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# FULLY NONLINEAR FETI METHOD

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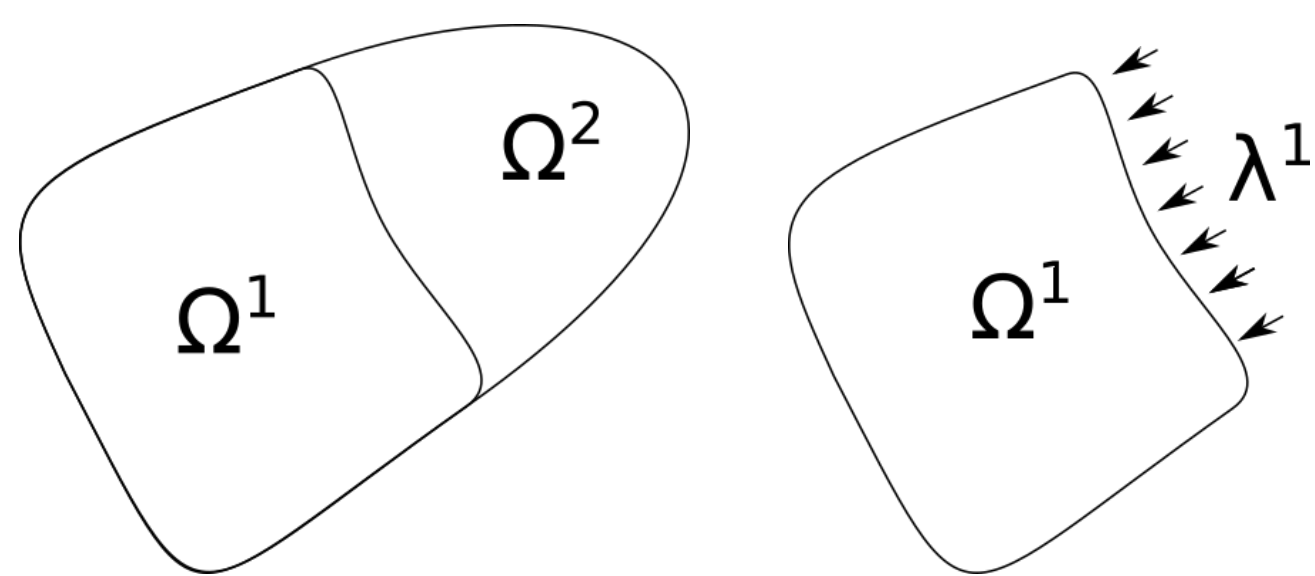
## DDM for nonlinear problems

- (NKS) Newton solver in outer loop, linear DD (Krylov) in inner loop
  - Schwarz DD (Cai, Gropp, et al. 1994);
  - Schur DD (FETI) (Bhardwaj et al. 1999)...
  - Coupling with inexact Newton technique (Biros and Ghattas 2003).
- Primal/dual/mixed approaches (Negrello, Gosselet, Rey, and Pebrel 2016), Improvement of the impedance of the mixed approach (Negrello, Gosselet, and Rey 2017).

- (SNK) Solve the nonlinearity at the scale of the subdomains:

- Schwarz DD: ASPIN (Cai and Keyes 2002), RASPEN (Dolean et al. 2016), LATIN (Ladevèze, Néron, and Gosselet 2007), (Badea 2009), global/local non-invasive coupling (Gosselet et al. 2018).
- Schur DD, “nonlinear relocalization”: primal (Dirichlet) and mixed (Robin) approach (Cresta et al. 2007), dual approach (Pebrel, Rey, and Gosselet 2008) and nonlinear FETIDP and BDDC (Klawonn, Lanser, and Rheinbach 2014) with large scale assessments in (Klawonn, Lanser, Rheinbach, and Uran 2017a).

## Nonlinear PDE + FE + DD



Nonlinear PDE representative of quasi-static structure mechanics:

$$g_{int}(u) + g_{ext} = 0 \text{ in } \Omega$$

Assumptions:

- $g_{int}$  is differentiable, tangent matrices SPsD
- Global problem is well posed
- Classical Newton-Raphson would converge.

Decomposed version (block notations):

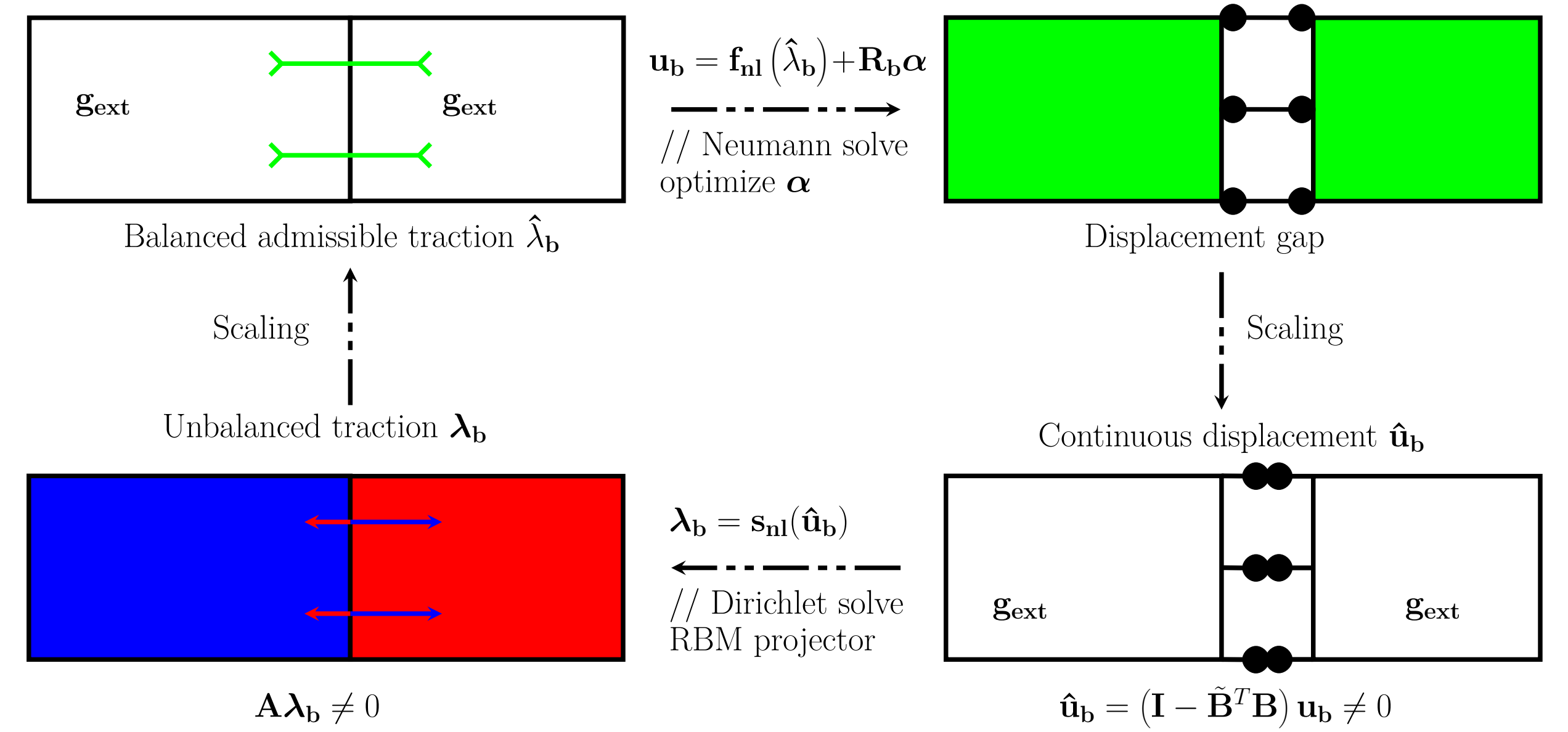
$$\mathbf{g}_{int}(\mathbf{u}) + \mathbf{g}_{ext} + \mathbf{T}^T \boldsymbol{\lambda}_b = 0 \text{ with } \begin{cases} \mathbf{B} \mathbf{u}_b = 0 \\ \mathbf{A} \boldsymbol{\lambda}_b = 0 \end{cases}$$

- $\boldsymbol{\lambda}_b$ : unknown local nodal reaction,
- $\mathbf{T}$ : trace operator,
- subscript  $b$ : boundary dof
- $\mathbf{A} / \mathbf{B}$ : primal/dual assembly,
- $\tilde{\mathbf{A}} / \tilde{\mathbf{B}}$ : scaling  $\mathbf{A} \tilde{\mathbf{A}}^T = \mathbf{I}, \mathbf{B} \tilde{\mathbf{B}}^T \mathbf{B} = \mathbf{B}$   
 $\mathbf{A}^T \tilde{\mathbf{A}} + \tilde{\mathbf{B}}^T \mathbf{B} = \mathbf{I}$
- Small strain hypothesis:  
 $\exists$  rigid body motions  $\mathbf{R}$   
 $\forall \mathbf{u}, \boldsymbol{\alpha}, \mathbf{g}_{int}(\mathbf{u} + \mathbf{R}\boldsymbol{\alpha}) = \mathbf{g}_{int}(\mathbf{u})$   
 $\mathbf{R}^T \mathbf{g}_{int} = 0$   
Let  $\mathbf{G}_B = \mathbf{B}^T \mathbf{R}$

## Fully nonlinear FETI

Nonlinear fixed point

$$\hat{\boldsymbol{\lambda}}_b = (\mathbf{I} - \tilde{\mathbf{A}}^T \mathbf{A}) \boldsymbol{\lambda}_b; 0 = \mathbf{R}^T (\mathbf{g}_{ext} + \mathbf{T}^T \boldsymbol{\lambda}_b)$$



(Rixen and Farhat 1999; Parret-Fréaud et al. 2010)

$$\text{Find } \tilde{\boldsymbol{\lambda}}_B \text{ s.t. } \hat{\boldsymbol{\lambda}}_b = \mathbf{B}^T (\boldsymbol{\lambda}_{B_0} + \mathbf{P}_B \tilde{\boldsymbol{\lambda}}_B) \text{ satisfies fixed point } (\mathbf{I} - \tilde{\mathbf{A}}^T \mathbf{A}) \mathbf{s}_{nl}([\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\hat{\boldsymbol{\lambda}}_b)) - \hat{\boldsymbol{\lambda}}_b = 0$$

- Identical to FETI with Dirichlet preconditioner in linear
- No contraction  $\Rightarrow$  Krylov solver in linear, Newton in nonlinear
- One Newton iteration, starting from  $\hat{\boldsymbol{\lambda}}_b = \mathbf{B}^T (\boldsymbol{\lambda}_{B_0} + \mathbf{P}_B \tilde{\boldsymbol{\lambda}}_B)$ :  
// Solve independent nonlinear Neumann problems  $\mathbf{u}_b = \mathbf{f}_{nl}(\hat{\boldsymbol{\lambda}}_b)$   
– Assemble displacement gap, compute continuous displacement  $\hat{\mathbf{u}}_b$   
// Solve independent nonlinear Dirichlet problems  $\boldsymbol{\lambda}_b = \tilde{\mathbf{B}} \mathbf{s}_{nl}(\hat{\mathbf{u}}_b)$   
– Assemble reaction lack-of-balance, compute balanced reaction  $\tilde{\boldsymbol{\lambda}}_B$   
– Krylov solver for the (distributed) interface tangent system  
 $\tilde{\mathbf{B}}(D\mathbf{s}_{nl}')([\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}](D\mathbf{f}_{nl})) \mathbf{B}^T \mathbf{P}_B \boldsymbol{\delta} = -\mathbf{P}_B (\tilde{\boldsymbol{\lambda}}_B' - \tilde{\boldsymbol{\lambda}}_B)$   
– Update reaction  $\hat{\boldsymbol{\lambda}}_b \leftarrow \hat{\boldsymbol{\lambda}}_b + \mathbf{B}^T \mathbf{P}_B \boldsymbol{\delta}$
- Equivalent to classical NKS if no inner Newton iterations
- Equivalent to FETI-NL if no inner Newton iterations for Dirichlet problem

## Condensed NL formulations

Primal approach. Local Dirichlet–Neumann operator:  $\boldsymbol{\lambda}_b = \mathbf{s}_{nl}(\mathbf{u}_b; \mathbf{g}_{ext})$

$$\text{Find } \mathbf{u}_A \in \mathbb{R}^{n_A} \text{ such that } : \mathbf{A} \mathbf{s}_{nl}(\mathbf{A}^T \mathbf{u}_A; \mathbf{g}_{ext}) = 0$$

Dual approach. Compatibility condition:  $\mathbf{R}^T (\mathbf{g}_{ext} + \mathbf{T}^T \boldsymbol{\lambda}_b) = 0$

$$\text{Local Neumann-Dirichlet operator } \mathbf{u}_b = \mathbf{f}_{nl}(\boldsymbol{\lambda}_b; \mathbf{g}_{ext}) + \mathbf{T} \mathbf{R} \boldsymbol{\alpha}$$

$$\boldsymbol{\lambda}_B = \boldsymbol{\lambda}_{B_0} + \mathbf{P}_B \tilde{\boldsymbol{\lambda}}_B \text{ with: } \begin{cases} \mathbf{P}_B = \mathbf{I} - \mathbf{Q}_B \mathbf{G}_B (\mathbf{G}_B^T \mathbf{Q}_B \mathbf{G}_B)^{-1} \mathbf{G}_B^T \\ \boldsymbol{\lambda}_{B_0} = -\mathbf{Q}_B \mathbf{G}_B (\mathbf{G}_B^T \mathbf{Q}_B \mathbf{G}_B)^{-1} \mathbf{R}^T \mathbf{g}_{ext} \end{cases}, \mathbf{Q}_B \text{ SPD matrix}$$

$\boldsymbol{\alpha}$  can be computed by minimizing the  $\mathbf{Q}_B$ -norm of displacement jump.

$$\text{Find } \tilde{\boldsymbol{\lambda}}_B \in \mathbb{R}^{n_B} \text{ such that: } \mathbf{P}_B^T \mathbf{B} \mathbf{f}_{nl}(\mathbf{B}^T (\boldsymbol{\lambda}_{B_0} + \mathbf{P}_B \tilde{\boldsymbol{\lambda}}_B); \mathbf{g}_{ext}) = 0$$

→ Newton solver, tangent system is usual FETI (Krylov solver + Dirichlet preconditioner)

## Illustration: nonlinear thermal problem

Heat equation:

$$\nabla \cdot (K(T) \nabla T) = 0$$

Non-classical behavior allowing “rigid body motions”:

$$K(T) = (1 + \nabla T \cdot \nabla T)^\alpha$$

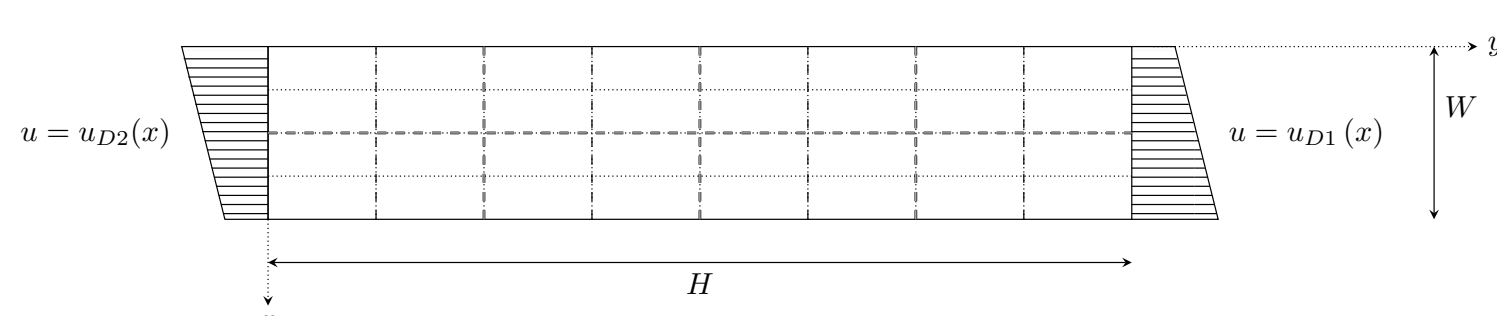


Fig. 2: Geometry and Dirichlet BCs



Fig. 3: Temperature map ( $\alpha = 0.6$ )

	# Global Newton iterations														
	BDD-NL			FETI-NL			FETI-precNL			Gains of FETI-precNL (%) vs. BDD-NL vs. FETI-NL					
#SD	8	16	32	8	16	32	8	16	32	8	16	32	8	16	32
$\alpha = 0$	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
$\alpha = 0.2$	6	6	7	5	6	5	3	3	4	50	50	43	40	50	20
$\alpha = 0.4$	7	6	8	7	8	9	3	3	4	57	50	50	57	63	56
$\alpha = 0.6$	7	6	8	8	7	9	4	4	5	43	33	38	50	43	44
	# Krylov iterations														
	BDD-NL			FETI-NL			FETI-precNL			Gains of FETI-precNL (%) vs. BDD-NL vs. FETI-NL					
#SD	8	16	32	8	16	32	8	16	32	8	16	32	8	16	32
$\alpha = 0$	8	8	14	8	8	15	8	8	15	0	0	-6	0	0	0
$\alpha = 0.2$	20	21	30	17	24	32	17	17	29	15	19	3	0	29	9
$\alpha = 0.4$	22	22	31	20	25	40	16	15	27	27	32	13	20	40	33
$\alpha = 0.6$	21	21	38	25	23	43	22	20	33	-5	5	13	12	13	23
	max # Local Newton iterations														
	BDD-NL			FETI-NL			FETI-precNL			Ratio of FETI-precNL vs. BDD-NL vs. FETI-NL					
#SD	8	16	32	8	16	32	8	16	32	8	16	32	8	16	32
$\alpha = 0$	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
$\alpha = 0.2$	11	13	17	11	14	14	18	14	17	1.6	1.1	1	1.6	1	1.2
$\alpha = 0.4$	15	15	29	18	20	33	22	18	20	1.5	1.2	1.1	1.2	0.9	0.6
$\alpha = 0.6$	18	17	20	24	24	34	29	24	29	1.6	1.4	1.5	1.2	1	0.9

In most configurations,

- Significant decrease of outer Newton (better prediction of nonlinear behavior) and inner Krylov iterations (all-to-all communications)
- Limited extra parallel computations (inner Newtons)

## Work in progress

- Large scale implementation for significant assessment and more general nonlinearities.
- Introducing robustness-enhancing techniques (adaptive coarse space, multipreconditioning) in this framework.
- Improvement of the load-balancing.