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Abstract. In the present paper by using the tools developed for finite volume schemes, we adapt the strategy used to deal with the existence of a renormalized solution for parabolic problems with L^1 data. We prove here the convergence of the finite volume approximate solution of a noncoercive and nonlinear parabolic equation to a renormalized solution. Under additional assumptions on the problem, the uniqueness of the renormalized solution implies that the whole sequence converges.

1. Introduction

In this work we consider a finite volume implicit in time discretization (with an upstream choice for the convective flux) of the following parabolic problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}(\lambda(u)\nabla u) + \operatorname{div}(\boldsymbol{v}u) = f \text{ in } Q, \\ &u = 0 \text{ on } \partial\Omega \times (0,T), \\ &u(x,0) = u^0(x) \ \forall x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$, T is a positive number and $Q = \Omega \times (0,T)$. Moreover λ is a continuous function such that $\lambda(u) \geq \mu > 0$, with μ a positive real number, $\boldsymbol{v} \in (L^{d+2}(Q))^d$, $u^0 \in L^1(\Omega)$ and $f \in L^1(Q)$.

The main difficulties in dealing with the existence and the uniqueness of a solution to problem (1.1) are due to the noncoercive and nonlinear characters of the operator and to the L^1 data f. These difficulties naturally have an impact on any finite volume or finite element scheme to handle approximation of solution of (1.1). Indeed, due to the noncoercive character of the equation, we have to control a discrete variational formulation of $\operatorname{div}(\boldsymbol{v}u)$ which is not simple (see Proposition 3.3). Moreover, since the data is not regular enough, residual terms appear when controling the discrete time derivative: we use a nonlinear test function and the chain rule formula does not apply in the discrete case. We thus have to control these extra terms.

When $\boldsymbol{v} = 0$, as far as parabolic equations with L^1 or measure data are concerned Boccardo and Gallouët have proved in [11] the existence of a distributional solution for nonlinear and coercive equations. Since the solution in the sense of distribution is not unique in general (see [32] in the elliptic case and [31] in the parabolic case) different notions of solution have been developed for parabolic (or elliptic) equations with L^1 data (and measure data): the notion of renormalized solution which have been introduced by DiPerna and Lions in [16] for first order equations. This notion has been adapted to elliptic equations ([27], see also [13] for measure

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data) and to parabolic equations with L^1 or measure data ([8, 9, 29]). Renormalized solutions allow one to obtain in general existence, stability and uniqueness results when the data belongs to L^1. Nonlinear version of (1.1) with $p-$growth operator is adressed in [14, 15]: existence and uniqueness are proved (under more restrictive assumptions). A large number of papers is devoted to renormalized or entropy solutions of parabolic problems with L^1 data: see, among others [7, 10, 18, 30, 31].

As far as finite volume schemes are concerned we refer to the book [19] in which the authors study finite volume approximation for linear or nonlinear elliptic, parabolic and hyperbolic equations. As in the continuous case one of the main obstacle for time dependent equations is to derive some compactness (and then the pointwise convergence) for the approximate solution which is necessary (but not sufficient) to pass to the limit. One of the methods is to derive with the equation an estimate on differences of time and space translates (see [19] and [1, 2, 3, 4, 20] in the context of degenerate parabolic equations). Another method is to use the compactness result proved in [22] which is a discrete version of Aubin-Lions Lemma (see [33]).

When $f \in L^1$, the convergence of a finite volume scheme for the linear version of (1.1) (i.e. $\lambda = 1$) and when $\boldsymbol{v}$ is more regular ($\boldsymbol{v} \in \mathcal{C}^1(\bar{\Omega} \times [0,T])$ and $\mathrm{div}(\boldsymbol{v}) = 0$) has been studied in [21] (see also [17] for the stationary case). They proved that the solution of the scheme for equation $\partial_t u + \mathrm{div}(u\boldsymbol{v}) - \Delta u = f$ converges to a weak solution of this equation in the sense

$$
\begin{cases}
u \in \bigcap\limits_{1 \le q < (d+2)/(d+1)} L^q(0,T; W_0^{1,q}(\Omega)), \\
\int_Q u\, \partial_t \varphi \, dx\, dt + \int_Q \nabla u \cdot \nabla \varphi \, dx\, dt - \int_Q u \boldsymbol{v} \cdot \nabla \varphi \, dx\, dt \\
\qquad\qquad = \int_Q f\, \varphi \, dx\, dt + \int_\Omega u_0 \varphi(0)\, dx \ \forall \varphi \in \mathcal{C}_c^\infty(\Omega \times [0,T)).
\end{cases} \tag{1.2}
$$

The case where λ is a continuous and bounded function is dealt in [22, 24], where the authors prove the convergence of the solution of the finite volume scheme to a weak solution (in the sense of (1.2)) of (1.1). However due to the low regularity of u the uniqueness of such a solution in the sense of (1.2) remains an open question even in the linear case.

Using the notion of renormalized solution, the existence of a solution of (1.1) is established in [15] for a non linear version with p-growth condition while the uniqueness is dealt in [14].

The novelty of the present paper is to mix the tools developed for finite volume schemes ([19, 22, 21]) with the strategy to deal with the existence of a renormalized solution for parabolic equations with L^1 data ([9, 10]). The main result (see Theorem 2.8) is that the finite volume approximate solution of (1.1) converges (up to a subsequence) to a renormalized solution of (1.1). Moreover when λ is a locally Lipschitz continuous or when $\boldsymbol{v}$ does not depend on t, since the renormalized solution is unique we deduce that the whole sequence converges to the unique renormalized solution of (1.1). To our knowledge and with respect to the above mentioned references the result is new (in particular for nonlinear parabolic equation with L^1 data). Let us mention that such an analysis is performed in [25] for the stationnary case, namely the equation $-\mathrm{div}(\lambda(u)\nabla u) + \mathrm{div}(\boldsymbol{v}u) = f$ with $f \in L^1(\Omega)$, $\boldsymbol{v} \in L^p(\Omega)^d$, $2 < p < +\infty$ if $d = 2$, $p = d$ if $d \ge 3$ and with Dirichlet boundary conditions. It is worth noting that time dependent equations give additional difficulties due to the lack of regularity of $\frac{\partial u}{\partial t}$. Moreover in the case where λ is not bounded there is no reason to have a solution in the sense of distributions for $f \in L^1(Q)$. This is why the framework of renormalized solutions appears to be more convenient.

Recall (see [9, 10]) that a renormalized solution of (1.1) is a measurable function u defined from Q to $\mathbb{R}$ such that

$$
u \in L^\infty(0,T; L^1(\Omega)), \tag{1.3}
$$

$$\forall k > 0,\ T_k(u) \in L^2(0,T;H_0^1(\Omega)), \tag{1.4}$$

$$\lim_{k\to+\infty} \frac{1}{k} \int_0^T \int_\Omega \lambda(u)|\nabla T_k(u)|^2 \, dx\, dt = 0, \tag{1.5}$$

and for every functions $\xi \in \mathcal{C}_c^\infty(\Omega \times [0,T))$ and $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support and $S'(u)\xi \in L^2(0,T;H_0^1(\Omega))$.

$$\begin{aligned} &- \int_0^T \int_\Omega \xi_t S(u)\, dx\, dt - \int_\Omega \xi(x,0) S(u^0)\, dx \\ &+ \int_0^T \int_\Omega \lambda(u) \nabla u \cdot \nabla(\xi S'(u))\, dx\, dt - \int_0^T \int_\Omega u\, \boldsymbol{v} \cdot \nabla(\xi S'(u))\, dx\, dt \\ &= \int_0^T \int_\Omega f\, \xi S'(u)\, dx\, dt. \end{aligned} \tag{1.6}$$

Here T_k is the truncate function at height k:

$$T_k(s) = \begin{cases} s \text{ if } |s| \le k, \\ \operatorname{sgn}(s) k \text{ if } |s| > k. \end{cases}$$

The paper is organized as follows. In Section 2, we present the finite volume scheme, the time discretization and the properties of the discrete gradient. Section 3 is devoted to prove several estimates, especially an estimate on the time derivative and the discrete equivalent to (1.5) which is crucial to pass to the limit in the scheme. In Section 4, we prove the convergence of the finite volume scheme, passing to the limit and using both a density argument and Steklov averages.

2. The finite volume scheme and the time discretization

Let us define the admissibility of the mesh in the present work.

Definition 2.1. (Admissible mesh) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. An admissible finite volume mesh of Ω is given by a finite partition $\mathcal{T}$ of Ω in polygonal convex sets, called the "control volumes", by a finite family $\mathcal{E}$ of disjoint subsets of $\bar\Omega$ contained in affine hyperplanes, called the "edges", and by a family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ of points in Ω such that*

- *each $\sigma \in \mathcal{E}$ is a non-empty open subset of ∂K for some $K \in \mathcal{T}$,*
- *by denoting $\mathcal{E}(K) = \{\sigma \in \mathcal{E}\,;\ \sigma \in \partial K\}$, one has $\partial K = \cup_{\sigma\in\mathcal{E}(K)} \bar\sigma$ for all $K \in \mathcal{T}$,*
- *for all $K \neq L$ in $\mathcal{T}$, either the $(d-1)$-dimensional measure of $\bar K \cap \bar L$ is null, or $\bar K \cap \bar L = \bar\sigma$ for some $\sigma \in \mathcal{E}$, that we denote then $\sigma = K|L$,*
- *for all $K \in \mathcal{T}$, x_K is in the interior of K,*
- *for all $\sigma = K|L \in \mathcal{E}$, the line (x_K, x_L) intersects and is orthogonal to σ,*
- *for all $\sigma \in \mathcal{E}$, $\sigma \subset \partial\Omega \cap \partial K$, the line which is orthogonal to σ and going through x_K intersects σ.*

We denote by $|K|$ (resp. $|\sigma|$) the Lebesgue measure of $K \in \mathcal{T}$ (resp. $\sigma \in \mathcal{E}$). The unit normal to $\sigma \in \mathcal{E}(K)$ outward to K is denoted by $\eta_{K,\sigma}$. $\mathcal{E}_{int}$ (resp. $\mathcal{E}_{ext}$) is defined as the set of interior (resp. boundary) edges.
For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$ we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. For any $\sigma \in \mathcal{E}$, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{int}$ (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K)$.
The size of the mesh, denoted by $h_\mathcal{T}$, is defined by $h_\mathcal{T} = \sup_{K\in\mathcal{T}} \operatorname{diam}(K)$.

In the continuous case, Poincaré and Sobolev inequalities are necessary to find a priori estimates, and then to solve the problem. Since, in the discrete case we also need to find such estimates, we will need discrete versions of these two inequalities (see [12]). Before that, we have to define the discrete $W_0^{1,q}$ norm.

Definition 2.2. (Discrete $W_0^{1,q}$ norm) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d \geq 2$, and let $\mathcal{T}$ be an admissible mesh. Define $X(\mathcal{T})$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh. For $v_\mathcal{T} \in X(\mathcal{T})$ and $q \in [1, +\infty[$, we define the discrete $W_0^{1,q}$ norm by*

$$\|v_\mathcal{T}\|_{1,q,\mathcal{T}}^q = \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} |\sigma| d_\sigma \left| \frac{v_K - v_L}{d_\sigma} \right|^q + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}(K)}} |\sigma| d_\sigma \left| \frac{v_K}{d_\sigma} \right|^q$$

where v_K denotes the value taken by v on the control volume K.

Let us also define

$$\|v_\mathcal{T}\|_{1,\infty,\mathcal{T}} = \max \Big\{ \{ \frac{|v_K - v_L|}{d_\sigma},\ \sigma \in \mathcal{E}_{int},\ \sigma = K|L \} \cup \{ \frac{|v_K|}{d_\sigma},\ \sigma \in \mathcal{E}_{ext},\ \sigma \in \mathcal{E}(K) \} \cup \{ |v_K|,\ K \in \mathcal{T} \} \Big\}.$$

Definition 2.3. (Discrete dual norm) *Let $q > 1$ and let q' be given by $\frac{1}{q} + \frac{1}{q'} = 1$ if q is finite and $q' = 1$ if $q = +\infty$. We define the dual norm by*

$$\|v\|_{-1,q',\mathcal{T}} = \sup_{\substack{w \in X(\mathcal{T}) \\ w \neq 0}} \frac{1}{\|w\|_{1,q,\mathcal{T}}} \int_\Omega v\, w\, dx.$$

Proposition 2.4. (Discrete Poincaré inequality) *Let $\mathcal{T}$ be an admissible mesh and $v_\mathcal{T} \in X(\mathcal{T})$. Then, if $1 \leq q \leq 2$,*

$$\|v_\mathcal{T}\|_{L^q(\Omega)} \leq \mathrm{diam}(\Omega) \|v_\mathcal{T}\|_{1,q,\mathcal{T}}.$$

Proposition 2.5. (Discrete Sobolev inequality) *Let $1 \leq q \leq 2$, $\mathcal{T}$ be an admissible mesh and $\xi > 0$ satisfying*

$$\text{for all } K \in \mathcal{T} \text{ and all } \sigma \in \mathcal{E}(K),\ d_{K,\sigma} \geq \xi d_\sigma.$$

Then, with $q* = \frac{dq}{d-q}$ if $q < d$ and $q* < \infty$ if $q = d = 2$, there exists $C > 0$ only depending on $(\Omega,\ q,\ q*,\ \xi)$ such that, for all $v_\mathcal{T} \in X(\mathcal{T})$,

$$\|v_\mathcal{T}\|_{L^{q*}(\Omega)} \leq C \|v_\mathcal{T}\|_{1,q,\mathcal{T}}.$$

As far as the time dicretization is concerned, we suppose given a uniform partition of the interval $[0,T)$ such that $[0,T] = \cup_{1 \leq i \leq N} [t^{i-1}, t^i]$ where $t^i = i\delta t$ and $\delta t = \frac{T}{N}$. We define $X_\mathcal{D}$ the space of piecewise constant functions over each $K \times (t^{i-1}, t^i)$ for $K \in \mathcal{T}$ and $1 \leq i \leq N$. To each sequence $(v_\mathcal{T}^i)_{1 \leq i \leq N}$ of functions of $X(\mathcal{T})$, we associate the function $v_\mathcal{D} \in X_\mathcal{D}$ defined by :

$$v_\mathcal{D}(x,t) = v_\mathcal{T}^i(x)\ \forall x \in \Omega,\ \forall t \in (t^{i-1}, t^i),\ 1 \leq i \leq N,$$

$$\text{i.e. } v_\mathcal{D}(x,t) = v_K^i\ \forall x \in K,\ K \in \mathcal{T},\ \forall t \in (t^{i-1}, t^i),\ 1 \leq i \leq N.$$

We now define the discrete time derivative of $v_\mathcal{D}$ by

$$\partial_{t,\mathcal{D}} v^i(x) = \frac{v^i(x) - v^{i-1}(x)}{\delta t} \quad \forall 1 \leq i \leq N,$$

$$\text{i.e. } \partial_t v_K^i = \frac{v_K^i - v_K^{i-1}}{\delta t} \quad \forall K \in \mathcal{T},\ \forall 1 \leq i \leq N.$$

Before writing the finite volume scheme, let us recall the definition of the discrete finite volume gradient (see [24]).

Definition 2.6. (Discrete finite volume gradient) *For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$, we define the volume $D_{K,\sigma}$ as the cone of basis σ and of opposite vertex x_K. Then, we define the "diamond-cell" D_σ (see Figure 1) by*

$$D_\sigma = D_{K,\sigma} \cup D_{L,\sigma} \qquad \text{if } \sigma = K|L \in \mathcal{E}_{int},$$
$$D_\sigma = D_{K,\sigma} \qquad \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K).$$

For any $v_\mathcal{D} \in X_\mathcal{D}$, we define the discrete gradient $\nabla_\mathcal{D} v_\mathcal{D}$ as the piecewise constant function over each $D_\sigma \times (t^{i-1}, t^i)$ and given by

$$\forall \sigma \in \mathcal{E}_{int},\ \sigma = K|L, \forall 1 \le i \le N,$$
$$\nabla_\mathcal{D} v_\mathcal{D}(x,t) = d\frac{v_L^i - v_K^i}{d_\sigma}\eta_{K,\sigma},\ \forall x \in D_\sigma,\ \forall t \in (t^{i-1}, t^i],$$
$$\forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K),\ \forall 1 \le i \le N,$$
$$\nabla_\mathcal{D} v_\mathcal{D}(x,t) = d\frac{0 - v_K^i}{d_\sigma}\eta_{K,\sigma},\ \forall x \in D_\sigma,\ \forall t \in (t^{i-1}, t^i].$$

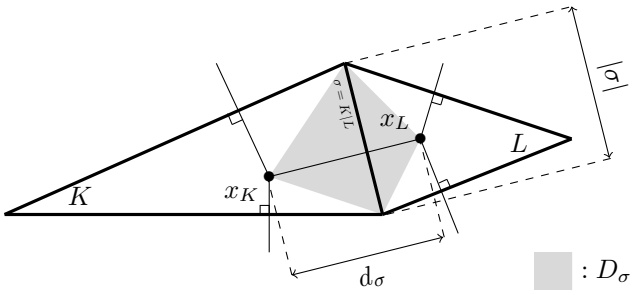

FIGURE 1. The diamond D_σ

Lemma 2.7. (Weak convergence of the finite volume gradient) *Let $(\mathcal{T}_m)_{m\ge1}$ be a sequence of admissible meshes such that there exists $\xi > 0$ satisfying*

$$\text{for all } m \ge 1 \text{ for all } K \in \mathcal{T}_m \text{ and all } \sigma \in \mathcal{E}(K),\ d_{K,\sigma} \ge \xi d_\sigma,$$

and such that $h_{\mathcal{T}_m} \to 0$ and let δt_m be a sequence of time steps such that $\delta t_m \to 0$. Let $v_{\mathcal{D}_m} \in X_{\mathcal{D}_m}$ and let us assume that there exists $C > 0$ such that $\sum_{i=1}^N \delta t_m \|v^i_{\mathcal{T}_m}\|^2_{1,2,\mathcal{T}_m} \le C$, and that $v_{\mathcal{D}_m}$ converges weakly in $L^1(Q)$ to $v \in L^2(0,T;H_0^1(\Omega))$. Then $\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}$ converges to ∇v weakly in $L^2(Q)^d$.

The proof of this lemma is an easy adaptation of the proof done in [24] in the elliptic case. The only difference is the presence of a time integral.

Let $\mathcal{T}$ be an admissible mesh, we are now in a position to give the finite volume discretization of (1.1). For $1 \le i \le N$, $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}(K)$, we define

$$\boldsymbol{v}^i_{K,\sigma} = \frac{1}{\delta t |D_\sigma|}\int_{t_{i-1}}^{t_i}\int_{D_\sigma} \boldsymbol{v}\cdot\eta_{K,\sigma}\,dx\,dt, \tag{2.1}$$

$$f^i_K = \frac{1}{\delta t |K|}\int_{t_{i-1}}^{t_i}\int_{K} f\,dx\,dt, \tag{2.2}$$

$$u^0_K = \frac{1}{|K|}\int_K u^0(x)\,dx. \tag{2.3}$$

By defining $f_\mathcal{D} = (f^i_\mathcal{T})_{1\le i\le N}$, we observe that if f belongs to $L^1(Q)$ standard analysis arguments allow one to obtain that $f_\mathcal{D}$ converges to f in $L^1(Q)$ as δt and $h_\mathcal{T}$ go to zero.

We also need to define the upstream choice of u by:

$$\forall \sigma = K|L \in \mathcal{E}_{int},\ u^i_{\sigma,+} = u^i_K \qquad \text{if } \boldsymbol{v}^i_{K,\sigma} \geq 0,\ u^i_{\sigma,+} = u^i_L \qquad \text{otherwise,} \tag{2.4}$$

$$\forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K),\ u^i_{\sigma,+} = u^i_K \qquad \text{if } \boldsymbol{v}^i_{K,\sigma} \geq 0,\ u^i_{\sigma,+} = 0 \qquad \text{otherwise.} \tag{2.5}$$

We denote $u^i_{\sigma,-}$ the downstream choice of u, i.e. $u^i_{\sigma,-}$ is such that $\{u^i_{\sigma,+}, u^i_{\sigma,-}\} = \{u^i_K, u^i_L\}$ (with $u^i_L = 0$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K)$).

For all $1 \leq i \leq N$ we define $\lambda(u)^i_\sigma$ in the following way

$$\forall \sigma = K|L \in \mathcal{E}_{int},\ \min[\lambda(u^i_K), \lambda(u^i_L)] \leq \lambda(u)^i_\sigma \leq \max[\lambda(u^i_K), \lambda(u^i_L)], \tag{2.6}$$

$$\forall \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K),\ \lambda(u)^i_\sigma = \lambda(u^i_K), \tag{2.7}$$

where $\lambda(u)^i_\sigma$ is for example the mean value of $\lambda(u^i_K)$ and $\lambda(u^i_L)$.

We are now in a position to write the scheme as the following set of equations:

For all $1 \leq i \leq N$ and for all $K \in \mathcal{T}$

$$|K|\frac{u^i_K - u^{i-1}_K}{\delta t} + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma (u^i_K - u^i_L) + \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{v}^i_{K,\sigma} u^i_{\sigma,+} = |K| f^i_K, \tag{2.8}$$

with $u^i_L = 0$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}(K)$. It is worth noting that the above implicit scheme is nonlinear. The existence of a solution of (2.8) will be studied later in Section 4 while Section 3 is devoted to a priori estimates.

Let us now state the main result of this paper. We assume in the whole paper that λ is a continuous function such that

$$\lambda(u) \geq \mu > 0 \text{ with } \mu \text{ a positive real number,} \tag{2.9}$$

$$\boldsymbol{v} \in (L^{d+2}(Q))^d, \tag{2.10}$$

$$u^0 \in L^1(\Omega), \tag{2.11}$$

$$f \in L^1(Q). \tag{2.12}$$

Theorem 2.8. *If $\mathcal{T}$ is an admissible mesh and δt a time step, then there exists at least one solution to (2.8).*
If $(\mathcal{T}_m)_{m\geq 1}$ is a sequence of admissible meshes such that there exists $\xi > 0$ satisfying

$$\text{for all } m \geq 1 \text{ for all } K \in \mathcal{T}_m \text{ and all } \sigma \in \mathcal{E}(K),\ d_{K,\sigma} \geq \xi d_\sigma, \tag{2.13}$$

and such that $h_{\mathcal{T}_m} \to 0$ and if $(\delta t_m)_{m\geq 1}$ is a sequence of time steps such that $\delta t_m \to 0$, then if $u_{\mathcal{D}_m} = (u^i_{\mathcal{T}_m})_{1\leq i\leq N_m}$ is one solution to (2.8) with $\mathcal{T} = \mathcal{T}_m$, $\delta t = \delta t_m$ and $N = N_m$, up to a subsequence, $u_{\mathcal{D}_m}$ converges to u in the sense that $u_{\mathcal{D}_m}$ converges to u almost everywhere in Q, where u is a renormalized solution of (1.1) and for all $n > 0$, $\nabla_{\mathcal{D}} T_n(u_{\mathcal{D}_m})$ converges weakly to $\nabla T_n(u)$ in $L^2(Q)^d$.

Moreover if

(H1): *λ is a local Lipschitz continuous function, or*
(H2): *$\boldsymbol{v}(x,t) = \boldsymbol{v}(x)$ is independent of t*

then the whole sequence $u_{\mathcal{D}_m}$ converges almost everywhere to u where u is the unique renormalized solution of (1.1).

For parabolic equations with L^1 data the classical results of convergence for finite volume schemes ([21, 22]) establish a convergence to a solution on the sense of distributions which is not unique in general (see [31]). This result presents the interest to obtain the convergence of the approximate solution to the unique renormalized solution. Let us mention that the proof of this theorem is not a straightforward adaptation of the result obtained in [25] for the elliptic case. We

first show a discrete log-type estimate; the proof of which being classical in the elliptic case (see [17]). We then have to establish a discrete version of the decay of the energy (1.5) which is necessary to pass to the limit in the discrete renormalized formulation. In a third time we need a compactness result. In [21, 22] the authors derive estimates on $u_{\mathcal{D}}$ and apply a discrete version of the Aubin-Simon Lemma to extract a convergent subsequence. Here, since the data is not regular enough and since λ is not bounded, we derive estimates on $H_k(u_{\mathcal{D}})$, where H_k is a regularized version of T_k, instead of $u_{\mathcal{D}}$ itself. Then using the strategy developed in [5] we extract a convergent subsequence. In Section 4 we prove the theorem, adapting to the discrete case the techniques used to deal with the existence of a renormalized solution.

3. Estimates

This section is devoted to derive estimates on the sequence $u_{\mathcal{D}}$. The very first estimate, $u_{\mathcal{D}}$ bounded in $L^\infty(0,T;L^1(\Omega))$, is proved in Proposition 3.1. Since we do not assume that $\mathrm{div}\boldsymbol{v} = 0$ we use a different method than the one developed in [21]. A log-type estimate is first obtained in the same spirit that in the elliptic case (see [17]) which in turn gives the $L^\infty(L^1)$ bound. We then prove in Proposition 3.3 uniform estimates on the truncated discrete energy which are discrete versions of the continuous renormalized case. Using Proposition 3.3, the discrete equation and the discrete compactness result proved in [22] we are then able to obtain in Proposition 3.8 the pointwise convergence of $u_{\mathcal{D}}$ which is crucial to perform the convergence analysis.

Proposition 3.1. *Let $\mathcal{T}$ be an admissible mesh and let δt be a time step. If $u_{\mathcal{D}} = (u^i_{\mathcal{T}})_{1\le i\le N}$ is a solution to (2.8), then*

$$\sum_{i=1}^{N} \delta t \| \ln(1+|u^i_{\mathcal{T}}|)\|^2_{1,2,\mathcal{T}} \le C. \tag{3.1}$$

and

$$\|u_{\mathcal{D}}\|_{L^\infty(0,T;L^1(\Omega))} \le C. \tag{3.2}$$

Proof. We only sketch the ideas of the proof. Estimate (3.1) is proved in [17] in the elliptic case. Adapting this proof to the parabolic case and using the convexity argument of [21] we are in a position to show that for all $1 \le M \le N$

$$\sum_{i=1}^{M} \delta t \| \ln(1+|u^i_{\mathcal{T}}|)\|^2_{1,2,\mathcal{T}} + \|u^M_{\mathcal{T}}\|_{L^1(\Omega)} \le C.$$

Since this inequality provides a bound of $u_{\mathcal{D}}$ in $L^\infty(0,T;L^1(\Omega))$, we get (3.1) and (3.2). □

Corollary 3.2. *Let $\mathcal{T}$ be an admissible mesh and let δt be a time step. If $u_{\mathcal{D}} = (u^i_{\mathcal{T}})_{1\le i\le N}$ is a solution to (2.8) and, for $n > 0$, $E_n = \{(x,t) \in \Omega\times(0,T); |u_{\mathcal{D}}(x,t)| \ge n\}$, then there exists $C > 0$ only depending on $(\Omega, \boldsymbol{v}, f, d, p)$ such that*

$$|E_n| \le \frac{C}{(\ln(1+n))^2}. \tag{3.3}$$

Proof. Using Proposition 3.1 and the discrete Poincaré inequality, we get that

$$\sum_{i=1}^{N} \delta t \, \| \ln(1+|u^i_{\mathcal{T}}|)\|^2_{L^2(\Omega)} \le C \quad \text{i.e.} \quad \| \ln(1+|u_{\mathcal{D}}|)\|^2_{L^2(0,T;L^2(\Omega))} \le C,$$

where C only depends on $\mathrm{diam}(\Omega)$, d, $\|f\|_{L^1(Q)}$, $\|u^0\|_{L^1(\Omega)}$ and $\|\boldsymbol{v}\|_{L^2(Q)^d}$. Thus,

$$(\ln(1+n))^2 |E_n| \le C.$$ □

In the following proposition we prove a uniform estimate on the truncate energy of $u_{\mathcal{D}}$ (see (3.5)) which is crucial to pass to the limit in the approximate problem. We explicitely observe that (3.5) is the discrete version of (1.5) which is imposed in the definition of the renormalized solution. In the elliptic case, in order to prove such estimate, we use the convergence of $u_{\mathcal{D}}$ (see [25]) which we do not have yet. We will see that Corollary 3.2 and an equi-integrability argument are sufficient to obtain the desired estimate. Since the equation contains the term $\mathrm{div}(\boldsymbol{v}\, u)$ we also have to uniformly control the discrete version of $\frac{1}{n}\int_0^T\int_\Omega \boldsymbol{v}\, u\, \nabla T_n(u)\, dx\, dt$ which is stated in (3.6). Estimate (3.7) is a control of a rest which will be needed to pass to the limit.

Proposition 3.3. (Discrete estimate on the energy)
Let $(\mathcal{T}_m)_{m\geq 1}$ be a sequence of admissible meshes such that there exists $\xi > 0$ satisfying

$$\text{for all } m \geq 1 \text{ for all } K \in \mathcal{T}_m \text{ and all } \sigma \in \mathcal{E}(K),\ d_{K,\sigma} \geq \xi d_\sigma,$$

and $(\delta t_m)_{m\geq 1}$ be a sequence of time steps.
If $u_{\mathcal{D}_m} = (u^i_{\mathcal{T}_m})_{1\leq i\leq N}$ is a solution to (2.8), then

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}_m}\to 0\\ \delta t_m\to 0}}} \frac{1}{n} \sup_{1\leq i\leq N} \|u^i_{\mathcal{T}} T_n(u^i_{\mathcal{T}})\|_{L^1(\Omega)} = 0 \tag{3.4}$$

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}_m}\to 0\\ \delta t_m\to 0}}} \frac{1}{n} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma (u^i_K - u^i_L)(T_n(u^i_K) - T_n(u^i_L)) = 0, \tag{3.5}$$

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}_m}\to 0\\ \delta t_m\to 0}}} \frac{1}{n} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\, |\boldsymbol{v}^i_{K,\sigma}|\, |u^i_{\sigma,+}|\, |T_n(u^i_{\sigma,+}) - T_n(u^i_{\sigma,-})| = 0, \tag{3.6}$$

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}_m}\to 0\\ \delta t_m\to 0}}} \frac{1}{n} \sum_{i=1}^N \sum_{K\in\mathcal{T}} |K| \int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r) T_n'(r)\, dr = 0, \tag{3.7}$$

where $u^i_L = 0$ if $\sigma \in \mathcal{E}_{ext}$

Proof. We first establish (3.5). Let $\mathcal{T}$ be an admissible mesh and δt a time step and let $u_{\mathcal{D}}$ be a solution of (2.8). Multiplying each equation of the scheme by $\frac{T_n(u^i_K)}{n}$, summing on $K \in \mathcal{T}$, gathering by edges and summing over the discrete times from 1 to M where $1 \leq M \leq N$ lead to

$$T_1^M + T_2^M + T_3^M = T_4^M \tag{3.8}$$

with

$$T_1^M = \frac{1}{n} \sum_{i=1}^M \delta t \sum_{K\in\mathcal{T}} |K| \frac{u^i_K - u^{i-1}_K}{\delta t} T_n(u^i_K), \tag{3.9}$$

$$T_2^M = \frac{1}{n} \sum_{i=1}^M \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma (u^i_K - u^i_L)(T_n(u^i_K) - T_n(u^i_L)), \tag{3.10}$$

$$T_3^M = \frac{1}{n} \sum_{i=1}^M \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}^i_{K,\sigma} u^i_{\sigma,+} (T_n(u^i_K) - T_n(u^i_L)), \tag{3.11}$$

$$T_4^M = \frac{1}{n} \sum_{i=1}^M \delta t \sum_{K\in\mathcal{T}} |K| f^i_K T_n(u^i_K). \tag{3.12}$$

We define the function $\widetilde{T}_n$ by $\widetilde{T}_n(r) = \int_0^r T_n(s)\,ds$ for all $r \in \mathbb{R}$, which satisfies the following properties (which are used in the sequel):

$$\widetilde{T}_n \text{ is a convex Lipschitz continuous function,} \tag{3.13}$$

$$\forall r \in \mathbb{R},\ n|r| \geq \widetilde{T}_n(r) \geq \frac{1}{2} r T_n(r) \geq \frac{1}{2}|T_n(r)|^2, \tag{3.14}$$

$$\widetilde{T}_n(b) = \widetilde{T}_n(a) + (b-a)T_n(a) + \int_a^b (b-r)T_n'(r)dr, \quad \forall a, b \in \mathbb{R}. \tag{3.15}$$

As far as T_1^M is concerned, thanks to (3.15) we find that

$$T_1^M = \frac{1}{n}\sum_{K\in\mathcal{T}} |K|\widetilde{T}_n(u_K^M) - \frac{1}{n}\sum_{K\in\mathcal{T}} |K|\widetilde{T}_n(u_K^0) + \frac{1}{n}\sum_{i=1}^{M}\sum_{K\in\mathcal{T}} |K| \int_{u_K^i}^{u_K^{i-1}} (u_K^{i-1} - r)T_n'(r)\,dr.$$

Since

$$\widetilde{T}_n(u_K^M) \geq \frac{1}{2} u_K^M T_n(u_K^M) \geq \frac{1}{2}|T_n(u_K^M)|^2$$

we have

$$\frac{1}{n}\sum_{K\in\mathcal{T}} |K|\widetilde{T}_n(u_K^M) \geq \frac{1}{2n}\|u_{\mathcal{T}}^M T_n(u_{\mathcal{T}}^M)\|_{L^1(\Omega)}. \tag{3.16}$$

Moreover the convexity of $\widetilde{T}_n$ ($T_n' \geq 0$ a.e. in $\mathbb{R}$) yields that

$$\frac{1}{n}\sum_{i=1}^{M}\sum_{K\in\mathcal{T}} |K| \int_{u_K^i}^{u_K^{i-1}} (u_K^{i-1} - r)T_n'(r)\,dr \geq 0. \tag{3.17}$$

We now study T_3^M. If $\sigma \in \mathcal{E}$, by the definition (2.4) of $u_{\sigma,+}^i$ and recalling that $u_{\sigma,-}^i$ is the downstream choice of $u_{\mathcal{T}}^i$, if $\boldsymbol{v}_{K,\sigma}^i \geq 0$ we have

$$\boldsymbol{v}_{K,\sigma}^i(T_n(u_K^i) - T_n(u_L^i)) = \boldsymbol{v}_{K,\sigma}^i(T_n(u_{\sigma,+}^i) - T_n(u_{\sigma,-}^i)),$$

and if $\boldsymbol{v}_{K,\sigma}^i < 0$ we have

$$\boldsymbol{v}_{K,\sigma}^i(T_n(u_K^i) - T_n(u_L^i)) = -\boldsymbol{v}_{K,\sigma}^i(T_n(u_{\sigma,+}^i) - T_n(u_{\sigma,-}^i)).$$

It follows that T_3^M can be rewritten as

$$-T_3^M = \frac{1}{n}\sum_{i=1}^{M} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\,|\boldsymbol{v}_{K,\sigma}^i|\,u_{\sigma,+}^i(T_n(u_{\sigma,-}^i) - T_n(u_{\sigma,+}^i)).$$

As in [17], we define for all $1 \leq i \leq N$ the subsets $\mathcal{A}^i$ of edges by

$$\mathcal{A}^i = \{\sigma \in \mathcal{E}\,;\ u_{\sigma,+}^i \geq u_{\sigma,-}^i,\ u_{\sigma,+}^i < 0\} \cup \{\sigma \in \mathcal{E}\,;\ u_{\sigma,+}^i < u_{\sigma,-}^i,\ u_{\sigma,+}^i \geq 0\}.$$

Since T_n is nondecreasing we have

$$\forall \sigma \notin \mathcal{A}^i,\ u_{\sigma,+}^i(T_n(u_{\sigma,-}^i) - T_n(u_{\sigma,+}^i)) \leq 0,$$
$$\forall \sigma \in \mathcal{A}^i,\ u_{\sigma,+}^i(T_n(u_{\sigma,-}^i) - T_n(u_{\sigma,+}^i)) \geq 0.$$

It allows one to split $-T_3^M$ into two sums, the first one with non negative terms and the second one with non positive terms. It follows that for any $1 \leq M \leq N$

$$-T_3^M \leq \frac{1}{n}\sum_{i=1}^{M} \delta t \sum_{\sigma\in\mathcal{A}^i} |\sigma|\,|\boldsymbol{v}_{K,\sigma}^i|\,u_{\sigma,+}^i(T_n(u_{\sigma,-}^i) - T_n(u_{\sigma,+}^i)). \tag{3.18}$$

In view of (3.8), (3.16), (3.17) and (3.18) we obtain for any $1 \leq M \leq N$

$$0 \leq \frac{1}{2n}\|u_{\mathcal{T}}^M T_n(u_{\mathcal{T}}^M)\|_{L^1(\Omega)} + T_{1,3}'^M + T_2^M \leq T_4'^M + T_{1,2}' - T_3'^M, \tag{3.19}$$

where

$$T_{1,2}^{'} = \frac{1}{n} \sum_{K\in\mathcal{T}} |K| \widetilde{T}_n(u_K^0), \tag{3.20}$$

$$T_{1,3}^{'M} = \frac{1}{n} \sum_{i=1}^{M} \sum_{K\in\mathcal{T}} |K| \int_{u_K^i}^{u_K^{i-1}} (u_K^{i-1} - r) T_n'(r)\, dr, \tag{3.21}$$

$$-T_3^{'M} = \frac{1}{n} \sum_{i=1}^{M} \delta t \sum_{\sigma\in A^i} |\sigma|\, |\boldsymbol{v}_{K,\sigma}^i| u_{\sigma,+}^i (T_n(u_{\sigma,-}^i) - T_n(u_{\sigma,+}^i)), \tag{3.22}$$

$$T_4^{'M} = \frac{1}{n} \sum_{i=1}^{M} \delta t \sum_{K\in\mathcal{T}} |K|\, |f_K^i|\, |T_n(u_K^i)|. \tag{3.23}$$

Since $T_{1,3}^{'M}$, T_2^M, $T_4^{'M}$ and $-T_3^{'M}$ are non negative and nondecreasing with respect to M we obtain that

$$T_{1,1}' = \frac{1}{2n} \sup_{1\le i\le N} \|u_{\mathcal{T}}^i T_n(u_{\mathcal{T}}^i)\|_{L^1(\Omega)} \le T_4^{'N} + T_{1,2}^{'} - T_3^{'N}.$$

Writing (3.19) for $M = N$ and summing to the previous inequality leads to

$$0 \le T_{1,1}' + T_{1,3}^{'N} + T_2^N \le 2T_4^{'N} + 2T_{1,2}^{'} - 2T_3^{'N}, \tag{3.24}$$

We now pass to the limit in (3.24) as $n \to +\infty$, by studying the terms $T_4^{'N}$, $T_{1,2}^{'}$ and $T_3^{'N}$. Observe that dealing with $T_3^{'N}$ is more intricate and requires as in the continuous case a splitting argument (see (3.27)).

Due to the definition of $u_{\mathcal{D}}$ we have

$$T_4^{'N} = \frac{1}{n} \int_0^T \int_\Omega |f_{\mathcal{D}}|\, |T_n(u_{\mathcal{D}})|\, dx\, dt,$$

from which it follows $\forall k > 0$

$$\begin{aligned} T_4^{'N} &= \frac{1}{n} \int_{E_k} |f_{\mathcal{D}}|\, |T_n(u_{\mathcal{D}})| + \frac{1}{n} \int_{Q\setminus E_k} |f_{\mathcal{D}}|\, |T_n(u_{\mathcal{D}})| \\ &\le \int_{E_k} |f_{\mathcal{D}}| + \frac{k}{n} \int_{Q\setminus E_k} |f_{\mathcal{D}}|. \end{aligned}$$

Recalling that $f_{\mathcal{D}}$ converges strongly to f in $L^1(Q)$, Corollary 3.2 and an equi-integrability argument lead to

$$\lim_{n\to+\infty} \lim_{\substack{h_{\mathcal{T}}\to 0 \\ \delta t\to 0}} T_4^{'N} = 0. \tag{3.25}$$

As far as $T_{1,2}'$ is concerned since $(u_K^0)_{K\in\mathcal{T}}$ converges to u^0 in $L^1(\Omega)$ and since $\widetilde{T}_n$ is a Lipschitz continuous function, we have

$$\lim_{\substack{h_{\mathcal{T}}\to 0 \\ \delta t\to 0}} T_{1,2}' = \frac{1}{n} \int_\Omega \widetilde{T}_n(u^0)\, dx.$$

Recalling that $\frac{1}{n}\widetilde{T}_n(u^0)$ goes to 0 almost everywhere in Ω while $|\frac{1}{n}\widetilde{T}_n(u^0)| \le |u^0| \in L^1(\Omega)$ the Lebesgue theorem implies that

$$\lim_{n\to+\infty} \lim_{\substack{h_{\mathcal{T}}\to 0 \\ \delta t\to 0}} T_{1,2}' = 0. \tag{3.26}$$

We now turn to $T_3^{'N}$ and we prove that

$$-T_3^{'N} \le \frac{1}{4} T_2^N + \frac{1}{4} T_{1,1}' + R(n) \tag{3.27}$$

with R verifying $\lim_{n\to+\infty} R(n) = 0$.

Because for any $\sigma \in \mathcal{A}^i$, $u^i_{\sigma,+}$ and $u^i_{\sigma,-}$ have the same sign and $|u^i_{\sigma,+}| \geq n$ implies $|u^i_{\sigma,-}| \geq n$, we can split T_3' into a sum on $\{|u^i_{\sigma,+}| < r\}$ and a sum on $\{r \leq |u^i_{\sigma,+}| \leq n\}$ where $r > 0$ will be chosen later. It leads to $-T_3'^N \leq I_1 + I_2$ with

$$I_1 = \frac{1}{n}\sum_{i=1}^{N} \delta t \sum_{\substack{\sigma\in\mathcal{A}^i \\ |u^i_{\sigma,+}|<r}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,u^i_{\sigma,+}(T_n(u^i_{\sigma,-}) - T_n(u^i_{\sigma,+})), \tag{3.28}$$

$$I_2 = \frac{1}{n}\sum_{i=1}^{N} \delta t \sum_{\substack{\sigma\in\mathcal{A}^i \\ r\leq|u^i_{\sigma,+}|\leq n}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,u^i_{\sigma,+}(T_n(u^i_{\sigma,-}) - T_n(u^i_{\sigma,+})). \tag{3.29}$$

For any $\sigma \in \mathcal{A}$ $|u^i_{\sigma,+}| \geq r$ implies $|u^i_{\sigma,-}| \geq r$, thus using Hölder inequality:

$$I_2 \leq \frac{1}{n} I_{2,1} \cdot I_{2,2} \cdot I_{2,3},$$

where

$$I_{2,1} = \left(\sum_{i=1}^{N} \delta t \sum_{\substack{\sigma\in\mathcal{A}^i \\ r\leq|u^i_{\sigma,+}|\leq n \\ r\leq|u^i_{\sigma,-}|}} |\sigma|\,d_\sigma\,|\boldsymbol{v}^i_{K,\sigma}|^{d+2}\right)^{\frac{1}{d+2}},$$

$$I_{2,2} = \left(\sum_{i=1}^{N} \delta t \sum_{\substack{\sigma\in\mathcal{A}^i \\ r\leq|u^i_{\sigma,+}|\leq n \\ r\leq|u^i_{\sigma,-}|}} |\sigma|\,d_\sigma\,|T_n(u^i_{\sigma,+})|^{\frac{2(d+2)}{d}}\right)^{\frac{d}{2(d+2)}},$$

$$I_{2,3} = \left(\sum_{i=1}^{N} \delta t \sum_{\substack{\sigma\in\mathcal{A}^i \\ r\leq|u^i_{\sigma,+}|\leq n \\ r\leq|u^i_{\sigma,-}|}} \frac{|\sigma|}{d_\sigma}[T_n(u^i_{\sigma,-}) - T_n(u^i_{\sigma,+})]^2\right)^{\frac{1}{2}}.$$

Using Jensen's inequality we have

$$|\boldsymbol{v}^i_{K,\sigma}|^{d+2} \leq \frac{1}{\delta t |D_\sigma|}\int_{t_{i-1}}^{t_i}\int_{D_\sigma} |\boldsymbol{v}|^{d+2} dx\,dt,$$

thus recalling that $E_r = \{(x,t) \in \Omega \times (0,T)\,;\ |u_{\mathcal{D}}(x,t)| \geq r\}$ and taking into account that in $I_{2,1}$ the sum is on $|u_K| \geq r$ we obtain

$$I_{2,1} \leq d^{\frac{1}{d+2}} \|\boldsymbol{v}\|_{L^{d+2}(E_r)^d}. \tag{3.30}$$

We now control $I_{2,2}$. As in [17], in view of the definition of $\mathcal{A}^i$ and since $d_{K,\sigma} \geq \xi d_\sigma$ we have, $\forall 1 \leq i \leq N$,

$$\sum_{\substack{\sigma\in\mathcal{A}^i \\ r\leq|u^i_{\sigma,+}|\leq n \\ r\leq|u^i_{\sigma,-}|}} |\sigma|\,d_\sigma\,|T_n(u^i_{\sigma,+})|^{\frac{4}{d}+2} \leq \sum_{K\in\mathcal{T}} |T_n(u^i_K)|^{\frac{4}{d}+2} \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{A}^i} |\sigma| d_\sigma$$

$$\leq \frac{d}{\xi}\sum_{K\in\mathcal{T}} |K||T_n(u^i_K)|^{\frac{4}{d}+2}.$$

If $d \geq 3$, using Hölder and Sobolev inequalities (see Proposition 2.5) and (3.14) we get, $\forall 1 \leq i \leq N$,

$$\sum_{K\in\mathcal{T}} |K||T_n(u_K^i)|^{\frac{4}{d}+2} \leq \Big(\sum_{K\in\mathcal{T}} |K||T_n(u_K^i)|^2\Big)^{\frac{2}{d}} \Big(\sum_{K\in\mathcal{T}} |K||T_n(u_K^i)|^{2^*}\Big)^{\frac{2}{2^*}}$$
$$\leq C\Big(\sup_{1\leq i\leq N} \sum_{K\in\mathcal{T}} |K|u_K^i T_n(u_K^i)\Big)^{\frac{2}{d}} \Big(\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}|T_n(u_K^i) - T_n(u_L^i)|^2\Big).$$

Summing over the time steps and raising to the power $\frac{d}{2(d+2)}$ we obtain

(3.31)
$$I_{2,2} \leq C\Big(\sup_{1\leq i\leq N} \|u_{\mathcal{T}}^i T_n(u_{\mathcal{T}}^i)\|_{L^1(\Omega)}\Big)^{\frac{1}{d+2}} \Big(\sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}|T_n(u_K^i) - T_n(u_L^i)|^2\Big)^{\frac{d}{2(d+2)}},$$

where $C > 0$ is a constant depending on Ω, d and ξ.

In the case of dimension $d = 2$, using a discrete version of Gagliardo-Nirenberg inequality (see Theorem 4 in [6]) together with Poincaré inequality we have

$$\sum_{K\in\mathcal{T}} |K||T_n(u_K^i)|^4 \leq C\Big(\sum_{K\in\mathcal{T}} |K||T_n(u_K^i)|^2\Big)\Big(\sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}|T_n(u_K^i) - T_n(u_L^i)|^2\Big)$$

where $C > 0$ is a constant depending on Ω, d and ξ. Using similar arguments it follows that (3.31) holds for $d = 2$.

As far as $I_{2,3}$ is concerned we observe that

$$I_{2,3} \leq \Big(\sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}|T_n(u_K^i) - T_n(u_L^i)|^2\Big)^{\frac{1}{2}}.$$

Then from (3.30) and (3.31), introducing $\mu^{\frac{d+1}{d+2}}$ where $\lambda(u) \geq \mu > 0$, and using Young's inequality it follows that

$$|I_2| \leq \frac{C\|\boldsymbol{v}\|_{L^{d+2}(E_r)^d}}{n}\Bigg(\frac{1}{\mu^{d+1}(d+2)} \sup_{1\leq i\leq N} \|u_{\mathcal{T}}^i T_n(u_{\mathcal{T}}^i)\|_{L^1(\Omega)}$$
$$+\frac{d+1}{d+2}\mu \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}|T_n(u_K^i) - T_n(u_L^i)|^2\Bigg).$$

By Corollary 3.2 and since $\boldsymbol{v} \in L^{d+2}(Q)^d$, the absolute continuity of the integral implies that there exists $r > 0$ (independent of $\mathcal{T}$ and δt) such that for all admissible mesh $\mathcal{T}$ and all time step δt

$$\max\left\{C\|\boldsymbol{v}\|_{L^{d+2}(E_r)^d} \frac{1}{\mu^{d+1}(d+2)}, C\|\boldsymbol{v}\|_{L^{d+2}(E_r)^d} \frac{d+1}{d+2}\right\} \leq \frac{1}{8}. \tag{3.32}$$

Therefore

$$|I_2| \leq \frac{1}{n}\Bigg(\frac{1}{8} \sup_{1\leq i\leq N} \|u_{\mathcal{T}}^i T_n(u_{\mathcal{T}}^i)\|_{L^1(\Omega)}$$
$$+ \frac{\mu}{8} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}|T_n(u_K^i) - T_n(u_L^i)|^2\Bigg). \tag{3.33}$$

We now turn to I_1. Using Cauchy-Schwarz inequality and Young inequality we have by the definition of $\mathcal{A}_i$

$$|I_1| = \frac{1}{n}\sum_{i=1}^{N}\delta t \sum_{\substack{\sigma\in\mathcal{A}^i\\ |u^i_{\sigma,+}|\le r}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,u^i_{\sigma,+}(T_n(u^i_{\sigma,-}) - T_n(u^i_{\sigma,+}))$$

$$\le \frac{r}{n}\left(\sum_{i=1}^{N}\delta t \sum_{\substack{\sigma\in\mathcal{A}^i\\ |u^i_{\sigma,+}|\le r}} |\sigma| d_\sigma |\boldsymbol{v}^i_{K,\sigma}|^2\right)^{\frac12}\left(\sum_{i=1}^{N}\delta t \sum_{\substack{\sigma\in\mathcal{A}^i\\ |u^i_{\sigma,+}|\le r}} \frac{|\sigma|}{d_\sigma}(T_n(u^i_{\sigma,-}) - T_n(u^i_{\sigma,+}))^2\right)^{\frac12}$$

(3.34)

$$\le \frac{1}{n}\left[\frac{2r^2 d\,\|\,\boldsymbol{v}\,\|^2_{L^2(Q)^d}}{\mu} + \frac{\mu}{8}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}(T_n(u^i_{\sigma,-}) - T_n(u^i_{\sigma,+}))^2\right].$$

Gathering (3.33) and (3.34) and since $\lambda(u)^i_\sigma \ge \mu$ we get

$$\begin{aligned}
-T_3^{'N} \le |I_1 + I_2| &\le \frac{2}{n}\frac{r^2 d\,\|\,\boldsymbol{v}\,\|^2_{L^2(Q)^d}}{\mu} + \frac{1}{8n}\sup_{1\le i\le N}\|u^i_{\mathcal{T}} T_n(u^i_{\mathcal{T}})\|_{L^1(\Omega)}\\
&\qquad + \frac{\mu}{4n}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}(u^i_K - u^i_L)(T_n(u^i_K) - T_n(u^i_L))\\
&\le \frac{2}{n}\frac{r^2 d\,\|\,\boldsymbol{v}\,\|^2_{L^2(Q)^d}}{\mu} + \frac{1}{8n}\sup_{1\le i\le N}\|u^i_{\mathcal{T}} T_n(u^i_{\mathcal{T}})\|_{L^1(\Omega)}\\
&\qquad + \frac{1}{4n}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)^i_\sigma(u^i_K - u^i_L)(T_n(u^i_K) - T_n(u^i_L))\\
&\le \frac{2}{n}\frac{r^2 d\,\|\,\boldsymbol{v}\,\|^2_{L^2(Q)^d}}{\mu} + \frac{1}{4}T'_{1,1} + \frac{1}{4}T_2^N,
\end{aligned}$$

which is (3.27). From (3.24) it follows that

$$0 \le \frac{1}{2}T'_{1,1} + T_{1,3}^{'N} + \frac{1}{2}T_2^N \le 2T_4^{'N} + 2T'_{1,2} + \frac{4}{n}\frac{r^2 d\,\|\,\boldsymbol{v}\,\|^2_{L^2(Q)^d}}{\mu},$$

where $T'_{1,1}$, $T_{1,3}^{'N}$ and T_2^N are non negative. Thus by (3.25) and (3.26) we get (3.4), (3.5) and (3.7).

We now turn to (3.6). By (3.8) for $M = N$ we have $T_1^N + T_2^N + T_3^N = T_4^N$. Due to the definition of T_1^N we have $T_1^N \ge -T_{1,2}^{'} + T_{1,3}^{'N}$ so that the non negative character of T_2^N and $T_{1,3}^{'N}$ leads to

$$T_3^N \le T_{1,2}^{'} + T_4^N \le T_{1,2}^{'} + T_4^{'N}.$$

If we denote $\mathcal{B}^i$ as the complement of $\mathcal{A}^i$ we have

$$\forall\sigma\in\mathcal{B}^i \quad u^i_{\sigma,+}(T_n(u^i_{\sigma,+}) - T_n(u^i_{\sigma,-})) > 0.$$

It follows that writing T_3^N as $T_3^N = T_3^{'N} + T_3^{''N}$ with

$$0 \le T_3^{''N} = \frac{1}{n}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{B}^i}|\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|u^i_{\sigma,+}(T_n(u^i_{\sigma,+}) - T_n(u^i_{\sigma,-})),$$

we deduce that

$$0 \le T_3^{''N} \le T_{1,2}^{'} + T_4^{'N} - T_3^{'N}.$$

Then using (3.25), (3.26) and (3.27) we obtain

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}} T_3^{''N} = 0.$$

At last remarking

$$T_3^{''N} - T_3^{'N} = \frac{1}{n}\sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\, |\boldsymbol{v}^i_{K,\sigma}|\, |u^i_{\sigma,+}|\, |T_n(u^i_{\sigma,+}) - T_n(u^i_{\sigma,-})|$$

gives (3.5). □

Remark 3.4. *In the present paper the data f is given and the "discrete" right-hand side f^i_K is such that $f_{\mathcal{D}}$ converges strongly in $L^1(Q)$ to f. If we replace $f_{\mathcal{D}}$ by a L^1 bounded sequence $f_{\mathcal{D}}$, that is if we assume that $f_{\mathcal{D}}$ is only bounded in $L^1(Q)$, the attentive reader may notice that Proposition 3.1 and then Corollary 3.2 hold true. However as in the continuous case, Proposition 3.3 should be replaced by a weaker one. Indeed except the precise behavior of $T_4^{'N}$ all the computations and the arguments of the proof hold. Since $T_4^{'N}$ is (only) bounded we obtain $\forall n \geq 1$, there exists C depending only on $(\Omega, \boldsymbol{v}, f, n, d, \xi)$ such that*

$$\frac{1}{n}\sup_{1\leq i\leq N} \|u^i_{\mathcal{T}} T_n(u^i_{\mathcal{T}})\|_{L^1(\Omega)} \leq C, \tag{3.35}$$

$$\sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma (u^i_K - u^i_L)(T_n(u^i_K) - T_n(u^i_L)) \leq C, \tag{3.36}$$

$$\sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\, |\boldsymbol{v}^i_{K,\sigma}|\, |u^i_{\sigma,+}|\, |T_n(u^i_{\sigma,+}) - T_n(u^i_{\sigma,-})| \leq C, \tag{3.37}$$

$$\sum_{i=1}^{N} \sum_{K\in\mathcal{T}} |K| \int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r) T_n'(r)\, dr \leq C. \tag{3.38}$$

Such estimates are useful when dealing with strongly coupled PDE systems in fluid or solid thermomechanic containing a balance energy equation with an expected L^1 data (see e.g. [22] *for discrete approximate solutions for a turbulence model and* [23] *for a thermoviscoelastic model).*

Corollary 3.5. (Estimate on $T_n(u_{\mathcal{D}})$) *Let $\mathcal{T}$ be an admissible mesh, let δt be a time step and $\xi > 0$ such that* (2.13) *holds. If $u_{\mathcal{D}} = (u^i_{\mathcal{T}})_{1\leq i\leq N}$ is a solution to (2.8), then for all $n > 0$ there exists $C > 0$ only depending on $(\Omega, \boldsymbol{v}, f, n, d, \xi)$ such that*

$$\sum_{i=1}^{N} \delta t \|T_n(u^i_{\mathcal{T}})\|_{1,2,\mathcal{T}} \leq C. \tag{3.39}$$

Proof. Thanks to (3.5) and the fact that $\lambda(u) \geq \mu$, we have

$$\frac{\mu}{2}\sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} (u^i_K - u^i_L)(T_n(u^i_K) - T_n(u^i_L)) \leq C.$$

Since $|T_n(u^i_K) - T_n(u^i_L)| \leq |u^i_K - u^i_L|$, we obtain

$$\frac{\mu}{2}\sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} (T_n(u^i_K) - T_n(u^i_L))^2 \leq C,$$

which is (3.39). □

As explained in the introduction, we use the discrete version of Aubin-Simon Lemma proved in [22, 21], which requires in particular estimate on the discrete time derivative. Since λ is not bounded we cannot expect to have estimate on the time derivative of $u_{\mathcal{D}}$ as in [22, 21]. To overcome this additionnal difficulty we propose to use the discrete Aubin-Simon Lemma on a regularized truncate of $u_{\mathcal{D}}$. Let us introduce the functions h_n and $\widetilde{h}_n$ defined by

$$h_n(r) = \begin{cases} 0 & \text{if } r \leq -2n, \\ \frac{r}{n} + 2 & \text{if } -2n \leq r \leq -n, \\ 1 & \text{if } -n \leq r \leq n, \\ \frac{-r}{n} + 2 & \text{if } n \leq r \leq 2n, \\ 0 & \text{if } r \geq 2n. \end{cases} \tag{3.40}$$

and $\widetilde{h}_n(s) = \int_0^s h_n(r)\,dr$. These two functions satisfy the following properties:

$$\widetilde{h}_n \text{ is a continuous Lipschitz function,} \tag{3.41}$$

$$h_n \text{ is bounded by 1,} \tag{3.42}$$

$$\forall r, s \in \mathbb{R},\ h_n(r)(1 - h_{2n}(s)) = h_n(r)(h_{2n}(r) - h_{2n}(s)), \tag{3.43}$$

$$\forall r, s \in \mathbb{R},\ |h_n(r) - h_n(s)| \leq \frac{1}{n}|T_{2n}(r) - T_{2n}(s)|. \tag{3.44}$$

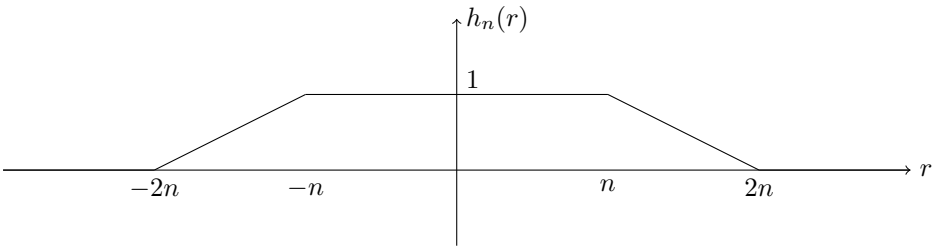

FIGURE 2. The function h_n

Proposition 3.6. (Estimate on $\widetilde{h}_n(u_{\mathcal{D}})$) *Let $\mathcal{T}$ be an admissible mesh, δt be a time step and $\xi > 0$ such that (2.13) holds. If $u_{\mathcal{D}} = (u^i_{\mathcal{T}})_{1\leq i\leq N}$ is a solution to (2.8), then for all $n > 0$ there exists $C > 0$ only depending on $(\Omega, \boldsymbol{v}, f, n, d, \xi)$ such that*

$$\sum_{i=1}^{N} \delta t \|\widetilde{h}_n(u^i_{\mathcal{T}})\|^2_{1,2,\mathcal{T}} \leq C. \tag{3.45}$$

Proof. We can perform the proof thanks to (3.39) and the fact that

$$\begin{aligned}
\sum_{i=1}^{N} \delta t \|\widetilde{h}_n(u^i_{\mathcal{T}})\|^2_{1,2,\mathcal{T}} &= \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} |\widetilde{h}_n(u^i_K) - \widetilde{h}_n(u^i_L)|^2 \\
&= \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} |\widetilde{h}_n(T_{2n}(u^i_K)) - \widetilde{h}_n(T_{2n}(u^i_L))|^2 \\
&\leq \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} |T_{2n}(u^i_K) - T_{2n}(u^i_L)|^2 \\
&= \sum_{i=1}^{N} \delta t \|T_{2n}(u^i_{\mathcal{T}})\|^2_{1,2,\mathcal{T}}.
\end{aligned}$$

□

Proposition 3.7. (Estimate on $\partial_t \widetilde{h}_n(u_\mathcal{D})$) *Let $\mathcal{T}$ be an admissible mesh, let δt be a time step and $\xi > 0$ such that (2.13) holds. If $u_\mathcal{D} = (u^i_\mathcal{T})_{1\le i\le N}$ is a solution to (2.8), then for all $n > 0$ there exists $C > 0$ such that*

$$\sum_{i=1}^{N} \delta t \|\partial_t \widetilde{h}_n(u^i_\mathcal{T})\|_{-1,1,\mathcal{T}} \le C. \tag{3.46}$$

Proof. Let $w \in X(\mathcal{T})$. Multiplying each equation of the scheme by $h_n(u^i_K)w_K$, summing over the control volumes and gathering by edges, we get $T^i_1+T^i_2+T^i_3 = T^i_4$ with

$$T^i_1 = \sum_{K\in\mathcal{T}} |K| \frac{u^i_K - u^{i-1}_K}{\delta t} h_n(u^i_K) w_K, \tag{3.47}$$

$$T^i_2 = \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma (u^i_K - u^i_L)(h_n(u^i_K)w_K - h_n(u^i_L)w_L), \tag{3.48}$$

$$T^i_3 = \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}^i_{K,\sigma} u^i_{\sigma,+}(h_n(u^i_K)w_K - h_n(u^i_L)w_L), \tag{3.49}$$

$$T^i_4 = \sum_{K\in\mathcal{T}} |K| f^i_K h_n(u^i_K) w_K. \tag{3.50}$$

As far as T^i_1 is concerned we have $T^i_1 = T^i_{1,1} + T^i_{1,2}$ with

$$T^i_{1,1} = \sum_{K\in\mathcal{T}} |K| \frac{\widetilde{h}_n(u^i_K) - \widetilde{h}_n(u^{i-1}_K)}{\delta t} w_K,$$

$$T^i_{1,2} = \sum_{K\in\mathcal{T}} \frac{|K|}{\delta t} w_K \int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r) h_n'(r)\, dr,$$

where $\widetilde{h}_n(s) = \int_0^s h_n(r)\, dr$.

By Definition 2.3 we have

$$\sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}} \frac{1}{\|w\|_{1,\infty,\mathcal{T}}} T^i_{1,1} = \|\partial_t \widetilde{h}_n(u^i_\mathcal{T})\|_{-1,1,\mathcal{T}}. \tag{3.51}$$

We now study the terms T^i_2, T^i_3, T^i_4 and $T^i_{1,2}$ in order to control (3.51).

Since h_n is bounded by 1 we have $|T^i_4| \le \|w\|_{L^\infty(\Omega)} \||f|^i\|_{L^1(\Omega)}$ where $|f|^i = \frac{1}{\delta t}\int_{t_{i-1}}^{t_i} |f|\, dt$. We thus get

$$\sum_{i=1}^{N} \delta t \sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}} \frac{1}{\|w\|_{1,\infty,\mathcal{T}}} |T^i_4| \le \|f\|_{L^1(Q)}. \tag{3.52}$$

As far as T^i_2 is concerned we have $T^i_2 = T^i_{2,1} + T^i_{2,2}$ with

$$T^i_{2,1} = \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma h_n(u^i_K)(u^i_K - u^i_L)(w_K - w_L),$$

$$T^i_{2,2} = \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma w_L (u^i_K - u^i_L)(h_n(u^i_K) - h_n(u^i_L)).$$

In view of the definition of h_n we get

$$|T^i_{2,2}| \le \frac{\|w\|_{L^\infty(\Omega)}}{n} \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)^i_\sigma (u^i_K - u^i_L)(T_{2n}(u^i_K) - T_{2n}(u^i_L)),$$

so that taking the supremum on $X(\mathcal{T})$ and summing over the time steps give

$$\begin{aligned}&\sum_{i=1}^{N}\delta t\sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}}\frac{1}{\|w\|_{1,\infty,\mathcal{T}}}|T_{2,2}^i|\\ &\quad\leq\frac{1}{n}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)_\sigma^i(u_K^i-u_L^i)(T_{2n}(u_K^i)-T_{2n}(u_L^i)),\end{aligned}\tag{3.53}$$

which is bounded thanks to the estimate (3.5). We now turn to $T_{2,1}^i$, which reads as

$$T_{2,1}^i=T_{2,1,1}^i+T_{2,1,2}^i \text{ where}$$

$$T_{2,1,1}^i=\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)_\sigma^i h_n(u_K^i)h_{2n}(u_L^i)(u_K^i-u_L^i)(w_K-w_L),$$

$$T_{2,1,2}^i=\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)_\sigma^i h_n(u_K^i)(1-h_{2n}(u_L^i))(u_K^i-u_L^i)(w_K-w_L).$$

Since $h_n(u_K^i)(1-h_{2n}(u_L^i))=h_n(u_K^i)(h_{2n}(u_K^i)-h_{2n}(u_L^i))$ we have

$$|T_{2,1,2}^i|\leq\frac{\|w\|_{L^\infty(\Omega)}}{n}\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)_\sigma^i(T_{4n}(u_K^i)-T_{4n}(u_L^i))(u_K^i-u_L^i),$$

which gives thanks to (3.5)

$$\sum_{i=1}^{N}\delta t\sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}}\frac{1}{\|w\|_{1,\infty,\mathcal{T}}}|T_{2,1,2}^i|\leq C.\tag{3.54}$$

As far as $T_{2,1,1}^i$ is concerned, we have

$$T_{2,1,1}^i=\sum_{\sigma\in\mathcal{E}}|\sigma|\lambda(u)_\sigma^i h_n(u_K^i)h_{2n}(u_L^i)(T_{4n}(u_K^i)-T_{4n}(u_L^i))\frac{w_K-w_L}{d_\sigma},$$

and by the definition of $\lambda(u)_\sigma^i$, we can write

$$|T_{2,1,1}^i|\leq\|w\|_{1,\infty,\mathcal{T}}\left(\max_{r\in[-4n,4n]}|\lambda(r)|\right)\|T_{4n}(u_{\mathcal{T}}^i)\|_{1,1,\mathcal{T}}.$$

Thus, we deduce that

$$\sum_{i=1}^{N}\delta t\sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}}\frac{1}{\|w\|_{1,\infty,\mathcal{T}}}|T_{2,1,1}^i|\leq C.\tag{3.55}$$

Gathering (3.54) and (3.55) we get

$$\sum_{i=1}^{N}\delta t\sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}}\frac{1}{\|w\|_{1,\infty,\mathcal{T}}}|T_2^i|\leq C.\tag{3.56}$$

The term T_3^i can be rewritten as

$$T_3^i=\sum_{\sigma\in\mathcal{E}}|\sigma|\,|\boldsymbol{v}_{K,\sigma}^i|\,u_{\sigma,+}^i(h_n(u_{\sigma,+}^i)w_{\sigma,+}-h_n(u_{\sigma,-}^i)w_{\sigma,-}),$$

where $w_{\sigma,+} = w_K$ if $\boldsymbol{v}^i_{K,\sigma} \geq 0$, w_L otherwise and $(w_{\sigma,+}, w_{\sigma,-}) = (w_K, w_L)$. We then have $T^i_3 = T^i_{3,1} + T^i_{3,2}$ with

$$T^i_{3,1} = \sum_{\sigma\in\mathcal{E}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,u^i_{\sigma,+} h_n(u^i_{\sigma,+})(w_{\sigma,+} - w_{\sigma,-}),$$
$$T^i_{3,2} = \sum_{\sigma\in\mathcal{E}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,u^i_{\sigma,+} w_{\sigma,-}(h_n(u^i_{\sigma,+}) - h_n(u^i_{\sigma,-})).$$

By the definition of h_n we have

$$\begin{aligned}|T^i_{3,1}| &\leq \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma |\boldsymbol{v}^i_{K,\sigma}|\,|T_{2n}(u^i_{\sigma,+})|\,\frac{|w_{\sigma,+} - w_{\sigma,-}|}{d_\sigma}\\ &\leq \|w\|_{1,\infty,\mathcal{T}} \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma |\boldsymbol{v}^i_{K,\sigma}|\,|T_{2n}(u^i_{\sigma,+})|.\end{aligned}$$

Using Hölder and Poincaré inequalities we get

$$\begin{aligned}\sum_{i=1}^N \delta t \sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}} \frac{1}{\|w\|_{1,\infty,\mathcal{T}}} |T^i_{3,1}| &\leq \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma |\boldsymbol{v}^i_{K,\sigma}|\,|T_{2n}(u^i_{\sigma,+})|\\ &\leq \left(\sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma |\boldsymbol{v}^i_{K,\sigma}|^2\right)^{\frac12} \left(\sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma |T_{2n}(u^i_{\sigma,+})|^2\right)^{\frac12}\\ &\leq d^{\frac12} \|\boldsymbol{v}\|_{L^2(Q)^d} C \left(\sum_{i=1}^N \delta t \|T_{2n}(u^i_{\mathcal{T}})\|^2_{1,2,\mathcal{T}}\right)^{\frac12}. \qquad (3.57)\end{aligned}$$

For $T^i_{3,2}$ it is sufficient to observe that

$$|T^i_{3,2}| \leq \frac{\|w\|_{L^\infty(\Omega)}}{n} \sum_{\sigma\in\mathcal{E}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,|u^i_{\sigma,+}|\,|T_{2n}(u^i_{\sigma,+}) - T_{2n}(u^i_{\sigma,-})|,$$

from which it follows

$$\sum_{i=1}^N \delta t \sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}} \frac{1}{\|w\|_{1,\infty,\mathcal{T}}} |T^i_{3,2}| \leq \frac{1}{n} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\,|\boldsymbol{v}^i_{K,\sigma}|\,|u^i_{\sigma,+}|\,|T_{2n}(u^i_{\sigma,+}) - T_{2n}(u^i_{\sigma,-})|, \qquad (3.58)$$

which is bounded thanks to (3.6).

It remains to control $T^i_{1,2}$. Noticing that

$$\left|\int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r) h_n'(r)\,dr\right| \leq \int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r) T_{2n}'(r)\,dr, \qquad (3.59)$$

(3.59) and (3.7) yield that

$$\sum_{i=1}^N \delta t \sup_{\substack{w\in X(\mathcal{T})\\ w\neq 0}} \frac{1}{\|w\|_{1,\infty,\mathcal{T}}} |T^i_{1,2}| \leq C. \qquad (3.60)$$

Gathering (3.51) to (3.60), we obtain the estimate

$$\sum_{i=1}^N \delta t \|\partial_t \widetilde{h}_n(u^i_{\mathcal{T}})\|_{-1,1,\mathcal{T}} \leq C. \qquad \square$$

We are now in a position to extract a convergent subsequence to $T_n(u_{\mathcal{D}})$.

Proposition 3.8. *Let $(\mathcal{T}_m)_{m\geq 1}$ be a sequence of admissible meshes such that there exists $\xi > 0$ satisfying*

$$\text{for all } m \geq 1 \text{ for all } K \in \mathcal{T} \text{ and all } \sigma \in \mathcal{E}(K),\ d_{K,\sigma} \geq \xi d_\sigma,$$

and let δt_m be a sequence of time steps. If $u_{\mathcal{D}_m} = (u^i_{\mathcal{T}_m})_{1\leq i\leq N_m}$ is a solution to (2.8) with $\mathcal{T} = \mathcal{T}_m$, $\delta t = \delta t_m$ and $N = N_m$, then there exists a measurable function u finite a.e. in Q such that, up to a subsequence,

$$u_{\mathcal{D}_m} \to u \text{ almost everywhere in } Q, \tag{3.61}$$

$$u \in L^1(Q) \tag{3.62}$$

$$T_n(u) \in L^2(0,T;H^1_0(\Omega)), \quad \forall n > 0 \tag{3.63}$$

$$\nabla_{\mathcal{D}} T_n(u_{\mathcal{D}_m}) \rightharpoonup \nabla T_n(u) \text{ in } (L^2(Q))^d \quad \forall n > 0. \tag{3.64}$$

In order to prove this proposition we use a discrete version of the Aubin-Simon Lemma proved in [21] and in [22].

Proof. For a fixed $n > 0$ let us consider the sequence $\widetilde{h}_n(u_{\mathcal{D}_m})$. Due to estimates (3.45) and (3.46), we can apply the discrete version of the Aubin-Simon Lemma (see [21, 22]) which states that (up to a subsequence) $\widetilde{h}_n(u_{\mathcal{D}_m})$ converges in $L^1(Q)$ to a function belonging to $L^2(0,T;H^1_0(\Omega))$.

Using the diagonal process, we can deduce that there exists a subsequence of $u_{\mathcal{D}_m}$, still indexed by m, such that for any n, $\widetilde{h}_n(u_{\mathcal{D}_m})$ converges almost everywhere to a measurable function w_n belonging to $L^2(0,T;H^1_0(\Omega))$. Due to the definition of $\widetilde{h}_n$, the sequence $T_n(u_{\mathcal{D}_m}) = T_n(\widetilde{h}_n(u_{\mathcal{D}_m}))$ converges almost everywhere in Q to the measurable function $T_n(w_n)$ which belongs to $L^2(0,T;H^1_0(\Omega))$.

Using Corollary 3.2 we proceed as in [5] and we obtain that $u_{\mathcal{D}_m}$ is a Cauchy sequence in measure. Thus there exists a measurable function u such that $u_{\mathcal{D}_m} \to u$ a.e. in Q. Due to Corollary 3.2 u is finite almost everywhere. Using (3.4) for $n = 1$ we have

$$\|u_{\mathcal{D}_m} T_1(u_{\mathcal{D}_m})\|_{L^1(Q)} \leq T \sup_{1\leq i\leq N} \|u^i_{\mathcal{T}_m} T_1(u^i_{\mathcal{T}_m})\|_{L^1(\Omega)} \leq C$$

where $C > 0$ is a constant independent of m. The inequality $|r| \leq 1 + rT_1(r)$ for any $r \in \mathbb{R}$ gives that $u_{\mathcal{D}_m}$ is bounded in $L^1(Q)$ so that the point-wise convergence of $u_{\mathcal{D}_m}$ and Fatou Lemma lead to (3.62). Moreover from the convergences already obtained and using Lemma 2.7 we get (3.63) and (3.64). □

Remark 3.9. *Following Remark 3.4 it is worth noting that Corollaries 3.5, 3.6 and Proposition 3.7 hold true if instead of (3.5), (3.6) and (3.7) (see Proposition 3.3) we have (3.36), (3.37) and (3.38) (which are weaker). It follows that if we replace in the scheme $f_{\mathcal{D}}$ (an approximation of a fixed data f) by a sequence $f_{\mathcal{D}}$ which is only bounded in $L^1(Q)$ Proposition 3.8 holds true.*

Since passing to the limit in the scheme involve functions depending on $u_{\mathcal{D}_m}$ which are constant on the diamond meshes we prove the following convergence results.

Proposition 3.10. *Under the assumptions of Proposition 3.8 and up to a subsequence, let u verifying (3.63)–(3.64). For $n \geq 1$, let us define $\overline{T_n(u)}^m$, $\overline{h}^m_n$ and $\overline{\lambda}^m_n$ which are constant on each diamond meshes by $\forall t \in [t_{i-1}, t_i),\ 1 \leq i \leq N_m$,*

$\forall x \in D_\sigma,\ \sigma \in \mathcal{E}_m$

$$\overline{T_n(u)}^m(x,t) = \frac{T_n(u_K^i) + T_n(u_L^i)}{2}, \tag{3.65}$$

$$\overline{h}_n^m(x,t) = \frac{h_n(u_K^i)h_{2n}(u_L^i) + h_n(u_L^i)h_{2n}(u_K^i)}{2}, \tag{3.66}$$

$$\overline{\lambda}_n^m(x,t) = \lambda(T_n(u))_\sigma^i. \tag{3.67}$$

Then the functions $\overline{T_n(u)}^m$, $\overline{h}_n^m$ and $\overline{\lambda}_n^m$ converge respectively to $T_n(u)$, $h_n(u)$ and $\lambda(T_n(u))$ in $L^q(Q)$ for any $1 \le q < +\infty$ and in L^∞ weak-$\star$.

Proof. Let $n > 0$ and let us consider $\overline{T_n(u)}^m$. As in [24], since $T_n(u_{\mathcal{D}_m})$ converges to $T_n(u)$ in $L^1(Q)$, proving $\|\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m})\|_{L^1(Q)}$ goes to zero implies that $\overline{T_n(u)}^m$ converges to $T_n(u)$ in $L^1(Q)$. By writing

$$\begin{aligned}
\|\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m})\|_{L^1(Q)} &= \sum_{i=1}^N \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |D_{K,\sigma}| \frac{|T_n(u_K^i) - T_n(u_L^i)|}{2} \\
&\le \frac{1}{2}\sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |D_\sigma| |T_n(u_K^i) - T_n(u_L^i)| \\
&\le \frac{1}{2d}\sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma |T_n(u_K^i) - T_n(u_L^i)| \\
&\le \frac{h_\tau}{d}\sum_{i=1}^N \delta t \|T_n(u_{\mathcal{T}}^i)\|_{1,1,\tau}
\end{aligned}$$

and by using (3.39) (see Corollary 3.5) we deduce that $\|\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m})\|_{L^1(Q)}$ goes to zero as m goes to infinity. Since $\overline{T_n(u)}^m$ is uniformly bounded with respect to m standard analysis arguments yield that $\overline{T_n(u)}^m$ converges to $T_n(u)$ in $L^q(Q)$ for any $1 < q < +\infty$ and in L^∞ weak-$\star$.

As far as $\overline{h}_n^m$ is concerned it is sufficient to remark that

$$\left|\frac{h_n(u_K^i)h_{2n}(u_L^i) + h_n(u_L^i)h_{2n}(u_K^i)}{2} - h_n(u_K^i)\right| \le |h_n(u_K) - h_n(u_L)|,$$

to use the Lipschitz character of h_n and similar arguments.

We now turn to the convergence of $\overline{\lambda}_n^m(x,t)$. We firstly remark that $2\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m})$ converges to $T_n(u)$ in $L^q(Q)$ for any $1 \le q < +\infty$ and in L^∞ weak-$\star$ as m goes to infinity. Moreover assumption (2.6) together with the definition of the functions on diamond meshes give

$$\begin{aligned}
&\min(\lambda(T_n(u_{\mathcal{D}_m})), \lambda(2\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m}))) \\
&\qquad \le \overline{\lambda}_n^m(x,t) \le \max(\lambda(T_n(u_{\mathcal{D}_m})), \lambda(2\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m}))).
\end{aligned}$$

Since λ is a continuous function the convergence of the two functions $T_n(u_{\mathcal{D}_m})$ and $2\overline{T_n(u)}^m - T_n(u_{\mathcal{D}_m})$ (which are bounded uniformly in L^∞ with respect to m) then imply that $\overline{\lambda}_n^m$ converge to $\lambda(T_n(u))$ in $L^q(Q)$ for any $1 \le q < +\infty$ and in L^∞ weak-$\star$. □

4. Proof of Theorem 2.8

Proof. Concerning the existence of a solution to the scheme, using the Brouwer theorem, the proof is quite usual when λ is a bounded function (see [19]). In the case where λ is not bounded, we replace λ by $\lambda(T_{1/\varepsilon})$ and we show with the Brouwer theorem that the modified scheme admits at least one solution $u_{\mathcal{D}}^\varepsilon$. Since we have

a bound on $u_{\mathcal{D}}^{\varepsilon}$ (uniform in ε), for ε small enough, $\lambda(T_{1/\varepsilon}(u_{\mathcal{D}}^{\varepsilon})) = \lambda(u_{\mathcal{D}}^{\varepsilon})$. Then the scheme (2.8) admits at least a solution. We then split the rest of the proof in four steps. In Step 1 we pass to the limit in the scheme as the size of the mesh and the time step tend to zero. It is worth noting that we take in the scheme a discrete version of what is a test function in the renormalized formulation. This is to say that we multiply each equation of the scheme by a discrete version of $h_n(u)\varphi$ where $\varphi \in \mathcal{C}_c^\infty(\Omega \times [0,T))$ and h_n is defined by (3.40). Step 2 is devoted to pass to the limit as n tends to infinity and to show that the renormalized formulation (1.6) holds. As in the continuous case the main difficulty is the lack of regularity of the time derivative $\frac{\partial u}{\partial t}$. We use here Steklov averages and the method developed in [10] (see also [30]). In Step 3 we prove that the limit u satisfies the estimate on the energy (1.5) and in Step 4 we recall that under assumption **H1** or **H2** the uniqueness of the renormalized solution provides that the whole sequence converges to the renormalized solution.

Step 1 We prove that the limit u satisfies (4.14). Let $\varphi \in \mathcal{C}_c^\infty(\Omega\times[0,T))$ and h_n the function defined by (3.40). We denote by $\varphi_{\mathcal{D}}$ the function defined by $\varphi_K^i = \varphi(x_K, t_i)$ for all $K \in \mathcal{T}$ and all $0 \le i \le N$.

Multiplying each equation of the scheme by $\varphi_K^{i-1} h_n(u_K^i)$ (which is a discrete version of the test function used in the renormalized formulation), summing over the control volumes, gathering by edges and summing over the time steps, we get $T_1 + T_2 + T_3 = T_4$ with

$$
\begin{aligned}
T_1 &= \sum_{i=1}^{N} \sum_{K\in\mathcal{T}} |K|(u_K^i - u_K^{i-1}) h_n(u_K^i)\varphi_K^{i-1},\\
T_2 &= \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i (u_K^i - u_L^i)(h_n(u_K^i)\varphi_K^{i-1} - h_n(u_L^i)\varphi_L^{i-1}),\\
T_3 &= \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i (h_n(u_K^i)\varphi_K^{i-1} - h_n(u_L^i)\varphi_L^{i-1}),\\
T_4 &= \sum_{i=1}^{N} \delta t \sum_{K\in\mathcal{T}} |K| f_K^i h_n(u_K^i)\varphi_K^{i-1}.
\end{aligned}
$$

We now pass to the limit as $h_{\mathcal{T}} \to 0$ and $\delta t \to 0$.

As far as the lower order term T_4 is concerned, we have $T_4 = T_{4,1} + T_{4,2}$ with

$$
\begin{aligned}
T_{4,1} &= \sum_{i=1}^{N} \sum_{K\in\mathcal{T}} \int_{t_{i-1}}^{t_i} \int_K f h_n(u_K^i)\varphi_K^i dx\, dt,\\
T_{4,2} &= \sum_{i=1}^{N} \sum_{K\in\mathcal{T}} \int_{t_{i-1}}^{t_i} \int_K f h_n(u_K^i)(\varphi_K^{i-1} - \varphi_K^i) dx\, dt.
\end{aligned}
$$

By the regularity of φ, we have $\varphi_{\mathcal{D}} \to \varphi$ uniformly on Q as $h_{\mathcal{T}} \to 0$ and $\delta t \to 0$. Since $h_n(u_{\mathcal{D}}) \to h_n(u)$ a.e. and L^∞ weak-$\star$, $\varphi_{\mathcal{D}} \to \varphi$ uniformly and $|f\,\varphi_{\mathcal{T}}\, h_n(u_{\mathcal{D}})| \le C_\varphi |f| \in L^1(Q)$, the Lebesgue theorem ensures that

$$
\begin{aligned}
\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}} T_{4,1} &= \lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}} \int_\Omega f\,\varphi_{\mathcal{D}}\, h_n(u_{\mathcal{D}})\, dx\, dt\\
&= \int_0^T \int_\Omega f\,\varphi\, h_n(u)\, dx\, dt. \qquad (4.1)
\end{aligned}
$$

In view of the definition of h_n and the regularity of φ,

$$|T_{4,2}| \leq c\|\varphi_t\|_\infty \delta t \|f\|_{L^1(Q)},$$

thus

$$\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}} T_{4,2} = 0. \tag{4.2}$$

We now study the convergence of the diffusion term. We have

$$\begin{aligned} T_2 &= \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i (u_K^i - u_L^i)(h_n(u_K^i)\varphi_K^{i-1} - h_n(u_L^i)\varphi_L^{i-1}) \\ &= T_{2,1} + T_{2,2} \end{aligned}$$

with

$$T_{2,1} = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i \, h_n(u_K^i)(u_K^i - u_L^i)(\varphi_K^{i-1} - \varphi_L^{i-1}),$$

$$T_{2,2} = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i \, \varphi_L^{i-1}(u_K^i - u_L^i)(h_n(u_K^i) - h_n(u_L^i)).$$

Not assuming any bound on the function λ gives additional technical difficulties in order to control $T_{2,1}$ and $T_{2,2}$. In view of the definition of h_n and λ we get

$$|T_{2,2}| \leq \frac{c\|\varphi\|_\infty}{n} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i (u_K^i - u_L^i)\,(T_{2n}(u_K^i) - T_{2n}(u_L^i)).$$

From the discrete estimate on the energy (see (3.5) in Proposition 3.3), it follows that

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}} T_{2,2} = 0. \tag{4.3}$$

As far as $T_{2,1}$ is concerned we have $T_{2,1} = T_{2,1,1} + T_{2,1,2}$ with

$$T_{2,1,1} = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i \, h_n(u_K^i) h_{2n}(u_L^i)(u_K^i - u_L^i)(\varphi_K^{i-1} - \varphi_L^{i-1}),$$

$$T_{2,1,2} = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i \, h_n(u_K^i)(1 - h_{2n}(u_L^i))(u_K^i - u_L^i)(\varphi_K^{i-1} - \varphi_L^{i-1}).$$

Observing that $h_n(u_K^i)(1 - h_{2n}(u_L^i)) = h_n(u_K^i)(h_{2n}(u_K^i) - h_{2n}(u_L^i))$ gives

$$|T_{2,1,2}| \leq \frac{c\|\varphi\|_\infty}{n} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i (T_{4n}(u_K^i) - T_{4n}(u_L^i))(u_K^i - u_L^i),$$

and from (3.5) it follows that

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}} T_{2,1,2} = 0. \tag{4.4}$$

For $T_{2,1,1}$ we write $T_{2,1,1} = I + II$ with

$$I = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i \frac{h_n(u_K^i)h_{2n}(u_L^i) + h_n(u_L^i)h_{2n}(u_K^i)}{2} (u_K^i - u_L^i)(\varphi_K^{i-1} - \varphi_L^{i-1}),$$

$$II = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma} \lambda(u)_\sigma^i \frac{h_n(u_K^i)h_{2n}(u_L^i) - h_n(u_L^i)h_{2n}(u_K^i)}{2} (u_K^i - u_L^i)(\varphi_K^{i-1} - \varphi_L^{i-1}).$$

Since

$$h_n(u_K^i)h_{2n}(u_L^i) - h_n(u_L^i)h_{2n}(u_K^i) \\ = h_n(u_K^i)(h_{2n}(u_L^i) - h_{2n}(u_K^i)) + (h_n(u_K^i) - h_n(u_L^i))h_{2n}(u_K^i),$$

and since

$$|h_{2n}(u_L^i) - h_{2n}(u_K^i)| \le \frac{1}{2n}\,|T_{4n}(u_l^i) - T_{4n}(u_K^i)|,$$
$$|h_n(u_L^i) - h_n(u_K^i)| \le \frac{1}{n}\,|T_{2n}(u_L^i) - T_{2n}(u_K^i)|,$$

estimate (3.5) leads to

$$|II| \le \|\varphi\|_\infty \omega(n, h_\mathcal{T}, \delta t), \tag{4.5}$$

with

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_\mathcal{T}\to 0\\ \delta t\to 0}}}\, \omega(n, h_\mathcal{T}, \delta t) = 0.$$

As far as I is concerned, we have $I = I_1 + I_2$ with

$$I_1 = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}\lambda(u)_\sigma^i \frac{h_n(u_K^i)h_{2n}(u_L^i) + h_n(u_L^i)h_{2n}(u_K^i)}{2}(u_K^i - u_L^i)(\varphi_K^i - \varphi_L^i),$$

$$I_2 = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} \frac{|\sigma|}{d_\sigma}\lambda(u)_\sigma^i \frac{h_n(u_K^i)h_{2n}(u_L^i) + h_n(u_L^i)h_{2n}(u_K^i)}{2} \\ \times (u_K^i - u_L^i)(\varphi_K^{i-1} - \varphi_K^i + \varphi_L^i - \varphi_L^{i-1}),$$

Both u_K^i and u_L^i being truncated in I_2 due to the regularity of φ we get

$$|I_2| \le C_\varphi(h_\mathcal{T} + \delta t)\left(\max_{r\in[-4n,4n]} |\lambda(r)|\right)\sum_{i=1}^N \delta t \|T_{4n}(u_\mathcal{T}^i)\|_{1,1,\mathcal{T}},$$

thus

$$\lim_{\substack{h_\mathcal{T}\to 0\\ \delta t\to 0}} I_2 = 0. \tag{4.6}$$

In view of the definitions of $\overline{h}_n^m$ and $\overline{\lambda}_n^m$ (see (3.66) and (3.67)) we then write $I_1 = I_{1,1} + I_{1,2}$ with

$$I_{1,1} = \int_0^T \int_\Omega \overline{\lambda}_{4n}^m\, \overline{h}_n^m \nabla_\mathcal{D} T_{4n}(u_\mathcal{D}) \cdot \nabla\varphi \, dx\, dt,$$

$$I_{1,2} = \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} d|D_\sigma|\lambda(u)_\sigma^i \frac{h_n(u_K^i)h_{2n}(u_L^i) + h_n(u_L^i)h_{2n}(u_K^i)}{2} \\ \frac{T_{4n}(u_K^i) - T_{4n}(u_L^i)}{d_\sigma} C(\varphi, \mathcal{T}, \delta t),$$

where $C(\varphi, \mathcal{T}, \delta t) = \frac{1}{\delta t |D_\sigma|}\int_{t_{i-1}}^{t_i}\int_{D_\sigma} \nabla\varphi\cdot\eta_{K,\sigma}\, dx\, dt - \frac{\varphi_L^i - \varphi_K^i}{d_\sigma}$. Thanks to the regularity of φ, we have

$$|I_{1,2}| \le C_\varphi(h_\mathcal{T} + \delta t)\left(\max_{r\in[-4n,4n]} |\lambda(r)|\right)\sum_{i=1}^N \delta t \|T_{4n}(u_\mathcal{T}^i)\|_{1,1,\mathcal{T}},$$

which implies

$$\lim_{\substack{h_\mathcal{T}\to 0\\ \delta t\to 0}} I_{1,2} = 0. \tag{4.7}$$

In view of Lemma 3.10 $\overline{h}_n^m$ (resp. $\overline{\lambda}_{4n}^m$) converges to $h_n(u)$ (resp. $\lambda(T_{4n}(u))$) while $\nabla_{\mathcal{D}} T_{4n}(u_{\mathcal{D}})$ tends to $\nabla T_{4n}(u)$ weakly in $L^2(0,T;L^2(\Omega)^d)$. Since φ belongs to $\mathcal{C}_c^\infty(\Omega\times[0,T))$ and since the support of h_n is $[-2n,2n]$ we obtain

$$\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}} I_1 = \int_0^T \int_\Omega \lambda(u) h_n(u) \nabla T_{4n}(u)\cdot\nabla\varphi\, dx\, dt. \tag{4.8}$$

For the convection term we have

$$\begin{aligned}
T_3 &= \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i (\varphi_K^{i-1} h_n(u_K^i) - \varphi_L^{i-1} h_n(u_L^i))\\
&= \sum_{i=1}^N \delta t \sum_{\substack{\sigma\in\mathcal{E}\\ \boldsymbol{v}_{K,\sigma}^i\geq 0}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i h_n(u_{\sigma,+}^i)(\varphi_K^{i-1} - \varphi_L^{i-1})\\
&\quad + \sum_{i=1}^N \delta t \sum_{\substack{\sigma\in\mathcal{E}\\ \boldsymbol{v}_{K,\sigma}^i\geq 0}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i \varphi_L^{i-1} (h_n(u_{\sigma,+}^i) - h_n(u_{\sigma,-}^i))\\
&\quad - \sum_{i=1}^N \delta t \sum_{\substack{\sigma\in\mathcal{E}\\ \boldsymbol{v}_{K,\sigma}^i< 0}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i h_n(u_{\sigma,+}^i)(\varphi_L^{i-1} - \varphi_K^{i-1})\\
&\quad - \sum_{i=1}^N \delta t \sum_{\substack{\sigma\in\mathcal{E}\\ \boldsymbol{v}_{K,\sigma}^i< 0}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i \varphi_K^{i-1} (h_n(u_{\sigma,+}^i) - h_n(u_{\sigma,-}^i))\\
&= T_{3,1} + T_{3,2} + T_{3,3}
\end{aligned}$$

with

$$\begin{aligned}
T_{3,1} &= \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i h_n(u_{\sigma,+}^i)(\varphi_K^{i-1} - \varphi_L^{i-1})\\
T_{3,2} &= \sum_{i=1}^N \delta t \sum_{\substack{\sigma\in\mathcal{E}\\ \boldsymbol{v}_{K,\sigma}^i\geq 0}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i \varphi_L^{i-1} (h_n(u_{\sigma,+}^i) - h_n(u_{\sigma,-}^i))\\
T_{3,3} &= -\sum_{i=1}^N \delta t \sum_{\substack{\sigma\in\mathcal{E}\\ \boldsymbol{v}_{K,\sigma}^i< 0}} |\sigma| \boldsymbol{v}_{K,\sigma}^i u_{\sigma,+}^i \varphi_K^{i-1} (h_n(u_{\sigma,+}^i) - h_n(u_{\sigma,-}^i)).
\end{aligned}$$

Since

$$|T_{3,2}+T_{3,3}| \leq \frac{c\|\varphi\|_\infty}{n} \sum_{i=1}^N \delta t \sum_{\sigma\in\mathcal{E}} |\sigma|\, |\boldsymbol{v}_{K,\sigma}^i|\, |u_{\sigma,+}^i|\, |T_{2n}(u_{\sigma,+}^i) - T_{2n}(u_{\sigma,-}^i)|,$$

using (3.6) we show that

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}} T_{3,2} + T_{3,3} = 0. \tag{4.9}$$

Concerning $T_{3,1}$ we have $T_{3,1} = T_{3,1,1} + T_{3,1,2}$ with

$$T_{3,1,1} = \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}^i_{K,\sigma} u^i_{\sigma,+} h_n(u^i_{\sigma,+})(\varphi^i_K - \varphi^i_L),$$
$$T_{3,1,2} = \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}^i_{K,\sigma} u^i_{\sigma,+} h_n(u^i_{\sigma,+})(\varphi^{i-1}_K - \varphi^i_K + \varphi^i_L - \varphi^{i-1}_L).$$

Still by the regularity of φ and using Hölder's inequality we obtain

$$|T_{3,1,2}| \le C_\varphi (h_\mathcal{T} + \delta t) \|\boldsymbol{v}\|_{L^2(Q)^d} \|T_{2n}(u_\mathcal{D})\|_{L^2(Q)},$$

which implies that

$$\lim_{\substack{h_\mathcal{T}\to 0\\ \delta t\to 0}} T_{3,1,2} = 0. \tag{4.10}$$

As far as $T_{3,1,1}$ is concerned we have

$$\begin{aligned} T_{3,1,1} &= \sum_{i=1}^{N} \delta t \sum_{\sigma\in\mathcal{E}} |\sigma| \boldsymbol{v}^i_{K,\sigma} u^i_{\sigma,+} h_n(u^i_{\sigma,+})(\varphi^i_K - \varphi^i_L) \\ &= \sum_{i=1}^{N} \sum_{\sigma\in\mathcal{E}} \frac{|\sigma| d_\sigma}{d|D_\sigma|}\, u^i_{\sigma,+}\, h_n(u^i_{\sigma,+})\, d\frac{\varphi^i_K - \varphi^i_L}{d_\sigma} \int_{t_{i-1}}^{t_i} \int_{D_\sigma} \boldsymbol{v}\cdot\eta_{K,\sigma}\, dx\, dt \\ &= -\sum_{i=1}^{N} \sum_{\sigma\in\mathcal{E}} \int_{D_\sigma} T_{2n}(u^i_{\sigma,+})\, h_n(u^i_{\sigma,+}) \boldsymbol{v}\cdot\nabla_\mathcal{D}\varphi_\mathcal{D}\, dx\, dt. \end{aligned}$$

Let us define $\overline{T_{2n}(u)h_n(u)}^{\mathcal{D}_m}$, $\forall t\in[t_{i-1},t_i)$, $\forall\sigma\in\mathcal{E}$, $\forall x\in D_\sigma$, by

$$\overline{T_{2n}(u)h_n(u)}^{\mathcal{D}_m}(x,t) = T_{2n}(u^i_{\sigma,+})\, h_n(u^i_{\sigma,+}).$$

We write

$$T_{3,1,1} = -\int_0^T \int_\Omega \overline{T_{2n}(u)h_n(u)}^{\mathcal{D}_m} \boldsymbol{v}\cdot\nabla_\mathcal{D}\varphi_\mathcal{D}\, dx.$$

By Lemma 2.7, $\nabla_\mathcal{D}\varphi_\mathcal{D}$ converges weakly to $\nabla\varphi$ in $(L^2(Q))^d$, and then in $(L^{(d+2)'}(Q))^d$. Moreover, in view of the definition of $\overline{T_{2n}(u)h_n(u)}^{\mathcal{D}_m}$, we can show that it tends to $T_{2n}(u)h_n(u)$ in L^∞ weak-$\star$ as $h_\mathcal{T}\to 0$ and $\delta t\to 0$. Then we obtain

$$\lim_{\substack{h_\mathcal{T}\to 0\\ \delta t\to 0}} T_{3,1,1} = -\int_0^T \int_\Omega T_{2n}(u)\, h_n(u)\, \boldsymbol{v}\cdot\nabla\varphi\, dx\, dt. \tag{4.11}$$

It remains to study the time derivative term. In view of the definition of $\widetilde{h}_n(v)$ we have

$$\widetilde{h}_n(u^{i-1}_K) - \widetilde{h}_n(u^i_K) = (u^{i-1}_K - u^i_K)h_n(u^i_K) + \int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r)h'_n(r)dr,$$

we write $T_1 = T_{1,1} + T_{1,2}$ with

$$T_{1,1} = \sum_{i=1}^{N} \delta t \sum_{K\in\mathcal{T}} |K| \frac{\widetilde{h}_n(u^i_K) - \widetilde{h}_n(u^{i-1}_K)}{\delta t} \varphi^{i-1}_K,$$
$$T_{1,2} = \sum_{i=1}^{N} \sum_{K\in\mathcal{T}} |K| \varphi^{i-1}_K \int_{u^i_K}^{u^{i-1}_K} (u^{i-1}_K - r)h'_n(r)dr.$$

Reordering the sums leads to

$$\begin{aligned} T_{1,1} &= -\sum_{i=1}^{N} \delta t \sum_{K\in\mathcal{T}} |K| \widetilde{h}_n(u_K^i) \frac{\varphi_K^i - \varphi_K^{i-1}}{\delta t} - \sum_{K\in\mathcal{T}} |K| \widetilde{h}_n(u_K^0)\varphi_K^0 \\ &= -\int_0^T \int_\Omega \widetilde{h}_n(u_\mathcal{T}) \partial_t \varphi \, dx \, dt + R_1 - \int_\Omega \widetilde{h}_n(u_\mathcal{T}^0)\varphi(x,0)\, dx + R_2, \end{aligned}$$

where

$$\begin{aligned} R_1 &= -\sum_{i=1}^{N} \delta t \sum_{K\in\mathcal{T}} |K| \widetilde{h}_n(u_K^i) \left[\frac{\varphi_K^i - \varphi_K^{i-1}}{\delta t} - \frac{1}{|K|\delta t} \int_{t_{i-1}}^{t_i} \int_K \partial_t \varphi \, dx \, dt \right], \\ R_2 &= -\sum_{K\in\mathcal{T}} \int_K \widetilde{h}_n(u_\mathcal{T}^0) \left[\varphi_K^0 - \varphi(x,0) \right] dx. \end{aligned}$$

By the regularity of φ we have

$$\begin{aligned} |R_1| &\le C_\varphi (h_\mathcal{T} + \delta t) \sum_{i=1}^{N} \delta t \sum_{K\in\mathcal{T}} |K| \, |\widetilde{h}_n(u_K^i)| \\ &\le C_\varphi (h_\mathcal{T} + \delta t) 2nT|\Omega| \end{aligned}$$

and $|R_2| \le C_\varphi h_\mathcal{T} 2nT|\Omega|$. Since $u_\mathcal{D}$ converges a.e. in Q to u and since $u_\mathcal{T}^0 = (u_K^0)_{K\in\mathcal{T}}$ converges in $L^1(\Omega)$ to u^0, recalling that $\widetilde{h}_n$ is bounded continuous function, Lebesgue theorem leads to

$$\lim_{\substack{h_\mathcal{T}\to 0 \\ \delta t\to 0}} T_{1,1} = -\int_0^T \int_\Omega \widetilde{h}_n(u) \partial_t \varphi \, dx \, dt - \int_\Omega \widetilde{h}_n(u^0)\varphi(x,0)\, dx. \tag{4.12}$$

Since

$$|T_{1,2}| \le \|\varphi\|_\infty \sum_{i=1}^{N} \sum_{K\in\mathcal{T}} |K| \int_{u_K^i}^{u_K^{i-1}} (u_K^{i-1} - r) T_{2n}'(r)\, dr,$$

(3.7) implies that

$$\lim_{n\to+\infty} \overline{\lim_{\substack{h_\mathcal{T}\to 0 \\ \delta t\to 0}}} T_{1,2} = 0. \tag{4.13}$$

We are now in a position to pass to the limit as $h_\mathcal{T} \to 0$ and $\delta t \to 0$ in the scheme. Gathering equations (4.1) to (4.13), we obtain that

$$\begin{aligned} &-\int_0^T \int_\Omega \widetilde{h}_n(u) \partial_t \varphi \, dx \, dt - \int_\Omega \widetilde{h}_n(u^0)\varphi(x,0)\, dx \\ &\quad + \int_0^T \int_\Omega \lambda(u)\, h_n(u)\, \nabla u \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_\Omega u\, h_n(u)\, \boldsymbol{v} \cdot \nabla \varphi \, dx \, dt \\ &\qquad\qquad - \int_0^T \int_\Omega f\, \varphi\, h_n(u)\, dx \, dt = \lim_{\substack{h_\mathcal{T}\to 0 \\ \delta t\to 0}} T(n,\varphi) \end{aligned} \tag{4.14}$$

where $\lim\limits_{\substack{h_\mathcal{T}\to 0 \\ \delta t\to 0}} |T(n,\varphi)| \le C(\|\varphi\|_{L^\infty(Q)})\omega(n)$ with $\omega(n) \to 0$ as $n \to +\infty$, where $C(\|\varphi\|_{L^\infty(Q)})$ depends on $\|\varphi\|_{L^\infty(Q)}$.

Step 2 In this step we obtain that the renormalized equation (1.6) holds.
It is worth noting that due to the time derivative $\partial_t\varphi$ in (4.14) equation (1.6) is not a straightforward consequence of (4.14) and a density argument. Using Steklov averages, Lemmas 2.1 and 2.3 of [10] (see also [7, 30]) and appropriate test functions

in (4.14) we can obtain that the renormalized equation (1.6) holds. The proof is postponed in Appendix, Proposition 5.1.

Step 3 We now prove that u satisfies the decay (1.5) of the truncate energy. Thanks to the discrete estimate on the energy (3.5) we have

$$\lim_{n\to\infty} \overline{\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}} \frac{1}{n}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)^i_\sigma(T_{2n}(u^i_K)-T_{2n}(u^i_L))^2=0,$$

and

$$\begin{aligned}
&\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}\frac{|\sigma|}{d_\sigma}\lambda(u)^i_\sigma(T_{2n}(u^i_K)-T_{2n}(u^i_L))^2\\
&\qquad=\sum_{i=1}^{N}\delta t\frac{1}{d}\sum_{\sigma\in\mathcal{E}}|D_\sigma|\lambda(u)^i_\sigma\left(d\frac{T_{2n}(u^i_K)-T_{2n}(u^i_L)}{d_\sigma}\right)^2\\
&\qquad\geq\frac{1}{d}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}|D_{K,\sigma}|h_n(u^i_K)\frac{\lambda(u^i_K)}{2}\left(d\frac{T_{2n}(u^i_K)-T_{2n}(u^i_L)}{d_\sigma}\right)^2\\
&\qquad+\frac{1}{d}\sum_{i=1}^{N}\delta t\sum_{\sigma\in\mathcal{E}}|D_{L,\sigma}|h_n(u^i_L)\frac{\lambda(u^i_L)}{2}\left(d\frac{T_{2n}(u^i_K)-T_{2n}(u^i_L)}{d_\sigma}\right)^2\\
&\qquad=\frac{1}{d}\int_0^T\int_\Omega h_n(u_{\mathcal{D}})\frac{\lambda(T_{2n}(u_{\mathcal{D}}))}{2}|\nabla_{\mathcal{D}}T_{2n}(u_{\mathcal{D}})|^2\,dx\,dt,
\end{aligned}$$

thus, $\lim_{n\to\infty}\overline{\lim_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}}\frac{1}{n}\int_0^T\int_\Omega h_n(u_{\mathcal{D}})\frac{\lambda(u_{\mathcal{D}})}{2}|\nabla_{\mathcal{D}}T_{2n}(u_{\mathcal{D}})|^2\,dx\,dt=0$. Since $\nabla T_{2n}(u_{\mathcal{D}})$ converges weakly in $L^2(Q)^d$, $\lambda(T_{2n}(u_{\mathcal{D}}))$ converges to $\lambda(T_{2n}(u))$ L^∞ weak$-\star$ and $h_n(u_{\mathcal{D}})$ converge to $h_n(u)$ almost everywhere in Q we also have

$$\frac{1}{n}\int_0^T\int_\Omega h_n(u)\lambda(u)|\nabla T_{2n}(u)|^2\,dx\,dt\leq\liminf_{\substack{h_{\mathcal{T}}\to 0\\ \delta t\to 0}}\frac{1}{n}\int_0^T\int_\Omega h_n(u_{\mathcal{D}})\lambda(u_{\mathcal{D}})|\nabla_{\mathcal{D}}T_{2n}(u_{\mathcal{D}})|^2\,dx\,dt,$$

which leads to (1.5).

Step 4 We prove here the uniqueness of the renormalized solution. In view of the convergence analysis, if the renormalized solution of (1.1) u is unique usual arguments imply that the whole sequence $u_{\mathcal{D}_m}$ converges to u. To our knowledge due to the nonlinearity of the operator and the presence of the term $-\mathrm{div}(\boldsymbol{v}(x,t)u)$ the uniqueness is proved under additional assumptions on λ or $\boldsymbol{v}$.

If λ is local Lipschitz continuous, namely

$$\forall n\in\mathbb{N},\ \exists C_n>0 \text{ such that } \forall|r|,|t|\leq n,\ |\lambda(r)-\lambda(t)|\leq C_n|r-t|,$$

then we are in the scope of [14] (see also [8]): the renormalized solution is unique.

If λ is a continuous function the price to paid to have uniqueness is to assume more regularity on $\boldsymbol{v}$. If $\boldsymbol{v}$ depends only on x then the time doubling variable method, see e.g. [28] (which is not the method developed in [8]), allows one to prove the uniqueness of the renormalized solution. Indeed the change of variable $w=\widetilde{\lambda}(u)$ where $\widetilde{\lambda}(r)=\int_0^r\lambda(s)\,ds$ implies that w is a renormalized solution of

$$\begin{aligned}
&\frac{\partial\widetilde{\lambda}^{-1}(w)}{\partial t}-\Delta w+\mathrm{div}(\boldsymbol{v}\widetilde{\lambda}^{-1}(w))=f \text{ in } Q,\\
&w=0 \text{ on } \partial\Omega\times(0,T),\ w(x,0)=\widetilde{\lambda}(u^0(x))\ \forall x\in\Omega.
\end{aligned}$$

Since the function $\widetilde{\lambda}(r)$ is local Lipschitz continuous and since $\boldsymbol{v}$ does not depend on t the results of [10] (see Section 5 Comments and Remarks) give the uniqueness of the renormalized solution w which in turn gives the uniqueness of u. □

5. Appendix

Proposition 5.1. *Let Ω be a bounded domain of $\mathbb{R}^N$, $T>0$. Let us assume that (2.9)–(2.12) hold and that u is a measurable function defined from $Q=\Omega\times(0,T)$ which verifies*

$$u\in L^1(Q);\quad T_n(u)\in L^2(0,T;H_0^1(\Omega)),\quad \forall n>0, \tag{5.1}$$

and for any $\varphi\in\mathcal{C}_c^\infty(\Omega\times[0,T))$, for any $n>0$

$$\begin{aligned}-\int_0^T\int_\Omega \widetilde{h}_n(u)\partial_t\varphi\,dx\,dt-\int_\Omega \widetilde{h}_n(u^0)\varphi(x,0)\,dx\\ +\int_0^T\int_\Omega \lambda(u)\,h_n(u)\,\nabla u\cdot\nabla\varphi\,dx\,dt-\int_0^T\int_\Omega u\,h_n(u)\,\boldsymbol{v}\cdot\nabla\varphi\,dx\,dt\\ -\int_0^T\int_\Omega f\,\varphi\,h_n(u)\,dx\,dt=\omega(n,\varphi)\end{aligned} \tag{5.2}$$

where $|\omega(n,\varphi)|\le C(\|\varphi\|_{L^\infty(Q)})\omega(n)$ with $\omega(n)\to 0$ as $n\to+\infty$ and where $C(\|\varphi\|_{L^\infty(Q)})$ depends on $\|\varphi\|_{L^\infty(Q)}$.

Then u verifies equation (1.6).

Proof. Due to the presence of the time derivative $\partial_t\varphi$ in (5.2) equation (1.6) is not a straightforward consequence of (4.14) and a density argument. We use here the Steklov averages which were used in [10] to deal with Stefan problems with L^1 data and renormalized solutions.

Step 1. Let ξ be a non negative function belonging to $\mathcal{C}_c^\infty(\Omega\times[0,T))$ and S a function belonging to $W^{2,\infty}(\mathbb{R})$ such that S' is nondecreasing, S'' has a compact support and $S'(u)\xi\in L^2(0,T;H_0^1(\Omega))$. It is worth noting that for $k>0$ large enough, $S'(u)$ read as $S'(T_k(u))$.

Our aim is now to prove that

$$\begin{aligned}-\int_0^T\int_\Omega \xi_t\int_0^u h_n(r)S'(r)drdx\,dt-\int_\Omega\xi(x,0)\int_0^{u_0}h_n(r)S'(r)drdx\\ +\int_0^T\int_\Omega\lambda(u)\,h_n(u)\nabla u\cdot\nabla\big(\xi S'(u)\big)dx\,dt-\int_0^T\int_\Omega u\,h_n(u)\,\boldsymbol{v}\cdot\nabla\big(\xi S'(u)\big)\,dx\,dt\\ -\int_0^T\int_\Omega f\,h_n(u)\,\xi S'(u)\,dx\,dt=\omega(n,\xi,S'),\end{aligned} \tag{5.3}$$

where $|\omega(n,\xi,S')|\le C(\|\xi\|_{L^\infty(Q)},\|S'\|_\infty)\omega(n)$ with $\omega(n)\to 0$ as $n\to+\infty$, where $C(\|\xi\|_{L^\infty(Q)},\|S'\|_\infty)$ depends on $\|\xi\|_{L^\infty(Q)}$ and $\|S'\|_{L^\infty(\mathbb{R})}$.

By density we can take $\varphi=\xi(S'(u))_h$ as test function in (5.2) where $(S'(u))_h$ is the (forward) Steklov average

$$(S'(u))_h(x,t)=\frac{1}{h}\int_t^{t+h}S'(u(x,\tau))\,d\tau,$$

so that we obtain for h small enough

$$
\begin{aligned}
(5.4)\quad & -\int_0^T\int_\Omega \partial_t\big(\xi(S'(u))_h\big)\big(\widetilde{h}_n(u)-\widetilde{h}_n(u^0)\big)\,dx\,dt \\
&+\int_0^T\int_\Omega \lambda(u)\,h_n(u)\nabla u\cdot\nabla\big(\xi(S'(u))_h\big)\,dx\,dt-\int_0^T\int_\Omega u\,h_n(u)\,\boldsymbol{v}\cdot\nabla\big(\xi(S'(u))_h\big)\,dx\,dt \\
&\qquad -\int_0^T\int_\Omega f\,h_n(u)\,\xi(S'(u))_h dx\,dt=\omega(n,\xi,S',h),
\end{aligned}
$$

where $|\omega(n,\xi,S',h)|\le C(\|\xi\|_{L^\infty(Q)},\|S'\|_\infty)\omega(n)$ with $\omega(n)\to 0$ as $n\to+\infty$ and where $C(\|\xi\|_{L^\infty(Q)},\|S'\|_\infty)$ depends on $\|\xi\|_{L^\infty(Q)}$ and on $\|S'\|_{L^\infty(\mathbb{R})}$.

Since $\xi(S'(u))_h$ converges to $\xi S'(u)$ strongly in $L^2(0,T;H^1_0(\Omega))$ and in L^∞ weak-$\star$ as h tends to 0, in view of the regularity (5.1) the main difficulty in dealing with the asymptotic behavior of the left hand-side of (5.4) is the time derivative term. We now apply Lemma 2.1 of [10] with $w=u$, $B(r)=\widetilde{h}_n(r)$, $\beta=B(u)$, $\beta_0=B(u_0)$, $w_0=u_0$ and $F(\lambda)=S'(\lambda)$ (S' being nondecreasing). It gives

$$
\begin{aligned}
&-\int_0^T\int_\Omega \partial_t\big(\xi(S'(u))_h\big)\big(\widetilde{h}_n(u)-\widetilde{h}_n(u^0)\big)\,dx\,dt \\
&\ge -\int_0^T\int_\Omega \xi_t\left(\widetilde{h}_n(u)\frac{1}{h}\int_t^{t+h}S'(u(\tau))d\tau-\frac{1}{h}\int_t^{t+h}\int_0^{u(\tau)}S''(r)\widetilde{h}_n(r)\,dr\,d\tau\right)dx\,dt \\
&\quad -\int_\Omega \xi(x,0)\left(\widetilde{h}_n(u^0)S'(u^0)-\int_0^{u^0}S''(r)\widetilde{h}_n(r)\,dr\right)dx,
\end{aligned}
$$

and letting h tends to zero we deduce that

$$
\begin{aligned}
&\liminf_{h\to 0}\left\{-\int_0^T\int_\Omega \partial_t\big(\xi(S'(u))_h\big)\big(\widetilde{h}_n(u)-\widetilde{h}_n(u^0)\big)\,dx\,dt\right\} \\
&\ge -\int_0^T\int_\Omega \xi_t\left(\widetilde{h}_n(u)S'(u)-\int_0^u S''(r)\widetilde{h}_n(r)\,dr\right)dx\,dt \\
&\quad -\int_\Omega \xi(x,0)\left(\widetilde{h}_n(u^0)S'(u^0)-\int_0^{u^0}S''(r)\widetilde{h}_n(r)\,dr\right)dx.
\end{aligned}
$$

Using the identity $\widetilde{h}_n(r)S'(r)-\int_0^r S''(s)\widetilde{h}_n(s)ds=\int_0^r h_n(s)S'(s)ds$ and injecting this previous inequality in (5.4), letting h tends to zero we get

$$
\begin{aligned}
(5.5)\quad & -\int_0^T\int_\Omega \xi_t\int_0^u h_n(r)S'(r)drdx\,dt-\int_\Omega \xi(x,0)\int_0^{u_0}h_n(r)S'(r)drdx \\
&+\int_0^T\int_\Omega \lambda(u)\,h_n(u)\nabla u\cdot\nabla\big(\xi S'(u)\big)dx\,dt-\int_0^T\int_\Omega u\,h_n(u)\,\boldsymbol{v}\cdot\nabla\big(\xi S'(u)\big)\,dx\,dt \\
&\qquad -\int_0^T\int_\Omega f\,h_n(u)\,\xi S'(u)\,dx\,dt\le C(\|\xi\|_{L^\infty(Q)},\|S'\|_\infty)\omega(n)
\end{aligned}
$$

where $\omega(n)\to 0$ as $n\to+\infty$ and where $C(\|\xi\|_{L^\infty(Q)},\|S'\|_\infty)$ depends on $\|\xi\|_{L^\infty(Q)}$ and on $\|S'\|_{L^\infty(\mathbb{R})}$.

We now prove the reverse inequality. We take $\varphi=\xi(S'(u))_{-h}$ as a test function in (5.2) where $(S'(u))_{-h}$ is the (backward) Steklov average

$$
(S'(u))_{-h}(x,t)=\frac{1}{h}\int_{t-h}^t S'(\hat{u}(x,\tau))\,d\tau
$$

with

$$\hat{u}(x,t) = \begin{cases} u(x,t) \text{ if } t \geq 0, \\ U_j \text{ if } t < 0, \end{cases}$$

and where U_j is a sequence in $\mathcal{C}_c^\infty(\Omega)$ such that $U_j \to u^0$ strongly in $L^1(\Omega)$. It follows that

$$\begin{aligned} (5.6) \quad & -\int_0^T \int_\Omega \partial_t\big(\xi(S'(u))_{-h}\big)(\widetilde{h}_n(u) - \widetilde{h}_n(u^0))\,dx\,dt \\ & + \int_0^T \int_\Omega \lambda(u)\, h_n(u) \nabla u \cdot \nabla\big(\xi(S'(u))_{-h}\big)\,dx\,dt - \int_0^T \int_\Omega u\, h_n(u)\, \boldsymbol{v} \cdot \nabla\big(\xi(S'(u))_{-h}\big)\,dx\,dt \\ & \qquad - \int_0^T \int_\Omega f\, h_n(u)\, \xi(S'(u))_{-h} dx\,dt = \omega_1(n, \xi, S', -h), \end{aligned}$$

where $|\omega_1(n,\xi,S',-h)| \leq C(\|\xi\|_{L^\infty(Q)}, \|S'\|_\infty)\omega(n)$ with $\omega(n) \to 0$ as $n \to +\infty$ and where $C(\|\xi\|_{L^\infty(Q)}, \|S'\|_\infty)$ depends on $\|\xi\|_{L^\infty(Q)}$ and on $\|S'\|_{L^\infty(\mathbb{R})}$.

We now apply Lemma 2.3 of [10] with $w = u$, $B(r) = \widetilde{h}_n(r)$, $\beta = B(u)$, $\beta_0 = B(u_0)$, $w_0 = U_j$ and $F(\lambda) = S'(\lambda)$. It gives

$$\begin{aligned} & -\int_0^T \int_\Omega \partial_t\big(\xi(S'(u))_{-h}\big)(\widetilde{h}_n(u) - \widetilde{h}_n(u^0))\,dx\,dt \\ & \leq -\int_0^T \int_\Omega \xi_t \left(\widetilde{h}_n(u) \frac{1}{h} \int_{t-h}^t S'(\hat{u}(\tau))d\tau - \frac{1}{h} \int_{t-h}^t \int_0^{\hat{u}(\tau)} S''(r) \widetilde{h}_n(r)\, dr\, d\tau \right) dx\,dt \\ & \quad - \int_\Omega \xi(x,0) \left(\widetilde{h}_n(u^0) S'(U_j) - \int_0^{U_j} S''(r) \widetilde{h}_n(r)\, dr \right) dx. \end{aligned}$$

We now let h tend to zero and then j tend to infinity to deduce that

$$\begin{aligned} & \limsup_{j \to +\infty} \limsup_{h \to 0} \left\{ -\int_0^T \int_\Omega \partial_t\big(\xi(S'(u))_{-h}\big)(\widetilde{h}_n(u) - \widetilde{h}_n(u^0))\,dx\,dt \right\} \\ & \leq -\int_0^T \int_\Omega \xi_t \left(\widetilde{h}_n(u) S'(u) - \int_0^u S''(r) \widetilde{h}_n(r)\, dr \right) dx\,dt \\ & \quad - \int_\Omega \xi(x,0) \left(\widetilde{h}_n(u^0) S'(u^0) - \int_0^{u^0} S''(r) \widetilde{h}_n(r)\, dr \right) dx. \end{aligned}$$

Since $\xi(S'(u))_{-h}$ converges to $\xi S'(u)$ strongly in $L^2(0,T; H_0^1(\Omega))$ and in L^∞ weak-$\star$, injecting this last inequality in (5.6) and letting h tends to zero (and then j tend to infinity) we get

$$\begin{aligned} & -C(\|\xi\|_{L^\infty(Q)}, \|S'\|_\infty)\omega(n) \leq -\int_0^T \int_\Omega \xi_t \int_0^u h_n(r) S'(r) dr dx\,dt \\ & \quad - \int_\Omega \xi(x,0) \int_0^{u_0} h_n(r) S'(r) dr dx + \int_0^T \int_\Omega \lambda(u)\, h_n(u) \nabla u \cdot \nabla\big(\xi S'(u)\big) dx\,dt \\ & \qquad - \int_0^T \int_\Omega u\, h_n(u)\, \boldsymbol{v} \cdot \nabla\big(\xi S'(u)\big)\,dx\,dt - \int_0^T \int_\Omega f\, h_n(u)\, \xi S'(u)\,dx\,dt, \end{aligned}$$

where $\omega(n) \to 0$ as $n \to +\infty$ and where $C(\|\xi\|_{L^\infty(Q)}, \|S'\|_\infty)$ depends on $\|\xi\|_{L^\infty(Q)}$ and on $\|S'\|_{L^\infty(\mathbb{R})}$.

It follows that the previous inequality and (5.5) yield (5.3).

Step 2. Let $\xi \in \mathcal{C}_c^\infty(\Omega \times [0,T))$ with $\xi \geq 0$ and let $S \in W^2(\mathbb{R})$ such that S' has a compact support. By writing for example $S'(r) = S'(0) + \int_0^r (S''(s))^+ ds -$

$\int_0^r (S''(s))^- ds$ we can decompose the function S into $S = S_1 - S_2$ where S_1, S_2 belongs to $W^{2,\infty}(\mathbb{R})$, S_1'' and S_2'' have compact support and S_1', S_2' are nondecreasing.

Using the previous step for S_1 and S_2 and subtracting the two inequalities it follows that

$$
\begin{aligned}
(5.7)\quad & -\int_0^T \int_\Omega \xi_t \int_0^u h_n(r) S'(r) dr dx\, dt - \int_\Omega \xi(x,0) \int_0^{u_0} h_n(r) S'(r) dr dx \\
& + \int_0^T \int_\Omega \lambda(u)\, h_n(u) \nabla u \cdot \nabla\big(\xi S'(u)\big) dx\, dt - \int_0^T \int_\Omega u\, h_n(u)\, \boldsymbol{v} \cdot \nabla\big(\xi S'(u)\big)\, dx\, dt \\
& - \int_0^T \int_\Omega f\, h_n(u)\, \xi S'(u)\, dx\, dt = \omega(n, \xi, S'),
\end{aligned}
$$

where $|\omega(n,\xi,S')| \le C(\|\xi\|_{L^\infty(Q)}, \|S'\|_\infty)\omega(n)$ with $\omega(n) \to 0$ as $n \to +\infty$, where $C(\|\xi\|_{L^\infty(Q)}, \|S'\|_\infty)$ depends on $\|\xi\|_{L^\infty(Q)}$ and $\|S'\|_{L^\infty(\mathbb{R})}$.

Since S' has a compact support, in view of the regularity of u we are in a position to pass to the limit in (5.5) as n tends to infinity. It follows that

$$
\begin{aligned}
& -\int_0^T \int_\Omega \xi_t \int_0^u S'(r) dr dx\, dt - \int_\Omega \xi(x,0) \int_0^{u_0} S'(r) dr dx \\
& + \int_0^T \int_\Omega \lambda(u)\, \nabla u \cdot \nabla\big(\xi S'(u)\big) dx\, dt - \int_0^T \int_\Omega u\, \boldsymbol{v} \cdot \nabla\big(\xi S'(u)\big)\, dx\, dt \\
& - \int_0^T \int_\Omega f\, \xi S'(u)\, dx\, dt = 0,
\end{aligned}
$$

which is (1.6) for $\xi \ge 0$. By replacing ξ by $-\xi$, S by $-S$ we obtain that (1.6) holds for $\xi \le 0$ and any $S \in W^2(\mathbb{R})$ such that S' has a compact support.

At last since (1.6) is linear with respect to ξ we obtain that (1.6) holds true. □

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