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On Existence, Uniqueness and Qualitative Properties of a system of Partial Differential Equations involved in a Statistical Learning Tool intending livestock Data Assimilation.

- Working-Paper Version -

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Abstract

In the article Flourent *et al.* (Submitted), a Biomimetic Statistical Learning tool was developed. This tool is based on a system of Partial Differential Equations.

Some applications showed that this tool provides accurate predictions and has an extrapolation capability. Hence, the mathematical analysis of the integrated PDE system becomes mandatory.

Yet, the two first equations of this system contain a diffusion operator depending on the sum of two unknown densities. This fact, coupled with the presence of a convection term, makes impossible the direct application of the classical results about parabolic equations. Therefore, the mathematical analysis of this model constitutes a new challenge for PDE analysis.

Some properties typically verified in the framework of the parabolic PDEs are not obtainable for the studied PDE system. Hence, we worked on two toy models to put in evidence the lost properties and the ones which can be expected despite the presence of an atypical diffusion term.

keywords : Statistical Learning, PDE, Estimates, Fourier Transform, Change of variable, Model-Data Coupling, Existence, Uniqueness.

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1 Introduction

In Flourent *et al.* (Submitted), an Artificial Intelligence (AI) tool was developed to perform forecasting and Data Assimilation in the framework of Smart Farming issues. Some applications of this tool, on fictitious and real data, showed that it provides accurate predictions and has interesting capabilities such as the capability to be fitted on a given range of data and then to be accurately applied on a wider range of data.

This AI tool, falling within the Data-Model Coupling approach, is based on a PDE system. The two first equations of this system are:

$$\begin{aligned} \frac{\partial \Phi_f}{\partial t}(t, x) + \omega \frac{\partial \Phi_f}{\partial x}(t, x) - \frac{\partial \left[c\chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x} \right]}{\partial x}(t, x) \\ = \frac{1}{2}Q(t, x) - fF(x)\Phi_f(t, x) - r\Phi_f(t, x), \end{aligned} \quad (1.1)$$

and,

$$\begin{aligned} \frac{\partial \Phi_b}{\partial t}(t, x) - \omega \frac{\partial \Phi_b}{\partial x}(t, x) - \frac{\partial \left[c\chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x} \right]}{\partial x}(t, x) \\ = \frac{1}{2}Q(t, x) - fF(x)\Phi_b(t, x) + r\Phi_b(t, x). \end{aligned} \quad (1.2)$$

These equations are set for $x \in (0, 1)$ and $t \in (0, +\infty)$, $\Phi_f(0, x)$, $\Phi_b(0, x)$, $\Psi(0, x)$ and $\Xi(0, x)$ are given for all x in $(0, 1)$, and the following boundary conditions are imposed:

$$\forall t \in (0, +\infty), \Phi_f(t, 0) = \Phi_b(t, 0), \quad (1.3)$$

and,

$$\forall t \in (0, +\infty), \Phi_b(t, 1) = \Phi_f(t, 1). \quad (1.4)$$

Moreover, χ is a non-negative function depending exclusively on x and being continuously differentiable and compactly supported in $(0, 1)$:

$$\chi \geq 0, \chi \in C_0^1((0, 1)). \quad (1.5)$$

Function F is a «Location» function and Q is an «Entry» function. They are non-negative, square integrable on $(0, 1)$ and have their support included in the support of χ :

$$F \in L^2((0, 1)), Q \in L^\infty\left((0, +\infty), L^2((0, 1))\right). \quad (1.6)$$

$$F \geq 0, Q \geq 0. \quad (1.7)$$

$$\text{Supp}(F) \subset \text{Supp}(\chi), \text{Supp}(Q) \subset \text{Supp}(\chi). \quad (1.8)$$

We refer to Flourent *et al.* (Submitted) for the detailed explanations of the different terms of Equations (1.1) and (1.2).

In these equations ω , c , f , r and u may be either constants or functions depending on x and t but also on Φ_f and Φ_b . Hence, these equations enter the framework of linear and non-linear PDEs.

Because of the accurate forecast capabilities of the AI tool embedding it, the mathematical analysis of this PDE system becomes mandatory. Yet, in the two above equations the «diffusion operator» depends on the sum of two unknown densities, Φ_f and Φ_b . This fact, coupled with the presence of a convection term, makes impossible the direct application of the classical results about parabolic equations provided for instance by Ladyženskaja *et al.* (1968). Therefore, the mathematical analysis of this model constitutes a new challenge for PDE analysis.

In front of this situation, this paper provides preliminary results concerning some properties of the PDE system made of Equations (1.1), (1.2) and three others given in the sequel.

Up to now, we consider ω , c , f , r and u are positive constants.

Our first contribution is the pertinent Variational Formulation of (1.1) and (1.2). It consists in seeking Φ_f and Φ_b in Space \mathbf{H} ,

$$\mathbf{H} = H^1((0, +\infty) \times (0, 1)) \cap C^0((0, +\infty), H^1(0, 1)) \cap C^1((0, +\infty), H^{-1}(0, 1)), \quad (1.9)$$

such that for any $\varphi_f \in \mathbf{H}$ and $\varphi_b \in \mathbf{H}$, the following equality holds true for any $T > 0$,

$$\begin{aligned} & - \int_0^T \int_0^1 \Phi_f(t, x) \frac{\partial \varphi_f}{\partial t}(t, x) dx dt - \int_0^T \int_0^1 \Phi_b(t, x) \frac{\partial \varphi_b}{\partial t}(t, x) dx dt \\ & + \int_0^1 \Phi_f(0, x) \varphi_f(0, x) dx - \int_0^1 \Phi_f(T, x) \varphi_f(T, x) dx + \int_0^1 \Phi_b(0, x) \varphi_b(0, x) dx \\ & - \int_0^1 \Phi_b(T, x) \varphi_b(T, x) dx - \int_0^T \omega \Phi_f(t, 1) \varphi_f(t, 1) dt + \int_0^T \omega \Phi_f(t, 0) \varphi_f(t, 0) dt \\ & + \int_0^T \omega \Phi_f(t, 1) \varphi_b(t, 1) - \int_0^T \omega \Phi_f(t, 0) \varphi_b(t, 0) \\ & = \int_0^T \int_0^1 \omega \Phi_f(t, x) \frac{\partial \varphi_f}{\partial x}(t, x) dx dt - \int_0^T \int_0^1 \omega \Phi_b(t, x) \frac{\partial \varphi_b}{\partial x}(t, x) dx dt \\ & - 2 \int_0^T \int_0^1 c \chi \left(\frac{\partial(\Phi_f + \Phi_b)}{\partial x}(t, x) \right) \left(\frac{\partial(\varphi_b + \varphi_f)}{\partial x}(t, x) \right) dx dt \\ & + \int_0^T \int_0^1 Q(t, x) (\varphi_f(t, x) + \varphi_b(t, x)) dx dt \\ & - \int_0^T \int_0^1 f F(x) (\Phi_f(t, x) \varphi_f(t, x) + \Phi_b(t, x) \varphi_b(t, x)) dx dt \\ & - \int_0^T \int_0^1 r \Phi_f(t, x) \varphi_f(t, x) dx dt + \int_0^T \int_0^1 r \Phi_b(t, x) \varphi_b(t, x) dx dt. \quad (1.10) \end{aligned}$$

From this Variational Formulation, we established classical a priori estimates remaining valid even in our situation. Among them, there is the L^2 a priori estimate that stands as follow:

Theorem 1.1. *Under assumptions (1.5) and (1.6), if $\Phi_f \in \mathbf{H}$ and $\Phi_b \in \mathbf{H}$ are solutions to Variational Formulation (1.10), then, for any $T > 0$, there exist constants $L_{2a}(T)$, $L_{2b}(T)$ and $L_3(T)$ such that for all $t \in (0, T)$,*

$$\|\Phi_f\|_2^2(t) \leq L_{2a}(T), \quad (1.11)$$

$$\|\Phi_b\|_2^2(t) \leq L_{2b}(T), \quad (1.12)$$

and,

$$\left\| \left\| \sqrt{\chi} \frac{\partial(\Phi_f + \Phi_b)}{\partial x} \right\| \right\|_2^2 \leq L_3(T), \quad (1.13)$$

where,

$$\|\Phi\|_2^2(t) = \int_0^1 \Phi^2(t, x) dx \quad (1.14)$$

and,

$$\left\| \left\| \frac{\partial \Phi}{\partial x} \right\| \right\|_2^2 = \int_0^T \int_0^1 \frac{\partial \Phi^2}{\partial x}(t, x) dx dt \quad (1.15)$$

The proof of this theorem is given in Section 2.3.

The model developed in Flourent *et al.* (Submitted) contains three other equations that are not real issues for the mathematical analysis. They involve function $\Psi(t, x)$, solution to:

$$\frac{\partial \Psi}{\partial t}(t, x) = fF(x) \left[\Phi_b(t, x) + \Phi_f(t, x) \right] - u\Psi(t, x), \quad (1.16)$$

function $\Xi(t, x)$, solution to:

$$\frac{\partial \Xi}{\partial t}(t, x) = u\Psi(t, x) \left(\frac{L - O(t)}{L} \right), \quad (1.17)$$

for $x \in (0, 1)$ and $t \in (0, +\infty)$ and for positive constants u and L . And finally the Outcome $O(t)$ of the model is given by:

$$O(t) = \int_{\Omega} \Xi(t, x) dx, \quad (1.18)$$

where Ω is subset of $(0, 1)$.

In Flourent *et al.* (Submitted), an alternative equation to Equation (1.17) without limiter is also used. In this case, function $\Xi(t, x)$ is solution to:

$$\frac{\partial \Xi}{\partial t}(t, x) = u\Psi(t, x). \quad (1.15.b)$$

The mathematical model made of Equations (1.1), (1.2), (1.16), (1.15.b) and (1.18), has a conservation property:

Property 1.2. *Under assumptions (1.5) and (1.6), if $\Phi_f \in \mathbf{H}$ and $\Phi_b \in \mathbf{H}$ are solutions to Equation (1.10), then there exists $\Psi \in C^1((0, +\infty); L^1((0, 1)))$ solution to Equation (1.16), $\Xi \in C^1((0, +\infty); L^1((0, 1)))$ solution to Equation (1.15.b), and $O \in C^1((0, +\infty))$ solution to (1.18). Moreover, for any $T > 0$, there exists $K_Q(T) > 0$ such that for any $t \in (0, T)$,*

$$\frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} \leq K_Q(T). \quad (1.19)$$

Remark. *Estimate (1.19) would give a L^1 estimate in cases when we can prove the non-negativity of functions Φ_f , Φ_b , Ψ and Ξ .*

Yet, as we shall see later on, this question of the non-negativity is a real issue for our model. Moreover, obtaining Estimate (1.19) with Equation (1.17) in place of Equation (1.15.b) seems to be not obvious.

The proof of this property is given in Section 2.

Beyond the question of the non-negativity, other properties typically verified in the framework of the parabolic PDEs are not obtainable for the studied PDE system. Therefore, we worked on two toy models to put in evidence the lost properties and the ones which can be expected despite the presence of a «diffusion term» depending on the sum of two unknown densities.

The first studied toy model is made of two equations. It considers Φ_f and Φ_b , defined for $x \in (0, 1)$ and $t \in (0, +\infty)$, are respectively solutions to,

$$\frac{\partial \Phi_f}{\partial t}(t, x) - c \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) = 0, \quad (1.20)$$

and,

$$\frac{\partial \Phi_b}{\partial t}(t, x) - c \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) = 0. \quad (1.21)$$

In this toy model we imposed for all $x \in (0, 1)$,

$$\Phi_f(0, x) \geq 0 \quad \text{and} \quad \Phi_b(0, x) \geq 0, \quad (1.22)$$

and for all $t \in (0, +\infty)$ and for non-negative functions $\overline{\Phi}_f^0(t)$, $\overline{\Phi}_b^0(t)$, $\overline{\Phi}_f^1(t)$ and $\overline{\Phi}_b^1(t)$,

$$\begin{aligned} \Phi_f(t, 0) &= \overline{\Phi}_f^0(t) \geq 0 \quad \text{and} \quad \Phi_b(t, 0) = \overline{\Phi}_b^0(t) \geq 0 \\ \Phi_f(t, 1) &= \overline{\Phi}_f^1(t) \geq 0 \quad \text{and} \quad \Phi_b(t, 1) = \overline{\Phi}_b^1(t) \geq 0 \end{aligned} \quad (1.23)$$

The loss of the regularizing effect was put in evidence through this first toy model. It is illustrated by the fact that if $\overline{\Phi}_f^0(t) = 0$, $\overline{\Phi}_f^1(t) = 1$, $\overline{\Phi}_b^0(t) = 1$ and $\overline{\Phi}_b^1(t) = 0$, then, for any $\tilde{x} \in (0, 1)$, the couple of functions

$$\Phi_f(t, x) = \begin{cases} 1, & \text{if } x < \tilde{x} \\ 0, & \text{otherwise} \end{cases} \quad (1.24)$$

and,

$$\Phi_b(t, x) = \begin{cases} 0, & \text{if } x < \tilde{x} \\ 1, & \text{otherwise} \end{cases} \quad (1.25)$$

is solution to Equations (1.20), (1.21), (1.22) and (1.23), and is clearly not regular.

Nevertheless, for this first toy model we proved that:

Property 1.3. *If $\overline{\Phi}_f^0(t) = \overline{\Phi}_f^1(t) = \overline{\Phi}_b^0(t) = \overline{\Phi}_b^1(t) = 0$, $\Phi_f(0, \cdot) \in L^2((0, 1))$, $\Phi_b(0, \cdot) \in L^2((0, 1))$ and $(\Phi_f(0, \cdot) - \Phi_b(0, \cdot)) \in H^1(0, 1)$, then, there exists a unique solution to (1.20), (1.21), (1.22) and (1.23). This solution belongs to $C^1((0, +\infty), H^1(0, 1))$.*

This property, demonstrated in Section 3.2, set a framework to ensure regularity.

Through this toy model we also showed that the non-negativity of Φ_f and Φ_b is not insured, even in the framework set by Property 1.3. This is illustrated by an example given in Section 3.3.

Then, reformulating (1.6), (1.7), (1.20), (1.21), (1.22) and (1.23) and using Fourier Transform, we demonstrated some properties of a second toy model, closer to the initial model developed in Flourent *et al.* (Submitted). This second toy model corresponds to the initial model without the convection term and with boundary conditions (1.23) replaced by periodicity conditions for Φ_f and Φ_b with respect to variable x . To do so we introduced Space $\mathbf{H}_\#$ defined as:

$$\mathbf{H}_\# = \{\Phi \in \mathbf{H}, \forall t \in (0, +\infty), \Phi(t, 0) = \Phi(t, 1)\} \quad (1.26)$$

Space $\mathbf{H}_\#$ can be seen as the functions of \mathbf{H} that are periodic consistently with the topology of \mathbf{H} .

Hence, in this model $\Phi_f \in \mathbf{H}_\#$ is a periodic function of period 1 with respect to x , solution to,

$$\frac{\partial \Phi_f}{\partial t}(t, x) - c \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) = \frac{1}{2} Q(t, x) - f \Phi_f(t, x) - r \Phi_f(t, x), \quad (1.27)$$

and, $\Phi_b \in \mathbf{H}_\#$ is a periodic function of period 1 with respect to x , solution to,

$$\frac{\partial \Phi_b}{\partial t}(t, x) - c \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) = \frac{1}{2} Q(t, x) - f \Phi_b(t, x) + r \Phi_f(t, x). \quad (1.28)$$

The three other equations of this model remain Equations (1.16), (1.18) and (1.17) or (1.15.b).

In the framework of this second toy model we assumed that,

$$Q \in L^\infty((0, +\infty), L^2((0, 1))), Q \geq 0, \quad (1.29)$$

$$\Phi_f(0, \cdot) \geq 0 \text{ and } \Phi_b(0, \cdot) \geq 0 \quad (1.30)$$

$$\Phi_f(0, \cdot) + \Phi_b(0, \cdot) \leq \frac{1}{f} \sup_{x \in (0, 1)} (Q(0, x)), \quad (1.31)$$

$$\Phi_f(0, \cdot) \in L^\infty((0, 1)), \Phi_b(0, \cdot) \in L^\infty((0, 1)), \quad (1.32)$$

$$\Phi_f(0, \cdot) - \Phi_b(0, \cdot) \in H^1((0, 1)), \quad (1.33)$$

$$\Phi_f(0, 0) - \Phi_b(0, 0) = \Phi_f(0, 1) - \Phi_b(0, 1), \quad (1.34)$$

$$\Psi(0, \cdot) \in L^\infty((0, 1)), \Psi(0, \cdot) \geq 0, \quad (1.35)$$

$$\Xi(0, \cdot) \in L^\infty((0, 1)), \Xi(0, \cdot) \geq 0, \quad (1.36)$$

$$0 \leq O(0) \leq L, \quad (1.37)$$

and,

$$\exists T_F > 0, \forall t > T_F, \forall x \in (0, 1), Q(t, x) = 0. \quad (1.38)$$

Through this second toy model we demonstrated that:

Theorem 1.4. *Under Assumptions (1.29) to (1.38), there exists unique $\Phi_f \in \mathbf{H}_\#$, $\Phi_b \in \mathbf{H}_\#$, $\Psi \in C^1((0, +\infty); L^1((0, 1)))$, $\Xi \in C^1((0, +\infty); L^1((0, 1)))$ and $O \in C^1((0, +\infty))$, solutions to the PDE system made of (1.27), (1.28), (1.16), (1.18) and Equation (1.17) or (1.15.b). Moreover, this PDE system has a conservation property. Hence, for any $T > 0$, there exists $K_{Q_2}(T) > 0$ such that for any $t \in (0, T)$,*

$$\frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} \leq K_{Q_2}(T). \quad (1.39)$$

The demonstration of Theorem 1.4 is presented in Section 4.

In section 2, we will present the demonstration of the conservation of the model developed in Flourent *et al.* (Submitted). In this section, the Variational Formulation principle applied to the studied model will be also explained. After giving sense to the equations of the PDE system by defining the suitable mathematical space in which the solution is sought, the results of a priori estimates will be presented. Those a priori estimates permitted to put in evidence some peculiarities of the initial studied mathematical model, preventing from the direct application of the classical results about parabolic equations to prove the wellposedness of the PDE system.

In Section 3 we will present the properties lost due to the structure of the «diffusion term» by analyzing a first simplified model.

In Section 4 we will present the analysis of a second toy model that does not contain convection term and by using Variable Substitutions and Fourier Transforms. The goal of this analysis is to put in evidence the properties preserved despite the structure of the «diffusion term». In this section we will present the demonstration of the existence and the uniqueness of the solution of this second toy model. We will also demonstrate the conservation of this second toy model and the non-negativity of some components of the solution.

2 Structure and analysis of the initial developed mathematical model

The mathematical model developed in Flourent *et al.* (Submitted), made of Equations (1.1), (1.2), (1.16), (1.15.b) and (1.18), is biomimetic and describes the evolution and the action over time $t \in (0, +\infty)$, of an information circulating and diffusing in a geometrical space that is interval $(0, 1)$. For further details we refer readers to Flourent *et al.* (Submitted).

2.1 Conservation property of the initial developed mathematical model

Proof of Property 1.2: To demonstrate this property we assumed the existence of Φ_f , Φ_b , Ψ , Ξ and O respectively solutions to (1.1), (1.2), (1.16), (1.15.b) and (1.18).

We summed and integrated those four equations over space. We obtained the following equality:

$$\int_0^1 \frac{\partial(\Phi_f + \Phi_b + \Psi + \Xi)}{\partial t}(t, x) dx = \int_0^1 2 \frac{\partial \left[c\chi \frac{\partial[\Phi_f + \Phi_b]}{\partial x} \right]}{\partial x}(t, x) dx + \int_0^1 Q(t, x) dx. \quad (2.1)$$

According to (1.5), Function χ is null at $x = 0$ and $x = 1$. Hence, integrating the first term of the right-hand side, we obtained:

$$\frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} = \int_0^1 Q(t, x) dx \leq \sup_{t \in \mathbb{R}^+} \left(\int_0^1 Q(t, x) dx \right). \quad (2.2)$$

Therefore, with or without the limiter coefficient, for all x belonging to $(0, 1)$ and for all t belonging to $(0, T)$, T belonging to $(0, +\infty)$, there exists a constant $K_Q(T) > 0$ satisfying Inequality (1.19). Therefore, we demonstrated Property 1.2. \blacksquare

2.2 Principle of the Variational Formulation

To prove the existence and the uniqueness of the solution of the PDE system presented in Introduction, it is essential to give a sense to the equations. To do so, it is necessary to define the mathematical space in which the solution is sought.

Given the structure of Equations (1.1) and (1.2), the two first components of the solution of the PDE system could be intuitively sought in spaces such as $C^1((0, +\infty), C^2(0, 1))$. Yet, as it can be read in Menza (2009) and David and Gosselet (2015), it is well known that Sobolev spaces are more suitable to study and prove the existence and the regularity of the solutions of PDEs. For this reason, the notion of «solution» needs to be weakened by searching the two first components of the solution associated to Equations (1.1) and (1.2) in a larger Banach space presenting a lower regularity. To do that it is necessary to write a weak formulation of the equations.

The objective of the Variational Formulation is to weaken the notion of solution by applying Test Functions to the PDE system and performing integrations in the sense of distributions. The goal is to search the solution in a mathematical space presenting a lower regularity but providing a suitable framework to establish the existence and the uniqueness of the solution.

There are several ways to get this formulation. Yet, we chose to use the distributions framework. Therefore, we considered Equations (1.1) and (1.2) as distribution equalities and we applied (1.1) to $\varphi_f \in \mathcal{D}((0, +\infty) \times (0, 1))$ and (1.2) to $\varphi_b \in \mathcal{D}((0, +\infty) \times (0, 1))$,

where space $\mathcal{D}((0, +\infty) \times (0, 1))$ is the topological space of the functions, infinitely continually differentiable and compactly supported in $((0, +\infty) \times (0, 1))$. Then by using the definition of equality and derivation in the distributions framework, we obtained respectively:

$$\begin{aligned} \forall \varphi_f \in \mathcal{D}((0, +\infty) \times (0, 1)), \\ -\langle \Phi_f, \partial_t \varphi_f \rangle - \omega \langle \Phi_f, \partial_x \varphi_f \rangle + c \left\langle \chi \left(\partial_x \Phi_f + \partial_x \Phi_b \right), \partial_x \varphi_f \right\rangle \\ = \frac{1}{2} \langle Q, \varphi_f \rangle - f \langle \Phi_f, \varphi_f \rangle - r \langle \Phi_f, \varphi_f \rangle, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \forall \varphi_b \in \mathcal{D}((0, +\infty) \times (0, 1)), \\ -\langle \Phi_b, \partial_t \varphi_b \rangle + \omega \langle \Phi_b, \partial_x \varphi_b \rangle + c \langle \chi (\partial_x \Phi_f + \partial_x \Phi_b), \partial_x \varphi_b \rangle \\ = \frac{1}{2} \langle Q, \varphi_b \rangle - f \langle \Phi_b, \varphi_b \rangle + r \langle \Phi_f, \varphi_b \rangle. \end{aligned} \quad (2.4)$$

In the light of the Variational Formulations (2.3) and (2.4), it seems reasonable to suppose that weak solutions of (1.1) and (1.2) are functions belonging to $L^2((0, +\infty) \times (0, 1))$, such as their first derivatives also belong to $L^2((0, +\infty) \times (0, 1))$. Therefore, we sought the two first components of the solution in Sobolev space $H^1((0, +\infty) \times (0, 1))$, such as Equalities (1.3) and (1.4) are respected in the sense of traces in Sobolev spaces. In this framework, Equations (2.3) and (2.4) can be rewritten such as:

$$\begin{aligned} \forall \varphi_f \in H_0^1((0, +\infty) \times (0, 1)), \\ - \int_0^{+\infty} \int_0^1 \Phi_f(t, x) \frac{\partial \varphi_f}{\partial t}(t, x) dx dt - \int_0^{+\infty} \int_0^1 \omega \Phi_f(t, x) \frac{\partial \varphi_f}{\partial x}(t, x) dx dt \\ + \int_0^{+\infty} \int_0^1 c \chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial \varphi_f}{\partial x}(t, x) dx dt \\ = \int_0^{+\infty} \int_0^1 \frac{1}{2} Q(t, x) \varphi_f(t, x) dx dt - \int_0^{+\infty} \int_0^1 f F(x) \Phi_f(t, x) \varphi_f(t, x) dx dt \\ - \int_0^{+\infty} \int_0^1 r \Phi_f(t, x) \varphi_f(t, x) dx dt, \end{aligned} \quad (2.5)$$

and $\forall \varphi_b \in H_0^1((0, +\infty) \times (0, 1))$,

$$\begin{aligned} - \int_0^{+\infty} \int_0^1 \Phi_b(t, x) \frac{\partial \varphi_b}{\partial t}(t, x) dx dt + \int_0^{+\infty} \int_0^1 \omega \Phi_b(t, x) \frac{\partial \varphi_b}{\partial x}(t, x) dx dt \\ + \int_0^{+\infty} \int_0^1 c \chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial \varphi_b}{\partial x}(t, x) dx dt \\ = \int_0^{+\infty} \int_0^1 \frac{1}{2} Q(t, x) \varphi_b(t, x) dx dt - \int_0^{+\infty} \int_0^1 f F(x) \Phi_b(t, x) \varphi_b(t, x) dx dt \\ + \int_0^{+\infty} \int_0^1 r \Phi_f(t, x) \varphi_b(t, x) dx dt, \end{aligned} \quad (2.6)$$

Yet, to take the boundary and the initial conditions into account, we enriched formulations (2.5) and (2.6) to get formulations for test functions having non-vanishing traces on $\{(0, x), x \in (0, 1)\}$, $\{(t, 0), t \in (0, +\infty)\}$ and $\{(t, 1), t \in (0, +\infty)\}$.

In view of the Variational Formulation (2.5) and (2.6), Φ_f , Φ_b , $\frac{\partial \Phi_f}{\partial x}$ and $\frac{\partial \Phi_b}{\partial x}$ have to belong to $L^2((0, +\infty) \times (0, 1))$. Hence, Φ_f and Φ_b have to belong to $H^1((0, +\infty) \times (0, 1))$.

Yet, some boundary conditions are imposed for $x = 0$ and $x = 1$. Therefore, the continuity of the functions Φ_f and Φ_b and their existence at $x = 0$ and $x = 1$ have to be ensured at each instant t belonging to $(0, +\infty)$. Hence, Φ_f and Φ_b have also to belong to $C^0((0, +\infty), H^1(0, 1))$.

The initial formulations (1.1) and (1.2) involve the Laplacian of Φ_f and Φ_b . Those terms is then multiplied by Test Functions which can be Φ_f or Φ_b . Yet, the Laplacian of a function belonging to $H^1(0, 1)$, belongs to $H^{-1}(0, 1)$. Hence, $\Phi_f(t, \cdot)$ and $\Phi_b(t, \cdot)$ have to be continuous and differentiable functions belonging to $H^{-1}(0, 1)$, for each $t \in (0, +\infty)$. Hence, Φ_f and Φ_b have to belong to $C^1((0, +\infty), H^{-1}(0, 1))$.

Therefore, it seems correct to think that Φ_f and Φ_b belong to \mathbf{H} .

To sum up, Space \mathbf{H} is the space of functions belonging to $L^2((0, +\infty) \times (0, 1))$, such as their first derivatives also belong to $L^2((0, +\infty) \times (0, 1))$, and at each instant $t \in (0, +\infty)$, continuous, differentiable and belonging to $H^1(0, 1)$ and $H^{-1}(0, 1)$.

If Test Functions φ_f and φ_b , also belong to \mathbf{H} , the weak formulations would be close to (2.5) and (2.6) except that the expressions of Φ_f , Φ_b , φ_f and φ_b at the edges of the domain, appear in the equations. Hence, we considered that we sought Φ_f , Φ_b , φ_f and φ_b in \mathbf{H} such that:

$\forall \varphi_f \in \mathbf{H}$,

$$\begin{aligned}
& - \int_0^T \int_0^1 \Phi_f(t, x) \frac{\partial \varphi_f}{\partial t}(t, x) dx dt + \int_0^1 \Phi_f(0, x) \varphi_f(0, x) dx - \int_0^1 \Phi_f(T, x) \varphi_f(T, x) dx \\
& - \int_0^T \int_0^1 \omega \Phi_f(t, x) \frac{\partial \varphi_f}{\partial x}(t, x) dx dt - \int_0^T \omega \Phi_f(t, 1) \varphi_f(t, 1) dt + \int_0^T \omega \Phi_f(t, 0) \varphi_f(t, 0) dt \\
& \quad + \int_0^T \int_0^1 c \chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial \varphi_f}{\partial x}(t, x) dx dt \\
& = \int_0^T \int_0^1 \frac{1}{2} Q(t, x) \varphi_f(t, x) dx dt - \int_0^T \int_0^1 f F(x) \Phi_f(t, x) \varphi_f(t, x) dx dt \\
& \quad - \int_0^T \int_0^1 r \Phi_f(t, x) \varphi_f(t, x) dx dt, \quad (2.7)
\end{aligned}$$

and $\forall \varphi_b \in \mathbf{H}$,

$$\begin{aligned}
& - \int_0^T \int_0^1 \Phi_b(t, x) \frac{\partial \varphi_b}{\partial t}(t, x) dx dt + \int_0^1 \Phi_b(0, x) \varphi_b(0, x) dx - \int_0^1 \Phi_b(T, x) \varphi_b(T, x) dx \\
& + \int_0^T \int_0^1 \omega \Phi_b(t, x) \frac{\partial \varphi_b}{\partial x}(t, x) dx dt + \int_0^T \omega \Phi_b(t, 1) \varphi_b(t, 1) - \int_0^T \omega \Phi_b(t, 0) \varphi_b(t, 0) \\
& \quad + \int_0^T \int_0^1 c \chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial \varphi_b}{\partial x}(t, x) dx dt \\
& = \int_0^T \int_0^1 \frac{1}{2} Q(t, x) \varphi_b(t, x) dx dt - \int_0^T \int_0^1 f F(x) \Phi_b(t, x) \varphi_b(t, x) dx dt \\
& \quad + \int_0^T \int_0^1 r \Phi_f(t, x) \varphi_b(t, x) dx dt. \quad (2.8)
\end{aligned}$$

Finally, by summing (2.7) and (2.8) we obtained the global Variational Formulation (1.10).

2.3 The first a priori estimates

In this section, the first a priori estimates will be presented. They permitted to demonstrate Theorem 1.1.

Proof of Theorem 1.1: We noticed that, if Φ_f and Φ_b , solutions to (1.1) and (1.2), exist and belong to \mathbf{H} , they can be used as test functions in formulations (2.7) and (2.8). Hence, in a first time, we chose $\varphi_f = \Phi_f$ and $\varphi_b = \Phi_b$. Equations (2.7) and (2.8) became respectively:

$$\begin{aligned}
& - \int_0^T \int_0^1 \Phi_f(t, x) \frac{\partial \Phi_f}{\partial t}(t, x) dx dt + \int_0^1 \Phi_f^2(0, x) dx - \int_0^1 \Phi_f^2(T, x) dx \\
& - \int_0^T \int_0^1 \omega \Phi_f(t, x) \frac{\partial \Phi_f}{\partial x}(t, x) dx dt - \int_0^T \omega \Phi_f^2(t, 1) dt + \int_0^T \omega \Phi_f^2(t, 0) dt \\
& \quad + \int_0^T \int_0^1 c \chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial \Phi_f}{\partial x}(t, x) dx dt \\
& = \int_0^T \int_0^1 \frac{1}{2} Q(t, x) \Phi_f(t, x) dx dt - \int_0^T \int_0^1 f F(x) \Phi_f(t, x) \Phi_f(t, x) dx dt \\
& \quad - \int_0^T \int_0^1 r \Phi_f(t, x) \Phi_f(t, x) dx dt, \quad (2.9)
\end{aligned}$$

and,

$$\begin{aligned}
& - \int_0^T \int_0^1 \Phi_b(t, x) \frac{\partial \Phi_b}{\partial t}(t, x) dx dt + \int_0^1 \Phi_b^2(0, x) dx - \int_0^1 \Phi_b^2(T, x) dx \\
& \quad + \int_0^T \int_0^1 \omega \Phi_b(t, x) \frac{\partial \Phi_b}{\partial x}(t, x) dx dt + \int_0^T \omega \Phi_b^2(t, 1) dt - \int_0^T \omega \Phi_b^2(t, 0) dt \\
& \quad \quad + \int_0^T \int_0^1 c\chi \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial \Phi_b}{\partial x}(t, x) dx dt \\
& = \int_0^T \int_0^1 \frac{1}{2} Q(t, x) \Phi_b(t, x) dx dt - \int_0^T \int_0^1 fF(x) \Phi_b(t, x) \Phi_b(t, x) dx dt \\
& \quad \quad \quad + \int_0^T \int_0^1 r \Phi_f(t, x) \Phi_b(t, x) dx dt. \quad (2.10)
\end{aligned}$$

Then, the objective was to verify from (2.9) and (2.10) that Φ_f and Φ_b belong to $L^{+\infty}((0, T), L^2((0, 1)))$, for any $T \in (0, +\infty)$, which would be consistent with the fact that Φ_f and Φ_b belong to \mathbf{H} .

Equations (2.9) and (2.10) depend of each other. Hence, it is necessary to study the solutions of these two equations at the same time. Therefore, by summing (2.9) and (2.10) we obtained the following equality:

$$\begin{aligned}
& - \int_0^T \int_0^1 \Phi_f(t, x) \frac{\partial \Phi_f}{\partial t}(t, x) dx dt - \int_0^T \int_0^1 \Phi_b(t, x) \frac{\partial \Phi_b}{\partial t}(t, x) dx dt \\
& \quad + \int_0^1 \Phi_f^2(0, x) dx - \int_0^1 \Phi_f^2(T, x) dx + \int_0^1 \Phi_b^2(0, x) dx - \int_0^1 \Phi_b^2(T, x) dx \\
& \quad - \int_0^T \omega \Phi_f^2(t, 1) dt + \int_0^T \omega \Phi_f^2(t, 0) dt + \int_0^T \omega \Phi_b^2(t, 1) dt - \int_0^T \omega \Phi_b^2(t, 0) dt \\
& = \int_0^T \int_0^1 \omega \Phi_f(t, x) \frac{\partial \Phi_f}{\partial x}(t, x) dx dt - \int_0^T \int_0^1 \omega \Phi_b(t, x) \frac{\partial \Phi_b}{\partial x}(t, x) dx dt \\
& \quad - 2 \int_0^T \int_0^1 c\chi \left(\frac{\partial \Phi_f + \Phi_b}{\partial x}(t, x) \right)^2 dx dt \\
& \quad + \int_0^T \int_0^1 Q(t, x) (\Phi_f(t, x) + \Phi_b(t, x)) dx dt \\
& \quad - \int_0^T \int_0^1 fF(x) (\Phi_f^2(t, x) + \Phi_b^2(t, x)) dx dt \\
& \quad - \int_0^T \int_0^1 r \Phi_f^2(t, x) dx dt + \int_0^T \int_0^1 r \Phi_f(t, x) \Phi_b(t, x) dx dt. \quad (2.11)
\end{aligned}$$

From Equation (2.11), we calculated the integrals and we obtained the following equality:

$$\begin{aligned}
& \int_0^T \left(\frac{\partial \|\Phi_f\|_2^2}{\partial t}(t, \cdot) + \frac{\partial \|\Phi_b\|_2^2}{\partial t}(t, \cdot) \right) dt = \\
& \int_0^1 2 \left(\Phi_f^2(0, x) dx - \Phi_f^2(T, x) dx + \Phi_b^2(0, x) dx - \Phi_b^2(T, x) \right) dx \\
& + \int_0^T 2\omega \left(-\Phi_b^2(t, 1) + \Phi_b^2(t, 0) - \Phi_f^2(t, 1) + \Phi_f^2(t, 0) \right) dt \\
& - 2 \int_0^T \int_0^1 c\chi \left(\frac{\partial(\Phi_f + \Phi_b)}{\partial x}(t, x) \right)^2 dx dt \\
& + 2 \int_0^T \int_0^1 Q(t, x) (\Phi_f(t, x) + \Phi_b(t, x)) dx dt \\
& - 2 \int_0^T \int_0^1 fF(x) (\Phi_f^2(t, x) + \Phi_b^2(t, x)) dx dt \\
& - 2 \int_0^T \int_0^1 r\Phi_f^2(t, x) dx dt + 2 \int_0^T \int_0^1 r\Phi_f(t, x)\Phi_b(t, x) dx dt, \quad (2.12)
\end{aligned}$$

where $\|\cdot\|_2$ is the usual norm on $L^2((0, 1))$.

Considering the imposed boundary conditions and since the function χ is non-negative, we obtained from Equation (2.12) the following inequality:

$$\begin{aligned}
& \int_0^T \left(\frac{\partial \|\Phi_f\|_2^2}{\partial t}(t, \cdot) + \frac{\partial \|\Phi_b\|_2^2}{\partial t}(t, \cdot) \right) dt \leq -2 \int_0^T \int_0^1 c\chi \left(\frac{\partial(\Phi_f + \Phi_b)}{\partial x}(t, x) \right)^2 dx dt \\
& + 2 \int_0^T \int_0^1 Q(t, x) (\Phi_f(t, x) + \Phi_b(t, x)) dx dt \\
& + 2 \int_0^T \int_0^1 r\Phi_f(t, x)\Phi_b(t, x) dx dt + K_1, \quad (2.13)
\end{aligned}$$

where K_1 is a positive constant.

However, using Schwarz inequality, we obtained for any $t \in (0, T)$, $T \in (0, +\infty)$:

$$\left| \int_0^1 Q(t, x)\Phi_f(t, x) dx \right| \leq \|Q(t, \cdot)\|_2 \|\Phi_f(t, \cdot)\|_2, \quad (2.14)$$

$$\left| \int_0^1 Q(t, x)\Phi_b(t, x) dx \right| \leq \|Q(t, \cdot)\|_2 \|\Phi_b(t, \cdot)\|_2, \quad (2.15)$$

and,

$$\left| \int_0^1 \Phi_f(t, x) \Phi_b(t, x) dx \right| \leq \|\Phi_f(t, \cdot)\|_2 \|\Phi_b(t, \cdot)\|_2. \quad (2.16)$$

Therefore, from Inequality (2.13) and by using Inequalities (2.14), (2.15) and (2.16), we get:

$$\begin{aligned} \int_0^T \left(\frac{\partial \|\Phi_f\|_2^2}{\partial t}(t, \cdot) + \frac{\partial \|\Phi_b\|_2^2}{\partial t}(t, \cdot) \right) dt &\leq 2 \int_0^T \left(r \|\Phi_f(t, \cdot)\|_2 \|\Phi_b(t, \cdot)\|_2 \right. \\ &\quad \left. + \|Q(t, \cdot)\|_2 \left(\|\Phi_f(t, \cdot)\|_2 + \|\Phi_b(t, \cdot)\|_2 \right) \right) dt + K_1 \end{aligned} \quad (2.17)$$

From Inequality (2.17) and by using the binomial theorem we obtained:

$$\begin{aligned} \int_0^T \left(\frac{\partial \|\Phi_f\|_2^2}{\partial t}(t, \cdot) + \frac{\partial \|\Phi_b\|_2^2}{\partial t}(t, \cdot) \right) dt &\leq 2 \int_0^T \left(r \left(\|\Phi_f(t, \cdot)\|_2^2 + \|\Phi_b(t, \cdot)\|_2^2 \right) \right. \\ &\quad \left. + \|Q(t, \cdot)\|_2 \left(\sqrt{2 \left(\|\Phi_f(t, \cdot)\|_2^2 + \|\Phi_b(t, \cdot)\|_2^2 \right)} \right) \right) dt + K_1 \end{aligned} \quad (2.18)$$

According to the Gronwall lemma, for any $T \in (0, +\infty)$, there exist a constant $L_1(T)$, which can be seen as an increasing function of T , such that:

$$\|\Phi_f(T, \cdot)\|_2^2 + \|\Phi_b(T, \cdot)\|_2^2 \leq L_1(T). \quad (2.19)$$

Hence, for any $T \in (0, +\infty)$ and for any $t \in (0, T)$, we obtained:

$$\|\Phi_f(t, \cdot)\|_2^2 + \|\Phi_b(t, \cdot)\|_2^2 \leq L_1(T). \quad (2.20)$$

Therefore, for any $T \in (0, +\infty)$, there exist constants $L_{2a}(T)$ and $L_{2b}(T)$ such that Inequalities (1.11) and (1.12) are verified. Hence, we deduced from these two inequalities that Φ_f and Φ_b belong to $L^\infty\left((0, T), L^2(\mathbb{R})\right)$, $T \in (0, +\infty)$. Which is consistent with the fact that Φ_f and Φ_b belong to \mathbf{H} .

In the aim of demonstrating that Φ_f and Φ_b belong to $L^\infty\left((0, T), H^1((0, 1))\right)$, $T \in (0, +\infty)$, we took up Equation (2.13) and we used Inequalities (2.14), (2.15) and (2.16) to obtain the following inequality:

$$\begin{aligned} & \frac{\partial \|\Phi_f\|_2^2}{\partial t}(t, \cdot) + \frac{\partial \|\Phi_b\|_2^2}{\partial t}(t, \cdot) + 2 \int_0^1 c\chi \left(\frac{\partial(\Phi_f + \Phi_b)}{\partial x}(t, x) \right)^2 dx \\ & \leq 2 \|Q(t, \cdot)\|_2 \left(\|\Phi_f(t, \cdot)\|_2 + \|\Phi_b(t, \cdot)\|_2 \right) + 2r \|\Phi_f(t, \cdot)\|_2 \|\Phi_b(t, \cdot)\|_2 + K_1 \end{aligned} \quad (2.21)$$

To control the two first terms of Inequality (2.21) we integrated it over time. Then, by using (1.11) and (1.12) we obtained for any T belonging to $(0, +\infty)$:

$$\begin{aligned} & \int_0^T \frac{\partial \|\Phi_f\|_2^2}{\partial t}(t, \cdot) dt + \int_0^T \frac{\partial \|\Phi_b\|_2^2}{\partial t}(t, \cdot) dt \\ & \quad + 2 \int_0^T \int_0^1 c\chi \left(\frac{\partial(\Phi_f + \Phi_b)}{\partial x}(t, x) \right)^2 dx dt \leq L_{2c}(T), \end{aligned} \quad (2.22)$$

where $L_{2c}(T)$ is a positive constant.

In Inequality (2.22) the two first terms are bounded. Hence we obtained Inequality (1.13).

Therefore, we proved Theorem (1.1) and Inequality (1.13) shows that,

$$\forall T \in (0, +\infty), \quad \frac{\partial(\Phi_f + \Phi_b)}{\partial x} \in L^2((0, T) \times (0, 1)). \quad (2.23)$$

Hence, according to (2.20), $(\Phi_f + \Phi_b)$ belongs to $H^1((0, T) \times (0, 1))$, for any $T \in \mathbb{R}$. ■

To conclude this section, the obtained inequalities do not permit to deduce that Φ_f and Φ_b both belong to $L^\infty\left((0, T), H^1(0, 1)\right)$, $T \in (0, +\infty)$. Indeed, the dependence between Equations (1.1) and (1.2) requires to study the sum $(\Phi_f + \Phi_b)$ and we only bounded the average of $\chi(\Phi_f + \Phi_b)$ over time. Furthermore, Function χ , vanishing near $\partial\Omega$, masks the regularity or the irregularity of Φ_f and Φ_b at the edges of the domain. Yet, some boundary conditions are imposed to those functions. Hence, Function χ hinders the demonstration of the regularity of Φ_f and Φ_b at $\partial\Omega$.

The a priori estimates showed us that the different hypotheses and boundary conditions associated to the mathematical model developed in Flourent *et al.* (Submitted) complicate from demonstrating that Φ_f and Φ_b belong to $L^\infty\left((0, T), H^1(0, 1)\right)$, $T \in (0, +\infty)$.

We made some explorations and we concluded that the studied equations do not fall within the framework studied by Ladyženskaja *et al.* (1968). Therefore, we put in evidence the lost and the maintained properties on simplified models. The objective is to try to find some directions to achieve the demonstration of the existence and uniqueness of the solution of the initial model developed in Flourent *et al.* (Submitted).

3 Identification of the properties lost due to the «diffusion term» structure, through the use of a first toy model

In this section we will present the lost properties through the use of a first toy model made of Equations (1.20) and (1.21). The imposed initial conditions associated to this model are given by (1.22) and (1.23).

3.1 Uniqueness of the solution

We demonstrated that:

Property 3.1. *If the solution of Equations (1.20) and (1.21) exists, then it is unique.*

Proof of Property 3.1: We assumed that $\Phi_{f_1} \in \mathbf{H}$ and $\Phi_{f_2} \in \mathbf{H}$ are solutions to (1.20), and $\Phi_{b_1} \in \mathbf{H}$ and $\Phi_{b_2} \in \mathbf{H}$ are solutions to (1.21). The objective of this demonstration is to prove that

$$\Phi_{f_1} = \Phi_{f_2} \quad (3.1)$$

and,

$$\Phi_{b_1} = \Phi_{b_2} \quad (3.2)$$

By subtraction we obtained,

$$\frac{\partial \Phi_{f_1} - \Phi_{f_2}}{\partial t}(t, x) - c \frac{\partial \left[\chi \frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right]}{\partial x}(t, x) = 0 \quad (3.3)$$

and,

$$\frac{\partial \Phi_{b_1} - \Phi_{b_2}}{\partial t}(t, x) - c \frac{\partial \left[\chi \frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right]}{\partial x}(t, x) = 0 \quad (3.4)$$

We multiplied Equation (3.10) by Test Function $\Phi_{f_1} - \Phi_{f_2}$:

$$\begin{aligned} & \frac{\partial \Phi_{f_1} - \Phi_{f_2}}{\partial t}(t, x)(\Phi_{f_1} - \Phi_{f_2})(t, x) \\ & - c \frac{\partial \left[\chi \frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right]}{\partial x}(t, x)(\Phi_{f_1} - \Phi_{f_2})(t, x) = 0 \quad (3.5) \end{aligned}$$

Then, we multiplied Equation (3.4) by Test Function $\Phi_{b_1} - \Phi_{b_2}$:

$$\begin{aligned} & \frac{\partial \Phi_{b_1} - \Phi_{b_2}}{\partial t}(t, x)(\Phi_{b_1} - \Phi_{b_2})(t, x) \\ & - c \frac{\partial \left[\chi \frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right]}{\partial x}(t, x)(\Phi_{b_1} - \Phi_{b_2})(t, x) = 0 \quad (3.6) \end{aligned}$$

By integrating over space the first terms and by integrating by part the diffusion term in Equations (3.5) and (3.6) we obtained,

$$\begin{aligned} & \frac{1}{2} \frac{\partial \|\Phi_{f_1} - \Phi_{f_2}\|_2^2}{\partial t}(t, \cdot) \\ & + \int_0^1 c\chi \left(\frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right)^2(t, x) dx = 0 \quad (3.7) \end{aligned}$$

and,

$$\begin{aligned} & \frac{\partial \|\Phi_{b_1} - \Phi_{b_2}\|_2^2}{\partial t}(t, \cdot) \\ & + \int_0^1 c\chi \left(\frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right)^2(t, x) dx = 0 \quad (3.8) \end{aligned}$$

We summed Equations (3.7) and (3.8):

$$\begin{aligned} & \frac{1}{2} \frac{\partial \|\Phi_{f_1} - \Phi_{f_2}\|_2^2 + \|\Phi_{b_1} - \Phi_{b_2}\|_2^2}{\partial t}(t, \cdot) \\ & = - \int_0^1 2c\chi \left(\frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right)^2(t, x) dx \leq 0 \quad (3.9) \end{aligned}$$

Therefore, according to the Gronwall lemma we obtained:

$$\|\Phi_{f_1} - \Phi_{f_2}\|_2^2(t, \cdot) + \|\Phi_{b_1} - \Phi_{b_2}\|_2^2(t, \cdot) \leq 0 \quad (3.10)$$

We deduced that,

$$\|\Phi_{f_1} - \Phi_{f_2}\|_2^2(t, \cdot) = 0 \quad (3.11)$$

and,

$$\|\Phi_{b_1} - \Phi_{b_2}\|_2^2(t, \cdot) = 0 \quad (3.12)$$

Hence, we obtained Equalities (3.1) and (3.2). Therefore we proved Property 3.1. ■

3.2 The regularizing effect of the «diffusion operator»

In this subsection we will present the framework ensuring regularity, by demonstrating Property 1.3.

Proof of Property 1.3: We started by summing Equations (1.20) and (1.21) and we obtained,

$$\frac{\partial[\Phi_f + \Phi_b]}{\partial t}(t, x) - 2c \frac{\partial^2[\Phi_f + \Phi_b]}{\partial x^2}(t, x) = 0, \quad (3.13)$$

Then, we deduced Equation (1.21) from Equation (1.20) and we obtained,

$$\frac{\partial[\Phi_f - \Phi_b]}{\partial t}(t, x) = 0, \quad (3.14)$$

Equation (3.13) is the heat equation where the diffusion term provides a regularizing effect. Hence, if $(\Phi_f + \Phi_b)(0, \cdot) \in L^2((0, 1))$ and $(\Phi_f + \Phi_b)(\cdot, 1) = (\Phi_f + \Phi_b)(\cdot, 0) = 0$, then $(\Phi_f + \Phi_b)$ exists, is unique and belongs to \mathbf{H} .

However, in Equation (3.14) there is no regularizing effect. Nevertheless this equation shows that $(\Phi_f(t, x) - \Phi_b(t, x))$ is constant over time. Hence, as $(\Phi_f - \Phi_b)(\cdot, 1) = (\Phi_f - \Phi_b)(\cdot, 0) = 0$ and $\Phi_f(0, x) - \Phi_b(0, x)$ is in $H^1(0, 1)$, there is no irregularity generated by (3.14). Then $(\Phi_f - \Phi_b)$ belongs to $C^1((0, +\infty); H^1(0, 1))$. Hence, we proved the existence of Φ_f and Φ_b and that they belong to $C^1((0, +\infty); H^1(0, 1))$. Therefore, involving Property 3.1 ends the proof of Property 1.3. ■

This result showed us that the structure of the built model and especially the structure of the «diffusion operator» implied into it, seems to require to imposed some regularity on initial conditions.

3.3 The question of the non-negativity of Φ_f and Φ_b

We studied the question of the non-negativity of Φ_f and Φ_b solutions of Equations (1.20) and (1.21) by using this first toy model. We started by setting:

$$U(t, x) = \Phi_f(t, x) + \Phi_b(t, x), \quad (3.15)$$

and,

$$V(t, x) = \Phi_f(t, x) - \Phi_b(t, x). \quad (3.16)$$

Therefore, from Equations (3.15) and (3.16), Equation (3.13) became:

$$\frac{\partial U}{\partial t}(t, x) - 2c \frac{\partial^2 U}{\partial x^2}(t, x) = 0, \quad (3.17)$$

and, Equation (3.14) became:

$$\frac{\partial V}{\partial t}(t, x) = 0, \quad (3.18)$$

Hence,

$$V(t, x) = \Phi_f(0, x) - \Phi_b(0, x) = V_0(x). \quad (3.19)$$

We searched the elementary solutions of Equation (3.17) by resolving the heat equation and by using the method of the separation of variables. We obtained an elementary solution of the form,

$$U_n(t, x) = C_n \exp(2c(n\pi)^2 t) \sin(n\pi x). \quad (3.20)$$

According to the superposition principle, all the linear combinations based on Equation (3.20), are also solutions to Equation (3.15). Therefore, the general solution of (3.15) is :

$$U(t, x) = \sum_{n=1}^{\infty} \left(C_n \exp(2c(n\pi)^2 t) \sin(n\pi x) \right) \quad (3.21)$$

where,

$$C_n = 2 \int_0^1 U(0, y) \sin(n\pi y) dy \quad (3.22)$$

If there exists a time T_F such that, for all $t > T_F$, and for all $x \in (0, 1)$, $Q(t, x) = 0$, as soon as $t > T_F$, when t increase, Function U exponentially decreases to 0. Hence, if we fixed $V_0(x) > 0$ for all $x \in (0, 1)$, there exists a constant $T_{U_1} > 0$ such that for all $t > T_{U_1}$,

$$U(t, x) \leq V(t, x). \quad (3.23)$$

It implies that,

$$\Phi_b(t, x) \leq 0. \quad (3.24)$$

We also noticed that if we fixed $V_0(x) < 0$ for all $x \in (0, 1)$, there exists a constant $T_{U_2} > 0$ such that for all $t > T_{U_2}$,

$$U(t, x) \leq -V(t, x). \quad (3.25)$$

It implies that,

$$\Phi_f(t, x) \leq 0. \quad (3.26)$$

Therefore, we deduced from this very simple toy model that the non-negativity of Φ_f and Φ_b can be not verified.

By using this very simplified toy model we put in evidence that the involvement of a diffusion term depending on the sum of two unknown densities, induces the lost of properties as the regularizing effect relative to the diffusion term and the non-negativity of each component of the solution.

In the next section we will use a second toy model, closer to the model developed in Flourent *et al.* (Submitted), to put in evidence the properties which can be expected despite the structure of the «diffusion term».

4 Proof of the wellposedness of the second toy model: A mathematical model without convection

In this section we will present the a priori estimates performed on the second toy model.

This simplified model corresponds to the initial model presented in Introduction but without the convection terms. Moreover, we decided to work in a periodic framework. Therefore, in this framework, there are no edges and the function χ is not necessary anymore.

Hence, the two first equations of this second toy model are (1.27) and (1.28). The three other equations remain Equations (1.16), (1.18) and (1.17) or (1.15.b). In this model Φ_f and Φ_b are periodic functions with respect to x belonging to $\mathbf{H}_\#$ and we made Assumptions (1.29) to (1.38).

4.1 The exact expressions of Φ_f and Φ_b

In this section we will present the different steps permitting to find the exact expression of Φ_f and Φ_b . For this we will call upon Fourier Transform and we wrote:

$$Q(t, x) = \sum_{n \in \mathbb{Z}} \hat{Q}^n(t) \exp(2i\pi nx), \quad (4.1)$$

$$\Phi_f(t, x) = \sum_{n \in \mathbb{Z}} \hat{\Phi}_f^n(t) \exp(2i\pi nx), \quad (4.2)$$

and,

$$\Phi_b(t, x) = \sum_{n \in \mathbb{Z}} \hat{\Phi}_b^n(t) \exp(2i\pi nx), \quad (4.3)$$

we defined,

$$K_u^n = 8\pi^2 n^2 c + f, \quad (4.4)$$

and we obtained the following property:

Property 4.1. *Under Assumptions (1.32), (1.33) and (1.34), Equations (1.27) and (1.28) admit respectively a solution $\Phi_f \in \mathbf{H}_\#$ and $\Phi_b \in \mathbf{H}_\#$, such that for all $x \in (0, 1)$ and all $t \in (0, +\infty)$:*

$$\begin{aligned} \Phi_f(t, x) = & \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\int_0^t \hat{Q}^n(\tau) \exp\left(-K_u^n(t-\tau)\right) d\tau \right. \\ & + \int_0^t \int_0^\tau r \hat{Q}^n(\nu) \exp\left(-K_u^n(\tau-\nu)\right) \exp\left(-(r+f)(t-\tau)\right) d\nu d\tau \\ & + \int_0^t r \left(\hat{\Phi}_f^n(0) + \hat{\Phi}_b^n(0) \right) \exp\left(-K_u^n \tau\right) \exp\left(-(r+f)(t-\tau)\right) d\tau \\ & + \left(\hat{\Phi}_f^n(0) + \hat{\Phi}_b^n(0) \right) \exp\left(-K_u^n t\right) \\ & \left. + \left(\hat{\Phi}_f^n(0) - \hat{\Phi}_b^n(0) \right) \exp\left(-(r+f)t\right) \right) \exp(2i\pi nx), \quad (4.5) \end{aligned}$$

and,

$$\begin{aligned} \Phi_b(t, x) = & \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\int_0^t \hat{Q}^n(\tau) \exp\left(-K_u^n(t-\tau)\right) d\tau \right. \\ & - \int_0^t \int_0^\tau r \hat{Q}^n(\nu) \exp\left(-K_u^n(\tau-\nu)\right) \exp\left(-(r+f)(t-\tau)\right) d\nu d\tau \\ & - \int_0^t r \left(\hat{\Phi}_f^n(0) + \hat{\Phi}_b^n(0) \right) \exp\left(-K_u^n \tau\right) \exp\left(-(r+f)(t-\tau)\right) d\tau \\ & + \left(\hat{\Phi}_f^n(0) + \hat{\Phi}_b^n(0) \right) \exp\left(-K_u^n t\right) \\ & \left. - \left(\hat{\Phi}_f^n(0) - \hat{\Phi}_b^n(0) \right) \exp\left(-(r+f)t\right) \right) \exp(2i\pi nx). \quad (4.6) \end{aligned}$$

Proof of Property 4.1: As a first step, we summed Equations (1.27) and (1.28):

$$\begin{aligned} \frac{\partial \Phi_f + \Phi_b}{\partial t}(t, x) - 2c \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) \\ = Q(t, x) - f \left(\Phi_f(t, x) + \Phi_b(t, x) \right). \quad (4.7) \end{aligned}$$

Then, we deduced Equation (1.28) from Equations (1.27):

$$\frac{\partial \Phi_f - \Phi_b}{\partial t}(t, x) = -f \left(\Phi_f(t, x) - \Phi_b(t, x) \right) - 2r \Phi_f(t, x). \quad (4.8)$$

We made appear in Equations (4.7) and (4.8) several terms involving the difference and the sum of Φ_f and Φ_b . Therefore, we fixed:

$$U(t, x) = \Phi_f(t, x) + \Phi_b(t, x), \quad (4.9)$$

and,

$$V(t, x) = \Phi_f(t, x) - \Phi_b(t, x). \quad (4.10)$$

U and V respect periodicity boundary conditions with respect to x and $U(0, \cdot)$ and $V(0, \cdot)$ are periodic and belong to $L^2((0, 1))$.

Equation (4.7) became:

$$\frac{\partial U}{\partial t}(t, x) - 2c \frac{\partial^2 U}{\partial x^2}(t, x) = Q(t, x) - fU(t, x), \quad (4.11)$$

and Equation (4.8) became:

$$\frac{\partial V}{\partial t}(t, x) = -fV(t, x) - 2r\Phi_f(t, x), \quad (4.12)$$

or,

$$\frac{\partial V}{\partial t}(t, x) = -(r + f)V(t, x) - rU(t, x). \quad (4.13)$$

4.1.1 The exact expression of U

We applied the Fourier Transform to Equation (4.11) by setting:

$$U(t, x) = \sum_{n \in \mathbb{Z}} \hat{U}^n(t) \exp(2i\pi nx), \quad (4.14)$$

and then, from Equations (4.11) and (4.14), we obtained:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \left[\hat{U}^{n'}(t) \exp(2i\pi nx) + 2c \hat{U}^n(t) 4n^2 \pi^2 \exp(2i\pi nx) \right] \\ = \sum_{n \in \mathbb{Z}} \left[\hat{Q}^n(t) \exp(2i\pi nx) - f \hat{U}^n \exp(2i\pi nx) \right]. \end{aligned} \quad (4.15)$$

Then, by gathering some terms, we get:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \left[\hat{U}^{n'}(t) \exp(2i\pi nx) + (8cn^2\pi^2 + f)\hat{U}^n(t) \exp(2i\pi nx) \right] \\ = \sum_{n \in \mathbb{Z}} \left[\hat{Q}^n(t) \exp(2i\pi nx) \right]. \end{aligned} \quad (4.16)$$

The terms $(\exp(2i\pi nx))_{n \in \mathbb{Z}}$ form a Hilbert basis. Therefore, for all $n \in \mathbb{Z}$, we obtained:

$$\hat{U}^{n'}(t) \exp(2i\pi nx) + (8cn^2\pi^2 + f)\hat{U}^n(t) \exp(2i\pi nx) = \hat{Q}^n(t) \exp(2i\pi nx). \quad (4.17)$$

Hence,

$$\hat{U}^{n'}(t) + (8cn^2\pi^2 + f)\hat{U}^n(t) = \hat{Q}^n(t). \quad (4.18)$$

For all $n \in \mathbb{Z}$ this ODE is provided with an initial condition $\hat{U}^n(0)$ related with the Fourier coefficients of $\Phi_f(0, x)$ and $\Phi_b(0, x)$.

We resolved Equation (4.18) and we obtained:

$$\hat{U}^n(t) = \hat{U}^n(0) \exp(-K_u^n t) + \int_0^t \hat{Q}^n(\tau) \exp(-K_u^n(t - \tau)) d\tau, \quad (4.19)$$

where K_u^n is given by Equality (4.4).

4.1.2 The exact expression of V

We also applied the Fourier Transform to Equation (4.13) by fixing:

$$V(t, x) = \sum_{n \in \mathbb{Z}} \hat{V}^n(t) \exp(2i\pi nx) \quad (4.20)$$

Therefore Equation (4.13) became:

$$\sum_{n \in \mathbb{Z}} \left[\hat{V}^{n'}(t) \exp(2i\pi nx) + (r+f)\hat{V}^n(t) 4n^2\pi^2 \exp(2i\pi nx) \right] = \sum_{n \in \mathbb{Z}} \left[-r\hat{U}^n(t)(x) \exp(2i\pi nx) \right] \quad (4.21)$$

As previously, the terms $(\exp(2i\pi nx))_{n \in \mathbb{Z}}$ form a Hilbert basis. Therefore, for all $n \in \mathbb{Z}$,

$$\hat{V}^{n'}(t) + (r + f)\hat{V}^n(t)4n^2\pi^2 = -r\hat{U}^n(t) \quad (4.22)$$

For all $n \in \mathbb{Z}$ this ODE is provided with an initial condition $\hat{V}^n(0)$ related with the Fourier coefficients of $\Phi_f(0, x)$ and $\Phi_b(0, x)$.

We resolved Equation (4.22) and we obtained:

$$\hat{V}^n(t) = \hat{V}^n(0) \exp\left(- (r + f)t\right) + \int_0^t r\hat{U}^n(\tau) \exp\left(- (r + f)(t - \tau)\right) d\tau \quad (4.23)$$

From Equation (4.19), we get:

$$\begin{aligned} \hat{V}^n(t) &= \hat{V}^n(0) \exp\left(- (r + f)t\right) \\ &+ \int_0^t \int_0^\tau r\hat{Q}^n(\nu) \exp\left(- K_u^n(\tau - \nu)\right) \exp\left(- (r + f)(t - \tau)\right) d\nu d\tau \\ &+ \int_0^t r\hat{U}^n(0) \exp\left(- K_u^n\tau\right) \exp\left(- (r + f)(t - \tau)\right) d\tau. \end{aligned} \quad (4.24)$$

4.1.3 The exact expression of Φ_f and Φ_b

By summing Equations (4.9) and (4.10), we obtained:

$$\hat{U}^n(t) + \hat{V}^n(t) = 2\hat{\Phi}_f^n(t), \quad (4.25)$$

and by deducting Equation (4.10) to Equation (4.9), we obtained,

$$\hat{U}^n(t) - \hat{V}^n(t) = 2\hat{\Phi}_b^n(t), \quad (4.26)$$

where $\hat{\Phi}_f^n$ and $\hat{\Phi}_b^n$ are given by Equalities (4.2) and (4.3).

Therefore, from Equations (4.25), (4.26), (4.19) and (4.24), we obtained:

$$\begin{aligned} \hat{\Phi}_f^n(t) &= \frac{1}{2} \left(\int_0^t \hat{Q}^n(\tau) \exp\left(- K_u^n(t - \tau)\right) d\tau \right. \\ &+ \int_0^t \int_0^\tau r\hat{Q}^n(\nu) \exp\left(- K_u^n(\tau - \nu)\right) \exp\left(- (r + f)(t - \tau)\right) d\nu d\tau \\ &+ \int_0^t r\hat{U}^n(0) \exp\left(- K_u^n\tau\right) \exp\left(- (r + f)(t - \tau)\right) d\tau \\ &\left. + \hat{U}^n(0) \exp\left(- K_u^n t\right) + \hat{V}^n(0) \exp\left(- (r + f)t\right) \right), \end{aligned} \quad (4.27)$$

and,

$$\begin{aligned}
\hat{\Phi}_b^n(t) = & \frac{1}{2} \left(\int_0^t \hat{Q}^n(\tau) \exp(-K_u^n(t-\tau)) d\tau \right. \\
& - \int_0^t \int_0^\tau r \hat{Q}^n(\nu) \exp(-K_u^n(\tau-\nu)) \exp(-(r+f)(t-\tau)) d\nu d\tau \\
& - \int_0^t r \hat{U}^n(0) \exp(-K_u^n\tau) \exp(-(r+f)(t-\tau)) d\tau \\
& \left. + \hat{U}^n(0) \exp(-K_u^n t) - \hat{V}^n(0) \exp(-(r+f)t) \right). \quad (4.28)
\end{aligned}$$

By using Equations (4.2) and (4.27), and then Equations (4.3) and (4.28), we finally obtained the exact expressions of Φ_f and Φ_b .

4.1.4 Belonging of Φ_f and Φ_b to $\mathbf{H}_\#$

The expressions of Φ_f and Φ_b are given by Equations (4.5) and (4.6). Now, we have to prove that these two densities belong to $\mathbf{H}_\#$. To do so we will prove that Functions U and V , defined by Equations (4.9) and (4.10), belong to $\mathbf{H}_\#$ and we will deduce that Φ_f and Φ_b belong to $\mathbf{H}_\#$.

In a periodic context, Space $\mathbf{H}_\#$ can be defined such that,

$$\mathbf{H}_\# = H^1((0, +\infty) \times (0, 1)) \cap C^0((0, +\infty), H_\#^1(0, 1)) \cap C^1((0, +\infty), H_\#^{-1}(0, 1)). \quad (4.29)$$

Therefore, by definition, a function Φ defined such as,

$$\Phi(t, x) = \sum_{n \in \mathbb{Z}} \hat{\Phi}^n(t) \exp(2i\pi n x), \quad (4.30)$$

belongs to $\mathbf{H}_\#$ if for all $t \in (0, T)$, $T \in (0, +\infty)$:

$$\sum_{n \in \mathbb{Z}} \left(n \hat{\Phi}^n(t) \right)^2 \leq L_I, \quad (4.31)$$

and,

$$\sum_{n \in \mathbb{Z}} \frac{1}{|1 + n^2|} \hat{\Phi}^n(t)^2 \leq L_{II}, \quad (4.32)$$

where L_I and L_{II} are two positive constants.

Therefore, in this section we will prove that U and V verify Equations (4.31) and (4.32).

According to Equalities (4.14) and (4.19) and the binomial squares we obtained:

$$\left(n\hat{U}^n(t)\right)^2 \leq C_1 \left(n^2 \hat{U}^n(0)^2 \exp(-2K_u^n t) + n^2 \int_0^t \hat{Q}^n(\tau)^2 \exp(-2K_u^n(t-\tau)) d\tau \right), \quad (4.33)$$

where K_u^n is given by Equality (4.4) and C_1 is a positive constant.

We noticed that:

$$\begin{aligned} n^2 \int_0^t \hat{Q}^n(\tau)^2 \exp(-2K_u^n(t-\tau)) d\tau \\ = \int_0^t n^2 \hat{Q}^n(\tau)^2 \exp(-8\pi^2 c_d n^2(t-\tau)) \exp(-2f_d(t-\tau)) d\tau \\ \leq C_2 \int_0^t n^2 \hat{Q}^n(\tau)^2 \exp(-C_3 n^2(t-\tau)) d\tau, \end{aligned} \quad (4.34)$$

where C_2 and C_3 are two positive constants. Yet, according to assumption (1.29), $(\hat{Q}^n)_{n \in \mathbb{Z}}$ belongs to $L^\infty((0, T), l^2)$, for all $t \in (0, T)$, $T \in (0, +\infty)$. Therefore, there exists \overline{Q}^n such that for all $t \in (0, T)$, $T \in (0, +\infty)$:

$$\hat{Q}^n(t)^2 \leq \overline{Q}^n{}^2 \quad (4.35)$$

Moreover,

$$\begin{aligned} \int_0^t n^2 \exp(-C_3 n^2(t-\tau)) d\tau &= \left[\frac{1}{C_3} \exp(-C_3 n^2(t-\tau)) \right]_0^t \\ &= \frac{1}{C_3} \left(1 - \exp(-C_3 n^2 t) \right) \end{aligned} \quad (4.36)$$

Therefore, from Equations (4.34), (4.35) and (4.36), we obtained:

$$n^2 \int_0^t \hat{Q}^n(\tau)^2 \exp(-2K_u^n(t-\tau)) d\tau \leq C_4 \overline{Q}^n{}^2 \left(1 - \exp(-C_3 n^2 t) \right). \quad (4.37)$$

Since $0 \leq \exp(-C_3 n^2 t) \leq 1$, for all $t \in (0, T)$, $T \in (0, +\infty)$:

$$n^2 \int_0^t \hat{Q}^n(\tau)^2 \exp(-2K_u^n(t-\tau)) d\tau \leq C_5 \overline{Q}^n{}^2. \quad (4.38)$$

From Equations (4.33) and (4.38) we obtained

$$\sum_{n \in \mathbb{Z}} \left(n\hat{U}^n(t) \right)^2 \leq C_6 \left(\sum_{n \in \mathbb{Z}} n^2 \hat{U}^n(0)^2 \exp(-2K_u^n t) + \sum_{n \in \mathbb{Z}} \overline{Q}^n{}^2 \right), \quad (4.39)$$

where C_6 is a positive constant.

According to assumption (1.29), $(\hat{Q}^n(t))_{n \in \mathbb{Z}}$ belongs to l^2 , for all $t \in (0, T)$, $T \in (0, +\infty)$. Therefore, there exists $K_1 > 0$ such that for all $t \in (0, T)$, $T \in (0, +\infty)$:

$$\sum_{n \in \mathbb{Z}} \hat{Q}^n(t)^2 \leq K_1 \text{ and } \sum_{n \in \mathbb{Z}} \overline{Q}^{n^2} \leq K_1. \quad (4.40)$$

According to assumption (1.32), $U(0, \cdot)$ belongs to $L^\infty((0, 1))$. It means that there exists $K_2 > 0$ such that:

$$\sum_{n \in \mathbb{Z}} \hat{U}^n(0)^2 \leq K_2 \quad (4.41)$$

Therefore, since the sum $\sum_{n \in \mathbb{Z}} n^2 \exp(-2K_u^n t)$ converges, there exists $K_3 > 0$ such that:

$$\sum_{n \in \mathbb{Z}} n^2 \hat{U}^n(0)^2 \exp(-2K_u^n t) \leq K_3 \quad (4.42)$$

Finally, from Equations (4.39), (4.40) and (4.42) we proved that Function U verifies (4.31).

Function U verifies Inequality (4.31). Hence, for all $t \in (0, T)$, $T \in (0, +\infty)$:

$$\sum_{n \in \mathbb{Z}} \hat{U}^n(t)^2 \leq K_2 \quad (4.43)$$

Moreover, $0 \leq \frac{1}{|1 + n^2|} \leq 1$, for all $n \in \mathbb{Z}$. Hence, Function U verifies (4.32).

We finally proved that U belongs to $\mathbf{H}_\#$.

According to Equalities (4.20) and (4.24) and the binomial squares we obtained:

$$\begin{aligned} (n\hat{V}^n(t))^2 &\leq C_7 \left(n^2 \hat{V}^n(0)^2 \exp(-2(r+f)t) \right. \\ &\quad + n^2 \int_0^t \int_0^\tau r^2 \hat{Q}^n(\nu)^2 \exp(-2K_u^n(\tau-\nu)) \exp(-2(r+f)(t-\tau)) \, d\nu \, d\tau \\ &\quad \left. + n^2 \int_0^t r^2 \hat{U}^n(0)^2 \exp(-2K_u^n \tau) \exp(-2(r+f)(t-\tau)) \, d\tau \right), \end{aligned} \quad (4.44)$$

where K_u^n is given by Equality (4.4) and C_7 is a positive constant.

For all $t \in (0, T)$, $T \in (0, +\infty)$, $0 \leq \exp(-2(r+f)(t-\tau)) \leq 1$. Hence, from Inequality (4.44) and for all $t \in (0, T)$, $T \in (0, +\infty)$ we obtained:

$$\begin{aligned} (n\hat{V}^n(t))^2 &\leq C_7 \left(n^2 \hat{V}^n(0)^2 + n^2 \int_0^t \int_0^\tau r^2 \hat{Q}^n(\nu)^2 \exp(-2K_u^n(\tau-\nu)) \, d\nu \, d\tau \right. \\ &\quad \left. + n^2 \int_0^t r^2 \hat{U}^n(0)^2 \exp(-2K_u^n \tau) \, d\tau \right). \end{aligned} \quad (4.45)$$

We noticed that:

$$\begin{aligned}
n^2 \int_0^t r^2 \hat{U}^n(0)^2 \exp(-2K_u^n \tau) d\tau \\
&= \int_0^t n^2 r^2 \hat{U}^n(0)^2 \exp(-8\pi^2 c_d n^2 \tau) \exp(-2f_d \tau) d\tau \\
&\leq C_8 \hat{U}^n(0)^2 \int_0^t n^2 \exp(-C_9 n^2 \tau) d\tau, \quad (4.46)
\end{aligned}$$

where C_8 and C_9 are two positive constants.

Moreover, for all $t \in (0, T)$, $T \in (0, +\infty)$,

$$\int_0^t n^2 \exp(-C_9 n^2 \tau) d\tau = \left[\frac{-1}{C_9} \exp(-C_9 n^2 \tau) \right]_0^t = \frac{1}{C_9} (1 - \exp(-C_9 n^2 t)) \quad (4.47)$$

Therefore, from Inequalities (4.46) and (4.47), for all $t \in (0, T)$, $T \in (0, +\infty)$ we have:

$$\begin{aligned}
n^2 \int_0^t r^2 \hat{U}^n(0)^2 \exp(-2K_u^n \tau) d\tau \\
\leq C_{10} \hat{U}^n(0)^2 (1 - \exp(-C_9 n^2 t)) \leq C_{10} \hat{U}^n(0)^2, \quad (4.48)
\end{aligned}$$

where C_{10} is a positive constant.

Using Inequality (4.35) we have:

$$\begin{aligned}
n^2 \int_0^t \int_0^\tau r^2 \hat{Q}^n(\nu)^2 \exp(-2K_u^n(\tau - \nu)) d\nu d\tau \\
\leq C_{11} \overline{Q}^{n^2} \int_0^t \int_0^\tau n^2 \exp(-C_{12} n^2(\tau - \nu)) d\nu d\tau \quad (4.49)
\end{aligned}$$

where C_{11} and C_{12} are two positive constants.

We noticed that:

$$\begin{aligned}
\int_0^\tau n^2 \exp(-C_{12} n^2(\tau - \nu)) d\nu \\
= \left[\frac{1}{C_{12}} \exp(-C_{12} n^2(\tau - \nu)) \right]_0^\tau = \frac{1}{C_{12}} (1 - \exp(-C_{12} n^2 \tau)) \quad (4.50)
\end{aligned}$$

Therefore, from Inequalities (4.49) and (4.50) we obtained:

$$\begin{aligned}
n^2 \int_0^t \int_0^\tau r^2 \hat{Q}^n(\nu)^2 \exp(-2K_u^n(\tau - \nu)) d\nu d\tau \\
\leq C_{13} \overline{Q}^{n^2} \int_0^t (1 - \exp(-C_{12} n^2 \tau)) d\tau \leq C_{14} \overline{Q}^{n^2} \quad (4.51)
\end{aligned}$$

where C_{13} and C_{14} are two positive constants.

From Inequalities (4.45), (4.48) and (4.51) we obtained:

$$\sum_{n \in \mathbb{Z}} \left(n \hat{V}^n(t) \right)^2 \leq C_{14} \left(\sum_{n \in \mathbb{Z}} n^2 \hat{V}^n(0)^2 + \sum_{n \in \mathbb{Z}} \overline{Q}^{n^2} + \sum_{n \in \mathbb{Z}} \hat{U}^n(0)^2 \right). \quad (4.52)$$

According to assumptions (1.32) and (1.33) there exists $K_4 > 0$ such that:

$$\sum_{n \in \mathbb{Z}} n^2 \hat{V}^n(0)^2 + \sum_{n \in \mathbb{Z}} \hat{U}^n(0)^2 \leq K_4 \quad (4.53)$$

Therefore, from Inequalities (4.52), (4.53) and (4.40) we demonstrated that for all $t \in (0, T)$, $T \in (0, +\infty)$ Function V verifies (4.31).

Function V verifies (4.31). Hence, for all $t \in (0, T)$, $T \in (0, +\infty)$ there exists $K_5 > 0$ such that:

$$\sum_{n \in \mathbb{Z}} \hat{V}^n(t)^2 \leq K_5 \quad (4.54)$$

Moreover, $0 \leq \frac{1}{|1 + n^2|} \leq 1$, for all $n \in \mathbb{Z}$. Hence, Function V verifies (4.32).

We finally proved that V belongs to $\mathbf{H}_\#$.

In this section we proved that $\Phi_f + \Phi_b$ and $\Phi_f - \Phi_b$ belong to $\mathbf{H}_\#$. Hence, Φ_f and Φ_b belong to $\mathbf{H}_\#$.

Finally we proved Property 4.1. ■

4.2 Non-negativity of some components of the solution of the PDE system

In this section we will demonstrate the non-negativity of some components of the PDE system solution.

4.2.1 Non-negativity of $(\Phi_f + \Phi_b)$

In this section we will demonstrate that,

Property 4.2. *Under Assumption (1.29) to (1.34) there exists a solution (Φ_f, Φ_b) to (1.27) and (1.28). It satisfies $\Phi_f \in \mathbf{H}_\#$ and $\Phi_b \in \mathbf{H}_\#$ and for all $x \in (0, 1)$ and for all $t \in (0, +\infty)$,*

$$(\Phi_f + \Phi_b)(t, x) \geq 0 \quad (4.55)$$

Proof of Property 4.2:

In a first place, the existence of the solution (Φ_f, Φ_b) to (1.27) and (1.28) claimed in Property 4.2 is a direct consequence of Property 4.1.

Concerning the proof of Inequality (4.55), we started by fixing,

$$m_{\Phi}(t, x) = \min \left(0, (\Phi_f + \Phi_b)(t, x) \right), \quad (4.56)$$

and then we multiplied Equation (4.7) by m_{Φ} . We get:

$$\begin{aligned} \frac{\partial[\Phi_f + \Phi_b]}{\partial t}(t, x)m_{\Phi}(t, x) - 2c \frac{\partial^2[\Phi_f + \Phi_b]}{\partial x^2}(t, x)m_{\Phi}(t, x) \\ = Q(t, x)m_{\Phi}(t, x) - f\left(\Phi_f(t, x) + \Phi_b(t, x)\right)m_{\Phi}(t, x). \end{aligned} \quad (4.57)$$

Integrating by part Equation (4.57) with respect to x , we obtained,

$$\begin{aligned} \int_0^1 \frac{\partial[\Phi_f + \Phi_b]}{\partial t}(t, x)m_{\Phi}(t, x)dx + \int_0^1 2c \frac{\partial\Phi_f + \Phi_b}{\partial x}(t, x) \frac{\partial m_{\Phi}}{\partial x}(t, x)dx \\ = \int_0^1 \left(Q(t, x)m_{\Phi}(t, x) - f\left(\Phi_f(t, x) + \Phi_b(t, x)\right)m_{\Phi}(t, x) \right) dx. \end{aligned} \quad (4.58)$$

We have,

$$\frac{\partial[\Phi_f + \Phi_b]}{\partial t}(t, x)m_{\Phi}(t, x) = \frac{\partial m_{\Phi}}{\partial t}(t, x)m_{\Phi}(t, x) = \frac{1}{2} \frac{\partial m_{\Phi}^2}{\partial t}(t, x), \quad (4.59)$$

and,

$$\frac{\partial[\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial m_{\Phi}}{\partial x}(t, x) = \frac{\partial m_{\Phi}}{\partial x}(t, x) \frac{\partial m_{\Phi}}{\partial x}(t, x) = \left(\frac{\partial m_{\Phi}}{\partial x}(t, x) \right)^2, \quad (4.60)$$

Therefore, we obtained from Equations (4.58), (4.59) and (4.60),

$$\begin{aligned} \int_0^1 \frac{1}{2} \frac{\partial(m_{\Phi}^2)}{\partial t}(t, x)dx + \int_0^1 2c \left(\frac{\partial m_{\Phi}}{\partial x}(t, x) \right)^2 dx \\ = \int_0^1 \left(Q(t, x)m_{\Phi}(t, x) - f(m_{\Phi}(t, x))^2 \right) dx. \end{aligned} \quad (4.61)$$

according to Assumption (1.29) Q is non-negative and by definition Function m_{Φ} is negative for all $t \in (0, +\infty)$ and for all $x \in (0, 1)$. Hence, from Equation (4.61) we obtained the following inequality:

$$\frac{1}{2} \frac{\partial \|m_{\Phi}\|_2^2}{\partial t}(t, \cdot) \leq \int_0^1 Q(t, x)m_{\Phi}(t, x)dx \leq 0 \quad (4.62)$$

Therefore, Function $\|m_\Phi\|_2^2$ is decreasing and according to assumption (1.30) $m_\Phi^2(0, \cdot) = 0$. Hence,

$$\|m_\Phi(t, \cdot)\|_2^2 \leq 0 \quad (4.63)$$

Therefore,

$$\|m_\Phi(t, \cdot)\|_2 = 0 \quad (4.64)$$

Hence, Property 4.1 and Equality (4.64) prove Property 4.2. ■

4.2.2 Non-negativity of Ψ

Property 4.3. *Under Assumptions assumptions (1.29), (1.30) and (1.35) and if Φ_f and Φ_b belonging to $H_\#$ are solutions to (1.27) and (1.28), then Equation (1.16) admits a unique solution belonging to $C^1((0, +\infty), L^1((0, 1)))$. Moreover, this solution is non-negative.*

Proof of Property 4.3:

Functions $\Phi_f \in \mathbf{H}_\#$ and $\Phi_b \in \mathbf{H}_\#$ are solutions to Equations (1.27) and (1.28) and $\Psi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.16) which is a linear Ordinary Differential Equation (ODE). Hence, we have,

$$\Psi(t, x) = \Psi(0, x) \exp(-ut) \quad (4.65)$$

$$+ \int_0^t \exp(-u(t-s)) f(\Phi_b(s, x) + \Phi_f(s, x)) ds. \quad (4.66)$$

According to Property 4.2 ($\Phi_b(t, x) + \Phi_f(t, x)$) is non-negative for all x in $(0, 1)$ and for all t in $(0, T)$, T belonging to $(0, +\infty)$. Therefore, $\Psi(t, x)$ is non-negative, for all x in $(0, 1)$ and for all t in $(0, T)$, T belonging to $(0, +\infty)$. ■

4.2.3 Non-negativity and majoration of Function O

The objective of this subsection is to prove that:

Property 4.4. *Under assumptions (1.29), (1.30), (1.35) and (1.37) if Function $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.17) and $O \in C^1((0, +\infty))$ is solution to Equation (1.18), then for all t in $(0, T)$, T belonging to $(0, +\infty)$,*

$$0 \leq O(t) \leq L, \quad (4.67)$$

and

Property 4.5. *Under assumptions (1.29), (1.30), (1.35) and (1.37) if Function $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is non-negative and solution to Equation (1.15.b) and if $O \in C^1((0, +\infty))$ is solution to Equation (1.18), then for all t in $(0, T)$, T belonging to $(0, +\infty)$, Function O is non-negative.*

Proof of Property 4.4: To prove the non-negativity of Function O in case when Function $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.17) and $O \in C^1((0, +\infty))$ is solution to Equation (1.18) we posed,

$$m_O(t) = \min \left(0, O(t) \right). \quad (4.68)$$

From Equations (1.17) and (1.18) we obtained,

$$\frac{\partial O}{\partial t}(t) = \int_0^1 \frac{\partial \Xi}{\partial t}(t, x) dx = \int_0^1 u \Psi(t, x) \left(\frac{L - O(t)}{L} \right) dx \quad (4.69)$$

As a first step, we multiplied Equation (4.69) by m_O :

$$\begin{aligned} \frac{\partial O}{\partial t}(t) m_O(t) &= \int_0^1 u \Psi(t, x) \left(\frac{L - O(t)}{L} \right) m_O(t) dx \\ &= u \left(\frac{L - O(t)}{L} \right) m_O(t) \int_0^1 \Psi(t, x) dx \\ &= u m_O(t) \int_0^1 \Psi(t, x) dx - u \left(\frac{O(t)}{L} \right) m_O(t) \int_0^1 \Psi(t, x) dx \end{aligned} \quad (4.70)$$

By definition Function m_O is negative and according to Property 4.3 $\Psi(t, x)$ is non-negative, for all x in $(0, 1)$ and for all t in $(0, T)$, T belonging to $(0, +\infty)$. Therefore,

$$\frac{\partial O}{\partial t}(t) m_O(t) \leq -u \left(\frac{O(t)}{L} \right) m_O(t) \int_0^1 \Psi(t, x) dx \quad (4.71)$$

We have,

$$\frac{\partial O}{\partial t}(t) m_O(t) = \frac{\partial m_O}{\partial t}(t) m_O(t) = \frac{1}{2} \frac{\partial (m_O)^2}{\partial t}(t) \quad (4.72)$$

and,

$$O(t) m_O(t) = m_O^2(t) \quad (4.73)$$

Hence, by using (4.72) and (4.73), we obtained from Equation (4.71):

$$\frac{1}{2} \frac{\partial (m_O)^2}{\partial t}(t) \leq -\frac{u}{L} m_O^2(t) \int_0^1 \Psi(t, x) dx \leq 0 \quad (4.74)$$

Therefore, Function m_O^2 is decreasing and according to Assumption (1.37) $m_O^2(0) = 0$. Hence, for all t in $(0, T)$, T belonging to $(0, +\infty)$,

$$m_O^2(t) \leq 0 \quad (4.75)$$

Therefore, Function m_O^2 is null for all t in $(0, T)$, T belonging to $(0, +\infty)$. It means that Function O is non-negative.

To prove the majoration of Function O by the positive constant L we posed,

$$M_O(t) = \max\left(0, (O(t) - L)\right), \quad (4.76)$$

We multiplied Equation (4.69) by M_O :

$$\frac{\partial O}{\partial t}(t)M_O(t) = \int_0^1 u\Psi(t, x) \left(\frac{L - O(t)}{L}\right) M_O(t)dx \quad (4.77)$$

We have,

$$(L - O(t))M_O(t) = -M_O^2(t), \quad (4.78)$$

and,

$$\frac{\partial O}{\partial t}(t)M_O(t) = \frac{\partial M_O}{\partial t}(t)M_O(t) = \frac{1}{2} \frac{\partial M_O^2}{\partial t}(t) \quad (4.79)$$

Hence, according to Property 4.3 and from Equations (4.77), (4.78) and (4.79) we obtained:

$$\frac{1}{2} \frac{\partial M_O^2}{\partial t}(t) = -\frac{u}{L} M_O^2(t) \int_0^1 \Psi(t, x)dx \leq 0. \quad (4.80)$$

Therefore, Function M_O^2 is decreasing and according to Assumption (1.37) $M_O^2(0) = 0$. Hence,

$$\|M_O\|_2^2 \leq 0 \quad (4.81)$$

Therefore, Function M_O^2 is null for all t in $(0, T)$, T belonging to $(0, +\infty)$. It means that Function O is not upper than L .

We finally proved Property 4.4. ■

Proof of Property 4.5:

In case when $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.15.b) and $O \in C^1((0, +\infty))$ is solution to Equation (1.18), Function O only depends on Ξ . Hence, if Function Ξ is non-negative, O is also non-negative. ■

4.2.4 Non-negativity of Ξ

In the second toy model, Function Ξ can be solution to Equations (1.17) or (1.15.b). Therefore we proved that:

Property 4.6. *Under Assumptions (1.36) and (1.37) Function $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.17) or (1.15.b) and if $O \in C^1((0, +\infty))$ is solution to Equation (1.18), then for all t in $(0, T)$, T belonging to $(0, +\infty)$, Function Ξ is non-negative.*

Proof of Property 4.6:

In case when Function $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.17) and $O \in C^1((0, +\infty))$ is solution to Equation (1.18), Equation (1.17) is a nonlinear ODE. Hence, for all t in $(0, T)$ and for all x in $(0, 1)$ we have,

$$\Xi(t, x) = \Xi(0, x) + \int_0^t u\Psi(\tau, x) \left(\frac{L - O(\tau)}{L} \right) d\tau. \quad (4.82)$$

According to Properties 4.3 and 4.4, Functions $\Psi(t, x)$ and $O(t)$ are non-negative and $O(t)$ is not upper than L , for all x in $(0, 1)$ and for t in $(0, T)$, T belonging to $(0, +\infty)$. Therefore, in this case $\Xi(t, x)$ is non-negative, for all x in $(0, 1)$ and for all t in $(0, T)$, T belonging to $(0, +\infty)$.

In case when Function $\Xi \in C^1((0, +\infty), L^1((0, 1)))$ is solution to Equation (1.15.b) and Function $O \in C^1((0, +\infty))$ is solution to Equation (1.18), Equation (1.17) is a linear ODE. Hence, for all t in $(0, T)$, T belonging to $(0, +\infty)$ and for all x in $(0, 1)$ we have,

$$\Xi(t, x) = \Xi(0, x) + \int_0^t u\Psi(\tau, x)d\tau. \quad (4.83)$$

According to Property 4.3 $\Psi(t, x)$ is non-negative, for all x in $(0, 1)$ and for all t in $(0, T)$, T belonging to $(0, +\infty)$. Therefore, in this case $\Xi(t, x)$ is also non-negative, for all x in $(0, 1)$ and for all t in $(0, T)$, T belonging to $(0, +\infty)$.

We finally proved Property 4.6. ■

4.3 conservation of the second toy model

In this section we will prove that,

Property 4.7. *Under Assumptions (1.29) to (1.38) and according to Properties 4.2, 4.3, 4.4, 4.5 and 4.6, for $\Phi_f \in \mathbf{H}_\#$ and $\Phi_b \in \mathbf{H}_\#$ solutions to Equations (1.27) and (1.28), $\Psi \in C^1((0, +\infty); L^1((0, 1)))$ solution to Equation (1.16), $\Xi \in C^1((0, +\infty); L^1((0, 1)))$ solution to Equation (1.17) or (1.15.b), and $O \in C^1((0, +\infty))$ solution to (1.18), then, for any $T > 0$, there exists $K_{Q_2}(T) > 0$ such that for all x in $(0, 1)$ and for all t in $(0, T)$, Inequality (1.39) is satisfied.*

Proof of Property 4.7: In case when Functions Φ_f , Φ_b , Ψ , Ξ and O are respectively solutions to (1.27), (1.28), (1.16), (1.15.b) and (1.18). According to Assumption (1.29), Function Q is also non-negative. Hence, summing Equations (1.27), (1.28), (1.16), (1.15.b) and (1.18), and according to Properties 4.2, 4.3, 4.4, 4.5 and 4.6, we obtained,

$$\begin{aligned} \frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} \\ = 2c \int_0^1 \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) dx + \int_0^1 Q(t, x) dx \end{aligned} \quad (4.84)$$

Since Functions Φ_f and Φ_b belong to $\mathbf{H}_\#$, the integration of the first term of the right-hand side makes it vanish. Hence,

$$\begin{aligned} \frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} \\ = \int_0^1 Q(t, x) dx \leq \sup_{t \in \mathbb{R}^+} \left(\int_0^1 Q(t, x) dx \right) \end{aligned} \quad (4.85)$$

In the case when Functions Φ_f , Φ_b , Ψ , Ξ and O are respectively solutions to (1.27), (1.28), (1.16), (1.17) and (1.18). Hence, summing Equations (1.27), (1.28), (1.16), (1.17) and (1.18), and according to Properties 4.2, 4.3, 4.4, 4.5 and 4.6, we obtained,

$$\begin{aligned} \frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} \\ = 2c \int_0^1 \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) dx \\ + \int_0^1 Q(t, x) dx - \int_0^1 u \frac{O(t)}{L} \Psi(t, x) dx \end{aligned} \quad (4.86)$$

As previously, since Functions Φ_f and Φ_b belong to $\mathbf{H}_\#$, the integration of the first term of the right-hand side makes it vanish. Moreover, according to Assumption (1.29) and Property 4.3 Functions O and Ψ are positive. Hence,

$$\begin{aligned} \frac{\partial \left[\int_0^1 \Phi_f(t, x) dx + \int_0^1 \Phi_b(t, x) dx + \int_0^1 \Psi(t, x) dx + \int_0^1 \Xi(t, x) dx \right]}{\partial t} \\ \leq \sup_{t \in \mathbb{R}^+} \left(\int_0^1 Q(t, x) dx \right) \end{aligned} \quad (4.87)$$

Therefore, with or without a limiter coefficient, the model made of Equations (1.27), (1.28), (1.16), (1.18) and (1.17) or (1.15.b), has a conservation property.

We finally proved Property 4.7. ■

4.4 Majoration of the infinity norm of Φ_f , Φ_b and Ψ

The objective of this section is to prove that:

Property 4.8. *Under Assumptions (1.29) to (1.38), if $\Psi(0, \cdot) \in L^1((0, 1))$, then there exists a unique solution to Equation (1.16). Moreover, if there exists a time T_F such that, for all $t > T_F$, and for all $x \in (0, 1)$, $Q(t, x) = 0$, then there exists a positive constant $K_\Psi > 0$ such that,*

$$\sup_{\substack{t \in \mathbb{R}^+ \\ x \in (0, 1)}} \Psi = \|\Psi\|_\infty \leq K_\Psi. \quad (4.88)$$

Before that, we have to demonstrate that,

Property 4.9. *Under Assumptions (1.29) to (1.38), if there exists a time T_F such that, for all $t > T_F$, and for all $x \in (0, 1)$, $Q(t, x) = 0$, then there exists a constant $K_\Phi > 0$ such that $\Phi_f \in \mathbf{H}$ and $\Phi_b \in \mathbf{H}$, solutions to (1.27) and (1.28) and given by Property 4.2, satisfy,*

$$\sup_{\substack{t \in \mathbb{R}^+ \\ x \in (0, 1)}} (\Phi_f + \Phi_b) = \|\Phi_f + \Phi_b\|_\infty \leq K_\Phi. \quad (4.89)$$

Proof of Property 4.9: We posed,

$$M_\Phi(t, x) = \max\left(0, f(\Phi_f(t, x) + \Phi_b(t, x)) - K\right), \quad (4.90)$$

where,

$$K = \sup_{x \in (0, 1)} (Q(t, x)). \quad (4.91)$$

As a first step, we summed Equations (1.27) and (1.28), we multiplied the obtained equality by M_Φ and we integrated it over space:

$$\begin{aligned} & \int_0^1 \frac{\partial \Phi_f + \Phi_b}{\partial t}(t, x) M_\Phi(t, x) dx - \int_0^1 2c \frac{\partial^2 [\Phi_f + \Phi_b]}{\partial x^2}(t, x) M_\Phi(t, x) dx \\ &= \int_0^1 Q(t, x) M_\Phi(t, x) dx - \int_0^1 f(\Phi_f(t, x) + \Phi_b(t, x)) M_\Phi(t, x) dx. \end{aligned} \quad (4.92)$$

By integrating by part we obtained,

$$\begin{aligned} & \int_0^1 \frac{\partial \Phi_f + \Phi_b}{\partial t}(t, x) M_\Phi(t, x) dx + \int_0^1 2c \frac{\partial [\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial M_\Phi}{\partial x}(t, x) dx \\ &= \int_0^1 Q(t, x) M_\Phi(t, x) dx - \int_0^1 f(\Phi_f(t, x) + \Phi_b(t, x)) M_\Phi(t, x) dx. \end{aligned} \quad (4.93)$$

We have,

$$\frac{\partial[\Phi_f + \Phi_b]}{\partial t}(t, x)M_\Phi(t, x) = \frac{1}{f} \frac{\partial M_\Phi}{\partial t}(t, x)M_\Phi(t, x) = \frac{1}{2f} \frac{\partial(M_\Phi)^2}{\partial t}(t, x), \quad (4.94)$$

and,

$$\frac{\partial[\Phi_f + \Phi_b]}{\partial x}(t, x) \frac{\partial M_\Phi}{\partial x}(t, x) = \frac{1}{f} \frac{\partial M_\Phi}{\partial x}(t, x) \frac{\partial M_\Phi}{\partial x}(t, x) = \frac{1}{f} \left(\frac{\partial M_\Phi}{\partial x}(t, x) \right)^2. \quad (4.95)$$

Hence, by using (4.94) and (4.95), we obtained from Equation (4.93):

$$\begin{aligned} & \frac{1}{2f} \int_0^1 \frac{\partial(M_\Phi)^2}{\partial t}(t, x) dx + \int_0^1 \frac{2c}{f} \left(\frac{\partial(M_\Phi)}{\partial x} \right)^2(t, x) dx \\ & = \int_0^1 Q(t, x) M_\Phi(t, x) dx - \int_0^1 f (\Phi_f(t, x) + \Phi_b(t, x)) M_\Phi(t, x) dx. \end{aligned} \quad (4.96)$$

By definition, Function M_Φ is non-negative. Hence, from Assumption (1.29) and Equality (4.91) we have,

$$\begin{aligned} & \frac{1}{2f} \int_0^1 \frac{\partial(M_\Phi)^2}{\partial t}(t, x) dx \leq \int_0^1 K M_\Phi(t, x) dx \\ & \quad - \int_0^1 f (\Phi_f(t, x) + \Phi_b(t, x)) M_\Phi(t, x) dx. \end{aligned} \quad (4.97)$$

Hence, we have,

$$\frac{1}{2f} \frac{\partial \|M_\Phi\|_2^2}{\partial t}(t) dx \leq - \int_0^1 (M_\Phi(t, x))^2 dx \leq 0. \quad (4.98)$$

Therefore, Function $\|M_\Phi\|_2^2$ is decreasing and according to Assumption (1.31) $\|M_\Phi\|_2^2(0) = 0$. Hence,

$$\|M_\Phi\|_2^2(t, \cdot) \leq 0 \quad (4.99)$$

Therefore, for all t in $(0, +\infty)$ and for all $x \in (0, 1)$, function $M_\Phi(t, x)$ is null. It means that, for all t in $(0, +\infty)$ and for all $x \in (0, 1)$, $(\Phi_f + \Phi_b)$ is bounded by a positive constant K_Φ . Therefore, Property 4.9 is well verified. \blacksquare

Proof of Property 4.8: The exact expression of Ψ is given by Equation (4.65). From this equation and according to Assumption (1.35) and Property 4.9 there exists a positive constant K_Ψ such as Property 4.8 is verified. \blacksquare

4.5 The existence of the solution of Equations (1.17) and (1.15.b)

In this section we will prove that:

Property 4.10. *Under Assumptions (1.29) to (1.38), Equations (1.15.b) and (1.17), associated with Equation (1.18), admit a unique solution belonging to $C^1((0, T), L^1((0, 1)))$, $T \in (0, +\infty)$.*

Proof of Property 4.10:

Equation (1.15.b) is a linear ODE. Hence, this equation admits a unique solution.

The objective of this section is to prove that Equation (1.17) admits a unique solution.

To do so we posed:

$$G_{\xi_1}(t, x) = G_{\xi_1}(0, x) + \int_0^t g(\tau, x) \left(\frac{L - s_{\xi_1}(\tau)}{L} \right) d\tau, \quad s_{\xi_1}(t) = s_{\xi_1}(0) + \int_0^1 \xi_1(t, x) dx, \quad (4.100)$$

and,

$$G_{\xi_2}(t, x) = G_{\xi_2}(0, x) + \int_0^t g(\tau, x) \left(\frac{L - s_{\xi_2}(\tau)}{L} \right) d\tau, \quad s_{\xi_2}(t) = s_{\xi_2}(0) + \int_0^1 \xi_2(t, x) dx, \quad (4.101)$$

where,

$$g(t, x) = u\Psi(t, x), \quad (4.102)$$

$$G_{\xi_1}(0, x) = G_{\xi_2}(0, x), \quad (4.103)$$

and,

$$s_{\xi_1}(0) = s_{\xi_2}(0). \quad (4.104)$$

Functions ξ_1 and ξ_2 are positive functions belonging to $C^1((0, T), L^1((0, 1)))$ and Functions s_{ξ_1} and s_{ξ_2} are positive functions belonging to $C^1((0, T))$, $T \in (0, +\infty)$.

We deduced Equation (4.101) from Equation (4.100):

$$\begin{aligned} |G_{\xi_1} - G_{\xi_2}|(t, x) &= \left| \int_0^t g(\tau, x) \left(\frac{s_{\xi_2}(\tau) - s_{\xi_1}(\tau)}{L} \right) d\tau \right| \\ &= \left| \int_0^t \int_0^1 g(\tau, x) \left(\frac{\xi_1(\tau, \alpha) - \xi_2(\tau, \alpha)}{L} \right) d\tau d\alpha \right| \end{aligned} \quad (4.105)$$

and $\forall t \leq T_1$, $T_1 \in (0, T)$, we obtained:

$$\begin{aligned} |G_{\xi_1} - G_{\xi_2}|(t, x) &\leq \frac{T_1}{L} \int_0^1 \bar{g}(x) \left\| \xi_1(\alpha) - \xi_2(\alpha) \right\|_{L^\infty((0, T_1))} d\alpha, \\ &\leq \frac{T_1 \bar{g}}{L} \left\| \xi_1 - \xi_2 \right\|_{L^\infty((0, T_1), L^1((0, 1)))}, \end{aligned} \quad (4.106)$$

where,

$$\bar{g}(x) = u \left\| \Psi(x) \right\|_{L^\infty((0, T_1))} \quad \text{and} \quad \bar{g} = u \left\| \Psi \right\|_{L^\infty((0, T_1), L^1((0, 1)))} \quad (4.107)$$

$$\left\| \Psi(x) \right\|_{L^\infty((0, T_1))} = \sup_{t \in (0, T_1)} |\Psi(t, x)| \quad (4.108)$$

and,

$$u \left\| \Psi \right\|_{L^\infty((0, T_1) \times (0, 1))} = \sup_{\substack{t \in (0, T_1) \\ x \in (0, 1)}} \left\| \Psi(t, x) \right\| \quad (4.109)$$

Since, according to Property 4.8 \bar{g} is bounded, we chose $T_1 < \frac{L}{\bar{g}}$. Hence, we obtained for all $t \leq T_1$:

$$|G_{\xi_1} - G_{\xi_2}|(t, x) \leq k \left\| \xi_1 - \xi_2 \right\|_{L^\infty((0, T_1), L^1((0, 1)))}, \quad \text{where } k < 1 \quad (4.110)$$

Therefore, we have:

$$\left\| G_{\xi_1} - G_{\xi_2} \right\|_{L^\infty((0, T_1), L^1((0, 1)))} \leq k \left\| \xi_1 - \xi_2 \right\|_{L^\infty((0, T_1), L^1((0, 1)))}. \quad (4.111)$$

Equation (4.111) demonstrates that operator $G : \xi \mapsto G_\xi$, is k -contractive from $C^0((0, T_1), L^1((0, 1)))$ in $C^0((0, T_1), L^1((0, 1)))$. Yet, G is equivalent to Ξ which is solution to Equation (1.17). Hence, for all $t \in (0, T_1)$ and for all $x \in (0, 1)$, the solution of Equation (1.17) corresponds to the fixed-point of G . According to the Banach fixed-point theorem, this fixed-point exists and is unique. Therefore, Equation (1.17) admits a unique solution for all $t \in (0, T_1)$ and for all $x \in (0, 1)$.

The value of Function Ξ at $t = T_1$ is given. Then, for all t such as $T_1 \leq t \leq T_2$, $T_2 \in (0, T)$, we have:

$$\begin{aligned} |G_{\xi_1} - G_{\xi_2}|(t, x) &\leq \frac{T_2 - T_1}{L} \int_0^1 \bar{g}(x) \left\| \xi_1(\alpha) - \xi_2(\alpha) \right\|_{L^\infty((T_1, T_2))} d\alpha, \\ &\leq \frac{(T_2 - T_1) \bar{g}}{L} \left\| \xi_1 - \xi_2 \right\|_{L^\infty((T_1, T_2), L^1((0, 1)))}. \end{aligned} \quad (4.112)$$

We chose T_2 such that $T_2 - T_1 < \frac{L}{g}$. Hence, for all $t \in (T_1, T_2)$ and for all $x \in (0, 1)$, we obtained:

$$|G_{\xi_1} - G_{\xi_2}|(t, x) \leq k_2 \left\| \xi_1(\alpha) - \xi_2(\alpha) \right\|_{L^\infty((T_1, T_2), L^1((0, 1)))}, \quad \text{where } k_2 < 1 \quad (4.113)$$

Therefore, we have:

$$\left\| G_{\xi_1} - G_{\xi_2} \right\|_{L^\infty((T_1, T_2), L^1((0, 1)))} \leq k_2 \left\| \xi_1 - \xi_2 \right\|_{L^\infty((T_1, T_2), L^1((0, 1)))}. \quad (4.114)$$

Equation (4.114) demonstrates that Operator G is k -contractive from $C^0((T_1, T_2), L^1((0, 1)))$ in $C^0((T_1, T_2), L^1((0, 1)))$. As G is equivalent to Ξ which is solution to Equation (1.17), for all $t \in (T_1, T_2)$ and for all $x \in (0, 1)$, the solution of Equation (1.17) corresponds to the fixed-point of G . According to the Banach fixed-point theorem, this fixed-point exists and is unique. Therefore, Equation (1.17) admits a unique solution for all $t \in (T_1, T_2)$ and for all $x \in (0, 1)$.

By iteration and by proceeding in the same way we can range time-interval $(0, T)$, $T \in (0, +\infty)$. Indeed, by iteration we can demonstrate that G is k -contractive over a series of intervals (T_n, T_{n+1}) covering the interval $(0, T)$ and verifying $T_n - T_{n+1} \leq \frac{L}{g}$. According to the Banach fixed-point theorem, since G is equivalent to Ξ solution to Equation (1.17), this equation admits a unique solution on all those intervals of time and for all $x \in (0, 1)$.

We finally proved Property 4.10. ■

Since Equation (1.18) is an ODE the wellposedness of this equation is obvious.

Finally, we proved in this section the existence of a solution of the second toy model made of Equations (1.27), (1.28), (1.16), (1.18) and Equation (1.17) or (1.15.b).

4.6 Uniqueness of the solution of the PDE system

In this section we will prove the uniqueness of the solution of the PDE system.

We assumed that Φ_{f_1} and Φ_{f_2} solutions of (1.27), and Φ_{b_1} and Φ_{b_2} solutions of (1.28). The objective of this demonstration is to prove that,

$$\Phi_{f_1} = \Phi_{f_2} \quad (4.115)$$

and,

$$\Phi_{b_1} = \Phi_{b_2} \quad (4.116)$$

By subtraction we obtained,

$$\begin{aligned} \frac{\partial \Phi_{f_1} - \Phi_{f_2}}{\partial t}(t, x) + \omega \frac{\partial \Phi_{f_1} - \Phi_{f_2}}{\partial x}(t, x) - c \frac{\partial \left[\chi \frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right]}{\partial x}(t, x) \\ = -fF(x) (\Phi_{f_1} - \Phi_{f_2})(t, x) - r (\Phi_{f_1} - \Phi_{f_2})(t, x). \end{aligned} \quad (4.117)$$

and,

$$\begin{aligned} \frac{\partial \Phi_{b_1} - \Phi_{b_2}}{\partial t}(t, x) - \omega \frac{\partial \Phi_{b_1} - \Phi_{b_2}}{\partial x}(t, x) - c \frac{\partial \left[\chi \frac{\partial [\Phi_{f_1} - \Phi_{f_2} + \Phi_{b_1} - \Phi_{b_2}]}{\partial x} \right]}{\partial x}(t, x) \\ = -fF(x) (\Phi_{b_1} - \Phi_{b_2})(t, x) + r (\Phi_{f_1} - \Phi_{f_2})(t, x). \end{aligned} \quad (4.118)$$

We applied the same method used in Section 3 until the obtainment of an equation close to (3.10). Hence, from Equations (4.117) and (4.118) we obtained the following inequality:

$$\begin{aligned} \int_0^T \left(\frac{\partial \|\Phi_{f_1} - \Phi_{f_2}\|_2^2}{\partial t}(t) + \frac{\partial \|\Phi_{b_1} - \Phi_{b_2}\|_2^2}{\partial t}(t) \right) dt \\ \leq 2 \int_0^T \left(r \left(\|\Phi_{f_1} - \Phi_{f_2}\|_2^2(t) + \|\Phi_{b_1} - \Phi_{b_2}\|_2^2(t) \right) \right). \end{aligned} \quad (4.119)$$

According to the Gronwall lemma, for any $T \in (0, +\infty)$ and for any $t \in (0, T)$, we obtained:

$$\begin{aligned} \|\Phi_{f_1} - \Phi_{f_2}\|_2^2(t) + \|\Phi_{b_1} - \Phi_{b_2}\|_2^2(t) \\ \leq \|\Phi_{f_1} - \Phi_{f_2}\|_2^2(0) + \|\Phi_{b_1} - \Phi_{b_2}\|_2^2(0) = 0 \end{aligned} \quad (4.120)$$

We deduced from Equation (4.120) that:

$$\|\Phi_{f_1} - \Phi_{f_2}\|_2^2(t) + \|\Phi_{b_1} - \Phi_{b_2}\|_2^2(t) = 0 \quad (4.121)$$

We deduced that,

$$\|\Phi_{f_1} - \Phi_{f_2}\|_2^2(t, \cdot) = 0 \quad (4.122)$$

and,

$$\|\Phi_{b_1} - \Phi_{b_2}\|_2^2(t, \cdot) = 0 \quad (4.123)$$

Therefore,

$$\Phi_{f_1}(t, x) = \Phi_{f_2}(t, x) \quad (4.124)$$

and,

$$\Phi_{b_1}(t, x) = \Phi_{b_2}(t, x) \quad (4.125)$$

Therefore we proved the uniqueness of the solution of Equations (1.27) and (1.28).

Equation (1.16) is a linear ODE. Hence, the uniqueness of its solution of this equation is obvious. We demonstrated in Section 4.5 that in case when Function $\Xi \in C^1((0, +\infty); L^1((0, 1)))$ is solution to Equation (1.17) this function is k-contractive from $C^0((0, T) \times (0, 1))$ in $C^0((0, T) \times (0, 1))$. Therefore, according to the Banach fixed-point theorem, Equation (1.17) admits a unique solution.

Equation (1.15.b) is an ODE. Hence this equation admits a unique solution.

Function $O \in C^1((0, +\infty))$ solution to Equation (1.18) only depends on Ξ . Hence, Equation (1.18) also admits a unique solution. Finally, we proved the uniqueness of the solution of the PDE system.

In the section we proved the existence and the uniqueness of the solution of the PDE system. Moreover, we proved the conservation property of this PDE system and the non-negativity of some components of the solution. Hence, we finally prove Theorem 1.4. ■

5 Conclusion

The two first equations of the mathematical model developed in the article Flourent *et al.* (Submitted), contain a diffusion operator depending on the sum of two unknown densities. This fact, coupled with the presence of a convection term, makes impossible the direct application of the classical results about parabolic equations. Hence, we worked on two toy models. The loss of the regularizing effect and the positivity of each component of the solution due to the structure of the diffusion term was put in evidence through the first toy model.

The analysis of the second toy model which does not contain convection term and by using Variable Substitutions and Fourier Transforms put in evidence the properties preserved despite the structure of the diffusion term. We proved the existence and the uniqueness of the solution of the PDE system. Moreover, we also proved the conservation property of this PDE system and the non-negativity of some components of the solution.

References

- David, C. and P. Gosselet (2015). *Equations aux dérivées partielles-2e éd.: Cours et exercices corrigés*. Dunod.
- Flourent, H., E. Frénod, and V. Sincholle (Submitted). “An innovating Statistical Learning Tool based on Partial Differential Equations, intending livestock Data Assimilation”.
- Ladyženskaja, O. A., V. A. Solonnikov, and N. N. Uralceva (1968). *Linear and quasi-linear equations of parabolic type*. Translations of mathematical monographs. Providence, RI: American Mathematical Society. URL: <https://cds.cern.ch/record/2623252>.
- Menza, L. Di (2009). *Analyse numérique des équations aux dérivées partielles*. Cassini.