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STOCHASTIC CONTROL ON NETWORKS: WEAK DPP AND VERIFICATION THEOREM

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ABSTRACT. The purpose of this article is to study a stochastic control problem on a junction, with control at the junction point. The problem of control is formulated in the weak sense, using a relaxed control, namely a control which takes values in the space of probability measures on a compact set. We prove first the compactness of the admissible rules and the dynamic programming principle (DPP). We complete this article by giving a verification Theorem for the value function of the problem, using some recent results on quasi linear non degenerate PDE posed on a junction, with non linear Neumann boundary condition at the junction point. An example is given, where the optimal control at the junction point is solution of a convex quadratic optimization problem with linear constraints.

1. INTRODUCTION

Diffusions on graphs have attracted a lot of intention in the last 20 years. They were introduced in the seminal works of Freidlin and Sheu in [7] and Freidlin and Wetzell in [8]. More precisely, given a junction $\mathcal{J} = \bigcup_{i=1}^I J_i$, (σ_i, b_i) regular functions from \mathbb{R}_+ to \mathbb{R} , and $\alpha_1 \dots \alpha_I$ positive constants such that $\alpha_1 + \dots + \alpha_I = 1$, the authors in [8] have proved that there exists a continuous Markov process $X = (x, i)$ defined on \mathcal{J} . Thereafter in [7], it is shown that there exists a one dimensional Wiener process W defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to the natural filtration of $X = (x, i)$, such that the process x satisfies the following stochastic differential equation for a finite time horizon $T > 0$,

$$dx(t) = \sigma_{i(t)}(x(t))dW(t) + b_{i(t)}(x(t))dt + dl(t), \quad 0 \leq t \leq T, \quad (1)$$

where l starts from 0 and satisfies:

$$l \text{ is increasing and, } \mathbb{P}\left(\left(\int_0^t \mathbf{1}_{\{x(s)>0\}} dl(s)\right)_{0 \leq t \leq T} = 0\right) = 1. \quad (2)$$

Moreover, [7] gives the following Itô's formula:

$$df_{i(t)}(x(t)) = \left(b_{i(t)}(x(t))\partial_x f_{i(t)}(x(t)) + \frac{1}{2}\sigma_{i(t)}^2(x(t))\partial_x^2 f_{i(t)}(x(t)) \right) dt + \partial_x f_{i(t)}(x(t))\sigma_{i(t)}(x(t))dW(t) + \sum_{i=1}^I \alpha_i \partial_x f_i(0) dl(t), \quad (3)$$

for f regular enough. The process l can be interpreted as the local time of the process X at the vertex, whose quadratic approximation is given by:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}} \left[\left| \left(\frac{1}{2\varepsilon} \sum_{j=1}^I \int_0^\cdot \sigma_j^2(0) \mathbf{1}_{\{0 \leq x(s) \leq \varepsilon, j(s)=j\}} ds \right) - l(\cdot) \right|_{(0,T)}^2 \right] = 0. \quad (4)$$

Let us recall that initially introduced by J. Walsh in [26], the Walsh's Brownian motion is a diffusion process on a set of I rays in \mathbb{R}^2 emanating from 0. To each ray J_i is associated a weight α_i corresponding heuristically to the probability for the process to go in this ray, and on each ray, the process behaves like a Brownian motion. Obviously, due to the irregularities of the trajectories of the Brownian motion, this description is a non-sense. This process may be described by its excursions measure;

$$P = \sum_{i=1}^I \alpha_i Q_i,$$

where Q_i is the excursion measure of the reflected Brownian motion on the ray J_i . Diffusion on graphs generalizes the notion of Walsh's Brownian motion. This object has given rise to an abundant literature on Brownian filtrations, especially by giving a negative answer to the following problem:

- "if a Brownian motion is adapted to some filtration, is this filtration generated by a Brownian motion?" (See for instance [5] and [25], Sect. 17, p. 103)

Remark that, strong solutions have been established only for the case $I = 2$ on the line, where the process is called the skew Brownian motion, and has been studied by several researchers (see for instance [16], for a summary on the various ways for the construction of the skew Brownian motion).

There are several constructions of Walsh's Brownian motion in terms of resolvents, infinitesimal generators, semigroups, and excursion theory. Recently, in [12], the authors have

given a system of stochastic equations for Walsh's planar semimartingales, unique in distribution. Pathwise uniqueness fails, since that the Walsh's Brownian motion is a process whose filtration cannot be generated by any Brownian motion of any dimension. For this result see the celebrated paper: [24]. Thereafter, a stochastic optimal control/stopping problem of a Walsh's planar semimartingale has been studied in [14], but without control at the junction point, since it is assumed that the process is "immediately dispatched along some ray", when it reaches the origin.

In this work, we study a stochastic control problem with control at the junction point. Since the construction of a strong solution for diffusion of type (1) is still a fairly complicated open problem, we use here a weak martingale formulation, and the method of compactification of the controls, as it has been introduced in [15].

Let us mention that the control theory on stratified domains of networks have already been well-studied in the literature, for first order problems, and we refer for instance to [2], [3], [4], [9],[1], and [21].

In this problem, the method differs a little from what it has been already done in the literature: we add a more general relaxation at the junction point, due to the process l introduced in equation (1) and its paths properties. This new method of relaxation is introduced in Section 2.1 where we formulate the stochastic control martingale problem, with control at the junction point. We prove the compactness of the admissible rules in Section 3, and the dynamic programming principle is established in Section 4.

The second main target of this work, is to address a characterization of the value function of this problem of control, in term of non linear parabolic partial differential equations posed on a junction. Due to the process l and the quadratic approximation (4), we will get that the parabolic equation that characterized the value function, has non degenerate viscosity at the junction point $x = 0$, and satisfies a non linear Neumann and non dynamical boundary condition at $x = 0$, for example (without cost at the junction point):

$$F(u(t, 0), \partial_x u(t, 0)) = \inf_{\alpha_i \in [0, 1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i \partial_x u_i(t, 0) \right\} = 0. \quad (5)$$

Until now, the only result of existence and uniqueness of these type of equation has been given in [19], where the author has shown well-posedness of classical solutions for the

following problem:

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_x^2 u_i(t, x) + H_i(x, \partial_x u_i(t, x)) = 0, \\ \text{for all } x > 0, \text{ and for all } i \in \{1 \dots I\}, \\ F(u(t, 0), \partial_x u(t, 0)) = 0, \end{cases} \quad (6)$$

in suitable Hölder spaces: see Theorem 2.2 for the existence and Theorem 2.4 for the comparison in [19], and thus the uniqueness. The main assumptions are that the equation is uniformly parabolic with smooth coefficients, and the term $F = F(u, p)$ is increasing with respect to p , which is a natural assumption regarding to the set where the controls $(\alpha_1 \dots \alpha_I)$ are valued.

Therefore, in section 5, we will be able to state a verification theorem. A simple example of illustration is also given, where we consider Hamiltonians with quadratic growth on each edge. At the junction point, remark first that the solution of the convex optimization problem:

$$\inf_{\alpha_i \in [0,1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i \partial_x u_i(t, 0) \right\} = 0,$$

is equal to $\min_{i \in \{1 \dots I\}} \partial_x u_i(t, 0)$. It means that, heuristically, if no cost appears at the junction point, the optimal strategy is therefore to play at the junction the maximum weight α_i on the edge where the gradient $\partial_x u_i(t, 0)$ reaches its minimum, at each time we reach the junction point. We will give at the end of Section 5, an example with quadratic Hamiltonians on each edge, and at the junction point we will consider the following quadratic Hamiltonian:

$$\inf_{\alpha_i \in [0,1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i \partial_x u_i(t, 0) + \frac{1}{2} \sum_i \alpha_i^2 \sigma_i(0)^2 \right\} = 0$$

The optimal control at the junction point is then solution of a quadratic convex optimization problem, under linear constraints. Obviously, this is just a simple example, which can be improved in a more applied sense with another more general cost, since as explained in [17], this type of non linear PDE are involved in many applications in physics, chemistry, biology ...

Along this work, we work under the ellipticity assumption, which allows to state the uniqueness of non linear PDE involved for the value function. Moreover, the ellipticity

condition is used to prove that the "non-stickiness" assumption satisfied by the process:

$$E^P \left[\int_0^T \mathbf{1}_{\{x(s)=0\}} ds \right] = 0,$$

is closed under the weak convergence of probability measure (see Proposition 3.7). The dynamic programming principle (DPP) can be generated in the degenerate case, as it has been stated in [15], but here we do not focus on this technical point.

The paper is organized as follows. We introduce in Section 2 the stochastic control problem, with control at the junction point and we state our main results. Thereafter, in Section 3, we give a criterium of compactness for the controls at the junction point, and some path estimates to prove at the end of the section the compactness of the admissible rules. Section 4 is dedicated to the proof of the dynamic programming principle (DPP), where both stability properties of the set of rules by conditioning and concatenation at stopping times are shown. Finally, the last Section 5, is dedicated to the proof of the verification Theorem, with an example of illustration.

2. FORMULATION OF THE STOCHASTIC CONTROL PROBLEM AND MAIN RESULTS

In this section, we define our stochastic control problem and we state our main results: the dynamic programming principle (DPP), and the verification Theorem.

2.1. The stochastic control problem at the junction. In this sub section we define the stochastic control problem at the junction, using a weak martingale formulation. We use a classical relaxation on each edge.

Let \mathcal{J} be an unbounded junction defined for $I \in \mathbb{N}^*$ edges by:

$$\mathcal{J} = \bigcup_{i=1}^I J_i, \text{ with: } \forall i \in \{1 \dots I\} \quad J_i = [0, +\infty), \text{ and } \forall (i, j) \in \{1 \dots I\}^2, \quad i \neq j, \quad J_i \cap J_j = \{0\}.$$

The intersection of the $(J_i)_{1 \leq i \leq I}$ is called the junction point and is denoted by 0. We identify all the points of \mathcal{J} by the couples (x, i) (with $i \in \{1 \dots I\}, x \in [0, +\infty)$), such that we have: $(x, i) \in \mathcal{J}$ if and only if $x \in J_i$.

For $T > 0$, the time-space domain \mathcal{J}_T is defined by:

$$\mathcal{J}_T = [0, T] \times \mathcal{J}.$$

In the sequel, $\mathcal{C}_b^{1,2}(\mathcal{J}_T)$ is the class of function defined on \mathcal{J}_T with regularity $\mathcal{C}_b^{1,2}([0, T] \times [0, +\infty))$ on each edge, namely:

$$\mathcal{C}_b^{1,2}([0, T] \times [0, +\infty)) := \left\{ f : \mathcal{J}_T \rightarrow \mathbb{R}, \quad (t, (x, i)) \mapsto f_i(t, x), \quad \forall (i, j) \in \{1 \dots I\}^2, \quad \forall t \in (0, T), \right. \\ \left. f_i(t, 0) = f_j(t, 0), \quad \forall i \in \{1 \dots I\}, \quad (t, x) \mapsto f_i(t, x) \in \mathcal{C}_b^{1,2}([0, T] \times [0, +\infty)) \right\}.$$

We define in the sequel the controls (generalized actions) on the junction J .

We use the notations introduced in Appendix A, and for the convenience of the reader we recall that, for a giving compact K of \mathbb{R}^n ($n \in \mathbb{N}^*$):

$$L_{mc}^\infty([0, T] \times K) := \left\{ f \in L^\infty([0, T] \times K), \quad k \mapsto f(t, k) \in \mathcal{C}(K), \forall t \in [0, T] \right\}.$$

We denote by $\mathcal{M}_{mc}([0, T] \times K)$ the set consisting of non negative finite measures on $([0, T] \times K, \mathbb{B}([0, T]) \otimes \mathbb{B}(K))$, endowed with the finest topology making continuous the following family of linear forms $(\theta_f)_{f \in L_{mc}^\infty([0, T] \times K)}$,

$$\theta_f : \begin{cases} \mathcal{M}_{mc}([0, T] \times K) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{[0, T] \times K} f d\nu \end{cases}.$$

Let us introduce:

$$L[0, T] := \left\{ l : [0, T] \rightarrow \mathbb{R}, \text{ continuous nondecreasing such that } : l(0) = 0 \right\},$$

which is the space where the process $l(\cdot)$ introduced in (1) takes its value. Fixing $\underline{a} \in (0, 1)$, we define furthermore the following compact set A_0 of \mathbb{R}^I by

$$A_0 := \left\{ (\alpha_i) \in [\underline{a}, 1]^I, \quad \sum_{i=1}^I \alpha_i = 1 \right\},$$

which is the set where the controls α_i at the junction point appearing in the Ito's formula (3) are valued.

The set of generalized actions at the junction point 0, denoted $V([0, T] \times A_0)$ is defined

by:

$$V([0, T] \times A_0) := \left\{ \nu \in \mathcal{M}_{mc}([0, T] \times A_0), \exists l_\nu \in L[0, T], \nu^{[0, T]}(dt) = l_\nu(dt) \right\},$$

$$\text{where } \nu^{[0, T]}(dt) = \int_{A_0} \nu(dt, d\alpha_1, \dots, d\alpha_I).$$

As a consequence of the disintegration Theorem of a measure, (see for instance [13]), we will use the following notation for $\nu \in V([0, T] \times A_0)$:

$$\nu(dt, d\alpha_1 \dots d\alpha_I) = l_\nu(dt) \nu_t(d\alpha_1 \dots d\alpha_I),$$

where ν_t is a measurable kernel of mass 1 on $(A_0, \mathbb{B}(A_0))$.

As explained in the Introduction 1, we will establish a criterion of compactness of $V([0, T] \times A_0)$, for the weak topology $*\sigma(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0))$, in Theorem 3.1 of section 4, which will be useful in the proof of the compactness of the admissible rules in Section 3.

We turn now to define the set of controls, (generalized actions) in each edge J_i . Let $(K_i)_{1 \leq i \leq I}$, I compact sets of \mathbb{R} .

The set of generalized actions $U([0, T] \times K_i)$ on each edge J_i is defined by:

$$U([0, T] \times K_i) := \left\{ \nu \in \mathcal{M}_{mc}([0, T] \times K_i), \nu^{[0, T]}(dt) = \int_{K_i} \nu(dt, dk_i) = dt \right\}.$$

It is easy to show for each $i \in \{1 \dots I\}$, that $U([0, T] \times K_i)$ are compact for the weak topology $*\sigma(L_{mc}^\infty([0, T] \times K_i)', L_{mc}^\infty([0, T] \times K_i))$, and we will use the notation: for $\nu \in V([0, T] \times K_i)$;

$$\nu(dt, dk_i) = dt \nu_t(dk_i),$$

where ν_t is a measurable kernel of mass 1 on $(K_i, \mathbb{B}(K_i))$. Next we will formulate the stochastic problem of control, with control at the junction point. For this we introduce the following data:

$$\forall i \in \{1 \dots I\} : \begin{cases} \sigma_i \in L^\infty([0, +\infty) \times K_i) \\ b_i \in L^\infty([0, +\infty) \times K_i) \\ h_i \in \mathcal{C}_b([0, +\infty) \times K_i) \end{cases},$$

$$\begin{cases} h_0 \in \mathcal{C}_b(A_0) \\ g \in \mathcal{C}_b(\mathcal{J}, \mathbb{R}) \end{cases},$$

satisfying the following assumptions:

Assumption (\mathcal{H})

$$(i) \quad \exists c > 0, \quad \forall i \in \{1 \dots I\}, \quad \forall (x, k_i) \in [0, T] \times [0, +\infty) \times K_i, \quad \sigma_i(x, k_i) \geq c,$$

$$(ii) \quad \exists (|b|, |\sigma|) \in (0, +\infty)^2, \quad \forall i \in \{1 \dots I\},$$

$$\sup_{(x, k_i) \in [0, +\infty) \times K_i} |b_i(x, k_i)| + \sup_{(x, y) \in [0, +\infty), x \neq y, k_i \in K_i} \frac{|b_i(x, k_i) - b_i(y, k_i)|}{|x - y|} \leq |b|,$$

$$\sup_{(x, k_i) \in [0, +\infty) \times K_i} |\sigma_i(x, k_i)| + \sup_{(x, y) \in [0, +\infty), x \neq y, k_i \in K_i} \frac{|\sigma_i(x, k_i) - \sigma_i(y, k_i)|}{|x - y|} \leq |\sigma|.$$

The canonical space involved in the martingale formulation is the following one:

$$\Phi = \mathcal{C}^{\mathcal{J}}[0, T] \times \left(\prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0),$$

endowed with its Borel σ algebra $\mathbb{B}(\Phi)$. Here $\mathcal{C}^{\mathcal{J}}([0, T])$ is the Polish set of continuous maps defined in $[0, T]$, valued in the junction J , endowed the metric $d_{[0, T]}^{\mathcal{J}}$ defined by

$$\begin{aligned} & \forall \left((x(\cdot), i(\cdot)), (y(\cdot), j(\cdot)) \right) \in \mathcal{C}^{\mathcal{J}}([0, T])^2 : \\ & d_{[0, T]}^{\mathcal{J}} \left((x(\cdot), i(\cdot)), (y(\cdot), j(\cdot)) \right) = \sup_{t \in [0, T]} d^{\mathcal{J}} \left((x(t), i(t)), (y(t), j(t)) \right), \end{aligned}$$

where:

$$\forall \left((x, i), (y, j) \right) \in \mathcal{J}^2, \quad d^{\mathcal{J}} \left((x, i), (y, j) \right) = \begin{cases} |x - y| & \text{if } i = j, \\ x + y & \text{if } i \neq j. \end{cases}$$

The canonical process is then defined on the measurable space $(\Phi, \mathbb{B}(\Phi))$ by:

$$X : \begin{cases} [0, T] \times \Phi \rightarrow \mathcal{J} \times \left(\prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0) \\ \left(s, Y(\cdot) \right) \mapsto \left(X(s, Y(\cdot)) = \left(\left(y(s), j(s) \right), \nu_1(s) \dots \nu_I(s), \nu_0(s) \right) \right), \end{cases}$$

where for each $i \in \{1, \dots, I\}$, $\nu_i(s)(dt, dK_i) = \mathbf{1}_{[0, s]}(t) \nu_i(dt, dK_i)$, and $\nu_0(s)(dt, d\alpha_1, \dots, d\alpha_I) = \mathbf{1}_{[0, s]}(t) \nu_0(dt, d\alpha_1, \dots, d\alpha_I)$.

It is easy to check that the process $\left(X(s)\right)_{0 \leq s \leq T}$ has continuous paths.

We denote in the sequel by $(\Psi_t)_{0 \leq t \leq T}$ the right continuous filtration generated by this process.

Let $(t, (x, i)) \in [0, T] \times \mathcal{J}$.

We define the set of admissible rules $\mathcal{A}(t, (x, i))$, as the set of all the probability measures $P_t^{(x, i)}$ defined on the filtered probability space $(\Phi, (\Psi_t)_{0 \leq t \leq T})$ satisfying the following conditions:

Conditions (\mathcal{S}_0)

-(i) For each $u \leq t$, $X(u) = \left((x, i), \nu_1(t) \dots \nu_I(t), \nu_0(t) \right)$, $P_t^{(x, i)}$ a.s.

-(ii) For each $s \geq t$,

$$\int_t^s \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I) = \int_t^s \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(s)(du) = 0, \quad P_t^{(x, i)} \text{ a.s.}$$

-(iii) For any $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$, the following process $(M^f(s))_{0 \leq s \leq T}$ defined on the filtered probability space $(\Phi, \mathbb{B}(\Phi), (\Psi_t)_{0 \leq t \leq T}, P_t^{(x, i)})$ by:

$$\begin{aligned} \forall s \in [t, T], \quad M^f(s) - M^f(t) &= f(s, X(s)) - f(t, X(t)) - \\ &\sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} \left(\partial_i f_i(u, x(u)) + \frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 f_i(u, x(u)) + \right. \\ &\left. b_i(x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) - \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I), \end{aligned}$$

is a $(\Psi_s)_{t \leq s \leq T}$ continuous martingale under the probability measure $P_t^{(x, i)}$, after time t .

Remark 2.1. *The fact that $\mathcal{A}(t, (x, i))$ is non empty, is a consequence of Lemma 2.3 in [7]. More precisely, it is shown that there exists $P \in \mathcal{A}(t, (x, i))$, with a constant control at the junction point: namely for $(a_1 \dots a_I) \in A_0$,*

$$\forall s \in [0, T], \quad \forall \nu_0 \in V([0, T] \times A_0), \quad \nu_0(s)(dt, d\alpha_1 \dots d\alpha_I) = \mathbf{1}_{[0, s]}(t) \delta_{(a_1, \dots, a_I)}(\alpha_1, \dots, \alpha_I),$$

then:

$$\int_{A_0} \delta_{(a_1, \dots, a_I)}(\alpha_1, \dots, \alpha_I) = 1, \quad \text{and} \quad \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x f(u, 0) \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I) = \sum_{i=1}^I \int_t^s a_i \partial_x f_i(u, 0) l_{\nu_0(s)}(du), \quad P \text{ a.s.}$$

Finally, we define the following reward function Λ of the stochastic control problem, where h_0 is the cost at the junction point, and the h_i are the costs on each edge J_i by:

$$\Lambda : \begin{cases} \mathcal{A}(t, (x, i)) \rightarrow \mathbb{R} \\ P_t^{(x, i)} \mapsto \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) \right. \\ \left. + \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(du, d\alpha_1, \dots, d\alpha_I) + g(X_T) \right]. \end{cases} \quad (7)$$

The corresponding value function v is defined by:

$$v : \begin{cases} [0, T] \times \mathcal{J} \rightarrow \mathbb{R} \\ (t, (x, i)) \mapsto \inf_{P_t^{(x, i)} \in \mathcal{A}(t, (x, i))} \Lambda(P_t^{(x, i)}) \end{cases}. \quad (8)$$

2.2. Main results. In this section, we state the two main results of this work, which are the dynamic programming principle (DPP) Theorem 2.2 and the verification Theorem 2.4, related to the stochastic control problem with control at the junction point.

Theorem 2.2. *Dynamic Programming Principle equation : Assume (\mathcal{H}) , and let τ be a $(\Psi)_{t \leq s \leq T}$ stopping time, we have:*

$$v_i(t, x) = \inf_{P_t^{(x, i)} \in \mathcal{A}(t, (x, i))} \left\{ \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] \right\}. \quad (9)$$

In the sequel, we state a verification theorem. We use some recent results on uniqueness and solvability, for a class of quasi linear PDE posed on a junction, with non linear Neumann boundary (see Theorem 4.5 in [19]). Our main assumptions are that each hamiltonian on each edge have quadratic growth with respect to the gradient, and the

control do not appear in the diffusion terms.

Using the compactness of the sets $(K_i)_{i \in \{1 \dots I\}}$ and A_0 , we get that there exist:

$$\forall i \in \{1 \dots I\}, \quad \bar{k}_i := \begin{cases} \mathbb{R}_+ \times \mathbb{R} \rightarrow K_i, \\ (x, p) \mapsto \bar{k}_i(x, p) \end{cases}, \quad \bar{\alpha}_i := \begin{cases} \mathbb{R}^I \rightarrow A_0, \\ (p_1, \dots, p_I) \mapsto \bar{\alpha}_i(p_1, \dots, p_I) \end{cases},$$

such that:

$$\begin{aligned} \forall (x, p) \in \mathbb{R}_+ \times \mathbb{R}, \quad \forall i \in \{1 \dots I\}, \\ \inf_{k_i \in K_i} \{b_i(x, k_i)p + h_i(x, k_i)\} = b_i(x, \bar{k}_i(x, p))p + h_i(x, \bar{k}_i(x, p)), \\ \forall (p_1, \dots, p_I) \in \mathbb{R}^I, \\ \inf_{(\alpha_i)_{i \in \{1 \dots I\}} \in A_0} \left\{ \sum_{i=1}^I \alpha_i p_i + h_0(\alpha_0, \dots, \alpha_I) \right\} = \sum_{i=1}^I \bar{\alpha}_i(p_1, \dots, p_I) p_i + h_0(\bar{\alpha}_1(p_1, \dots, p_I), \dots, \bar{\alpha}_I(p_1, \dots, p_I)). \end{aligned}$$

We call in the sequel the following functions:

$$\left(H_i(\cdot, \cdot) := \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}, \quad (x, p) \mapsto b_i(x, \bar{k}_i(x, p))p + h_i(x, \bar{k}_i(x, p)) \right)_{i \in \{1 \dots I\}}$$

the Hamiltonians at each edge $i \in \{1 \dots I\}$, and

$$H_0(\cdot) := \mathbb{R}^I \rightarrow \mathbb{R}, \quad (p_1, \dots, p_I) \mapsto \sum_{i=1}^I \bar{\alpha}_i(p_1, \dots, p_I) p_i + h_0(\bar{\alpha}_1(p_1, \dots, p_I), \dots, \bar{\alpha}_I(p_1, \dots, p_I))$$

the Hamiltonian at the junction point.

In the sequel, we will make the following assumptions, which are weaker than assumption (\mathcal{H}) .

Assumption (\mathcal{P})

(i) The diffusions terms $(\sigma_i)_{i \in \{1 \dots I\}}$ do not depend on the controls $(k_i)_{i \in \{1 \dots I\}}$:

$$\forall i \in \{1 \dots I\}, \quad \forall (x, k_i) \in \mathbb{R}_+ \times K_i, \quad \sigma_i(x, k_i) = \sigma_i(x),$$

and the coefficients have the following regularity:

$$\begin{cases} \sigma_i \in \mathcal{C}_b^1(\mathbb{R}_+, \mathbb{R}), & i \in \{1 \dots I\}, \\ b_i \in \mathcal{C}_b^1(\mathbb{R}_+ \times K_i, \mathbb{R}), & i \in \{1 \dots I\}, \\ h_i \in \mathcal{C}_b^1(\mathbb{R}_+ \times K_i, \mathbb{R}), & i \in \{1 \dots I\}, \\ h_0 \in \mathcal{C}_b(A_0, \mathbb{R}), \\ g \in \mathcal{C}_b^2(\mathcal{J}) \end{cases} .$$

(ii) The Hamiltonians satisfy:

$$\forall i \in \{1, \dots, I\}, \quad H_i \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}).$$

Remark that using the ellipticity condition on the $(\alpha_i)_{1 \leq i \leq I} : \forall i \in \{1 \dots I\}, \quad 1 \geq \alpha_i \geq \underline{a} > 0$, it is easy to check first $H_0 \in \mathcal{C}(\mathbb{R}^I, \mathbb{R})$. We get easily too, using the boundness of h_0 , and the continuity of H_0 , that H_0 is increasing, and there exists $\bar{p} \in \mathbb{R}^I$, such that $H_0(\bar{p}) = 0$, namely assumption (i) b) and c) of Theorem 4.5 in [19] holds true.

(iii) The diffusions terms $(\sigma_i)_{i \in \{1 \dots I\}}$ are uniformly elliptic: there exists a constant $c > 0$, strictly positive such that:

$$\exists c > 0, \quad \forall i \in \{1 \dots I\}, \quad \forall x \in \mathbb{R}_+, \quad c \leq \sigma_i(x).$$

(iv) The growth of the Hamiltonians $(H_i)_{i \in 1 \dots I}$ on each edge with respect to p is quadratic, namely there exists $M_1 > 0$ a constant strictly positive such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x, p) \in \mathbb{R}_+ \times \mathbb{R}, \quad |b_i(x, \bar{k}_i(x, p))p + h_i(x, \bar{k}_i(x, p))| \leq M_1(1 + |p|)^2.$$

(v) We impose the following restrictions on the growth with respect to p of the derivatives for the Hamiltonians $(H_i)_{i \in 1 \dots I}$ on each edge, which are for all $i \in \{1 \dots I\}, \forall (x, p) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\begin{aligned} a) \quad & |\partial_p(b_i(x, \bar{k}_i(x, p))p + h_i(x, \bar{k}_i(x, p)))| \leq M_2(1 + |p|), \\ b) \quad & |\partial_x(b_i(x, \bar{k}_i(x, p))p + h_i(x, \bar{k}_i(x, p)))| \leq M_3(1 + |p|)^2, \end{aligned}$$

where $M_2 > 0$ and $M_3 > 0$ are strictly positive constants.

(vi) The terminal condition g satisfies the following compatibility condition:

$$\sum_{i=1}^I \bar{\alpha}_i(g_1(0), \dots, g_I(0))g_i(0) + h_0(\bar{\alpha}_1(g_1(0), \dots, g_I(0)), \dots, \bar{\alpha}_I(g_1(0), \dots, g_I(0))) = 0.$$

In the sequel, $\mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T)$ is the class of function with regularity $\mathcal{C}^{0,1}([0, T] \times [0, +\infty)) \cap \mathcal{C}^{1,2}((0, T) \times (0, +\infty))$ on each edge, continuous at the junction point. As a consequence Theorem 4.5 in [19] we have:

Theorem 2.3. *Assume (\mathcal{P}) . The following quasi linear backward parabolic problem with Neumann boundary condition at the junction point:*

$$\begin{cases} \partial_t u_i(t, x) + \frac{1}{2} \sigma_i(x)^2 \partial_x^2 u_i(t, x) + H_i(x, \partial_x u_i(t, x)) = 0, & \text{if } (t, x) \in (0, T) \times (0, +\infty), \\ H_0(\partial_x u(t, 0)) = 0, & \text{if } t \in (0, T], \\ \forall i \in \{1 \dots I\}, \quad u_i(T, x) = g_i(x), & \text{if } x \in [0, +\infty), \end{cases} \quad (10)$$

is uniquely solvable in the class $\mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T)$. Recall that:

$$\begin{aligned} \partial_x u(t, 0) &= (\partial_x u_1(t, 0), \dots, \partial_x u_I(t, 0)), \quad \forall (i, j) \in \{1 \dots I\}^2, \quad u_i(t, 0) = u_j(t, 0), \\ H_i(x, \partial_x u_i(t, x)) &= b_i(x, \bar{k}_i(x, \partial_x u_i(t, x))) \partial_x u_i(t, x) + h_i(x, \bar{k}_i(x, \partial_x u_i(t, x))) = \\ &= \inf_{k_i \in K_i} \{b_i(x, k_i) \partial_x u_i(t, x) + h_i(x, k_i)\}, \quad \text{if } (t, x) \in (0, T) \times (0, +\infty), \\ H_0(\partial_x u(t, 0)) &= \sum_{i=1}^I \bar{\alpha}_i(\partial_x u_1(t, 0), \dots, \partial_x u_I(t, 0)) \partial_x u_i(t, 0) + \\ &= h_0(\bar{\alpha}_1(\partial_x u_1(t, 0), \dots, \partial_x u_I(t, 0)), \dots, \bar{\alpha}_I(\partial_x u_1(t, 0), \dots, \partial_x u_I(t, 0))) \\ &= \inf_{(\alpha_i)_{i \in \{1 \dots I\}} \in A_0} \left\{ \sum_{i=1}^I \alpha_i \partial_x u_i(t, 0) + h_0(\alpha_0, \dots, \alpha_I) \right\}, \quad \text{if } t \in (0, T]. \end{aligned}$$

In the sequel we denote by u the unique solution of (10). We have the following verification Theorem.

Theorem 2.4. *For any $(t, (x, i)) \in [0, T] \times \mathcal{J}$, we have:*

$$u_i(t, x) \leq v_i(t, x).$$

Moreover, if there exists $\bar{P}_t^{(x,i)} \in \mathcal{A}(t, (x,i))$ such that (iii) of Condition (\mathcal{S}_0) is satisfied with the controls (where δ is the dirac measure):

on each edge J_i :

$$\forall t \in [0, T], \quad \nu_i(t)(dz, dk_i) = \delta_{\bar{k}_{i(t)}(x(t), \partial_x u_{i(t)}(t, x(t)))}(dk_i) dz, \quad \bar{P}_t^{(x,i)} \text{ a.s.},$$

and at the junction point:

$$\forall t \in [0, T], \quad \nu_0(t)(dz, d\alpha_1, \dots, d\alpha_I) = \delta_{(\bar{\alpha}_{i(t)}(\partial_x u_1(t,0), \dots, \partial_x u_I(t,0)))_{1 \leq i(t) \leq I}}(d\alpha_1, \dots, d\alpha_I) l_{\nu_0(t)}(dz), \quad \bar{P}_t^{(x,i)} \text{ a.s.},$$

which means that we have:

$$\begin{aligned} & \forall f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T), \quad f_{i(s)}(s, x(s)) - f_i(t, x) - \int_t^s \partial_t f_{i(z)}(z, x(z)) \\ & + \frac{1}{2} \sigma_{i(u)}^2(x(z)) \partial_x^2 f_i(z, x(z)) + b_{i(z)} \left(x(z), \bar{k}_{i(z)}(x(z), \partial_x u_{i(z)}(z, x(z))) \right) \partial_x f_i(z, x(z)) \Big) dz \\ & - \int_t^s \sum_{i(z)=1}^I \bar{\alpha}_{i(z)}(\partial_x u_1(z, 0), \dots, \partial_x u_I(z, 0)) \partial_x f_{i(z)}(z, 0) l_{\nu_0(s)}(dz), \end{aligned}$$

is a $(\Psi_s)_{t \leq s \leq T}$ continuous martingale under the probability measure $\bar{P}_t^{(x,i)}$, after time t , then $\bar{P}_t^{(x,i)}$ is optimal and we have for any $(t, (x, i)) \in [0, T] \times \mathcal{J}$:

$$\begin{aligned} v_i(t, x) = u_i(t, x) = E^{\bar{P}_t^{(x,i)}} & \left[\int_t^T h_{i(z)} \left(x(z), \bar{k}_{i(z)}(x(z), \partial_x u_{i(z)}(z, x(z))) \right) dz + \right. \\ & \left. \int_t^T h_0 \left(\left(\bar{\alpha}_{i(z)}(\partial_x u_1(z, 0), \dots, \partial_x u_I(z, 0)) \right)_{\{1 \leq i(z) \leq I\}} \right) l_{\nu_0(T)}(dz) + g(X_T) \right], \end{aligned}$$

3. COMPACTNESS OF THE ADMISSIBLE RULE

In this section, we will prove the compactness of the set of admissible rules $\mathcal{A}(t, (x, i))$, for the weak topology.

3.1. A criterium of compactness of the admissible rules at the junction point.

We first start by giving a criterium of compactness of the set of generalized actions $V([0, T] \times A_0)$ at the junction point, that will be useful in the sequel.

Theorem 3.1. *Let \mathcal{V} be a subset of $V([0, T] \times A_0)$. Assume that there exists a constant $C > 0$, and a modulus of continuity $w \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, with $w(0) = 0$, such that*

$$\forall \nu \in \mathcal{V}, \quad l_\nu(T) \leq C,$$

$$\forall \nu \in \mathcal{V}, \quad \forall (t, s) \in [0, T], \quad |l_\nu(t) - l_\nu(s)| \leq w(|t - s|),$$

then \mathcal{V} is compact for the weak topology $\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$.*

Proof. Since the σ Borel algebra $\mathbb{B}([0, T])$ of $[0, T]$ is countably generated, we get from Proposition A.3, that $\mathcal{M}_{mc}([0, T] \times A_0)$ is metrizable, therefore \mathcal{V} is metrizable and the compactness can be proved sequentially.

Let ν_n be a sequence of \mathcal{V} , we know that there exists a sequence l_{ν_n} of $L[0, T]$, such that

$$\nu_n^{[0, T]}(dt) = \int_{A_0} \nu_n(dt, d\alpha_1 \dots d\alpha_I) = l_{\nu_n}(dt).$$

Using the assumptions satisfied by the sequence l_{ν_n} , applying Ascoli's Theorem, we get that l_{ν_n} converges uniformly up to a sub sequence to $l \in \mathcal{C}[0, T]$, and since $L[0, T]$ is closed in $\mathcal{C}[0, T]$ for the uniform convergence, we deduce that $l \in L[0, T]$.

Let us now show that \mathcal{V} is relatively compact for $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$, and for this we are going to apply Theorem A.4.

We now show that $\nu_n^{[0, T]}$ and (resp. $\nu_n^{A_0} = \int_{[0, T]} \nu_n(dt, d\alpha_1 \dots d\alpha_I)$) are relatively compact in $\mathcal{M}_m([0, T])$ (resp. $\mathcal{M}_c(A_0)$), for the weak topologies $*\sigma\left(L^{\infty, 1}([0, T])', L^{\infty, 1}([0, T])\right)$, (resp. $*\sigma\left(\mathcal{C}(A_0)', \mathcal{C}(A_0)\right)$), where we recall that:

$$L^{\infty, 1}([0, T]) := \left\{ f \in L^\infty([0, T]), \exists B \in \mathbb{B}([0, T]), f(t) = \mathbf{1}_B(t) \right\},$$

and $\mathcal{M}_m([0, T])$, (resp. $\mathcal{M}_c(A_0)$), are the set of finite positive finite measures on $[0, T]$ (resp. A_0), endowed with the finest topology making continuous the following family of linear forms $(\theta_f)_{f \in L^{\infty, 1}([0, T])}$, defined by:

$$\theta_f : \begin{cases} \mathcal{M}_m([0, T]) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{[0, T]} f d\nu \end{cases} .$$

(resp. $(\theta_f)_{f \in \mathcal{C}(A_0)}$)

$$\theta_f : \left\{ \begin{array}{l} \mathcal{M}_c(A_0) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{A_0} f d\nu \end{array} \right\}.$$

Since l_n converges uniformly to l up to a sub sequence n_k , it is easy to get that for any $f \in L^{\infty,1}([0, T])$

$$\int_{[0, T]} f(t) l_{\nu_{n_k}}(dt) \xrightarrow[n_k]{k \rightarrow +\infty} \int_{[0, T]} f(t) l(dt),$$

namely $\nu_{n_k}^{[0, T]}(dt) \xrightarrow{*} l(dt)$ for $*\sigma(L^{\infty,1}([0, T])', L^{\infty,1}([0, T]))$.

On the other hand, we have

$$\|\nu_n^{A_0}\|_{\mathcal{C}(A_0)'} = \sup_{f \in \mathcal{C}(A_0), \|f\| \leq 1} \left| \int_{[0, T] \times A_0} f(t) \nu_n(dt, d\alpha_1 \dots d\alpha_I) \right| \leq l_n(T) \leq C,$$

and then we deduce that $\nu_n^{A_0}$ is relatively compact for the weak topology $*\sigma(\mathcal{C}(A_0)', \mathcal{C}(A_0))$.

We deduce finally using Theorem A.4, that ν_n is relatively compact, and then converges up to a sub sequence (denoted in the same way by n_k) to $\phi \in L_{mc}^{\infty,1}([0, T] \times A_0)'$, for $*\sigma(L_{mc}^{\infty,1}([0, T] \times A_0)', L_{mc}^{\infty,1}([0, T] \times A_0))$, where:

$$L_{mc}^{\infty,1}([0, T] \times A_0) := \left\{ f \in L^\infty([0, T] \times A_0), \exists B \in \mathbb{B}([0, T]), g \in \mathcal{C}(A_0), f(\cdot) = \mathbf{1}_B(\cdot)g(\cdot) \right\}.$$

We now turn to prove that ϕ can be represented by an element of $\nu \in \mathcal{M}_{mc}([0, T] \times A_0)$, namely

$$\begin{aligned} & \exists \nu \in \mathcal{M}_{mc}([0, T] \times A_0), \quad \forall f \in L_{mc}^{\infty,1}([0, T] \times A_0), \\ \phi(f) &= \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu(dt, d\alpha_1 \dots d\alpha_I). \end{aligned}$$

For this, we use a Riesz representation Theorem, and more precisely we are going to prove that ϕ satisfies (i) and (ii) of Theorem A.5.

Let $B \in \mathbb{B}([0, T])$, we have

$$(t, \alpha_1 \dots \alpha_I) \mapsto \mathbf{1}_B \otimes 1(t, \alpha_1 \dots \alpha_I) := \begin{cases} 1, & \text{if } t \in B, \\ 0, & \text{if } t \notin B, \end{cases}$$

belongs to $L^{\infty,1}([0, T] \times A_0)$, and

$$\begin{aligned} \nu_{n_k}(\mathbf{1}_B \otimes 1) &\xrightarrow{k \rightarrow +\infty} \phi(\mathbf{1}_B \otimes 1), \\ \nu_{n_k}(\mathbf{1}_B \otimes 1) &= l_{n_k}(B) \xrightarrow{k \rightarrow +\infty} l(B). \end{aligned}$$

By uniqueness of the weak limit, we get that $\phi(\mathbf{1}_B \otimes 1) = l(B)$, and since $l \in L[0, T]$, l defines a Borel measure on $([0, T], \mathbb{B}([0, T]))$, which means that (i) of Theorem A.5 holds true.

On the other hand, since A_0 is compact, we deduce easily that (ii) of Theorem A.5 holds true.

We deduce then that there exists $\nu \in \mathcal{M}_{mc}([0, T] \times A_0)$, such that:

$$\forall f \in L_{mc}^{\infty,1}([0, T] \times A_0), \quad \phi(f) = \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu(dt, d\alpha_1 \dots d\alpha_I).$$

Since ϕ is a continuous linear form on $Span(L_{mc}^{\infty,1}([0, T] \times A_0))$, which is dense in $L_{mc}^{\infty}([0, T] \times A_0)$ for the uniform convergence (see Lemma A.6), we deduce that

$$\forall f \in L_{mc}^{\infty}([0, T] \times A_0), \quad \phi(f) = \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu(dt, d\alpha_1 \dots d\alpha_I).$$

Finally, to complete the proof, it is enough to show that the projection $\nu^{[0, T]}(dt)$ is equal to $l(dt)$. For this we use that, for any $B \in \mathbb{B}([0, T])$

$$\begin{aligned} \int_{[0, T]} \mathbf{1}_B(t) l_{\nu_{n_k}}(dt) &\xrightarrow[n_k]{k \rightarrow +\infty} \int_{[0, T]} \mathbf{1}_B(t) l(dt), \\ \int_{[0, T] \times A_0} \mathbf{1}_B(t) \nu_{n_k}(dt, d\alpha_1 \dots d\alpha_I) &\xrightarrow[n_k]{k \rightarrow +\infty} \int_{[0, T] \times A_0} \mathbf{1}_B(t) \nu(dt, d\alpha_1 \dots d\alpha_I). \end{aligned}$$

Using the uniqueness of the weak limit, we get

$$\forall B \in \mathbb{B}([0, T]), \quad \int_{[0, T]} \mathbf{1}_B(t) l(dt) = \int_{[0, T] \times A_0} \mathbf{1}_B(t) \nu(dt, d\alpha_1 \dots d\alpha_I)$$

and then

$$l(dt) = \int_{A_0} \nu(dt, d\alpha_1 \dots d\alpha_I),$$

and that completes the proof. \square

Theorem 3.2. $V([0, T] \times A_0)$ endowed with the weak topology $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$ is Polish.

Proof. Recall that $\mathcal{M}_{mc}([0, T] \times A_0)$ endowed with the weak topology $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$ is separable since:

$$\mathcal{M}_{mc}([0, T] \times A_0) \subset \bigcup_{n \geq 0} \left\{ \phi \in L_{mc}^\infty([0, T] \times A_0)', \|\phi\| \leq n \right\},$$

and from Banach-Alaoglu-Bourbaki's Theorem:

$$\forall n \geq 0, \left\{ \phi \in L_{mc}^\infty([0, T] \times A_0)', \|\phi\| \leq n \right\}$$

is compact for $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$.

As a subset of $\mathcal{M}_{mc}([0, T] \times A_0)$, we deduce that $V([0, T] \times A_0)$ is separable for the weak topology $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$.

To conclude, let $\nu_n(dt, d\alpha_1 \dots d\alpha_I) := l_n(dt)\nu_{t,n}(dz)$ a Cauchy sequence of $V([0, T] \times A_0)$, we have then

$$\begin{aligned} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall p \geq 0, \forall f \in L_{mc}^\infty([0, T] \times A_0), \\ \left| \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu_{n+p}(dt, d\alpha_1 \dots d\alpha_I) - \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu_n(dt, d\alpha_1 \dots d\alpha_I) \right| \leq \varepsilon. \end{aligned}$$

Let $s \in [0, T]$, choosing $f(t, \alpha_1 \dots \alpha_I) = \mathbf{1}_{[0, s]}(t)$, we get that l_n is a Cauchy sequence of $L([0, T])$, and then converges uniformly to $l \in L([0, T])$. Therefore using the converse of Ascoli's Theorem, we get that the sequence l_n satisfies

$$\exists C > 0, \forall n \geq 0, l_n(T) \leq C,$$

$$\exists w \in \mathcal{C}([0, T]), w(0) = 0, \forall n \geq 0, \forall (t, s) \in [0, T], |l_n(t) - l_n(s)| \leq w(|t - s|).$$

We conclude then using Theorem 3.1, that ν_n converges to $\nu \in V([0, T] \times A_0)$ for the weak topology $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$, and that completes the proof. \square

3.2. Some estimates and paths properties of the process. This subsection is dedicated to give some estimates of the paths of the canonical process $X(\cdot)$, and the time

spent in the neighborhood of the junction point, which are key points to proof of the compacity of $\mathcal{A}(t, (x, i))$.

Proposition 3.3. *Define the following maps:*

$$\rho : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0) \rightarrow \mathbb{R} \\ \left((x(\cdot), i(\cdot)), \nu_0 \right) \mapsto \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1, \dots, d\alpha_I) \end{cases},$$

$$\rho_0 : \begin{cases} V([0, T] \times A_0) \rightarrow \mathbb{R} \\ \nu_0 \mapsto \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(du, d\alpha_1, \dots, d\alpha_I) \end{cases},$$

$$\forall i \in \{1 \dots I\} :$$

$$\rho_i : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times U([0, T] \times K_i) \rightarrow \mathbb{R} \\ \left((x(\cdot), i(\cdot)), \nu_i \right) \mapsto \int_t^T \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(du, dk_i) \end{cases}.$$

Then ρ , $(\rho_i)_{i \in \{1 \dots I\}}$ are lower semi continuous and ρ_0 is continuous.

Proof. We start by showing that ρ is lower semi continuous, and for this let $\left((x^n(\cdot), i^n(\cdot)), \nu_0^n(dt, d\alpha_1, \dots, d\alpha_I) \right)$ in $\mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0)$ converging to $\left((x(\cdot), i(\cdot)), \nu_0(dt, d\alpha_1 \dots d\alpha_I) \right)$. Let $p \geq 0$ and $\phi_p \in \mathcal{C}([0, +\infty))$ a sequence converging from below to $x \rightarrow \mathbf{1}_{\{x > 0\}}$ in the pointwise sense, as $p \rightarrow +\infty$. Since $\nu_n(dt, d\alpha_1, \dots, d\alpha_I) \xrightarrow{*} \nu(dt, d\alpha_1, \dots, d\alpha_I)$, we can find $\theta \in \mathcal{M}_{mc}([0, T] \times A_0)$, such that

$$\forall f \in L_{mc}^{\infty}([0, T] \times A_0), \quad \forall n \geq 0,$$

$$\int_{[0, T] \times A_0} |f(u, \alpha_1, \dots, \alpha_I)| \nu_n(du, d\alpha_1, \dots, d\alpha_I) \leq \int_{[0, T] \times A_0} |f(u, \alpha_1, \dots, \alpha_I)| \theta(du, d\alpha_1, \dots, d\alpha_I).$$

We write then:

$$\begin{aligned}
& \left| \int_t^T \int_{A_0} \phi_p(x^n(u)) \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) - \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1, \dots, d\alpha_I) \right| \leq \\
& \quad \int_t^T \int_{A_0} \left| \phi_p(x^n(u)) - \phi_p(x(u)) \right| \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) \\
& + \left| \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) - \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1, \dots, d\alpha_I) \right| \leq \\
& \quad \int_t^T \int_{A_0} \left| \phi_p(x^n(u)) - \phi_p(x(u)) \right| \theta(du, d\alpha_1, \dots, d\alpha_I) \\
& + \left| \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) - \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1, \dots, d\alpha_I) \right|.
\end{aligned}$$

Therefore we get that

$$\forall p \geq 0, \quad \lim_{n \rightarrow +\infty} \int_t^T \int_{A_0} \phi_p(x^n(u)) \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) = \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1, \dots, d\alpha_I).$$

Finally writing:

$$\int_t^T \int_{A_0} \mathbf{1}_{\{x^n(u) > 0\}} \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) \geq \int_t^T \int_{A_0} \phi_p(x^n(u)) \nu_0^n(du, d\alpha_1, \dots, d\alpha_I),$$

we get

$$\forall p \geq 0, \quad \liminf_{n \rightarrow +\infty} \int_t^T \int_{A_0} \mathbf{1}_{\{x^n(u) > 0\}} \nu_0^n(du, d\alpha_1, \dots, d\alpha_I) \geq \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1, \dots, d\alpha_I),$$

and hence

$$\begin{aligned}
\liminf_{n \rightarrow +\infty} \int_t^T \int_{A_0} \mathbf{1}_{\{x^n(u) > 0\}} \nu_0^n(du, d\alpha_1 \dots d\alpha_I) & \geq \limsup_{p \rightarrow +\infty} \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1, \dots, d\alpha_I) \\
& = \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1, \dots, d\alpha_I).
\end{aligned}$$

We conclude then that ρ is lower semi continuous. We use the same arguments to show that the $(\rho_i)_{i \in \{1 \dots I\}}$ are lower semi continuous and ρ_0 is continuous. \square

In the next Proposition, we characterize the paths of the process $x(\cdot)$, by showing that its martingale part can be represented by a Brownian integral.

Proposition 3.4. *Let $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$, and $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$, we have:*

$$\forall s \in [t, T], \quad d \langle f(\cdot, X(\cdot)) \rangle_s = \left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(\partial_x f_i(s, x(s)) \sigma_i(x(s), k_i) \right)^2 \nu_{i,s}(s)(dk_i) \right) ds, \quad P_t^{(x,i)} \text{ a.s.}$$

Moreover there exists a standard one dimensional Brownian motion $W(\cdot)$, $(\Psi_s)_{t \leq s \leq T}$ measurable, such that:

$$\begin{aligned} \forall s \in [t, T], \quad x(s) &= x + \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} b_i(x(u), k_i) \nu_i(s)(du, dk_i) \\ &+ \int_t^s \left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \sigma_i(x(u), k_i)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}} dW(u) \\ &+ l_{\nu_0(s)}(s), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned} \quad (11)$$

Proof. Let $g = g(x) \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R})$, we have using the classical Itô's formula:

$$\begin{aligned} \forall s \in [t, T], \quad g \circ f(s, X(s)) &= g \circ f(s, x) + \int_t^s \partial_x g \circ f(u, X(u)) df(u, X(u)) \\ &+ \frac{1}{2} \int_t^s \partial_x^2 g \circ f(u, X(u)) d \langle f(\cdot, X(\cdot)) \rangle_u, \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

On the other hand we have

$$\begin{aligned} \int_t^s \partial_x g \circ f(u, X(u)) df(u, X(u)) &= \sum_{i=1}^I \int_{K_i} \int_t^s \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \partial_x g \circ f(u, X(u)) \left(\partial_t f_i(u, x(u)) \right. \\ &+ \frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 f_i(u, x(u)) + b_i(x(u), k_i) \partial_x f_i(u, x(u)) \left. \right) \nu_i(s)(du, dk_i) \\ &+ \sum_{i=1}^I \int_{A_0} \int_t^s \alpha_i \partial_x g \circ f(u, X(u)) \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I) + \\ &\int_t^s \partial_x g \circ f(u, X(u)) dM^f(u), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

Using condition (\mathcal{S}_0) (ii), namely: $\int_t^s \int_{A_0} \mathbf{1}_{\{x(u)>0\}} \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I) = 0$, $P_t^{(x,i)}$ a.s., we get:

$$\begin{aligned} & \sum_{i=1}^I \int_t^s \int_{A_0} \partial_x g \circ f(u, X(u)) \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I) = \\ & \sum_{i=1}^I \int_t^s \int_{A_0} \partial_x g \circ f(u, 0) \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

On the other hand, using that $g \circ f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$, we know that

$$\begin{aligned} & \left(g \circ f(s, X(s)) - g \circ f(t, X(t)) - \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} \left(\partial_t(g \circ f_i)(u, x(u)) \right. \right. \\ & \left. \left. + \frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2(g \circ f_i)(u, x(u)) + b_i(x(u), k_i) \partial_x(g \circ f_i)(u, x(u)) \right) \nu_i(s)(du, dk_i) \right. \\ & \left. - \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x(g \circ f_i)(u, 0) \nu_0(s)(du, d\alpha_1, \dots, d\alpha_I) \right)_{t \leq s \leq T}, \end{aligned}$$

is a $(\Psi_s)_{t \leq s \leq T}$ continuous martingale under the probability measure $P_t^{(x,i)}$. Simple computations allows to get that, at each vertex, for all $x \in J_i^*$ and for all $s \in [t, T]$

$$\begin{aligned} & \partial_t(g \circ f)_i(s, x) + b_i(x, k_i) \partial_x(g \circ f)_i(s, x) + \frac{1}{2} \sigma_i^2(x, k_i) \partial_x^2(g \circ f)_i(s, x) = \\ & \partial_t f_i(s, x) \partial_x g \circ f_i(s, x) + b_i(x, k_i) \partial_x f_i(s, x) \partial_x g \circ f_i(s, x) + \\ & \frac{1}{2} \sigma_i^2(x, k_i) \left(\partial_x^2 f_i(s, x) \partial_x g \circ f_i(s, x) + \partial_x f_i(s, x)^2 \partial_x^2 g \circ f_i(s, x) \right). \end{aligned}$$

Identifying the martingale and finite variation terms, we get that:

$$\begin{aligned} & \forall s \geq t, \quad d \langle f(\cdot, X(\cdot)) \rangle_s = \\ & \left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} \left(\partial_x f_i(s, x(s)) \sigma_i(x(s), k_i) \right)^2 \nu_{i,s}(s)(dk_i) \right) ds, \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

Considering the special case when $f(x) = x$, if $x \in J_i^*$, after an argument of localization with stopping times, and (using the ellipticity assumption (i) (\mathcal{H})), if we set:

$$\forall s \geq t, \quad W(s) = \int_t^s \frac{1}{\left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} \sigma_i(x(u), k_i)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}}} df(u, X(u)), \quad P_t^{(x,i)} \text{ a.s.}$$

we get that

$$d \langle W(\cdot) \rangle_s = ds, \quad P_t^{(x,i)} \text{ a.s.}$$

Therefore using Paul Levy's Theorem characterization of the Brownian motion, $W(\cdot)$ is a standard one dimensional Brownian motion, $(\Psi_s)_{t \leq s \leq T}$ measurable and that completes the proof. \square

Next, we get upper bounds of the modulus of continuity of both processes $x(\cdot)$ and $l(\cdot)$, which are useful for the proof of the compactness of the admissible rules $\mathcal{A}(t, (x, i))$ for the weak topology.

Proposition 3.5. *Let $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$. There exists a constant C , depending only on the data $(T, |b|, |\sigma|)$, such that*

$$\begin{aligned} \forall s \in [t, T], \quad \mathbb{E}^{P_t^{(x,i)}} \left[\left| x(\cdot)^2 \right|_{(t,s)} \right] &\leq C(1 + x^2), \\ \forall s \in [t, T], \quad \mathbb{E}^{P_t^{(x,i)}} \left[\left| l_{\nu_0(\cdot)}(\cdot)^2 \right|_{(t,s)} \right] &\leq C(1 + x^2), \\ \mathbb{E}^{P_t^{(x,i)}} \left[\omega(X(\cdot), \theta)^2 \right] &\leq C\theta \ln\left(\frac{2T}{\theta}\right), \\ \mathbb{E}^{P_t^{(x,i)}} \left[\omega(l_{\nu_0(\cdot)}(\cdot), \theta)^2 \right] &\leq C\theta \ln\left(\frac{2T}{\theta}\right), \end{aligned}$$

where we have defined the following modulus of continuity

$$\begin{aligned} \omega(X, \theta) &= \sup \left\{ d^{\mathcal{J}}(X(s), X(u)), \quad (u, s) \in [t, T], \quad |u - s| \leq \theta, \quad \theta \in [0, T] \right\}, \\ \omega(l, \theta) &= \sup \left\{ |l(u) - l(s)|, \quad (u, s) \in [t, T], \quad |u - s| \leq \theta, \quad \theta \in [0, T] \right\}. \end{aligned}$$

Proof. We define the following map $f \in \mathcal{C}^{1,2}(\mathcal{J}^T)$, by $f(x, i) = x^2$, if $x \in J_i^*$, $i \in \{1 \dots I\}$. After an argument of localization with stopping times, and using condition (\mathcal{S}_0) (iii), we

get for all $s \in [t, T]$

$$\begin{aligned} & \frac{1}{2} \left| x(s)^2 - x^2 \right| \leq \\ & \left| \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(b_i(x(u), k_i)x(u) + \sigma_i(x(u), k_i) \right) \nu_i(s)(du, dk_i) \right| + |M^f(s)| \leq \\ & \left| \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(b_i(x(u), k_i)x(u) + \sigma_i(x(u), k_i) \right) \nu_i(\cdot)(du, dk_i) \right|_{(t,s)} + |M^f(\cdot)|_{(t,s)}. \end{aligned}$$

From Burkholder-Davis-Gundy inequality, and Proposition 3.4 we have

$$\begin{aligned} & \mathbb{E}^{P_t^{(x,i)}} \left[\left| M^f(\cdot) \right|_{(t,s)} \right] = \\ & \mathbb{E}^{P_t^{(x,i)}} \left[\left| \int_t^s \left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(2x(u)\sigma_i(x(u), k_i) \right)^2 \nu_{i,u}(\cdot)(dK_i) \right)^{\frac{1}{2}} dW(u) \right|_{(t,s)} \right] \\ & \leq 4 \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^s \left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(2x(u)\sigma_i(x(u), k_i) \right)^2 \nu_{i,u}(s)(dk_i) \right) du \right] \\ & \leq 16 \max_{i \in \{1 \dots I\}} |\sigma_i|^2 \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^s \left| x(\cdot)^2 \right|_{(t,u)} du \right]. \end{aligned}$$

On the other hand it is easy to see that there exists a constant C , depending only on the data $(T, |b|, |\sigma|)$, such that:

$$\begin{aligned} & \left| \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(b_i(x(u), k_i)x(u) + \sigma_i(x(u), k_i) \right) \nu_i(\cdot)(du, dk_i) \right|_{(t,s)} \\ & \leq C \left(1 + \int_t^s \left| x(\cdot)^2 \right|_{(t,u)} du \right). \end{aligned}$$

Therefore there exists a constant C , depending only on the data $(T, |b|, |\sigma|)$ such that:

$$\mathbb{E}^{P_t^{(x,i)}} \left[\left| x(\cdot)^2 \right|_{(t,s)} - x^2 \right] \leq C \left(1 + \int_t^s \mathbb{E}^{P_t^{(x,i)}} \left[\left| x(\cdot) \right|_{(t,u)}^2 du \right] \right).$$

Applying Gronwall's Lemma to the following measurable function:

$$\rho := \begin{cases} [t, T] \rightarrow \mathbb{R} \\ s \mapsto \mathbb{E}^{P_t^{(x,i)}} \left[\left| x(\cdot)^2 \right|_{(t,s)} \right] \end{cases},$$

we get that there exists a constant C , depending only on the data $(T, |b|, |\sigma|)$ such that:

$$\forall s \in [t, T], \quad \mathbb{E}^{P_t^{(x,i)}} \left[\left| x(\cdot)^2 \right|_{(t,s)} \right] \leq C(1 + x^2).$$

On the other hand, using (11), it is easy to see that there exists a constant C , depending only on the data $(T, |b|, |\sigma|)$ such that:

$$\forall s \in [t, T], \quad \mathbb{E}^{P_t^{(x,i)}} \left[\left| l_{\nu_0(\cdot)}(\cdot)^2 \right|_{(t,s)} \right] \leq C(1 + x^2).$$

We turn now to prove the required upper bounds for the modulus of continuity of the process $\left(x(s)\right)_{t \leq s \leq T}$, and $\left(l_{\nu_0(s)}(s)\right)_{t \leq s \leq T}$. For this end, let $\varepsilon > 0$, we introduce the following sequence of stopping times:

$$\begin{aligned} \theta_0^\varepsilon = t \quad ; \quad \tau_0^\varepsilon = \inf \left\{ t < u \leq T; \quad x(u) = 0 \right\} \quad ; \quad \theta_1^\varepsilon = \inf \left\{ \tau_0^\varepsilon < u \leq T; \quad x(u) = \varepsilon \right\} \\ \dots \\ \tau_n^\varepsilon = \inf \left\{ \theta_n^\varepsilon < u \leq T; \quad x(u) = 0 \right\} \quad ; \quad \theta_{n+1}^\varepsilon = \inf \left\{ \tau_n^\varepsilon < u \leq T; \quad x(u) = \varepsilon \right\}, \end{aligned}$$

and for each $u \in [t, T]$:

$$\bar{\theta}_u := \inf \left\{ \theta_n; \quad \theta_n^\varepsilon \geq u \right\}, \quad \text{and} \quad \underline{\theta}_u := \sup \left\{ \theta_n; \quad \theta_n^\varepsilon \leq u \right\}.$$

Let $(u, s) \in [t, T]^2$ such that $s \leq u$, and $u - s \leq \theta$, $\theta \in (0, T]$, we have (assuming that the process $X(\cdot)$ has reached the junction point between time $[s, u]$, (otherwise inequality (12) is still available)

$$d^{\mathcal{J}}(X(u), X(s)) \leq d^{\mathcal{J}}(X(u), X(\underline{\theta}_u)) + d^{\mathcal{J}}(X(\underline{\theta}_u), X(\bar{\theta}_s)) + d^{\mathcal{J}}(X(\bar{\theta}_s), X(s)), \quad P_t^{(x,i)} \text{ a.s.}$$

We get therefore for any $\varepsilon > 0$:

$$\omega(X, \theta) \leq 2\omega(\tilde{M}, \theta) + 2\varepsilon, \quad P_t^{(x,i)} \text{ a.s.} \quad (12)$$

where we have defined the process $\left(\tilde{M}(s)\right)_{t \leq s \leq T}$ by

$$\begin{aligned} \forall s \in [t, T], \quad \tilde{M}(s) = \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} b_i(x(u), k_i) \nu_i(s)(du, dk_i) \\ + \int_t^s \left(\sum_{i=1}^I \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \sigma_i(x(u), k_i)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}} dW(u), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

The process $\left(\tilde{M}(s)\right)_{t \leq s \leq T}$ satisfies assumptions of Theorem 3.1 of [6], therefore we know that there exists a constant C , depending only on the data $(T, |b|, |\sigma|)$ such that:

$$\forall \varepsilon > 0, \quad \mathbb{E}^{P_t^{(x,i)}} \left[\omega(X(\cdot), \theta)^2 \right] \leq C\theta \ln\left(\frac{2T}{\theta}\right) + 2\varepsilon,$$

and then

$$\mathbb{E}^{P_t^{(x,i)}} \left[\omega(X(\cdot), \theta)^2 \right] \leq C\theta \ln\left(\frac{2T}{\theta}\right).$$

We get the last upper bound for the modulus of continuity of the process $\left(l_{\nu_0(s)}(s)\right)_{t \leq s \leq T}$, using

$$\forall (u, s) \in [t, T]^2, \quad l_{\nu_0(u)}(u) - l_{\nu_0(s)}(s) = x(u) - x(s) - (\tilde{M}_u - \tilde{M}_s), \quad P_t^{(x,i)} \text{ a.s.}$$

□

Lemma 3.6. *Let $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$, and $M > 0$. There exists a constant $C > 0$, depending only on the data $(T, M, |b|, |\sigma|, x)$, introduced in assumption (\mathcal{H}) , such that*

$$\mathbb{E}^{P_t^{(x,i)}} \left[\exp(Mx(T)) \right] \leq C. \quad (13)$$

Proof. We define the following map ϕ by:

$$\phi := \begin{cases} [0, +\infty) \rightarrow \mathbb{R} \\ x \mapsto \exp(Mx) - Mx - 1 \end{cases}.$$

Let $k \geq 0$, we introduce the following stopping time:

$$\theta_k := \inf\{s \in [t, T], \quad x(s) \geq k\}.$$

Hence, using conditions (\mathcal{S}_0) (iii) with ϕ and Proposition 3.5, we get

$$\begin{aligned} \mathbb{E}^{P_t^{(x,i)}} \left[\exp(Mx(T \wedge \theta_k)) \right] &= \exp(Mx) - Mx + \mathbb{E}^{P_t^{(x,i)}} \left[Mx(T \wedge \theta_k) \right] + \\ &\mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^{T \wedge \theta_k} \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(\frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 \phi(x(u)) + \right. \right. \\ &\left. \left. b_i(x(u), k_i) \partial_x \phi(x(u)) \right) \nu_i(T \wedge \theta_k)(du, dk_i) \right] \leq C \left(1 + \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^{T \wedge \theta_k} \exp(Mx(u)) du \right] \right), \end{aligned}$$

where C is a constant depending only on $(T, M, |b|, |\sigma|, x)$. Hence sending $k \rightarrow +\infty$, we get using monotone convergence's Theorem and Fubini's Theorem

$$\mathbb{E}^{P_t^{(x,i)}} \left[\exp(Mx(T)) \right] \leq C \left(1 + \int_t^T \mathbb{E}^{P_t^{(x,i)}} \left[\exp(Mx(u)) \right] du \right).$$

We conclude finally using Gronwall's Lemma to the following measurable map

$$\rho := \begin{cases} [t, T] \rightarrow \mathbb{R} \\ s \mapsto \mathbb{E}^{P_t^{(x,i)}} \left[\exp(Mx(s)) \right] \end{cases}.$$

□

We state in the sequel a central estimate of the time spending by the process at the junction point. The following estimate, will be a key point to show that $\mathcal{A}(t, (x, i))$ is closed for the weak topology. The main assumption used is the ellipticity condition $\sigma_i \geq c > 0$, and will allows to state that the process does not spend time around the junction point.

Proposition 3.7. *Let $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$. There exists a constant $C > 0$, depending only on the data $(T, |b|, |\sigma|, c, x)$, introduced in assumption (\mathcal{H}) , such that:*

$$\forall \varepsilon > 0, \quad \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s) < \varepsilon\}} ds \right] \leq C\varepsilon. \quad (14)$$

Proof. Let $\varepsilon > 0$, and $\beta^\varepsilon \in \mathcal{C}([0, +\infty), \mathbb{R}_+)$ satisfying:

$$\forall x \geq 2\varepsilon, \quad \beta_\varepsilon(x) = 0, \quad \forall x \geq 0, \quad \mathbf{1}_{\{x < \varepsilon\}} \leq \beta_\varepsilon(x) \leq 1. \quad (15)$$

We define $u^\varepsilon \in \mathcal{C}^2([0, +\infty))$ as the unique solution of the following ordinary second order differential equation

$$\begin{cases} \partial_x^2 u^\varepsilon(x) - M\partial_x u^\varepsilon(x) = 2\beta^\varepsilon(x)/c^2, & \text{if } x \in (0, +\infty), \\ \partial_x u^\varepsilon(0) = 0, \\ u^\varepsilon(0) = 0. \end{cases} \quad (16)$$

where c is the constant of ellipticity defined in assumption $(\mathcal{H})(i)$, and M is given by:

$$M = \frac{|b|}{\frac{1}{2}c^2}.$$

The solution is:

$$u^\varepsilon(x) = \int_0^x \exp(Mz) \int_0^z \frac{2\beta^\varepsilon(u)}{c^2} \exp(-Mu) du dz.$$

By the assumption on β_ε , and assumption (\mathcal{H}) , we get:

$$\forall x \geq 0, \quad 0 \leq \partial_x u^\varepsilon(x) \leq 4\varepsilon/c^2 \exp(Mx), \quad 0 \leq u^\varepsilon(x) \leq \frac{4\varepsilon}{Mc^2} (\exp(Mx) - 1). \quad (17)$$

Hence applying condition (\mathcal{S}_0) (iii) (with $f = u^\varepsilon$, after an argument of localization with stopping times), we get using (15), (16) and (17):

$$\begin{aligned} & \mathbb{E}^{P_t^{(x,i)}} \left[u^\varepsilon(x(T)) - u^\varepsilon(x) \right] = \\ & \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(\frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 u^\varepsilon(x(u)) + \right. \right. \\ & \quad \left. \left. b_i(x(u), k_i) \partial_x u^\varepsilon(x(u)) \right) \nu_i(T)(du, dk_i) \right] = \\ & = \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \frac{1}{2} \sigma_i^2(x(u), k_i) \left(\partial_x^2 u^\varepsilon(x(u)) + \right. \right. \\ & \quad \left. \left. \frac{b_i(x(u), k_i)}{\frac{1}{2} \sigma_i^2(x(u), k_i)} \partial_x u^\varepsilon(x(u)) \right) \nu_i(T)(du, dk_i) \right] \geq \\ & \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \frac{1}{2} \sigma_i^2(x(u), k_i) \left(\partial_x^2 u^\varepsilon(x(u)) - M \partial_x u^\varepsilon(x(u)) \right) \nu_i(T)(du, dk_i) \right] \\ & \geq \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \frac{1}{2} c^2 \left(2\beta^\varepsilon(x(u))/c^2 \right) \nu_i(T)(du, dk_i) \right] \\ & \geq \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \beta^\varepsilon(x(u)) du \right] \geq \mathbb{E}^{P_t^{(x,i)}} \left[\int_0^T \mathbf{1}_{\{x(u) \leq \varepsilon\}} du \right]. \end{aligned}$$

Hence we get using (17):

$$\mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s) < \varepsilon\}} ds \right] \leq \frac{4\varepsilon}{Mc^2} \mathbb{E}^{P_t^{(x,i)}} \left[\exp(Mx(T)) - 1 \right].$$

We conclude using Lemma 3.6. □

3.3. Proof of the compactness of the admissible rules. We are able now to prove the main result of this section, namely the compactness of $\mathcal{A}(t, (x, i))$.

Theorem 3.8. *The set of probability measures $\mathcal{A}(t, (x, i))$, endowed with the weak topology is non empty, convex and compact. Moreover, the value function $v(\cdot, \cdot)$ attains its minimum. Finally the set of optimal rules is non empty convex and compact.*

Proof. We recall that the fact that $\mathcal{A}(t, (x, i))$ is non empty is a consequence of Remark 2.1. Let us show first that $\mathcal{A}(t, (x, i))$ is precompact for the weak topology.

It is enough to show that all the following projections

$$\begin{aligned} & \left\{ P_t^{(x,i)}|_{\mathcal{C}\mathcal{J}_{[0,T]}}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}, \\ & \left(\left\{ P_t^{(x,i)}|_{U([0,T] \times K_i)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\} \right)_{i \in \{1, \dots, I\}}, \\ & \left\{ P_t^{(x,i)}|_{V([0,T] \times A_0)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}, \end{aligned}$$

are precompact. The precompactness of $\left\{ P_t^{(x,i)}|_{\mathcal{C}\mathcal{J}_{[0,T]}}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}$ is a consequence of the upper bounds obtained in Proposition 3.5, and Ascoli's Theorem.

We focus on the precompactness of $\left\{ P_t^{(x,i)}|_{V([0,T] \times A_0)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}$. Let $\varepsilon > 0$. It follows from Proposition 3.5, that there exists a constant $C > 0$, depending only on the data $(T, |b|, |\sigma|)$ such that

$$\begin{aligned} \mathbb{E}^{P_t^{(x,i)}} \left[\left| l_{\nu_0(\cdot)}(\cdot)^2 \right|_{(t,T)} \right] &\leq C(1+x^2), \\ \forall \theta \in (0, T], \quad \mathbb{E}^{P_t^{(x,i)}} \left[\omega(l_{\nu_0(\cdot)}(\cdot), \theta)^2 \right] &\leq C\theta \ln\left(\frac{2T}{\theta}\right). \end{aligned}$$

Let us set:

$$K_\varepsilon := \left\{ \nu_0 \in V([0, T] \times A_0), \quad l_{\nu_0}(T) \leq \sqrt[2]{\frac{2C(1+x^2)}{\varepsilon}}, \quad \forall \theta \in (0, T]: \quad w(l_{\nu_0}, \theta) \leq \sqrt[2]{\frac{2C\theta \ln(\frac{2T}{\theta})}{\varepsilon}} \right\}.$$

Using Proposition 3.1, we know that K_ε is compact for the weak topology $*\sigma(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0))$. Moreover, using Tchebychev's inequality, we get that

$$P_t^{(x,i)}|_{V([0,T] \times A_0)} \left(\left(\nu_0(s) \right)_{t \leq s \leq T} \notin K_\varepsilon \right) \leq \varepsilon,$$

and that proves the precompactness of $\left\{ P_t^{(x,i)}|_{V([0,T] \times A_0)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}$. Finally, knowing that all $(U([0, T] \times K_i)_{1 \leq i \leq I})$ are compact, we can show that $\left(\left\{ P_t^{(x,i)}|_{U([0,T] \times K_i)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\} \right)_{i \in \{1, \dots, I\}}$ are precompact.

We turn now to prove that $\mathcal{A}(t, (x, i))$ is closed, and for this let $P_{t,n}^{(x,i)} \in \mathcal{A}(t, (x, i))$ converging weakly to $P_t^{(x,i)}$. We are going to show that $P_t^{(x,i)}$ satisfies condition (\mathcal{S}_0) .

Let $f_p \in \mathcal{C}_b(\Phi, R)$, uniformly bounded in p , converging to $\mathbf{1}_{(X(u)_{0 \leq u \leq t} = ((x,i), \nu_1(t) \dots \nu_I(t), \nu_0(t))}$ in the pointwise sense, and from above. We have:

$$\begin{aligned} \forall p \geq 0, \quad \mathbb{E}^{P_t^{(x,i)}} \left[f_p(X(\cdot)) \right] &= \lim_{n \rightarrow +\infty} \mathbb{E}^{P_{t,n}^{(x,i)}} \left[f_p(X(\cdot)) \right] \geq \\ &\lim_{n \rightarrow +\infty} \mathbb{E}^{P_{t,n}^{(x,i)}} \left[\mathbf{1}_{(X(u)_{0 \leq u \leq t} = ((x,i), \nu_1(t) \dots \nu_I(t), \nu_0(t))} \right] = 1 . \end{aligned}$$

Therefore we get:

$$\lim_{p \rightarrow +\infty} \mathbb{E}^{P_t^{(x,i)}} \left[f_p(X(\cdot)) \right] = 1,$$

and using Lebesgue's Theorem we have:

$$\mathbb{E}^{P_t^{(x,i)}} \left[\mathbf{1}_{(X(u)_{0 \leq u \leq t} = ((x,i), \nu_1(t) \dots \nu_I(t), \nu_0(t))} \right] = 1,$$

which means that (i) of conditions (\mathcal{S}_0) holds true.

Recall that from Proposition 3.3, the following map:

$$\rho : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0) \rightarrow \mathbb{R} \\ \left((x(\cdot), i(\cdot)), \nu_0 \right) \mapsto \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1 \dots d\alpha_I) \end{cases}$$

is lower semi continuous. Consequently, the following set O defined by

$$O := \left\{ \left((x(\cdot), i(\cdot)), \nu_0 \right) \in \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0), \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1 \dots d\alpha_I) > 0 \right\},$$

is open in $\mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0)$. We get then:

$$P_t^{(x,i)}(O) \leq \liminf_{n \rightarrow +\infty} P_{t,n}^{(x,i)}(O) = 0,$$

which means that (ii) of condition (\mathcal{S}_0) holds true.

Now let us show that (iii) of condition (\mathcal{S}_0) holds true. Remark first that:

$$\forall \varepsilon > 0, \sup_{n \geq 0} \mathbb{E}^{P_{t,n}^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s) < \varepsilon\}} ds \right] \leq C\varepsilon,$$

where C is a constant independent of ε . On the other the following map:

$$\theta := \begin{cases} \mathcal{A}(t, (x, i)) \rightarrow \mathbb{R} \\ P_t^{(x,i)} \mapsto \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s) < \varepsilon\}} ds \right] \end{cases},$$

is lower semi continuous for the weak topology. We get then:

$$\mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s) < \varepsilon\}} ds \right] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}^{P_{t,n}^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s) < \varepsilon\}} ds \right] \leq C\varepsilon,$$

which means:

$$\mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s)=0\}} ds \right] = 0.$$

To prove (iii) of condition (\mathcal{S}_0) , let $q \in \mathcal{C}_b(\Phi, \mathbb{R})$, Ψ_s measurable, and $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$. Using that:

$$\mathbb{E}^{P_{t,n}^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s)=0\}} ds \right] = \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^T \mathbf{1}_{\{x(s)=0\}} ds \right] = 0,$$

we have:

$$0 = \mathbb{E}^{P_{t,n}^{(x,i)}} \left[q(M^f(t) - M^f(s)) \right] \xrightarrow{n \rightarrow +\infty} \mathbb{E}^{P_t^{(x,i)}} \left[q(M^f(t) - M^f(s)) \right].$$

Therefore the process

$$\begin{aligned} \left(M^f(s) - M^f(t) = f(s, X(s)) - f(t, X(t)) - \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left(\partial_t f_i(u, x(u)) \right. \right. \\ \left. \left. + \frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 f_i(u, x(u)) + b_i(x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) \right. \\ \left. - \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \right)_{t \leq s \leq T}, \end{aligned}$$

is a $(\Psi_s)_{t \leq s \leq T}$ continuous martingale under the probability measure $P_t^{(x,i)}$, after time t , and that finally proves that $\mathcal{A}(t, (x, i))$ is closed for the weak topology.

We end the proof by showing that the value function $v(\cdot, \cdot)$ attains its minimum, and the set of optimal rules is convex and compact. Using Proposition 3.3, it is easy to check that the reward function Λ

$$\Lambda : \begin{cases} \mathcal{A}(t, (x, i)) \rightarrow \mathbb{R} \\ P_t^{(x, i)} \mapsto \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\left\{ (u, x(u), i(u)) \in J_i^* \right\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) \right. \\ \left. + \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right], \end{cases}$$

is lower semi continuous for the weak topology. Therefore the value function $v(\cdot, \cdot)$ attains its minimum on the compact set $\mathcal{A}(t, (x, i))$. Finally, the fact that the set of optimal rules is convex and compact, is a consequence of the compactness of $\mathcal{A}(t, (x, i))$, the lower semi continuity of Λ , and the linearity of $P_t^{(x, i)} \mapsto \Lambda(P_t^{(x, i)})$. \square

4. DYNAMIC PROGRAMMING PRINCIPLE

The following section is dedicated to the proof of the dynamic programming principle (DPP), Theorem 2.2. Both stability of the set $\mathcal{A}(t, (x, i))$ by conditioning and concatenation are proved.

We state first some propositions and a lemma of measurable selection, which will be useful in the sequel.

Proposition 4.1. *The following map:*

$$\begin{cases} [0, T] \times \mathcal{J} \rightarrow \mathcal{P}(\Phi, \Psi_T) \\ (t, (x, i)) \mapsto \mathcal{A}(t, (x, i)) \end{cases} \quad (18)$$

(where $\mathcal{P}(\Phi, \Psi_T)$ is the set of probability measures defined on Φ , is upper semi continuous.

Proof. We endow $\mathcal{P}(\Phi, \Psi_T)$ with the Hausdorff metric defined over all its compact sets. Since we have shown that $\mathcal{A}(t, (x, i))$ is compact for the weak topology, we follow then the same arguments of the proof of Proposition 5.10 in [15]. \square

Therefore as a consequence of the Proposition 4.1, Proposition 3.3 and Theorem 5.11 in [15], the value function defined in (8):

$$v := \begin{cases} [0, T] \times \mathcal{J} \rightarrow \mathbb{R} \\ (t, (x, i)) \mapsto v_i(t, x) \end{cases},$$

is lower semi continuous.

Proposition 4.2. (see for instance Corollary 5.4 in [15]). Let \mathcal{G}, \mathcal{H} be two separable metric spaces. Let w a lower semi continuous real function on $\mathcal{G} \times \mathcal{H}$ and $h \mapsto K_h$ a measurable map from \mathcal{H} into $\text{comp}(\mathcal{G})$, (the set of compact sets of \mathcal{G} , endowed with the Hausdorff metric). Then

-the map $v(h) := \inf \{w(g, h), g \in K_h\}$ is a Borel function and $h \mapsto M_h := \{g, v(h) = w(h, g), g \in K_h\}$ is a measurable map of \mathcal{H} into $\text{comp}(\mathcal{G})$.

-for each probability measure P on \mathcal{H} :

$$\begin{aligned} \int v(h) dP(h) &= \int \inf \{ w(g, h), g \in K_h \} dP(h) \\ &= \inf \left\{ \int w(\beta(h), h) dP(h), \beta : \mathcal{H} \rightarrow \mathcal{G}, \text{ measurable}, \beta(h) \in K_h \right\}. \end{aligned}$$

Proposition 4.3. Let τ a $(\Psi_t)_{0 \leq t \leq T}$ a stopping time, then:

$$\Psi_\tau = \sigma \left(X(s \wedge \tau), s \leq T \right),$$

and Ψ_τ is countably generated.

Proof. Recall that

$$\Psi_\tau = \left\{ B \in \Psi_T, B \cap \{\tau \leq t\} \in \Psi_t, \forall t \in [0, T] \right\},$$

and the space where is defined our canonical process $X(\cdot)$,

$$\Phi = \mathcal{C}^{\mathcal{J}}[0, T] \times \left(\prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0),$$

is Polish.

We can use then the same arguments of the proof of Lemma 1.3.3 in [23], to get the result.

□

In the sequel, we state a useful proposition, to prove both stability of the set $\mathcal{A}(t, (x, i))$ by conditioning and concatenation. For the convenience of the reader, we do not sketch the proof since it uses the same arguments of Lemma 6.1.1 in [23] for and Theorem 6.1.2 in [23].

Proposition 4.4. *Let τ is a $(\Psi_s)_{t \leq s \leq T}$ stopping time and $(Q_Y)_{Y \in \Phi}$, a transition probability kernel from (Φ, Ψ_τ) to (Φ, Ψ_T) satisfying:*

$$\forall Y \in \Phi, \quad Q_Y \left(X(\tau(Y), \cdot) = X(\tau(Y), Y) \right) = 1.$$

Then:

a) *there exists a unique transition probability kernel from (Φ, Ψ_τ) to (Φ, Ψ_T) denoted by $(\Pi_Y \otimes_{\tau(Y)} Q_Y)_{Y \in \Phi}$, such that:*

$$\begin{aligned} \forall Y \in \Phi, \quad \Pi_Y \otimes_{\tau(Y)} Q_Y \left(X(s, \cdot) = X(\tau(Y), Y) \quad , \forall s \in [0, \tau(Y)] \right) &= 1, \\ \forall A \in \sigma(X(s \vee \tau), 0 \leq s \leq T), \quad \Pi_Y \otimes_{\tau(Y)} Q_Y(A) &= Q_Y(A). \end{aligned}$$

b) *Moreover, if P is a probability measure on (Φ, Ψ_T) , then there exists a unique probability measure on (Φ, Ψ_τ) , denoted by $P \otimes_\tau Q$ such that:*

(i) *the restriction of $P \otimes_\tau Q$ with respect to Ψ_τ is equal to P ,*

(ii) *a r.c.p.d (regular conditional probability distribution) of $P \otimes_\tau Q$ with respect to Ψ_τ is equal to $(\Pi_Y \otimes_{\tau(Y)} Q_Y)_{Y \in \Phi}$.*

We start first by showing the stability of the set $\mathcal{A}(t, (x, i))$ by conditioning.

Proposition 4.5. $\mathcal{A}(t, (x, i))$ *is stable under conditioning, with the following meaning:*

Let $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$, and τ a $(\Psi_s)_{t \leq s \leq T}$ stopping time, then there exists a probability kernel from (Φ, Ψ_τ) to (Φ, Ψ_T) denoted by $(P_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))})_{Y \in \Phi}$, such that:

-there exists $N \subset \Psi_T$, with $P_t^{(x, i)}(N) = 0$ and

$$\forall Y(\cdot) \in \Phi \setminus N, \quad P_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))} \in \mathcal{A}\left(\tau, (x_\tau(Y(\cdot)), i_\tau(Y(\cdot)))\right),$$

-for all $f : \Phi \rightarrow \mathbb{R}$, $\sigma(X(s \vee \tau), 0 \leq s \leq T)$ measurable: $E^{P_t^{(x, i)}}[f | \Psi_\tau] = E^{P_\tau^{(x_\tau, i_\tau)}}[f]$, $P_t^{(x, i)}$

a.s.

Proof. Let τ be a $(\Psi_s)_{t \leq s \leq T}$ stopping time, and let $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$. Using Proposition 4.3, we get from Theorem 1.3.4 of [23], that there exist a r.c.p.d of $P_t^{(x,i)}$ respectively to the sub algebra

$$\Psi_\tau = \sigma\left(X(s \wedge \tau), 0 \leq s \leq T\right),$$

that we denote $(\overline{P}_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))})_{Y \in \Phi}$, which satisfies:

$$\forall Y \in \Phi, \quad \overline{P}_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))} \left(X(\tau(Y), \cdot) = X(\tau(Y), Y) \right) = 1.$$

Let $Y \in \Phi$, we are going to use the notations of Proposition 4.4 a), setting:

$$P_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))} = \Pi_Y \otimes_{\tau(Y)} \overline{P}_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))}.$$

First remark that it is easy to get, for all $f : \Phi \rightarrow \mathbb{R}$, $\sigma(X(s \vee \tau), 0 \leq s \leq T)$ measurable:

$$E^{P_t^{(x,i)}} [f | \Psi_\tau] = E^{P_\tau^{(x_\tau, i_\tau)}} [f], \quad P_t^{(x,i)} \text{ a.s.}$$

Now let us show that $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$, namely $P_\tau^{(x_\tau, i_\tau)}$ satisfies the conditions (\mathcal{S}_0) , $P_t^{(x,i)}$ almost surely.

Using the definition of $(\Pi_Y \otimes_{\tau(Y)} \overline{P}_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))})_{Y \in \Phi}$, we get that (i) of condition (\mathcal{S}_0) holds true, namely:

$$\forall Y \in \Phi, \quad P_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))} \left(X(s, \cdot) = X(\tau(Y), Y), \quad \forall s \in [0, \tau(Y(\cdot))] \right) = 1.$$

Using one more time the properties of $(\Pi_{Y(\cdot)} \otimes_{\tau(Y)} \overline{P}_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})})_{Y \in \Phi}$, stated in Proposition 4.4 a), we have:

$$\begin{aligned}
0 &= \mathbb{E}^{P_t^{(x,i)}} \left[\mathbf{1} \left\{ \int_{\tau}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right\} \right] \\
&= \mathbb{E}^{P_t^{(x,i)}} \left[\mathbb{E}^{\overline{P}_{\tau(\cdot)}^{(x_{\tau(\cdot)}, i_{\tau(\cdot)})}} \left[\mathbf{1} \left\{ \int_{\tau}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right\} \right] \right] \\
&= \mathbb{E}^{P_t^{(x,i)}} \left[\mathbb{E}^{\Pi \cdot \otimes_{\tau(\cdot)} \overline{P}_{\tau(\cdot)}^{(x_{\tau(\cdot)}, i_{\tau(\cdot)})}} \left[\mathbf{1} \left\{ \int_{\tau}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right\} \right] \right] \\
&= \mathbb{E}^{P_t^{(x,i)}} \left[\mathbb{E}^{\Pi \cdot \otimes_{\tau(\cdot)} \overline{P}_{\tau(\cdot)}^{(x_{\tau(\cdot)}, i_{\tau(\cdot)})}} \left[\mathbf{1} \left\{ \int_{\tau}^{\tau(\cdot)} \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) + \int_{\tau(\cdot)}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right\} \right] \right] \\
&= \mathbb{E}^{P_t^{(x,i)}} \left[\mathbb{E}^{\Pi \cdot \otimes_{\tau(\cdot)} \overline{P}_{\tau(\cdot)}^{(x_{\tau(\cdot)}, i_{\tau(\cdot)})}} \left[\mathbf{1} \left\{ \int_{\tau(\cdot)}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right\} \right] \right].
\end{aligned}$$

We get that there exists $\mathcal{E} \subset \Psi_T$, with $P_t^{(x,i)}(\mathcal{E}) = 0$

$$\forall Y(\cdot) \in \Phi \setminus \mathcal{E}, \quad P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})} \left(\int_{\tau(Y(\cdot))}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right) = 0,$$

and (ii) of condition (\mathcal{S}_0) holds true. Finally, let $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$. Using Theorem 1.2.10 of [23], we have that:

$$\begin{aligned}
\left(M^f(s) &:= f(s, X(s)) - f(\tau, X(\tau)) - \sum_{i=1}^I \int_{\tau}^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in \mathcal{J}_i^*\}} \left(\partial_t f_i(u, x(u)) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 f_i(u, x(u)) + b_i(x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) \right. \\
&\quad \left. - \int_{\tau}^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \right)_{\tau \leq s \leq T},
\end{aligned}$$

is a $(\Psi_s)_{\tau \leq s \leq T}$ continuous martingale under the probability measure $(\overline{P}_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})})_{Y \in \Phi}$, after the stopping time τ , and then under the probability measure $(P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})})_{Y \in \Phi}$, since the two last measures are equal for the measurable events after time τ . However the martingale property holds true $\forall Y(\cdot) \in \Phi \setminus \mathcal{N}(f)$ where $\mathcal{N}(f)$ is a negligible set depending on f , (namely $\mathcal{N}(f) \subset \Psi_T$ and $P_t^{(x,i)}(\mathcal{N}(f)) = 0$).

Assume then first that $f \in \mathcal{C}_0^{1,2}(\mathcal{J}_T)$: the class of continuous functions defined on $[0, T] \times \mathcal{J}$, having a regularity of class $\mathcal{C}^{1,2}([0, T] \times [0, +\infty))$ on each edge, and vanishing at each edge at $+\infty$. We get then that $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$ is separable with the following norm $\|\cdot\|_{\mathcal{C}_0^{1,2}(\mathcal{J}_T)}$, defined by:

$$\forall f \in \mathcal{C}_0^{1,2}(\mathcal{J}_T), \quad \|f\|_{\mathcal{C}_0^{1,2}(\mathcal{J}_T)} = \sum_{1 \leq i \leq I} \|f_i\|_{\mathcal{C}^{1,2}([0, T] \times [0, +\infty))},$$

$$\text{with : } \|f_i\|_{\mathcal{C}^{1,2}(\mathcal{J}_T)} = \sup_{(t,x) \in [0, T] \times [0, +\infty)} |f_i(t, x)| + \sup_{(t,x) \in [0, T] \times [0, +\infty)} |\partial_t f_i(t, x)| + \sup_{(t,x) \in [0, T] \times [0, +\infty)} |\partial_x f_i(t, x)| + \sup_{(t,x) \in [0, T] \times [0, +\infty)} |\partial_x^2 f_i(t, x)|.$$

Hence, let f_n a sequence of $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$, dense in $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$, we set:

$$\mathcal{N} = \bigcup_{n \geq 0} \mathcal{N}(f_n).$$

Thereafter, using that following functional:

$$\kappa := \begin{cases} \mathcal{C}_0^{1,2}(\mathcal{J}_T) \rightarrow \mathbb{R} \\ f \mapsto \left(f(s, X(s)) - \sum_{i=1}^I \int_{\tau}^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i\}} \left(\partial_t f_i(u, x(u)) \right. \right. \\ \left. \left. + \frac{1}{2} \sigma_i^2(x(u), k_i) \partial_x^2 f_i(u, x(u)) + b_i(x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) \right. \\ \left. - \int_{\tau}^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \right)_{0 \leq s \leq T}, \end{cases}$$

is continuous for any:

$$\left(\left(x(\cdot), i(\cdot) \right), \nu_1 \dots \nu_I, \nu_0 \right) \in \mathcal{C}^{\mathcal{J}}[0, T] \times \left(\prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0),$$

it is easy to check using Lebesgue's Theorem that $(M^f(s))_{\tau \leq s \leq T}$ is a $(\Psi_s)_{\tau \leq s \leq T}$ continuous martingale under the probability measure $(P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})})_{Y \in \Phi}$, after the stopping time τ , $\forall Y(\cdot) \in \Phi \setminus \mathcal{N}$, using once again that from Lemma 3.7:

$$\mathbb{E}^{P_{\tau}^{(x_{\tau}, i_{\tau})}} \left[\int_{\tau}^T \mathbf{1}_{\{x(s)=0\}} ds \right] = 0, \quad P_t^{(x,i)} \text{ a.s.}$$

To conclude, let $n \geq 0$, $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$, and $f_n \in \mathcal{C}_0^{1,2}(\mathcal{J}_T)$ a sequence converging in the pointwise sense to f , and equal to f on each edge $J_i \cap [0, n]$.

Let then θ a $(\Psi_s)_{\tau \leq s \leq T}$ stopping time after time τ , using Proposition 3.5, Tchebychev's

inequality and assumption (\mathcal{H}) , it is easy to get that there exists a constant $C > 0$ independent of n such that:

$$\left| \mathbb{E}^{\Pi \otimes_\tau P_\tau^{(x_\tau, i_\tau)}} \left[M^{f_n}(s) \mathbf{1}_{\{x(s) \geq n\}} \right] \right| \leq \frac{C}{n^2}, \quad P_t^{(x, i)} \text{ a.s.} \quad (19)$$

We write then:

$$\begin{aligned} \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[M^{f_n}(s) \middle| \Psi_\theta \right] &= \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[M^{f_n}(s) \mathbf{1}_{\{x(s) \leq n\}} \middle| \Psi_\theta \right] + \\ \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[M^{f_n}(s) \mathbf{1}_{\{x(s) \geq n\}} \middle| \Psi_\theta \right] &= M^{f_n}(\theta), \quad P_t^{(x, i)} \text{ a.s,} \end{aligned}$$

and we conclude using Lebesgue's Theorem and (19), setting:

$$N = \mathcal{N} \cup \mathcal{E}.$$

□

The second step is to prove the stability by concatenation.

Proposition 4.6. $\mathcal{A}(t, (x, i))$ is stable under concatenation with the following meaning: let $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$ and τ a $(\Psi_s)_{t \leq s \leq T}$ stopping time. Let $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$, such that:

$$\forall A \in \Psi_T, \quad P_\tau^{(x_\tau, i_\tau)}(A) := \begin{cases} \Phi \mapsto [0, 1] \\ Y \rightarrow P_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))}(A) \end{cases},$$

is $\Psi_T/\mathbb{B}([0, 1])$ measurable.

Then $P_t^{(x, i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(t, (x, i))$, (where $P_t^{(x, i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}$ is introduced in Proposition 4.4 b).)

Proof. We are going to prove that $P_t^{(x, i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}$ satisfies the conditions (\mathcal{S}_0) .

Since $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$, $P_t^{(x, i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}$ is equal to and $P_t^{(x, i)}$ on (Φ, Ψ_τ) , we obtain that condition (i) of (\mathcal{S}_0) holds true.

On the other hand we have:

$$\begin{aligned} & \mathbb{E}^{P_t^{(x,i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\int_t^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] = \\ & \mathbb{E}^{P_t^{(x,i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\int_t^\tau \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] + \mathbb{E}^{P_t^{(x,i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\int_\tau^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] = \\ & \mathbb{E}^{P_t^{(x,i)}} \left[\int_t^\tau \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] + \mathbb{E}^{P_t^{(x,i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\mathbb{E}^{\Pi \cdot \otimes_\tau (\cdot) P_{\tau(\cdot)}^{(x_\tau(\cdot), i_\tau(\cdot))}} \left[\int_\tau^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] \right]. \end{aligned}$$

First we remark that since $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$:

$$\mathbb{E}^{P_t^{(x,i)}} \left[\int_t^\tau \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] = 0.$$

Using Proposition 4.4 a), and that $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$, we get that:

$$\mathbb{E}^{\Pi \otimes_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\int_\tau^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] = \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[\int_\tau^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0}(T)(du) \right] = 0, \quad P_t^{(x,i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)} \text{ a.s.},$$

namely (ii) of (\mathcal{S}_0) holds true. We finish with the martingale conditions (iii) of (\mathcal{S}_0) . For this, we can use once again as in the proof of Proposition 4.5, the reverse of Theorem 1.2.10 of [23], Lemma 3.7, and the argument of separability of $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$, as soon as we have that there exists $N_f \subset \Psi_T$ with $P_t^{(x,i)} \otimes_\tau P_\tau^{(x_\tau, i_\tau)}(N_f) = 0$, such that:

$$\forall Y \in \Phi \setminus N_f, \quad \left(M^f(s) - M^f(s \wedge \tau(Y)), (\Psi_s)_{t \leq s \leq T}, \Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_\tau(Y), i_\tau(Y))} \right)_{t \leq s \leq T}$$

is a martingale. For this, since $(\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})})_{Y \in \Phi}$ is a r.c.p.d. of $(P_t^{(x, i)} \otimes_{\tau} P_{\tau}^{(x_{\tau}, i_{\tau})} | \Psi_{\tau})$, we get that:

$$\begin{aligned}
& \exists N_f \subset \Psi_T, \quad P_t^{(x, i)} \otimes_{\tau} P_{\tau}^{(x_{\tau}, i_{\tau})}(N_f) = 0, \quad \forall Y \in \Phi \setminus N_f, \quad \forall (s, u) \in [t, T], \quad u \leq s, \\
& E^{\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [M^f(s) - M^f(s \wedge \tau(Y)) | \Psi_u] \\
&= E^{\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [(M^f(s) - M^f(s \wedge \tau(Y))) \mathbf{1}_{\{s \geq \tau(Y)\}} | \Psi_u] \\
&= E^{\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [(M^f(s) - M^f(s \wedge \tau(Y))) \mathbf{1}_{\{s \geq u \geq \tau(Y)\}} | \Psi_u] + \\
& \quad E^{\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [(M^f(s) - M^f(s \wedge \tau(Y))) \mathbf{1}_{\{s \geq \tau(Y) \geq u\}} | \Psi_u] \\
&= E^{P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [(M^f(s) - M^f(s \wedge \tau(Y))) \mathbf{1}_{\{s \geq u \geq \tau(Y)\}} | \Psi_u] + \\
& \quad E^{\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [(M^f(s) - M^f(s \wedge \tau(Y))) \mathbf{1}_{\{s \geq \tau(Y) \geq u\}} | \Psi_u] \\
& \quad = M^f(u) - M^f(u \wedge \tau(Y)) + \\
& \quad E^{\Pi_Y \otimes_{\tau(Y)} P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [E^{P_{\tau(Y)}^{(x_{\tau(Y)}, i_{\tau(Y)})}} [(M^f(s) - M^f(s \wedge \tau(Y))) \mathbf{1}_{\{s \geq \tau(Y) \geq u\}} | \Psi_{\tau}] \Psi_u] \\
& \quad = M^f(u) - M^f(u \wedge \tau(Y)).
\end{aligned}$$

We can conclude that (iii) conditions of (\mathcal{S}_0) holds true and that completes the proof. \square

Now we have the necessary tools in order prove the main result of this Section, namely the dynamic programming principle (DPP).

4.1. Proof of Theorem 2.2.

Proof. Let τ be a $(\Psi_s)_{t \leq s \leq T}$ stopping time, and $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$, we have:

$$\begin{aligned}
& \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) + \right. \\
& \quad \left. \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] = \\
& \mathbb{E}^{P_t^{(x, i)}} \left[\mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) + \right. \right. \\
& \quad \left. \left. \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \mid \Psi_{\tau} \right] \right].
\end{aligned}$$

Using Proposition 4.5, namely the stability by conditioning, we know that there exists a transition probability kernel from (Φ, Ψ_τ) to (Φ, Ψ_T) , denoted by $(P_\tau^{(x_\tau(Y), i_\tau(Y))})_{Y \in \Phi}$, such that $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$, $P_t^{(x, i)}$ a.s, and for all $f : \Phi \rightarrow \mathbb{R}$, $\sigma(X(s \vee \tau), 0 \leq s \leq T)$ measurable: $E^{P_t^{(x, i)}}[f | \Psi_\tau] = E^{P_\tau^{(x_\tau, i_\tau)}}[f]$, $P_t^{(x, i)}$ a.s. We get therefore:

$$\begin{aligned}
 & \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) + \right. \\
 & \quad \left. \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \\
 &= \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\
 & \quad \left. \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + \right. \\
 & \quad \left. \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[\sum_{i=1}^I \int_\tau^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) + \right. \right. \\
 & \quad \left. \left. \int_\tau^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right] \\
 &\geq \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\
 & \quad \left. \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right].
 \end{aligned}$$

Taking the infimum over all the $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$, we get then the following first inequality:

$$\begin{aligned}
 & v_i(t, x) \geq \\
 & \inf_{P_t^{(x, i)} \in \mathcal{A}(t, (x, i))} \left\{ \mathbb{E}^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) \right. \right. \\
 & \quad \left. \left. + \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] \right\}.
 \end{aligned}$$

Let $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$. We focus now on the reverse inequality. For this we will use Proposition 4.2, with:

$$\mathcal{H} = (\Phi, \Psi_T), \quad \mathcal{G} = \mathcal{P}(\Phi, \Psi_T), \quad K : \mathcal{H} \rightarrow \text{comp}(\mathcal{G}), \quad Y \rightarrow K_Y = \mathcal{A}(\tau(Y), (x_{\tau(Y)}, i_{\tau(Y)})),$$

and

$$w : \begin{cases} \mathcal{P}(\Phi, \Phi_T) \times (\Phi, \Phi_T) \rightarrow \mathbb{R} \\ (P, Y) \mapsto \mathbb{E}^P \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(y(u), j(u)) \in J_i^*\}} h_i(y(u), K_i) \nu_i(T)(du, dk_i) + \right. \\ \left. \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(Y_T) \right] \end{cases},$$

From Proposition 3.3, we get that w is lower semi continuous. On the other hand, we know from Theorem 3.8, that for each $Y \in \Phi$, $K_Y = \mathcal{A}(\tau(Y), (x_{\tau(Y)}), i_{\tau(Y)})$ is compact for the weak topology. We get then that:

$$\begin{aligned} & \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^{\tau} \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\ & \quad \left. \int_t^{\tau} \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_{\tau}}(\tau, x_{\tau}) \right] = \\ & \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^{\tau} \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\ & \quad \left. \int_t^{\tau} \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) \right] + \\ \inf \{ & \mathbb{E}^{P_t^{(x,i)}} \left[\mathbb{E}^{P_{\tau}^{(x_{\tau}, i_{\tau})}} \left[\sum_{i=1}^I \int_{\tau}^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \right. \right. \\ & \quad \left. \left. \int_{\tau}^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right], \\ & Y \mapsto P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \text{ measurable, } P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \in \mathcal{A}(\tau(Y), (x(\tau(Y))), i(\tau(Y))) \} \\ & = \inf \{ \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^{\tau} \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\ & \quad \left. \int_t^{\tau} \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) \right] + \\ & \mathbb{E}^{P_t^{(x,i)}} \left[\mathbb{E}^{P_{\tau}^{(x_{\tau}, i_{\tau})}} \left[\sum_{i=1}^I \int_{\tau}^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \right. \right. \\ & \quad \left. \left. \int_{\tau}^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right], \\ & Y \mapsto P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \text{ measurable, } P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \in \mathcal{A}(\tau(Y), (x(\tau(Y))), i(\tau(Y))) \}. \end{aligned}$$

Using now the we the properties of the concatenated probability $P_t^{(x,i)} \oplus_\tau P_\tau^{(x_\tau, i_\tau)}$ introduced in Proposition 4.6, we get:

$$\begin{aligned}
 & \mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\
 & \quad \left. \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] = \\
 & = \inf \left\{ \mathbb{E}^{P_t^{(x,i)} \oplus_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) \right. \right. \\
 & + \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + \sum_{i=1}^I \int_\tau^T \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) \\
 & \quad \left. \left. + \int_\tau^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right\}, \\
 & Y \mapsto P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \text{ measurable, } P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \in \mathcal{A}\left(\tau(Y), (x(\tau(Y)), i(\tau(Y)))\right) \left. \right\} \\
 & = \inf \left\{ \mathbb{E}^{P_t^{(x,i)} \oplus_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) \right. \right. \\
 & \quad \left. \left. + \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right\}, \\
 & Y \mapsto P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \text{ measurable, } P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \in \mathcal{A}\left(\tau(Y), (x(\tau(Y)), i(\tau(Y)))\right) \left. \right\}.
 \end{aligned}$$

Or since $P_t^{(x,i)} \oplus_\tau P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(t, (x, i))$, we have then:

$$\begin{aligned}
 & \inf \left\{ \mathbb{E}^{P_t^{(x,i)} \oplus_\tau P_\tau^{(x_\tau, i_\tau)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(T)(du, dk_i) \right. \right. \\
 & \quad \left. \left. + \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right\}, \\
 & Y \mapsto P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \text{ measurable, } P_{\tau(Y)}^{x(\tau(Y)), i(\tau(Y))} \in \mathcal{A}\left(\tau(Y), (x(\tau(Y)), i(\tau(Y)))\right) \left. \right\}, \\
 & \geq v_i(t, x),
 \end{aligned}$$

which means therefore:

$$\mathbb{E}^{P_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\left\{ \left(x(u), i(u) \right) \in J_i^* \right\}} h_i(x(u), k_i) \nu_i(\tau)(du, dk_i) + \int_t^\tau \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] \geq v_i(t, x).$$

Taking the infimum over all the $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$, we conclude for the reverse inequality, and that completes the proof. \square

5. VERIFICATION THEOREM AND AN EXAMPLE OF ILLUSTRATION

Proof of Theorem 2.4.

Proof. Fix $(t, (x, i)) \in [0, T] \times \mathcal{J}$ and $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$. We start first by claiming that the unique solution of (10) u satisfies the Itô's formula: namely there exists a standard one dimensional Brownian motion $W(\cdot)$, $(\Psi_s)_{t \leq s \leq T}$ measurable such that:

$$\begin{aligned} \forall s \in [t, T], \quad u_{i(s)}(s, x(s)) - u_i(t, x) &= \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ \left(x(z), i(z) \right) \in J_i^* \right\}} \left(\partial_t u_i(z, x(z)) \right. \\ &\quad \left. + \frac{1}{2} \sigma_i^2(x(z)) \partial_x^2 u_i(z, x(z)) + b_i(x(z), k_i) \partial_x u_i(z, x(z)) \right) \nu_i(s)(dz, dk_i) \\ &+ \int_t^s \partial_x u_{i(z)}(z, x(z)) \sigma_{i(z)}(x(z)) dW(z) + \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x u_i(z, 0) \nu_0(s)(dz, d\alpha_1 \dots d\alpha_I), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned} \quad (20)$$

even if $u \in \mathcal{C}_b^{0,1}(\mathcal{J}_T) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}}_T)$. Recall that the existence of $W(\cdot)$ is a straight consequence of Proposition 3.4.

For the convenience of the reader, we do not give all the details that lead to the proof of (20), since they are very classical. The main idea is to regularize each function $(t, x) \mapsto u_i(t, x)$ on each edge $i \in \{1 \dots I\}$ by convolution on the domain $[0, T] \times [0, +\infty)$ with the same kernel ρ^n independent of $i \in \{1 \dots I\}$. Therefore the regularize sequence $u_i^n : (t, x) \mapsto (u_i \star \rho^n)(t, x)$ is in the class $\mathcal{C}_b^{1,2}(\mathcal{J}_T)$ satisfies (20), and we have:

$$\begin{aligned} \exists M > 0, \quad \max_{i \in \{1 \dots I\}} \sup_{n \geq 0} \left(\|\partial_t u_i^n(t, x)\|_{[0, T] \times [0, +\infty)} + \|\partial_x^2 u_i^n(t, x)\|_{[0, T] \times [0, +\infty)} \right) &\leq M, \\ \forall i \in \{1 \dots I\}, \quad u_{i,n} &\xrightarrow{n \rightarrow +\infty} u_i, \quad \text{in } \mathcal{C}^{0,1}([0, T] \times [0, R]), \quad \forall R > 0, \\ u_{i,n} &\xrightarrow{n \rightarrow +\infty} u_i, \quad \text{in } \mathcal{C}^{1,2}([\varepsilon, T - \varepsilon] \times [r, R]), \quad \forall r > 0, R > 0, r < R, \varepsilon > 0. \end{aligned}$$

The main idea is to argue by localization for x large and to use that the process does not spend time around the junction point, namely we know from Proposition 3.7, that we have:

$$\int_0^T \mathbf{1}_{\{x(s)=0\}} ds = 0, \quad P_t^{(x,i)} \text{ a.s.}$$

Passing to the limit up to a sub sequence, we then obtain (20) for u .

Let us prove now that $\forall (t, (x, i)) \in [0, T] \times \mathcal{J}$:

$$u_i(t, x) \leq v_i(t, x).$$

We set $s = T$. Using (20) and that u is solution of (10), we obtain:

$$\begin{aligned}
-u_i(t, x) &= -g(X_T) + \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(z), i(z)) \in J_i^*\}} \left(\partial_t u_i(z, x(z)) \right. \\
&+ \frac{1}{2} \sigma_i^2(x(z)) \partial_x^2 u_i(z, x(z)) + b_i(x(z), k_i) \partial_x u_i(z, x(z)) + h_i(x(z), k_i) \partial_x u_i(z, x(z)) \left. \right) \nu_i(T)(dz, dk_i) \\
&- \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(z), i(z)) \in J_i^*\}} h_i(x(z), k_i) \nu_i(T)(dz, dk_i) + \int_t^T \partial_x u_{i(z)}(z, x(z)) \sigma_{i(z)}(x(z)) dW(z) \\
&+ \sum_{i=1}^I \int_t^T \int_{A_0} \alpha_i \partial_x u_i(z, 0) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I) + \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I) \\
&\quad - \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I) \\
&\geq -g(X_T) + \sum_{i=1}^I \int_t^T \mathbf{1}_{\{(x(z), i(z)) \in J_i^*\}} \left(\partial_t u_i(z, x(z)) \right. \\
&+ \frac{1}{2} \sigma_i^2(x(z)) \partial_x^2 u_i(z, x(z)) + \inf_{k_i \in K_i} \left\{ b_i(x(z), k_i) \partial_x u_i(z, x(z)) + h_i(x(z), k_i) \partial_x u_i(z, x(z)) \right\} \left. \right) dz \\
&- \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(z), i(z)) \in J_i^*\}} h_i(x(z), k_i) \nu_i(s)(dz, dk_i) + \int_t^T \partial_x u_{i(z)}(z, x(z)) \sigma_i(x(z)) dW(z) \\
&\quad + \int_t^T \inf_{(\alpha_i)_{1 \leq i \leq I} \in A_0} \left\{ \sum_{i=1}^I \alpha_i \partial_x u_i(z, 0) + h_0(\alpha_1, \dots, \alpha_I) \right\} l_{\nu_0(T)}(dz) \\
&\quad - \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I) \\
&\geq -g(X_T) - \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(z), i(z)) \in J_i^*\}} h_i(x(z), k_i) \nu_i(T)(dz, dk_i) + \\
&+ \int_t^T \partial_x u_{i(z)}(z, x(z)) \sigma_{i(z)}(x(z)) dW(z) - \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I), \quad P_t^{(x, i)} \text{ a.s.}
\end{aligned}$$

Taking the expectation, we obtain then that:

$$\begin{aligned}
u_i(t, x) &\leq E^{P_t^{(x, i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(z), i(z)) \in J_i^*\}} h_i(x(z), k_i) \nu_i(T)(dz, dk_i) \right. \\
&\quad \left. \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I) + g(X_T) \right].
\end{aligned}$$

and finally taking the infimum over all the $P_t^{(x,i)} \in \mathcal{A}(t, (x,i))$, we have that:

$$u_i(t, x) \leq v_i(t, x).$$

To conclude, assume now that there exists $\bar{P}_t^{(x,i)} \in \mathcal{A}(t, (x,i))$ such that (iii) of Condition (\mathcal{S}_0) is satisfied with the controls:

on each edge J_i :

$$\forall t \in [0, T], \quad \nu_i(t)(dz, dk_i) = \delta_{\bar{k}_{i(t)}(x(t), \partial_x u_{i(t)}(t, x(t)))}(dk_i) dz, \quad \bar{P}_t^{(x,i)} \text{ a.s.},$$

and at the junction point:

$$\forall t \in [0, T], \quad \nu_0(t)(dz, d\alpha_1 \dots, d\alpha_I) = \delta_{(\bar{\alpha}_{i(t)}(\partial_x u_1(t,0), \dots, \partial_x u_I(t,0)))_{1 \leq i(t) \leq I}}(d\alpha_1, \dots, d\alpha_I) l_{\nu_0(t)}(dz), \quad \bar{P}_t^{(x,i)} \text{ a.s.},$$

We obtain using the same arguments above that:

$$\begin{aligned} u_i(t, x) &= E^{\bar{P}_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\left\{ \left(x(z), i(z) \right) \in J_i^* \right\}} h_i(x(z), k_i) \nu_i(T)(dz, dk_i) \right. \\ &\quad \left. \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T)(dz, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \leq v_i(t, x). \end{aligned}$$

We conclude that $\bar{P}_t^{(x,i)}$ is optimal and we have for any $(t, (x, i)) \in [0, T] \times \mathcal{J}$:

$$\begin{aligned} v_i(t, x) = u_i(t, x) &= E^{\bar{P}_t^{(x,i)}} \left[\int_t^T h_{i(z)} \left(x(z), \bar{k}_{i(z)}(x(z), \partial_x u_{i(z)}(z, x(z))) \right) dz + \right. \\ &\quad \left. \int_t^T h_0 \left(\left(\bar{\alpha}_{i(z)}(\partial_x u_1(z, 0), \dots, \partial_x u_I(z, 0)) \right)_{\{1 \leq i(z) \leq I\}} \right) l_{\nu_0(T)}(dz) + g(X_T) \right], \end{aligned}$$

which completes the proof. \square

Remark 5.1. *On the existence of the optimal measure $\bar{P}_t^{(x,i)}$:*

A natural question that arises is to get the existence of the optimal measure $\bar{P}_t^{(x,i)}$ which appears in the Theorem of verification 2.4. We remark first, that this measure exists if the controls are piecewise continuous. More precisely using the property of stability by concatenation stated in Proposition 4.6, and the theorem of existence with constant control in Lemma 2.3 in [7], it is easy to check that such a measure exists.

Naturally, once can construct a sequence of approximation $\bar{P}_t^{(x,i),n}$, with piecewise constant controls, and use an argument of tension to get the convergence to the required optimal

measure $\overline{P}_t^{(x,i)}$. The main estimates are given in Section 3.

To not surcharge this work, and for an easier reading, we do not sketch the proof of the existence of $\overline{P}_t^{(x,i)}$, but we will do it in a upcoming work [20], with more general controls at the junction point, depending on the local time. For this new type of diffusion, the definition of the operator and the uniqueness in distribution, will be the main interesting mathematical problems.

We finish this work by giving an example of illustration. As explained in introduction, at the junction point we consider the following cost:

$$\forall (\alpha_1, \dots, \alpha_I) \in A_0, \quad h_0(\alpha_1, \dots, \alpha_I) = \frac{1}{2} \sum_{i \in \{1 \dots I\}} \alpha_i^2 \sigma_i(0)^2.$$

Therefore we obtain the following Hamiltonian at the junction point:

$$\inf_{\alpha_i \in [\underline{a}, 1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i \partial_x u_i(t, 0) + \frac{1}{2} \sum_i \alpha_i^2 \sigma_i(0)^2 \right\} = 0.$$

The coefficients $(\overline{\alpha}_{i(t)}(\partial_x u_1(t, 0), \dots, \partial_x u_I(t, 0)))_{1 \leq i(t) \leq I}$ of the optimal control at the junction point:

$$\forall t \in [0, T], \quad \nu_0(t)(dz, d\alpha_1 \dots, d\alpha_I) = \delta_{(\overline{\alpha}_{i(t)}(\partial_x u_1(t, 0), \dots, \partial_x u_I(t, 0)))_{1 \leq i(t) \leq I}}(d\alpha_1 \dots, d\alpha_I) l_{\nu_0(t)}(dz), \quad \overline{P}_t^{(x,i)} \text{ a.s.},$$

are then solution of a quadratic convex optimization problem, with linear constraints, under the optimal measure $\overline{P}_t^{(x,i)}$.

This type of problem of optimization on the simplex, were introduced in the seminal paper of Harry Markowitz [18]. Thereafter many interest have been devolped in literature for solving this kind of problem, developing several algorithms (interior point, active set, augmented Lagrangian,...) in the theory of quadratic programming.

Fix a terminal condition $g \in \mathcal{C}_b^2(\mathcal{J})$ satisfying the following compatibility condition:

$$\inf_{\alpha_i \in [\underline{a}, 1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i g_i(0) + \frac{1}{2} \sum_i \alpha_i^2 \sigma_i(0)^2 \right\} = 0.$$

For each $i \in \{1 \dots I\}$, fix $\theta_i > 0$ and $(\gamma_i, \lambda_i, \rho_i) \in \mathbb{R}^3$. We consider on each edge the following Hamiltonians:

$$\forall (x, p) \in [0, +\infty) \times \mathbb{R}, \quad H_i(x, p) = -\frac{1}{2\theta_i} [p^2 \sin(x)^2 + 2\lambda_i p \sin(x) + \lambda_i^2] + \gamma_i \sin(x) + \rho_i.$$

From Theorem 2.3, we know that the following quasi linear backward parabolic problem with Neumann boundary condition at the junction:

$$\begin{cases} \partial_t u_i(t, x) + \frac{1}{2} \sigma_i(x)^2 \partial_x^2 u_i(t, x) + H_i(x, \partial_x u_i(t, x)) = 0, & \text{if } (t, x) \in (0, T) \times (0, +\infty), \\ H_0(\partial_x u(t, 0)) = 0, & \text{if } t \in (0, T], \\ \forall i \in \{1 \dots I\}, \quad u_i(T, x) = g_i(x), & \text{if } x \in [0, +\infty), \end{cases} \quad (21)$$

is uniquely solvable in the class $\mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T)$.

Moreover, the estimates of $\sup_{(t,x) \in [0,T] \times [0,+\infty)} |\partial_x u_i(t, x)|$ given in Lemma 4.1 and 4.2 of [19], allows to state that there exists a strictly positive constant $C_i = C_i(\|g\|_{\mathcal{C}_b^2(\mathcal{J})}, \|\sigma\|_{\mathcal{C}_b^2(\mathcal{J})}, c, \theta_i, \lambda_i, \gamma_i, \rho_i)$ such that:

$$\forall i \in \{1 \dots I\}, \quad \sup_{(t,x) \in [0,T] \times [0,+\infty)} |\partial_x u_i(t, x)| \leq C_i.$$

Hence as soon as we fixe $\kappa_i > 0$ such that:

$$\kappa_i \geq \frac{C_i + \lambda_i}{2\theta_i},$$

we get that:

$$\begin{aligned} \forall (t, x) \in (0, T] \times [0, +\infty), \quad H_i(x, \partial_x u_i(t, x)) = \\ -\frac{1}{2\theta_i} [\partial_x u_i(t, x)^2 \sin(x)^2 + 2\lambda_i \partial_x u_i(t, x) \sin(x) + \lambda_i^2] + \gamma_i \sin(x) + \rho_i = \\ \inf_{k_i \in [-\kappa_i, \kappa_i]} \left\{ \partial_x u_i(t, x) k_i \sin(x) + \theta_i k_i^2 + \gamma_i \sin(x) + \lambda_i k_i + \rho_i \right\}. \end{aligned}$$

Therefore regarding to Theorem 2.4 and Remark 5.1, if we set:

$$\forall i \in \{1 \dots I\}, \quad b_i(x) = k_i \sin(x), \quad h_i(x, k_i) = \theta_i k_i^2 + \gamma_i \sin(x) + \lambda_i k_i + \rho_i, \quad K_i = [-\kappa_i, \kappa_i],$$

$$\forall (\alpha_1, \dots, \alpha_I) \in A_0, \quad h_0(\alpha_1, \dots, \alpha_I) = \frac{1}{2} \sum_{i \in \{1 \dots I\}} \alpha_i^2 \sigma_i(0)^2,$$

we will get that the value function v is equal to the solution of (21), and is given by:

$$\begin{aligned} v_i(t, x) = & \mathbb{E}^{\overline{\mathcal{P}}_t^{(x,i)}} \left[\sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\left\{ \left(x(z), i(z) \right) \in J_i^* \right\}} h_i(x(z), k_i) \nu_i(T) (du, dk_i) \right. \\ & \left. + \int_t^T \int_{A_0} h_0(\alpha_1, \dots, \alpha_I) \nu_0(T) (dz, d\alpha_1, \dots, d\alpha_I) + g(X_T) \right], \end{aligned}$$

where $\overline{P}_t^{(x,i)} \in \mathcal{A}(t, (x,i))$ is the optimal measure, with the controls:
on each edge J_i :

$$\forall t \in [0, T], \quad \nu_i(t)(dz, dk_i) = \delta_{-\frac{\partial_x u_i(t,x) \sin(x) + \lambda_i}{2\theta_i}(dk_i)} dz, \quad \overline{P}_t^{(x,i)} \text{ a.s.},$$

and at the junction point:

$$\forall t \in [0, T], \quad \nu_0(t)(dz, d\alpha_1 \dots, d\alpha_I) = \delta_{(\overline{\alpha}_{i(t)}(\partial_x u_1(t,0), \dots, \partial_x u_I(t,0)))_{1 \leq i(t) \leq I}}(d\alpha_1 \dots, d\alpha_I) l_{\nu_0(t)}(dz), \quad \overline{P}_t^{(x,i)} \text{ a.s.},$$

where the vector $(\overline{\alpha}_{i(t)}(\partial_x u_1(t,0), \dots, \partial_x u_I(t,0)))$ solves the following quadratic problem with linear constraints:

$$\inf_{\alpha_i \in [\underline{\alpha}, 1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i \partial_x u_i(t,0) + \frac{1}{2} \sum_i \alpha_i^2 \sigma_i(0)^2 \right\} = 0.$$

APPENDIX A. SOME ANALYSIS TOOLS

We recall here some definitions and functional analysis tools. Let

$$\left\{ \begin{array}{l} (X, T) \text{ be a topological space and } \Sigma \text{ a } \sigma \text{ algebra on } X, \\ (E, \mathcal{E}) \text{ be a measurable space,} \\ (F, d) \text{ be a Polish space, endowed with its metric } d, \text{ and } \mathbb{B}(F) \text{ its Borel algebra.} \end{array} \right.$$

Definition A.1. *(E, E) is said to be countably generated, if there exists a countable base generating E. Namely there exists a sequence O_n of E, such that E = sigma(O_n, n in N).*

Since F is Polish, the measurable space (F, B(F)) is countably generated, (see for instance Proposition 3.1 in [22]).

Definition A.2. *Let P be a measure on (X, Sigma). We say that P is regular if for any measurable subset B in Sigma*

$$P(B) = \sup \left\{ P(K), K \text{ closed}, K \in \Sigma, K \subset B \right\} = \inf \left\{ P(O), O \text{ open}, O \in \Sigma, B \subset O \right\}.$$

We recall that any Borel probability measure, or in other terms any probability measure on a metric space endowed with its sigma-Borel algebra, is regular. (see for instance Proposition 2.3 in [22]).

We denote by :

- $L^\infty(E)$ the set consisting of all measurable real bounded maps on (E, \mathcal{E}) .

- $\mathcal{C}(F)$, (resp. $\mathcal{C}_u(F)$), are the set of continuous (resp. uniformly continuous) bounded functions on F .

- $L^\infty(E \times F)$ is the set of measurable bounded real functions defined on $(E \times F, \mathcal{E} \otimes \mathbb{B}(F))$.

- $\mathcal{M}(E)$ the set consisting of non negative finite measures on (E, \mathcal{E}) .

- $\mathcal{M}(F)$ the set consisting of non negative finite measures on $(F, \mathbb{B}(F))$.

- $\mathcal{M}(E \times F)$ the set consisting of non negative finite measures on $(E \times F, \mathcal{E} \otimes \mathbb{B}(F))$.

We set furthermore

$$\begin{aligned} L_{mc}^\infty(E \times F) &:= \left\{ f \in \mathcal{L}^\infty(E \times F), x \mapsto f(s, x) \in \mathcal{C}(F), \forall s \in E \right\}, \\ L_{mc}^{\infty,1}(E \times F) &:= \left\{ f \in L_{mc}^\infty(E \times F), \exists A \in \mathcal{E}, g \in \mathcal{C}_u(F), f(x, z) = \mathbf{1}_A(x)g(z) \right\}, \\ L_{mc}^{\infty,2}(E \times F) &:= \left\{ f \in L_{mc}^\infty(E \times F), \exists (A_n) \text{ a partition of } \mathcal{E}, \right. \\ &\quad \left. \text{and a sequence } (g_n) \text{ of } \in \mathcal{C}_u(F), f(x, z) = \sum_n \mathbf{1}_{A_n}(x)g_n(z) \right\}. \end{aligned}$$

On the other hand $\mathcal{M}(E)$ (resp. $\mathcal{M}(F)$, $\mathcal{M}(E \times F)$) are denoted by $\mathcal{M}_m(E)$, (resp. $\mathcal{M}_c(F)$, $\mathcal{M}_{mc}(E \times F)$) when they are endowed with the finest topology making continuous the following family of linear forms $(\theta_f)_{f \in L^\infty(E)}$, defined by

$$\theta_f : \begin{cases} \mathcal{M}(E) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_E f d\nu \end{cases} .$$

(resp. $(\theta_f)_{f \in \mathcal{C}(F)}$)

$$\theta_f : \begin{cases} \mathcal{M}(F) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_E f d\nu \end{cases} ,$$

$(\theta_f)_{f \in L_{mc}^\infty(E \times F)}$,

$$\theta_f : \begin{cases} \mathcal{M}(E \times F) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{E \times F} f d\nu \end{cases} .)$$

We identify $\mathcal{M}_{mc}(E \times F)$ (resp. $\mathcal{M}_m(E)$, $\mathcal{M}_c(F)$), as subsets of the dual spaces $L_{mc}^\infty(E \times F)'$ (resp. $L_m^\infty(E)'$, $\mathcal{C}(F)'$), endowed with the weak topologies $*\sigma(L_{mc}^\infty(E \times F)', L_{mc}^\infty(E \times F))$

$F)$ (resp. ${}^*\sigma(L_m^\infty(E)', L_m^\infty(E)), {}^*\sigma(\mathcal{C}(F)', \mathcal{C}(F))$).

We recall that a sequence ν_n of $L_{mc}^\infty(E \times F)'$ (resp. $L_m^\infty(E)', \mathcal{C}(F)'$), converges to $\nu \in L_{mc}^\infty(E \times F)'$, (resp. $L_m^\infty(E)', \mathcal{C}(F)'$) for the weak topology * , and we denote $\nu_n \xrightarrow{{}^*} \nu$, if and only if

$$\begin{aligned} & \forall f \in L_{mc}^\infty(E \times F), \quad \nu_n(f) \xrightarrow{n \rightarrow +\infty} \nu(f), \\ & \left(\text{resp. } \forall f \in L_m^\infty(E), \quad \nu_n(f) \xrightarrow{n \rightarrow +\infty} \nu(f), \quad \forall f \in \mathcal{C}(F), \quad \nu_n(f) \xrightarrow{n \rightarrow +\infty} \nu(f) \right). \end{aligned}$$

For any $\nu \in \mathcal{M}(E \times F)$, we denote by ν^E (resp. ν^F), the marginal of ν on E (resp. on F), defined by

$$\nu^E(dx) = \int_{z \in F} \nu(dz), \quad \nu^F(dz) = \int_{x \in E} \nu(dx).$$

Proposition A.3. *Suppose that E is countably generated, then $\mathcal{M}_{mc}(E \times F)$ is metrizable. (See for instance Proposition 2.10 in [11].)*

Theorem A.4. *Let \mathcal{N} be a subset of $\mathcal{M}_{mc}(E \times F)$. Then \mathcal{N} is relatively compact if and only if*

- (i) $\{\nu^F, \nu \in \mathcal{N}\}$ is relatively compact in $\mathcal{M}_m(E)$,
- (ii) $\{\nu^E, \nu \in \mathcal{N}\}$ is relatively compact in $\mathcal{M}_c(F)$.

(See for instance Proposition 2.10 in [11].)

Theorem A.5. *Let ϕ be a positive linear form defined on the vectorial space generated by $L_{mc}^{\infty,1}(E \times F)$ satisfying*

$$(i) \quad \begin{cases} \mathcal{E} \rightarrow \mathbb{R} \\ A \mapsto \phi(\mathbf{1}_A \otimes 1) \end{cases}$$

is a measure on (E, \mathcal{E}) , where we define for each $(x, z) \in E \times F$, $\mathbf{1}_A \otimes 1(x, z) = 1$, if $x \in A$ and $\mathbf{1}_A \otimes 1(x, z) = 0$, if $x \notin A$.

(ii) for each $\varepsilon > 0$, there exist a compact set K_ε of F such that $\phi(1) - \phi(1 \otimes f) \leq \varepsilon$, for any $f \in \mathcal{C}_u(F)$, satisfying $\mathbf{1}_{K_\varepsilon} \leq f \leq 1$, where we define for each $(x, z) \in E \times F$, $1(x, z) = 1$, and $1 \otimes f(x, z) = f(z)$.

Then there exists $\nu \in \mathcal{M}_{mc}(E \times F)$ such that

$$\forall f \in L_{mc}^{\infty,1}(E \times F), \quad \phi(f) = \int_{E \times F} f d\nu.$$

(See for instance Theorem 2.6 in [11]).

Lemma A.6. *Let K be a compact set of F , and $f \in L_{mc}^{\infty}(E \times F)$. Then there exist a sequence f_n of $L_{mc}^{\infty,2}(E \times F)$ converging to f uniformly on $E \times K$. (See for instance Lemma 2.5 in [11]).*

Definition A.7. *Let $(x = (x_1, \dots, x_I), y = (y_1 \dots y_I)) \in \mathbb{R}^{2I}$, we say that*

$$x \leq y, \text{ if } \forall i \in \{1 \dots I\}, \quad x_i \leq y_i,$$

and

$$x < y, \text{ if } x \leq y, \text{ and there exists } j \in \{1 \dots I\}, \quad x_j < y_j.$$

We say that $F \in \mathcal{C}(\mathbb{R}^I, \mathbb{R})$ is increasing if

$$\forall (x, y) \in \mathbb{R}^I, \text{ if } x \leq y, \text{ then } F(x) \leq F(y),$$

strictly increasing if

$$\forall (x, y) \in \mathbb{R}^I, \text{ if } x < y, \text{ then } F(x) < F(y).$$

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