3273, 1997.
- [41] T. Le Borgne, P.D. Huck, M. Dentz, and E. Villermaux. Scalar gradients in stirred mixtures and the deconstruction of random fields. *J. Fluid Mech.*, 812:578610, 2017.