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# Maximum independent sets in (pyramid, even hole)-free graphs

Maria Chudnovsky\*, Stéphan Thomassé †, Nicolas Trotignon† and Kristina Vušković‡

December 25, 2019

## Abstract

A *hole* in a graph is an induced cycle with at least 4 vertices. A graph is *even-hole-free* if it does not contain a hole on an even number of vertices. A *pyramid* is a graph made of three chordless paths  $P_1 = a \dots b_1$ ,  $P_2 = a \dots b_2$ ,  $P_3 = a \dots b_3$  of length at least 1, two of which have length at least 2, vertex-disjoint except at  $a$ , and such that  $b_1 b_2 b_3$  is a triangle and no edges exist between the paths except those of the triangle and the three edges incident with  $a$ .

We give a polynomial time algorithm to compute a maximum weighted independent set in a even-hole-free graph that contains no pyramid as an induced subgraph. Our result is based on a decomposition theorem and on bounding the number of minimal separators. All our results hold for a slightly larger class of graphs, the class of (square, prism, pyramid, theta, even wheel)-free graphs.

## 1 Introduction

In this article, graphs are finite and simple. A *hole* in a graph is an induced cycle with at least 4 vertices. The *length* of a hole is the number of vertices in it. A graph  $G$  *contains* a graph  $H$  if some induced subgraph of  $G$  is isomorphic to  $H$ . A graph  $G$  is *H-free* if it does not contain  $H$ . When  $\mathcal{H}$  is a set of graphs,  $G$  is *H-free* if it is  $H$ -free for all  $H$  in  $\mathcal{H}$ .

The class of even-hole-free graphs was the object of much research (see [10] for a survey). However, the complexity of computing a maximum independent set in an even-hole-free graph is not known.

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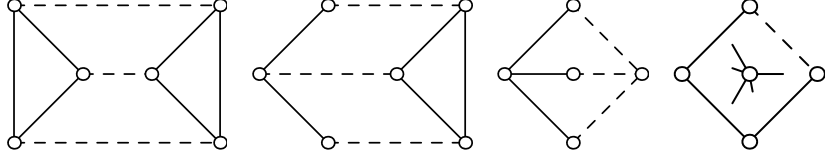


Figure 1: Prism, pyramid, theta and wheel (dashed lines represent paths)

A *pyramid* is a graph made of three chordless paths  $P_1 = a \dots b_1$ ,  $P_2 = a \dots b_2$ ,  $P_3 = a \dots b_3$  of length at least 1, two of which have length at least 2, vertex-disjoint except at  $a$ , and such that  $b_1 b_2 b_3$  is a triangle and no edges exist between the paths except those of the triangle and the three edges incident with  $a$ . See Fig. 1.

Our main result is a polynomial time algorithm to compute a maximum weighted independent set in an (even-hole, pyramid)-free graph. Our approach is by first proving a decomposition theorem for the class of (even-hole, pyramid)-free graph. This theorem might have other applications because the presence of a pyramid in an even-hole-free graphs places significant restrictions on its structure. The graphs seems to "organize itself" around the pyramid, in a way that can likely be exploited algorithmically. Results in this direction appear in Chudnovsky and Seymour [6]. So our result on the pyramid-free case might help to understand the full class of even-hole-free graphs. We use our decomposition theorem to prove that (even-hole, pyramid)-free graph contain polynomially many minimal separators (to be defined in the next section). In fact, we prove this property for a slightly larger class of graphs, namely the (theta, pyramid, prism, even wheel, square)-free graphs. And as we explain in the next section, this property implies the existence of a polynomial time algorithm to compute maximum weighted independent sets.

In section 2 we state formally our main results and motivate them further. In section 3, we prove the decomposition theorem. In section 4, we give several properties of minimal separators in our class of graphs. In section 5, we prove that graphs in our class contain polynomially many minimal separators.

## Notation

Let  $G$  be a graph. By a *path* we mean a chordless (or induced) path. When  $P$  is a path in  $G$ , we denote by  $P^*$  the path induced by the internal vertices of  $P$ . When  $a$  and  $b$  are vertices of a path  $P$ , we denote by  $aPb$  the subpath of  $P$  with ends  $a$  and  $b$ . A *clique* in a graph is a set of pairwise adjacent vertices.

When  $A, B \subseteq V(G)$ , we denote by  $N_B(A)$  the set of vertices of  $B \setminus A$  that have at least one neighbor in  $A$  and  $N(A)$  means  $N_{V(G)}(A)$ . Note that  $N_B(A)$  is disjoint from  $A$ . We write  $N(a)$  instead of  $N(\{a\})$  and  $N[a]$  for  $\{a\} \cup N(a)$ . We denote by  $G[A]$  the subgraph of  $G$  induced by  $A$ . To avoid too heavy notation, since there is no risk of confusion, when  $H$  is an induced subgraph of  $G$ , we write  $N_H$  instead of  $N_{V(H)}$ .

A vertex  $x$  is *complete* (resp. *anticomplete*) to  $A$  if  $x \notin A$  and  $x$  is adjacent to all vertices of  $A$  (resp. to no vertex of  $A$ ). We say that  $A$  is *complete* (resp. *anticomplete*) to  $B$  if every vertex of  $A$  is complete (resp. anticomplete) to  $B$  (note that this means in particular that  $A$  and  $B$  are disjoint).

## 2 Results

Let  $G$  be a graph and  $a, b \in V(G)$ . A set  $C \subseteq V(G)$  is an *minimal  $(a, b)$ -separator* if  $a$  and  $b$  are in distinct components of  $G \setminus C$  and  $C$  is minimal with this property. We say that  $C$  is a *minimal separator* if  $C$  is a minimal  $(a, b)$ -separator for some pair  $a, b$ .

It is easy to check that a minimal separator in a graph  $G$  can be equivalently defined as a set  $C \subseteq V(G)$  such that  $G \setminus C$  has a connected component  $L$  and a connected component  $R$  such that every vertex of  $C$  has neighbors in both  $L$  and  $R$ . Note that  $G \setminus C$  has possibly more connected components.

Say that a class  $\mathcal{C}$  of graphs has the *polynomial separator property* if there exists  $bc$  such that every graph  $G$  in  $\mathcal{C}$  has at most  $|V(G)|^{bc}$  minimal separators. As explained by Chudnovsky, Pilipczuk, Pilipczuk and Thomassé in [5] (see also the end of Section 5), it follows from results of Bouchité and Todinca [3, 4] that for any class of graphs, having the polynomial separator property implies that the Maximum Weighted Independent Set Problem can be solved in polynomial time.

We are therefore interested in finding classes of graphs where the number of minimal separators is bounded by some polynomial. To gain insight on this question, let us survey examples of graphs with exponentially many minimal separators.

For an integer  $k \geq 1$ , the  *$k$ -prism* is the graph consisting of two cliques on  $k$  vertices, and a  $k$ -edge matching between them. More precisely, the  $k$ -prism  $G$  has vertex set  $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ , each of the sets  $\{a_1, \dots, a_k\}$  and  $\{b_1, \dots, b_k\}$  is a clique,  $a_i b_i \in E(G)$  for every  $i \in \{1, \dots, k\}$ , and there are no other edges in  $G$ . See Fig. 2. As observed in [5], it is easy to check that a  $k$ -prism has  $2^k - 2$  minimal separators. This suggests that not containing a big matching plays a role in bounding the number of minimal separators, and indeed a simple theorem can be proved in this direction. Call  *$k$ -semi-induced matching* any graph whose vertex set can be partitioned into two sets  $X = \{x_1, \dots, x_k\}$  and  $Y = \{y_1, \dots, y_k\}$  such that the only edges between  $X$  and  $Y$  are the edges  $x_i y_i$  ( $i = 1, \dots, k$ ). The edges among vertices of  $X$  and vertices of  $Y$  are unrestricted.

**Theorem 2.1** *For every  $k$ , every graph  $G$  on  $n$  vertices that contains no  $k$ -semi-induced matching has at most  $O(n^{2k-2})$  minimal separators that can be enumerated in time  $O(n^{2k})$ .*

*Proof.* Let  $a$  and  $b$  be two non-adjacent vertices in a graph  $G$  that does not contain a  $k$ -semi-induced matching, and let  $C$  be a minimal separator separating them. Call  $A$  and  $B$  the components of  $G \setminus C$  that contain  $a$  and  $b$  respectively. By minimality of  $C$ , every vertex in  $C$  has a neighbor in  $A$ . It is therefore well defined to consider an inclusion-wise

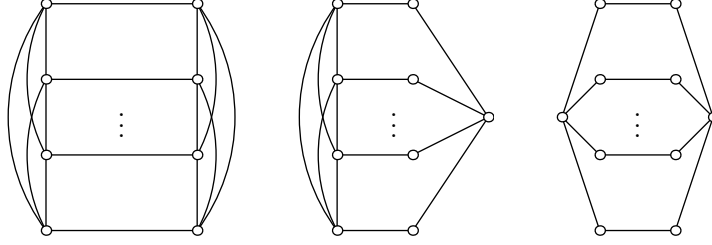


Figure 2:  $k$ -prism,  $k$ -pyramid,  $k$ -theta

minimal subset  $X_A$  of  $A$  such that  $C \subseteq N(X_A)$ . For every  $x \in X_A$ , there exists a vertex  $c \in C$  such that  $xc \in E(G)$  and no other vertex of  $X_A$  is adjacent to  $c$ . For otherwise,  $X_A \setminus \{x\}$  would contradict the minimality of  $X_A$ . It follows that  $G[X_A \cup C]$  contains an  $|X_A|$ -semi-induced matching, so  $|X_A| < k$ . We may define a similar set  $X_B$ , and we observe that  $C = N(X_A) \cap N(X_B)$ .

From the previous paragraph, the following algorithm enumerates all minimal separators of  $G$ : for every pair of sets  $X_A, X_B$  of cardinality less than  $k$ , compute  $C = N(X_A) \cap N(X_B)$  and check whether  $C$  is a minimal separator. Since  $\binom{n}{i} \leq n^i$ , we have  $\binom{n}{0} + \dots + \binom{n}{k-1} \leq kn^{k-1}$ , so the algorithm enumerates at most  $O(n^{2k-2})$  minimal separators in time  $O(n^{2k})$ .  $\square$

Results in other directions can be proved. Chudnovsky, Pilipczuk, Pilipczuk, and Thomassé [5] proved a graph  $G$  that contains no  $k$ -prism and no hole of length at least 5 has at most  $|V(G)|^{k+2}$  minimal separators. But since we are interested in even-hole-free graphs, we do not want to exclude odd holes.

In Fig. 2, variants of  $k$ -prisms are shown. They are obtained from  $k$ -prisms by subdividing the matching edges (once or twice) and contracting one or two of the cliques into a single vertex. We call these graphs  $k$ -pyramids and  $k$ -thetas. They are all easily checked to contain exponentially many minimal separators, and we do not define them more formally.

From these three examples, we can see that the so-called *3-path configurations* are maybe important to understand minimal separators. They are defined as being the pyramids (that we already know) and the thetas and prisms that we define now (see Fig. 1).

A *theta* is a graph made of three internally vertex-disjoint chordless paths  $P_1 = a \dots b$ ,  $P_2 = a \dots b$ ,  $P_3 = a \dots b$  of length at least 2 and such that no edges exist between the paths except the three edges incident with  $a$  and the three edges incident with  $b$ .

A *prism* is a graph made of three vertex-disjoint chordless paths  $P_1 = a_1 \dots b_1$ ,  $P_2 = a_2 \dots b_2$ ,  $P_3 = a_3 \dots b_3$  of length at least 1, such that  $a_1 a_2 a_3$  and  $b_1 b_2 b_3$  are triangles and no edges exist between the paths except those of the two triangles.

The examples of graphs with exponentially many separators that we have shown so far all contain a theta, a pyramid or a prism. But excluding them is not enough to guaranty

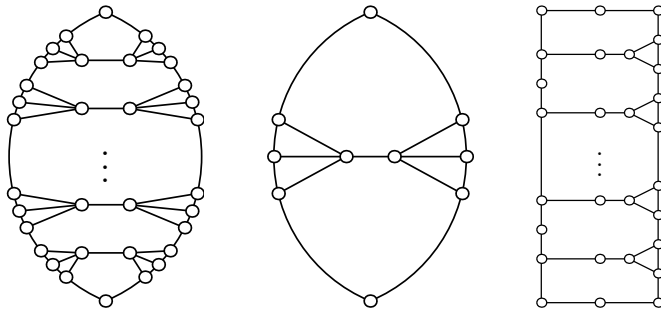


Figure 3:  $k$ -turtle, turtle and  $k$ -ladder

a polynomial number of minimum separators. In Fig. 3 a construction of graphs with no theta, no pyramid and no prism with exponentially many minimal separator is shown. We call this construction a  $k$ -turtle, and again it is easy to check that a  $k$ -turtle contains exponentially many minimal separators.

The  $k$ -turtles suggest defining a *turtle* as a graph made of two internally vertex-disjoint chordless paths  $P_1 = u \dots v$ ,  $P_2 = u \dots v$  that form a hole. Moreover, there are two adjacent vertices  $x, y$  not in the paths, such that  $x$  has at least three neighbors in  $P_1$  (and none in  $P_2$ ) and  $y$  has at least three neighbors in  $P_2$  (and none in  $P_1$ ), see Fig. 3. A  $k$ -turtle contains a turtle, in the same way as a  $k$ -prism contains a prism, a  $k$ -pyramid contains a pyramid and a  $k$ -theta contains a theta. In Fig. 3 is also represented a construction that we call the  $k$ -ladder, that provides examples of even-hole-free graphs with maximum degree 3 and exponentially many minimal separators.

Since we are not able to imagine examples of graphs with exponentially many minimal separators containing no prism, pyramid, theta or turtle, we propose the following conjecture, that would be in some sense the best possible statement regarding bounding the number of minimal separators.

**Conjecture 2.2** *There is a polynomial  $P$  such that every graph  $G$  that contains no prism, pyramid, theta or turtle has at most  $P(|V(G)|)$  minimal separators.*

Since we are interested in even-hole-free graphs, it is worth observing that every prism, theta and turtle contains an even hole. For theta and prism, this is because at least two of the three paths must have the same parity and therefore form an even hole. For turtles, it is because every turtle contains an even wheel. Let us define them.

A *wheel* is a graph made of a hole  $H$  called the *rim* and a vertex  $v$  called the *center* that has at least three neighbors in  $H$  (see Fig. 1). An *even wheel* is a wheel whose center has an even number of neighbors in the rim. It is easy to check that every turtle contains an even wheel, and that every even wheel contain an even hole.

A weakening of Conjecture 2.2 is therefore obtained by restricting it to (prism, pyramid, theta, even wheel)-free graphs. Note that (prism, theta, even wheel)-free graphs have been studied under the name of *odd-signable graphs* and they seem to capture essential properties of even-hole-free graphs, for more about them see the survey of Vušković [10]. Interestingly, prisms, pyramids, thetas and wheels are called *Truemper configurations* and they play an important role in many decomposition theorems for classes of graphs, see [11] for a survey. But we were not able to prove that (prism, pyramid, theta, even wheel)-free graphs have polynomially many minimal separators. However, we can prove that if we also exclude *squares* (holes of length 4), then the number of minimal separators is polynomially bounded.

We call  $\mathcal{C}$  the class of (square, prism, pyramid, theta, even wheel)-free graphs. Observe that  $\mathcal{C}$  is a superclass of the class of (even hole, pyramid)-free graphs. Here is our main result (proved in Section 5).

**Theorem 2.3** *Every graph in  $\mathcal{C}$  on  $n$  vertices contains at most  $O(n^8)$  minimal separators. There is an algorithm of complexity  $O(n^{10})$  that enumerates them. Consequently, there exists a polynomial time algorithm for the Maximum Weighted Independent Set restricted to  $\mathcal{C}$ .*

To prove Theorem 2.3, we rely on a decomposition theorem for  $\mathcal{C}$ . To state it, we need terminology. When  $H$  is a hole in some graph and  $u$  is a vertex not in  $H$  with at least two neighbors in  $H$ , we call  *$u$ -sector* of  $H$  any path of  $H$  of length at least 1, whose ends are adjacent to  $u$  and whose internal vertices are not. Observe that  $H$  is edgewise partitioned into its  $u$ -sectors. Let  $H$  be a hole in a graph and let  $u$  be a vertex not in  $H$ . We say that  $u$  is *major* w.r.t.  $H$  if  $N_H(u)$  is not included in a 3-vertex path of  $H$ . The decomposition theorem is the following (proved in Section 3).

**Theorem 2.4** *Let  $G$  be a graph in  $\mathcal{C}$ ,  $H$  a hole in  $G$  and  $w$  a major vertex w.r.t.  $H$ . If  $C$  is a connected component of  $G \setminus N[w]$ , then there exists a  $w$ -sector  $P = x \dots y$  of  $H$  such that  $N(C) \subseteq \{x, y\} \cup (N(w) \setminus V(H))$ .*

## Lower bounds for the number of minimal separators in $\mathcal{C}$

For every integer  $k$ , there exists a graph in  $\mathcal{C}$  with at least  $O(k^2)$  minimal separators, and this is the best lower bound that we have so far. A simple example of this phenomenon is a chordless cycle of length  $k \geq 5$  (any pair of non-adjacent vertices is a minimal separator). Another example  $G_k$  is maybe worth mentioning because it does not contain holes of length greater than 5. Let us describe  $G_k$ . Consider four cliques  $X$ ,  $Y$ ,  $X'$  and  $Y'$ , each on  $k$  vertices. Set  $X = \{x_1, \dots, x_k\}$ ,  $Y = \{y_1, \dots, y_k\}$ ,  $X' = \{x'_1, \dots, x'_k\}$ ,  $Y' = \{y'_1, \dots, y'_k\}$ . Add a vertex  $z$ . Add all possible edges between  $z$  and  $X \cup X'$ . Add all possible edges between  $Y$  and  $Y'$ . For every  $i = 1, \dots, k$ , add all possible edges from  $x_i$  to

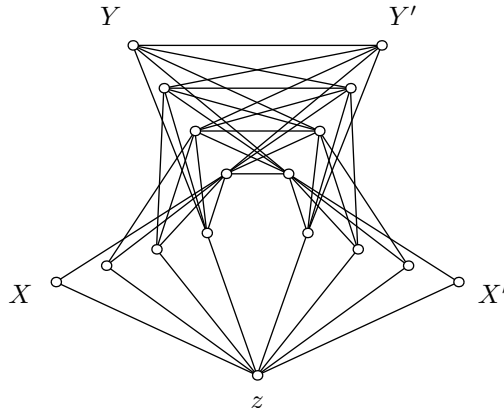


Figure 4: Graph  $G_4$  (edges of the cliques  $X$ ,  $Y$ ,  $X'$  and  $Y'$  are not represented)

$\{y_{k-i+1}, y_{k-i+2}, \dots, y_k\}$  and all possible edges from  $x'_i$  to  $\{y'_{k-i+1}, y'_{k-i+2}, \dots, y'_k\}$ . These are all the vertices and edges of  $G_k$ , see Fig. 4 where  $G_4$  is represented.

It is straightforward to check that  $G_k \in \mathcal{C}$ . To do so, it is convenient to note that every hole  $H$  in  $G_k$  must go through  $z$  and contains exactly one vertex in each of the sets  $X, X', Y$  and  $Y'$ . So every hole in  $G_k$  has length 5. Since squares, even wheels, thetas and prisms all contain even holes, the only obstruction that may exist in  $G_k$  is the pyramid. But  $G_k$  cannot contain it since in a pyramid there exists a vertex whose neighborhood contains three non-adjacent vertices, and this does not exist in  $G_k$ .

For every  $i \in \{1, \dots, k\}$ , set  $C_i = \{x_i, x_{i+1}, \dots, x_k\} \cup \{y_{k-i+2}, y_{k-i+3}, \dots, y_k\}$ . Note that for  $i \in \{1, \dots, k\}$ ,  $y_1 \notin C_i$  and  $C_1 \cap Y = \emptyset$ . For every  $j \in \{1, \dots, k\}$ , set  $C'_j = \{x'_j, x'_{j+1}, \dots, x'_k\} \cup \{y'_{k-j+2}, y'_{k-j+3}, \dots, y'_k\}$ . It is now easy to check that  $C_i \cup C'_j$  is a minimal separator (separating  $z$  from  $y_1$ ) for every pair  $(i, j)$  in  $\{1, \dots, k\}^2$ . Since  $|V(G_k)| = 4k + 1$  and there are  $k^2$  pairs  $(i, j)$  in  $\{1, \dots, k\}^2$ ,  $G_k$  has  $O(k^2)$  minimal separators.

## Rankwidth and semi-induced matchings in $\mathcal{C}$

One may suspect that graphs in  $\mathcal{C}$  are very “simple” in some sense (in which case our result would be less interesting). And it is not so easy to exhibit graphs from  $\mathcal{C}$  that are “complex”, so it is worth explaining here how to build such graphs. To measure the complexity of a graph we use the notion of rankwidth, that is equivalent to the notion of cliquewidth, in the sense that a class of graphs has unbounded rankwidth if and only if it has unbounded cliquewidth (see [7] for more about cliquewidth).

To provide graphs in  $\mathcal{C}$  of arbitrarily large rankwidth, it suffices to note that every hole-free graph (better known as *chordal graphs*) is in  $\mathcal{C}$ . Chordal graphs are known to have unbounded rankwidth. But chordal graphs are in some sense “simple”: they are all



complete graphs or have clique separators, and many problems can be solved in polynomial time for them (see [11] for more about that).

In [1] Adler, Le, Müller, Radovanović, Trotignon and Vušković describe even-hole-free graphs of arbitrarily large rankwidth. They are also diamond-free (where the *diamond* is the graph on vertices  $a, b, c, d$  with all possible edges except  $ab$ ) and they have no clique separator. So, to the best of our knowledge, they are “complex”. The only problem is that they are not in  $\mathcal{C}$  because they contain pyramids. We now explain how to modify graphs defined in [1] to obtain graphs in  $\mathcal{C}$ .

Graphs defined in [1] all vertex-wise partition into a path  $P$  and a clique  $K$ . An example is represented in Fig. 5, where  $|K| = 4$  and  $P$  is represented as a circle around  $K$ . Every vertex of  $K$  has neighbors in  $P$  and every vertex of  $P$  has at most one neighbor in  $K$ . These graphs contain pyramids that are built as follows: take three vertices  $a, b, c$  of  $K$  that induce a triangle, and consider neighbors  $a', b', c'$  of  $a, b, c$  respectively in  $P$ . Suppose that  $a', b', c'$  are chosen such that  $a'Pc'$  has no neighbor of  $a$  and  $c$  in its interior and  $b'$  is the unique neighbor of  $b$  in it. The pyramid is then formed by  $a'Pc'$  and  $abc$ .

To avoid pyramids in graphs from [1], take such a graph  $G$  and apply the following algorithm to it:

While there exist a 6-tuple  $(a, b, c, a', b', c')$  as above, denote by  $x$  and  $y$  the two neighbors of  $b'$  in  $P$ . Remove  $b'$  from  $P$ , replace it by a path  $xp_1p_2p_3p_4p_5p_6p_7y$  and add the following edges:  $bp_1$ ,  $bp_4$  and  $bp_7$ . Note that the obtained graph is still vertex-wise partitioned into a clique and a path, so that our procedure can be applied repeatedly.

Each time, the number of pyramids in the graph decreases so that the algorithm terminates. From the proofs in [1], it is easy to check that we obtain graphs in  $\mathcal{C}$  that have unbounded rankwidth. We omit further details that can be found in [1].

We observe that graphs from [1] may contain arbitrarily large semi-induced matchings, so that our main result cannot be a simple corollary of Theorem 2.1.

### 3 Decomposing graphs in $\mathcal{C}$

Recall that when  $H$  is a hole in some graph and  $u$  is a vertex not in  $H$  with at least two neighbors in  $H$ , we call  $u$ -*sector* of  $H$  any path of  $H$  of length at least 1, whose ends are adjacent to  $u$  and whose internal vertices are not. Observe that  $H$  is edgewise partitioned into its  $u$ -sectors.

Let  $H$  be a hole in a graph and let  $u$  be a vertex not in  $H$ . We say that  $u$  is *major* w.r.t.  $H$  if  $N_H(u)$  is not included in a 3-vertex path of  $H$ . We omit “w.r.t.  $H$ ” when  $H$  is clear from the context.

**Lemma 3.1** *In every graph in  $\mathcal{C}$ , every major vertex  $u$  w.r.t. a hole  $H$  has at least five neighbors in  $H$  or has exactly three neighbors in  $H$  that are pairwise non-adjacent.*

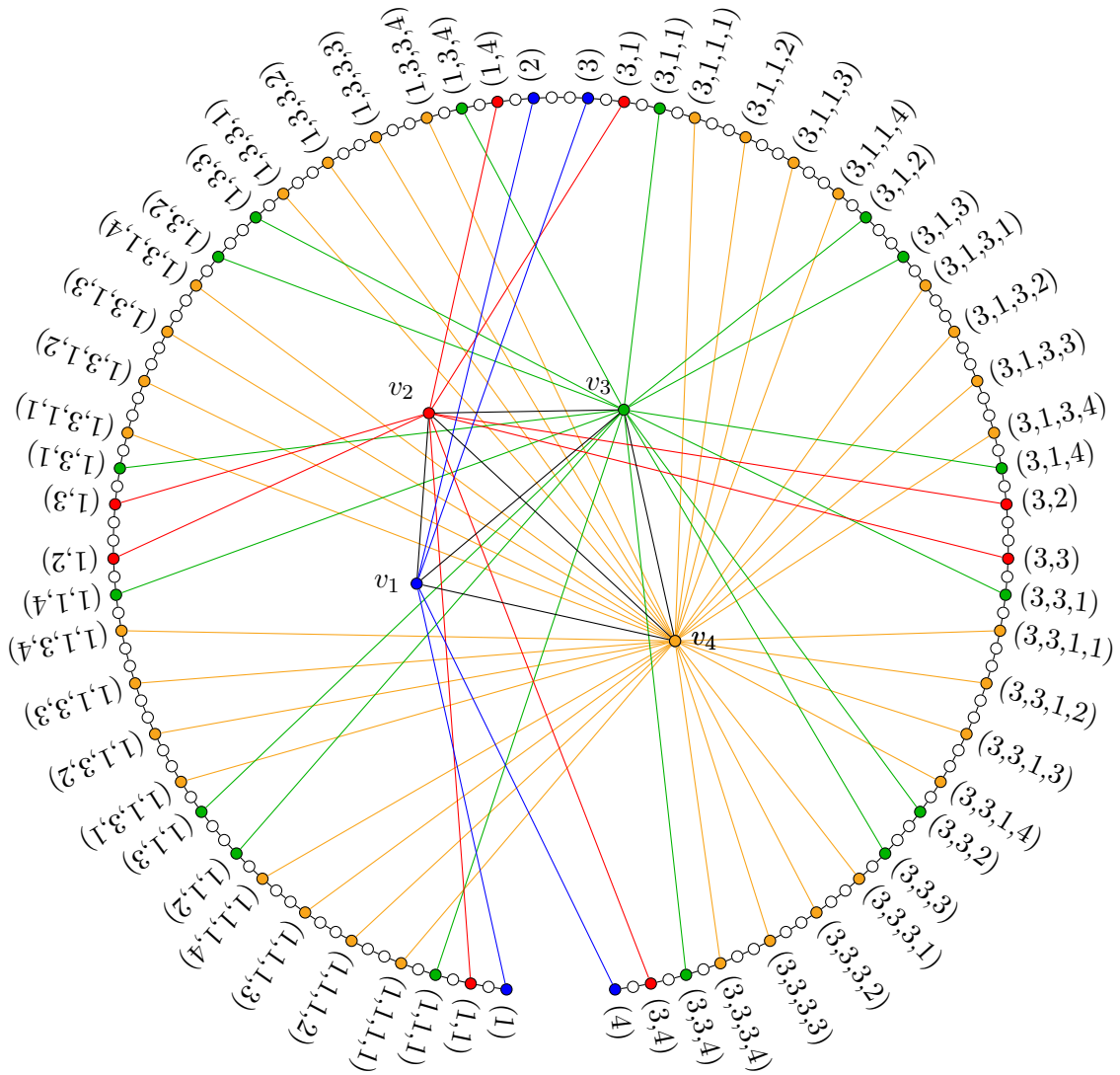


Figure 5: A graph from [1] (the clique contains 4 vertices)

*Proof.* If  $u$  has exactly two neighbors in  $H$ , since these two neighbors are not included in a 3-vertex path, they are non-adjacent. Hence,  $u$  and  $H$  form a theta, a contradiction. If  $u$  has exactly three neighbors in  $H$ , since these are not included a 3-vertex path, they are pairwise non-adjacent for otherwise  $u$  and  $H$  form a pyramid. If  $u$  has exactly four neighbors in  $H$ , then  $u$  and  $H$  form an even wheel, a contradiction.  $\square$

It follows from Lemma 3.1 that if  $u$  is major w.r.t. a hole  $H$ , then  $(H, u)$  is a wheel. We will use this fact throughout the paper. A vertex that is not major with respect to some hole  $H$  and still has neighbors in  $H$  is *minor* w.r.t.  $H$ .

**Lemma 3.2** *In a graph from  $\mathcal{C}$ , every minor vertex  $u$  w.r.t. a hole  $H$  satisfies one of the following.*

- $u$  has a unique neighbor in  $H$  (we then say that  $u$  is pending w.r.t.  $H$ ).
- $u$  has two neighbors in  $H$  which are adjacent (we then say that  $u$  is a cap w.r.t.  $H$ ).
- $u$  has three neighbors in  $H$  which induce a path  $xyz$  (we then say that  $u$  is a clone of  $y$  w.r.t.  $H$ ).

*Proof.* Otherwise,  $u$  has two non-adjacent neighbors in  $H$ , so  $u$  and  $H$  form a theta.  $\square$

When  $H$  is hole and  $u$  a clone of  $y$  w.r.t.  $H$ , we denote by  $H_{u \setminus y}$  the hole induced by  $\{u\} \cup V(H) \setminus \{y\}$ . Observe that  $y$  is a clone of  $u$  w.r.t.  $H_{u \setminus y}$ .

**Lemma 3.3** *Let  $H$  be a hole in a graph  $G \in \mathcal{C}$  and  $u$  be a clone of  $y$  w.r.t.  $H$ . Let  $v$  be a major vertex w.r.t.  $H$ . Then,  $vu \in E(G)$  if and only if  $vy \in E(G)$ . In particular, a vertex is major w.r.t.  $H$  if and only if it is major w.r.t.  $H_{u \setminus y}$ .*

*Proof.* Suppose that  $v$  is adjacent to exactly one of  $u, y$ . Since  $v$  is major w.r.t.  $H$ ,  $(H, v)$  is a wheel. If  $(H_{u \setminus y}, v)$  is also a wheel, then one of  $(H, v)$ ,  $(H_{u \setminus y}, v)$  is an even wheel, a contradiction. So,  $v$  has exactly two neighbors in  $H_{u \setminus y}$ , and hence exactly three neighbors in  $H$ . By Lemma 3.1 the neighbors of  $v$  in  $H$  are non-adjacent, but by Lemma 3.2, the neighbors of  $v$  in  $H_{u \setminus y}$  are adjacent, a contradiction.  $\square$

**Lemma 3.4** *Let  $u$  and  $v$  be two non-adjacent major vertices w.r.t. a hole  $H$  of a graph  $G \in \mathcal{C}$ . Let  $P = u' \dots u''$  be a  $u$ -sector of  $H$ . Then one of the following holds.*

- (i)  $P$  contains at most one neighbor of  $v$ , and if it has one, it is either  $u'$  or  $u''$ .
- (ii)  $u'u'' \in E(G)$  and  $v$  is adjacent to both  $u'$  and  $u''$ .
- (iii)  $P$  contains at least 3 neighbors of  $v$ .

*Proof.* Let  $R = x \dots y$  be the path induced by  $V(H) \setminus V(P)$ , with ends such that  $u'x \in E(G)$  and  $u''y \in E(G)$ .

(1)  $u$  has a neighbor in the interior of  $R$  (in particular,  $R$  has length at least 2).

Otherwise,  $N_H(u) \subseteq \{u', u'', x, y\}$ , contradicting Lemma 3.1. This proves (1).

Suppose first that  $P$  contains exactly one neighbor  $v'$  of  $v$ . Suppose for a contradiction that  $v'$  is not an end of  $P$ . If  $vx \in E(G)$  then  $u'v' \notin E(G)$  because  $G$  is square-free. Hence,  $V(P) \cup \{u, v, x\}$  induces a theta (if  $ux \notin E(G)$ ) or a pyramid (if  $ux \in E(G)$ ). So,  $vx \notin E(G)$ . Symmetrically,  $vy \notin E(G)$ . Hence,  $G$  contains a theta from  $u$  to  $v'$ : two paths use vertices of  $P$ , and the third one goes through  $v$ , some neighbor of  $v$  in the interior of  $R$  (which exists since  $v$  is major) and some neighbor of  $u$  in the interior of  $R$ , which exists by (1). So, (i) holds.

Suppose now that  $P$  contains exactly two neighbors  $v'$  and  $v''$  of  $v$ . If  $u'u'' \in E(G)$ , then (ii) holds, so we may assume that  $u'u'' \notin E(G)$ . Hence,  $u$  and  $P$  form a hole that we denote by  $H_u$ . We have  $v'v'' \in E(G)$  for otherwise,  $v$  and  $H_u$  form a theta. By Lemma 3.1,  $v$  has at least five neighbors in  $H$ , so at least one of them is in the interior of  $R$ . Also,  $u$  has a neighbor in the interior of  $R$  by (1). Hence,  $H_u$  together with a shortest path from  $u$  to  $v$  with interior in the interior of  $R$  form a pyramid, a contradiction.

Finally, if  $P$  contains at least three neighbors of  $v$ , then (iii) holds.  $\square$

Let  $H$  be a hole in a graph and let  $u$  and  $v$  be two vertices not in  $H$ . We say that  $u$  and  $v$  are *nested* w.r.t.  $H$  if  $H$  contains two distinct vertices  $a$  and  $b$  such that one  $(a, b)$ -path of  $H$  contains all neighbors of  $u$  in  $H$ , and the other one contains all neighbors of  $v$  in  $H$ . Observe that  $u$  and  $v$  may both be adjacent to  $a$  or to  $b$ . Observe that under the assumption that  $v$  has at least two neighbors in  $H$  (so that the notion of  $v$ -sector is defined),  $u$  and  $v$  are nested if and only if there exists a  $v$ -sector that contains every neighbor of  $u$  in  $H$ . If  $u$  is a cap, a pending vertex, or a vertex with no neighbor in  $H$ , then it is nested with all other vertices not in  $H$ .

**Lemma 3.5** *Let  $H$  be a hole in a graph  $G \in \mathcal{C}$ . If  $u$  and  $v$  are major or clones w.r.t.  $H$  and are nested, then  $uv \notin E(G)$ .*

*Proof.* Otherwise, let  $H_u$  be the hole formed by  $u$  and the  $u$ -sector of  $H$  that contains all neighbors of  $v$ . Then one of  $(H, v)$  or  $(H_u, v)$  is an even wheel, a contradiction.  $\square$

If  $u$  and  $v$  are two vertices not in  $H$  and not nested w.r.t.  $H$ , then they *cross* on  $H$ .

**Lemma 3.6** *Let  $H$  be a hole in a graph  $G \in \mathcal{C}$  and let  $u$  and  $v$  be two vertices not in  $H$ . If  $u$  and  $v$  cross, then one of the following holds.*

(i)  $H$  contains four vertices  $u', u'', v'$  and  $v''$  such that:

- $u', v', u''$  and  $v''$  are distinct and appear in this order along  $H$ ;

- $u', u'' \in N(u)$ ;
- $v', v'' \in N(v)$ .

(ii)  $N_H(u) = N_H(v)$ ,  $N_H(u)$  is an independent set and  $|N_H(u)| = 3$ .

(iii)  $N_H(u) = N_H(v)$  and both  $u$  and  $v$  are clones w.r.t.  $H$ .

*Proof.* Since a vertex with no neighbor in  $H$ , a cap or a vertex pending w.r.t.  $H$  is nested with any other vertex not in  $H$ , by Lemma 3.2,  $u$  and  $v$  are major or clones w.r.t.  $H$ . Hence, consider two non-adjacent neighbors  $a, b$  of  $u$  in  $H$ . Since  $v$  is major or clone,  $v$  has a neighbor in the interior of one  $(a, b)$ -path  $P_v$  of  $H$ . We suppose that  $a, b$  and  $P_v$  are chosen subject to these properties ( $ab \notin E(G)$ ,  $v$  has a neighbor in the interior of  $P_v$ ) and so that  $P_v$  is minimal. If  $v$  has neighbors in the interior of the other  $(a, b)$ -path of  $H$ , then (i) holds.

Otherwise,  $N_H(v) \subseteq V(P_v)$ . If  $P_v$  is a  $u$ -sector, then  $u$  and  $v$  are nested, so suppose that  $u$  has a neighbor  $u'$  that is an internal vertex of  $P_v$ . By the minimality of  $P_v$ ,  $N_H(v) \subseteq \{a, b, u'\}$ . Hence, either  $v$  is a clone of  $u'$  w.r.t.  $H$ , or by Lemma 3.1 applied to  $v$ ,  $N_H(v) = \{a, b, u'\}$  and  $N_H(v)$  is an independent set. So, if  $N_H(u) \geq 4$ , then (i) holds, and if  $N_H(u) = 3$ , then (ii) or (iii) holds.  $\square$

**Lemma 3.7** *Let  $H$  be a hole in a graph  $G \in \mathcal{C}$  and suppose that  $u$  and  $v$  are two major vertices w.r.t.  $H$ . Then  $uv \in E(G)$  if and only if  $u$  and  $v$  cross.*

*Proof.* If  $u$  and  $v$  are nested, then  $uv \notin E(G)$  by Lemma 3.5. It remains to prove the converse: if  $u$  and  $v$  cross, then they are adjacent. So suppose for a contradiction that they are not adjacent.

We apply Lemma 3.6. Since  $u$  and  $v$  are major, (iii) does not hold. If (ii) holds, then  $G$  contains a square, a contradiction. Hence we may assume that (i) holds: there exist in  $H$  two neighbors  $u', u''$  of  $u$  and two neighbors  $v', v''$  of  $v$  such that  $u', v', u'', v''$  are distinct and appear in this order along  $H$ . We choose them so that the path  $P$  from  $u'$  to  $u''$  in  $H \setminus v''$  is minimal. We now break into two cases.

**Case 1:**  $P$  is not a  $u$ -sector.

So, let  $u'''$  be a neighbor of  $u$  in the interior of  $P$ . By the minimality of  $P$ ,  $u'Pu'''$  and  $u'''Pu''$  are  $u$ -sectors and have no neighbor of  $v$  in their interior, so  $v' = u'''$ . Our goal in this case is to show the existence of three paths  $R_1, R_2$  and  $R_3$  forming a theta from  $u$  to  $v$ . We set  $R_1 = uv'v$ .

If  $v$  is adjacent to both  $u'$  and  $u''$ , then  $\{u, v, u', v'\}$  induces a square, a contradiction. So we may assume up to symmetry that  $v$  is not adjacent to  $u''$ . W.l.o.g. we may assume that  $v''$  is such that  $Q = v' \dots v''$  is a  $v$ -sector of  $H$  (that contains  $u''$ ). By Lemma 3.4,  $Q$  contains at least three neighbors of  $u$ . So, there exists a path  $R_2$  from  $u$  to  $v$  going through  $v''$  and the interior of  $R_2$  contains no neighbors of  $v'$ .

Let  $x$  be the neighbor of  $v''$  in  $H$  that is not in  $Q$  and let  $R$  be the path of  $H$  from  $x$  to  $u'$  that does not contain  $P$ .

We claim that  $v$  has a neighbor in the interior of  $R$  (which therefore has length at least 2). Otherwise,  $N_H(v) \subseteq \{u', v', v'', x\}$ . By Lemma 3.1,  $N_H(v) = \{u', v', v''\}$  and  $u'v' \notin E(G)$ . So,  $\{u, v, u', v'\}$  induces a square, a contradiction.

We claim that  $u$  has a neighbor in the interior of  $R$ . For suppose not. Since  $G$  contains no even wheel,  $u$  has an odd number of neighbors in  $Q$ , and since it also has an odd number of neighbors in  $H$ ,  $u$  must be adjacent to  $x$ . Let  $Q' = y \dots x$  be the  $u$  sector of  $H$  that contains  $v''$ . Since  $G$  is square-free,  $v$  is not adjacent to  $x$  or  $y$ . Hence,  $Q'$  contains a unique neighbor of  $v$ , that is in its interior, a contradiction to Lemma 3.4.

Now, by considering a path  $R_3$  from  $u$  to  $v$  with interior in the interior of  $R$  (which exists from the two claims we just proved), we see that  $R_1$ ,  $R_2$  and  $R_3$  form a theta.

**Case 2:**  $P$  is a  $u$ -sector.

We apply Lemma 3.4 to  $P$  and we observe that outcomes (i) and (ii) do not hold, so outcome (iii) holds:  $P$  contains at least three neighbors of  $v$ . It follows that there exist two internally vertex disjoint paths  $R_1$  and  $R_2$ , both from  $u$  to  $v$ , with interior in  $P$  and such that  $V(R_1) \cup V(R_2)$  induces a hole. Let  $x$  be the neighbor of  $u'$  in  $H$  that is not in  $P$ , and  $y$  be the neighbor of  $u''$  in  $H$  that is not in  $P$ . Let  $R$  be the path of  $H$  from  $x$  to  $y$  that does not contain  $P$ .

We claim that  $u$  has a neighbor in the interior of  $R$ . Otherwise,  $N_H(u) \subseteq \{u', u'', x, y\}$ , contradicting Lemma 3.1.

We claim that  $v$  has a neighbor in the interior of  $R$ . Otherwise,  $v''$  must be one of  $x$  or  $y$ , say  $x$  up to symmetry. But since there is no even wheel in  $G$ ,  $v$  has an odd number of neighbors in  $P$  and in  $H$ , so  $v$  must be adjacent to  $y$ . Since  $G$  is square-free,  $v$  cannot be adjacent to both  $u'$ ,  $u''$ , so suppose up to symmetry that  $v$  is not adjacent to  $u''$ . Hence,  $u''$  is the unique neighbor of  $u$  in some  $v$ -sector of  $H$  (moreover in its interior), a contradiction to Lemma 3.4.

Now, by considering a path  $R_3$  from  $u$  to  $v$  with interior in the interior of  $R$  (which exists from the two claims we just proved), we see that  $R_1$ ,  $R_2$  and  $R_3$  form a theta, a contradiction.  $\square$

**Lemma 3.8** *Let  $H$  be a hole in a graph  $G \in \mathcal{C}$ . If  $u$  and  $v$  are non-adjacent vertices of  $G \setminus H$  that cross, then  $u$  and  $v$  are both clones w.r.t.  $H$  and they have exactly two common neighbors on  $H$ .*

*Proof.* Since a vertex with no neighbor in  $H$ , a cap or a pendant vertex is nested with any other vertex outside  $H$ , by Lemma 3.2,  $u$  and  $v$  are major or clones. If they are both major, there is a contradiction by Lemma 3.7. If  $u$  is a clone of  $y$  and  $v$  is major (or vice versa), then by Lemma 3.3,  $vy \notin E(G)$ , and it follows that  $u$  and  $v$  are nested, a contradiction. If  $u$  and  $v$  are both clones, then they have two or three common neighbors on  $H$  (because they

cross). If they have three common neighbors, then  $G$  contains a square, a contradiction. Hence, they have two common neighbors as claimed.  $\square$

**Lemma 3.9** *Let  $H$  be a hole in a graph  $G \in \mathcal{C}$  and let  $P = u \dots v$  be a path of length at least 1, vertex-disjoint from  $H$ , and such that  $u$  and  $v$  have neighbors in  $H$  and no internal vertex of  $P$  has neighbor in  $H$ . If  $u$  and  $v$  are nested, then one of the following holds (up to a swap of  $u$  and  $v$ ):*

- (i)  $P$  has length 1,  $u$  is major or is a clone, and  $N_H(v)$  is an edge that contains exactly one neighbor of  $u$ .
- (ii)  $u$  is a major vertex or a clone,  $v$  is a cap and  $N_H(v) \subseteq N_H(u)$ .
- (iii)  $|N_H(v)| = 1$  and  $N_H(v) \subseteq N_H(u)$ .
- (iv)  $N_H(u) \cup N_H(v)$  is an edge of  $H$ .

*Proof.* By Lemma 3.2,  $u$  and  $v$  are major, clone, cap or pending. We may therefore consider four cases.

**Case 1.** At least one of  $u$  and  $v$  is major.

Up to symmetry, we suppose that  $u$  is major.

Suppose that  $v$  is also major. We apply Lemma 3.7 to  $u$  and  $v$ . Since  $u$  and  $v$  are nested,  $P$  has length at least 2. Hence  $G$  contains a theta, a contradiction. So, we may assume that  $v$  is minor.

Suppose that  $v$  is a clone of some vertex  $x \in V(H)$ . Since  $u$  and  $v$  are nested,  $ux \notin E(G)$ . By Lemma 3.5,  $P$  has length at least 2. So,  $N_H(v)$  is included in some  $u$ -sector  $Q$  of  $H$  and  $P$  and  $Q \setminus x$  form a theta, a contradiction.

Suppose that  $v$  is a cap and  $N_H(v) = xy$ . If  $x$  and  $y$  are in the interior of some  $u$ -sector  $Q$  of  $H$ , then  $P$  and  $Q$  form a pyramid. Hence, there exists a  $u$ -sector  $R = u' \dots u''$  such that w.l.o.g.  $x = u'$  and  $y \in V(R)$ . If  $y = u''$  then (ii) holds. If  $y \neq u''$ , then  $P$  has length 1, for otherwise  $P$  and  $R$  form a pyramid. Hence, (i) holds.

Suppose that  $v$  is pending. Then (iii) holds for otherwise  $N_H(v)$  is in the interior of some  $u$ -sector of  $H$  that together with  $P$  forms a theta.

**Case 2.** None of  $u, v$  is major, and at least one of  $u, v$  is a clone.

Up to symmetry, suppose that  $u$  is a clone of  $x$ .

Suppose that  $v$  is a clone of  $y$ . Since  $u$  and  $v$  are nested, we have  $x \neq y$  and  $xy \notin E(G)$ . By Lemma 3.5,  $P$  has length at least 2, so  $P$  and  $H \setminus \{x, y\}$  form a theta from  $u$  to  $v$ , a contradiction.

Suppose that  $v$  is a cap, and let  $yz$  be the two neighbors of  $v$ . If  $x \in \{y, z\}$ , then (ii) holds, so suppose  $x \notin \{y, z\}$ . Hence  $P$  and  $H \setminus x$  form a pyramid, unless (i) holds.

Suppose that  $v$  is pending. Then (iii) holds for otherwise  $H \setminus x$  and  $P$  form a theta.

**Case 3.** None of  $u, v$  is major or a clone, and at least one of  $u, v$  is a cap.

Up to symmetry, suppose that  $u$  is a cap.

Suppose that  $v$  is also a cap. Then (iv) holds, for otherwise  $H$  and  $P$  form a prism or an even wheel.

Suppose that  $v$  is pending. Then (iii) holds for otherwise  $H$  and  $P$  form a pyramid.

**Case 4.** Both  $u, v$  are pending vertices.

Then (iii) or (iv) holds, for otherwise  $H$  and  $P$  form a theta. □

**Lemma 3.10** *Let  $G$  be a graph in  $\mathcal{C}$ ,  $H$  a hole in  $G$  and  $w$  a major vertex w.r.t.  $H$ . Suppose that  $a, w', b, w''$  are four distinct vertices of  $H$  that appear in this order along  $H$  and such that  $w', w''$  are adjacent to  $w$ . Then every path  $P$  of  $G \setminus w$  from  $a$  to  $b$  has an internal vertex adjacent to  $w$ .*

*Proof.* Consider a counter-example such that  $P$  is of minimum length. Note that  $P$  has length at least 2. Let  $H_a$  (resp.  $H_b$ ) be the path of  $H$  from  $w'$  to  $w''$  that contains  $a$  (resp.  $b$ ). Let  $H_{w'}$  (resp.  $H_{w''}$ ) be the path of  $H$  from  $a$  to  $b$  that contains  $w'$  (resp.  $w''$ ).

(1)  $P^*$  is vertex-disjoint from  $V(H) \cup \{w\}$ .

Since  $P$  is a counterexample, its interior contains no neighbor of  $w$ , and since  $a, b, w'$  and  $w''$  are distinct, we have  $V(P) \cap \{w', w''\} = \emptyset$ . So, an internal vertex of  $P$  that is in  $H$  would yield a smaller counterexample, a contradiction to the minimality of  $P$ . This proves (1).

We set  $Q = P^* = u \dots v$ , where  $u$  is adjacent to  $a$  and  $v$  is adjacent to  $b$  (possibly,  $u = v$ ).

(2)  $u$  (resp.  $v$ ) and  $w$  are nested.

Since  $P$  is a counterexample,  $u$  and  $w$  are non-adjacent. Since  $w$  is major, by Lemma 3.8,  $u$  and  $w$  are nested. Similarly,  $v$  and  $w$  are nested. This proves (2).

(3)  $u$  and  $v$  are distinct and nested w.r.t.  $H$ .

By (2),  $N_H(u) \subseteq V(H_a)$  and  $N_H(v) \subseteq V(H_b)$ . So,  $u$  and  $v$  are distinct (because  $a \notin N(v)$ ) and nested. This proves (3).

(4) We may assume that  $H' = auQvbH_{w''}a$  is a hole that contains all neighbors of  $w$  in  $H$  except  $w'$ .

By the minimality of  $P$ , no internal vertex of  $Q$  has a neighbor in  $H_a^*$  or in  $H_b^*$ . It follows that  $N_H(Q^*) \subseteq \{w', w''\}$ .

Suppose first that  $N_H(Q^*) = \{w', w''\}$ . Then,  $H$  together with a path from  $w'$  to  $w''$  with interior in  $Q^*$  form a theta from  $w'$  to  $w''$ , a contradiction.



Suppose now that  $N_H(Q^*) = \emptyset$ . Then, by (3), we may apply Lemma 3.9 to  $Q$ . Since  $\{a\} \subseteq N_H(u) \subseteq V(H_a)$  and  $\{b\} \subseteq N_H(v) \subseteq V(H_b)$ , (ii), (iii) and (iv) of Lemma 3.9 cannot hold. So (i) of Lemma 3.9 must hold. Up to symmetry, we may therefore assume that  $w'$  is the unique common neighbor of  $u$  and  $v$  on  $H$  and  $\{b\} = N_H(v) \setminus \{w'\}$ . Since by (2)  $u$  and  $w$  are nested,  $w'$  is the unique neighbor of  $w$  in  $H_{w'}$ . Also,  $a$  may be chosen as close as possible to  $w''$  along  $H_a$ , so that  $H' = auQvbH_{w''}a$  is a hole that contains all neighbors of  $w$  in  $H$  except  $w'$ .

Suppose finally that  $|N_H(Q^*)| = 1$ . Up to symmetry we may assume  $N_H(Q^*) = \{w'\}$ . If  $w$  has a neighbor  $z$  in  $H_{w'}^* \setminus w'$ , then suppose up to symmetry that it is in  $aH_{w'}w'$ . We see that the four vertices  $a, z, w'$  and  $w''$  appear in this order along  $H$ , so that a path from  $a$  to  $w'$  with interior in  $Q$  contradicts the minimality of  $P$ . It follows that  $w$  has no neighbor in  $H_{w'}^* \setminus w'$ . We may choose  $a$  and  $b$  closest to  $w''$  along  $H_a$  and  $H_b$  respectively. Since by (2)  $u$  and  $w$  are nested (and  $v$  and  $w$  are nested), this implies that  $H' = auQvbH_{w''}a$  is a hole that contains all neighbors of  $w$  in  $H$  except  $w'$ . This proves (4).

If  $w$  has exactly three neighbors in  $H$ , then by Lemma 3.1 they are pairwise non-adjacent and  $H'$  (from (4)) and  $w$  form a theta, a contradiction. So,  $w$  has at least five neighbors in  $H$ , so that  $(H', w)$  is a wheel. But then, one of  $(H, w)$  or  $(H', w)$  is an even wheel, a contradiction.  $\square$

We can now prove Theorem 2.4 restated below.

**Theorem 2.4** *Let  $G$  be a graph in  $\mathcal{C}$ ,  $H$  a hole in  $G$  and  $w$  a major vertex w.r.t.  $H$ . If  $C$  is a connected component of  $G \setminus N[w]$ , then there exists a  $w$ -sector  $P = x \dots y$  of  $H$  such that  $N(C) \subseteq \{x, y\} \cup (N(w) \setminus V(H))$ .*

*Proof.* Set  $W = N[w] \cap V(H)$  and  $Z = N[w] \setminus V(H)$ . Clearly,  $N(C) \subseteq W \cup Z$ . We have to prove that there exists a  $w$ -sector  $P = x \dots y$  of  $H$  such that  $N_W(C) \subseteq \{x, y\}$ . Otherwise, we are in one of the following cases.

**Case 1:** there exists  $a, w', b, w''$  in  $W$ , appearing in this order along  $H$ , with  $a, b \in N_W(C)$ . In this case, a path from  $a$  to  $b$  with interior in  $C$  contradicts Lemma 3.10.

**Case 2:**  $|W| = 3$  and  $N_W(C) = W = \{x, y, z\}$  (and by Lemma 3.1,  $x, y$  and  $z$  are pairwise non-adjacent). In this case, suppose first that  $C$  contains a vertex  $a$  in  $H \setminus W$ . Up to symmetry, we may assume that  $a$  is in the  $w$ -sector of  $H$  from  $x$  to  $y$ . But then,  $x, a, y$  and  $z$  contradict Lemma 3.10 because  $C$  contains the interior of a path from  $a$  to  $z$ . Hence,  $C \cap V(H) = \emptyset$ . If some vertex  $v$  of  $C$  has more than one neighbor in  $\{x, y, z\}$ , then  $w$  and  $v$  are contained in a square of  $G$ , a contradiction. So, every vertex of  $C$  has at most one neighbor in  $W$ . Consider a path  $P$  with interior  $C$  and that is either from  $x$  to  $y$ , from  $y$  to  $z$ , or from  $z$  to  $x$ . Suppose that  $P$  has minimum length among all such paths. Up to symmetry,  $P = x \dots y$ , and by minimality,  $P$  contains no neighbor of  $z$ . It follows that  $P$  and  $H$  form a theta from  $x$  to  $y$ .  $\square$

## 4 Proper separators

A separator in a graph is *proper* if it is minimal and not a clique. In view of Theorem 2.3, we may restrict our attention to proper separators because it is known that in any graph  $G$  there exists at most  $O(|V(G)|)$  minimal clique separators and that they can be enumerated in time  $O(|V(G)||E(G)|)$ , see [2] for details.

Our goal is to prove that a graph in  $\mathcal{C}$  contains a “small” number of proper separators. This goal is achieved in the next section. Here we prove a series of technical lemmas telling where precisely the vertices of a proper separator are.

If  $C$  is a separator of  $G$ , a connected component  $D$  of  $G \setminus C$  is *full* if every vertex of  $C$  has a neighbor in  $D$ .

**Lemma 4.1** *If  $C$  is a proper separator of a graph  $G \in \mathcal{C}$ , then  $G \setminus C$  has exactly two full connected components.*

*Proof.* Otherwise, let  $c_1c_2$  be a non-edge in  $C$  and  $X, Y, Z$  be full components of  $G \setminus C$ . There exists a theta from  $c_1$  to  $c_2$ , made of three paths with interior in  $X, Y$  and  $Z$  respectively. This is a contradiction.  $\square$

In what follows, when  $C$  is a proper separator, we denote by  $L$  and  $R$  the two full components of  $G \setminus C$  that exist by Lemma 4.1. We call a  $C$ -hole any hole  $H$  such that  $V(H) \cap C = \{c_1, c_2\}$  where  $c_1, c_2$  are non-adjacent vertices, one component of  $H \setminus \{c_1, c_2\}$  is in  $L$  and the other one is in  $R$ . We then say that  $H$  is a  $(C, c_1, c_2)$ -hole.

For a  $(C, c_1, c_2)$ -hole  $H$ , we use notation  $H_L$  for the path of  $H$  from  $c_1$  to  $c_2$  with interior in  $L$  and  $H_R$  for the path of  $H$  from  $c_1$  to  $c_2$  with interior in  $R$ . We let  $l_1$  be the neighbor of  $c_1$  in  $H_L$ . We define similarly vertices  $r_1, l_2$ , and  $r_2$ .

A  $C$ -hole  $H$  is *clean* w.r.t.  $C$  if every major vertex w.r.t.  $H$  is in  $C$ . The next lemma shows that clean holes exist.

**Lemma 4.2** *Let  $C$  be a proper separator of a graph  $G \in \mathcal{C}$ . If  $c_1$  and  $c_2$  are non-adjacent vertices of  $C$ , then a shortest  $(C, c_1, c_2)$ -hole  $H$  is clean w.r.t.  $C$ .*

*Proof.* Consider a vertex  $v \notin C$  that is major w.r.t.  $H$ . If  $N_H(v) \subseteq V(H_L)$ , then a shorter  $(C, c_1, c_2)$ -hole exists (using  $v$  as a shortcut), a contradiction. Similarly, there is a contradiction if  $N_H(v) \subseteq V(H_R)$ . It follows that  $v$  has neighbors in both  $H_L^*$  and  $H_R^*$ , and in particular in both  $L$  and  $R$ , a contradiction. This proves that  $H$  is clean w.r.t.  $C$ .  $\square$

Let  $C$  be a proper separator of a graph  $G$  and  $H$  be a  $C$ -hole. A vertex in  $G$  is  $(C, H)$ -heavy if it is major w.r.t.  $H$  and has neighbors in the interiors of both  $H_L$  and  $H_R$ . Observe that a  $(C, H)$ -heavy vertex must be in  $C$ , because it has neighbors in both  $L$  and  $R$ .

**Lemma 4.3** *Let  $C$  be a proper separator of a graph  $G \in \mathcal{C}$ . Let  $H$  and  $H'$  be two  $(C, c_1, c_2)$ -holes that are clean w.r.t.  $C$ . A vertex in  $C$  is  $(C, H)$ -heavy if and only if it is  $(C, H')$ -heavy.*

*Proof.* Otherwise, suppose up to symmetry that some vertex  $v$  is  $(C, H)$ -heavy and not  $(C, H')$ -heavy. Hence,  $v$  has a neighbor  $v_L$  in the interior of  $H_L$  and a neighbor  $v_R$  in the interior of  $H_R$ .

(1) *We may assume that  $v$  is a clone of  $c_1$  w.r.t.  $H'$ .*

The vertices  $c_1, v_L, c_2, v_R$  are distinct and appear in this order along  $H$ . By Lemma 3.10, the path  $H'_L$  has an internal vertex adjacent to  $v$ . Similarly,  $H'_R$  has an internal vertex adjacent to  $v$ . Since  $v$  is not  $(C, H')$ -heavy, the only possibility is that  $v$  is a clone of  $c_1$  or  $c_2$  w.r.t.  $H'$ , and up to symmetry, we suppose it is a clone of  $c_1$ . This proves (1).

(2) *We may assume that  $v_L$  is an internal vertex of  $l_1 H_L c_2$  (in particular,  $H_L$  has length at least 3).*

Since  $v$  is not a clone w.r.t.  $H$ ,  $N_H(v) \not\subseteq \{c_1, l_1, r_1\}$ . Since by (1)  $vc_2 \notin E(G)$ ,  $v$  has a neighbor in the interior of either  $l_1 H_L c_2$  or  $r_1 H_R c_2$ . Up to symmetry, we may assume that  $v$  has a neighbor in the interior of  $l_1 H_L c_2$ . Hence,  $v_L$  can be chosen in the interior of  $l_1 H_L c_2$ . This proves (2).

(3)  $l_1 \neq l'_1$ .

Otherwise the vertices  $l_1, v_L, c_2, v_R$  are distinct and appear in this order along  $H$ . By Lemma 3.10, the path  $l_1 H'_L c_2$  has an internal vertex adjacent to  $v$ , a contradiction to (1). This proves (3).

(4)  $\{c_1\} \subseteq N_H(l'_1) \subseteq \{c_1, l_1\}$ .

Otherwise  $l'_1$  has two non-adjacent neighbors in  $H_L$ , and since  $H$  is clean w.r.t.  $C$ , by Lemma 3.2,  $l'_1$  is a clone of  $l_1$  w.r.t.  $H$ . Hence, the hole  $H_{l'_1 \setminus l_1}$  contains four distinct vertices (namely  $l'_1, v_L, c_2, v_R$ ). By Lemma 3.10,  $v$  has a neighbor in the interior of  $l'_1 H'_L c_2$ . This contradicts (1). This proves (4).

By (1),  $v$  is not adjacent to  $c_2$ . It follows that  $c_2$  is an internal vertex of some  $v$ -sector  $Q$  of  $H$ . We set  $Q = q_L \dots q_R$  with  $q_L \in L$  and  $q_R \in R$ . Note that by (2),  $q_L \neq l_1$ . Let  $x$  be the vertex of  $H'_L$  with a neighbor in  $Q$ , closest to  $c_1$  along  $H'_L$ . Note that  $x$  exists because of  $c_2$ . We set  $S = l'_1 H'_L x$ .

(5)  *$S$  has length at least 1.*

Otherwise  $S$  has length zero, so  $x = l'_1$  and  $l'_1$  has a neighbor in  $Q$ . This contradicts (4). This proves (5).

(6)  *$S$  is vertex disjoint from  $H$  and the only edges between  $S$  and  $H$  are  $l'_1 c_1$ , possibly  $l'_1 l_1$ , and the edges between  $x$  and  $Q$ .*

By (3),  $l'_1 \notin V(H)$  and by (4), the only edges between  $l'_1$  and  $Q$  are  $l'_1 c_1$  and possibly  $l'_1 l_1$ . Note that by the definition of  $x$ ,  $S$  is vertex disjoint from  $Q$  and  $x$  is the only vertex of  $S$  with neighbors in  $Q$ . Suppose that  $S$  contains any vertex  $b$  of  $H$  or that there is any

edge  $ab$  with  $a \in V(S)$ ,  $b \in V(H)$  and  $ab$  is not  $l'_1c_1$ ,  $l'_1l_1$  or an edge between  $x$  and  $Q$ . Then consider the four distinct vertices of  $H$ :  $c_2$ ,  $q_L$ ,  $b$  and  $v_R$ . We see that  $V(H'_L) \cup \{b\}$  contains a path  $P$  from  $c_2$  to  $b$ . By (1),  $P$  contains no internal vertex adjacent to  $v$ . This contradicts Lemma 3.10. This proves (6).

By (5) and (6),  $S$  and  $H$  contradict Lemma 3.9.  $\square$

By Lemma 4.3, for a vertex not in  $C$ , being heavy does not depend on the choice of a particular hole, but only on the choice of  $C$ ,  $c_1$  and  $c_2$ . The notion of  $(C, c_1, c_2)$ -heavy vertex is therefore relevant: a vertex is  $(C, c_1, c_2)$ -heavy if for some (or equivalently every) clean  $(C, c_1, c_2)$ -hole  $H$ , it is  $(C, H)$ -heavy.

Until the end of the section, we do not recall in the statements of the lemmas that we deal with a graph  $G$  in  $\mathcal{C}$ , a proper separator  $C$ , a clean  $(C, c_1, c_2)$ -hole  $H$  with the following notation:  $l_1$  is the neighbor of  $c_1$  in  $H_L$  and  $l'_1$  is the neighbor of  $l_1$  in  $H_L \setminus c_1$ . We define similarly vertices  $r_1$ ,  $r'_1$ ,  $l_2$ ,  $l'_2$ ,  $r_2$  and  $r'_2$ .

For  $i = 1, 2$ , we denote by  $L_i$  the set made of  $l_i$  and all the clones of  $l_i$  w.r.t.  $H$ . We denote by  $R_i$  the set made of  $r_i$  and all the clones of  $r_i$  w.r.t.  $H$ . We denote by  $C_i$  the set of vertices of  $G$  that are not  $(C, c_1, c_2)$ -heavy and have neighbors in both  $L_i$  and  $R_i$  (observe that  $c_i \in C_i$ ). Note that possibly  $L_1 = L_2$  or  $R_1 = R_2$  (not both since  $H$  is not a  $C_4$ ). Observe that  $L_i$  is possibly not included in  $L$ , because some vertices of  $L_i$  can be in  $C$ . Similarly,  $R_i$  is possibly not included in  $R$ . And  $C_i$  is possibly not included in  $C$  because some vertices of  $C_i$  might be in  $L$  or in  $R$  (not in both, because as we will see,  $C_i$  is clique and  $L$  is anticomplete to  $R$ ).

**Lemma 4.4** *For  $i \in \{1, 2\}$ ,  $L_i$ ,  $R_i$  and  $C_i$  are pairwise disjoint cliques. Moreover,  $L_i$  is anticomplete to  $R_i$ , and  $C_i$  is anticomplete to  $H \setminus \{c_i, l_i, r_i\}$ .*

*Proof.* We prove the lemma for  $i = 1$  ( $i = 2$  is similar). Clearly,  $L_1$  and  $R_1$  are disjoint. They are cliques for otherwise,  $G$  contains a square. By Lemma 3.5,  $L_1$  is anticomplete to  $R_1$ . It follows that  $C_1$  is disjoint from both  $L_1$  and  $R_1$ .

Let us prove that  $C_1$  is anticomplete to  $H \setminus \{c_1, l_1, r_1\}$ . Otherwise, let  $c \in C_1$  be a vertex with some neighbor in  $H \setminus \{c_1, l_1, r_1\}$ . Note that  $c \neq c_1$ . Also,  $cc_1 \in E(G)$  for otherwise  $G$  contains a square (with  $c$ ,  $c_1$ , and neighbors of  $c$  in  $L_1$  and  $R_1$ ). Let  $l \in L_1$  and  $r \in R_1$  be neighbors of  $c$  (they exist by definition of  $C_1$ ). We see that  $c$  is major w.r.t.  $H' = (H_{l/l_1})_{r/r_1}$ . Hence, by Lemma 3.3 (applied twice)  $c$  is major w.r.t.  $H$ , and therefore  $(C, c_1, c_2)$ -heavy, a contradiction to the definition of  $C_1$ .

It remains to prove that  $C_1$  is a clique, so suppose for a contradiction that  $c$  and  $c'$  are non-adjacent vertices of  $C_1$ . Let  $l, r$  be neighbors of  $c$  in  $L_1, R_1$  respectively, and  $l', r'$  be neighbors of  $c'$  in  $L_1, R_1$  respectively. If  $c$  and  $c'$  have common neighbors in both  $L_1$  and  $R_1$ , then  $G$  contains a square, a contradiction. Hence, we may assume that  $l \neq l'$  and that  $cl', c'l \notin E(G)$ .

If  $c$  and  $c'$  have a common neighbor  $r'' \in R_1$ , then the paths  $r''cl$ ,  $r''c'l'$  and  $r''r'_1Pl'_1$

form a pyramid. So,  $r \neq r'$ ,  $cr' \notin E(G)$  and  $c'r \notin E(G)$ . Hence, the paths  $rcl$ ,  $rc'l'$  and  $P$  form a prism.  $\square$

**Lemma 4.5** *For  $i \in \{1, 2\}$ ,  $L_i$ ,  $R_i$  and  $C_i$  are pairwise disjoint cliques. Moreover,  $L_i$  is anticomplete to  $R_i$ , and  $C_i$  is anticomplete to  $H \setminus \{c_i, l_i, r_i\}$ .*

*Proof.* We prove the lemma for  $i = 1$  ( $i = 2$  is similar). Clearly,  $L_1$  and  $R_1$  are disjoint. They are cliques for otherwise,  $G$  contains a square. By Lemma 3.5,  $L_1$  is anticomplete to  $R_1$ . It follows that  $C_1$  is disjoint from both  $L_1$  and  $R_1$ .

Let us prove that  $C_1$  is anticomplete to  $H \setminus \{c_1, l_1, r_1\}$ . Otherwise, let  $c \in C_1$  be a vertex with some neighbor in  $H \setminus \{c_1, l_1, r_1\}$ . Note that  $c \neq c_1$ . Also,  $cc_1 \in E(G)$  for otherwise  $G$  contains a square (with  $c$ ,  $c_1$ , and neighbors of  $c$  in  $L_1$  and  $R_1$ ). Let  $l \in L_1$  be a neighbor of  $c$  (it exists by definition of  $C_1$ ). We set  $P = r'_1 H_R c_2 H_L l'_1$  (it has length at least 1, and is induced by  $V(H) \setminus \{l_1, c_1, r_1\}$ ). So,  $c$  has a neighbor in  $P$  and we let  $x$  be the neighbor of  $c$  in  $P$  closest to  $l'_1$  along  $P$ . We see that if  $cl_1 \notin E(G)$ , then  $l \neq l_1$  and the hole  $cxPl'_1 l_1 c_1 c$  is the rim of an even wheel with center  $l$ , a contradiction. Hence,  $cl_1 \in E(G)$ , and symmetrically  $cr_1 \in E(G)$ . It follows that  $c$  is major w.r.t.  $H$  and has neighbors in both  $H_L^*$  and  $H_R^*$ , a contradiction to the definition of  $C_1$ .

It remains to prove that  $C_1$  is a clique, so suppose for a contradiction that  $c$  and  $c'$  are non-adjacent vertices of  $C_1$ . Let  $l, r$  be neighbors of  $c$  in  $L_1, R_1$  respectively, and  $l', r'$  be neighbors of  $c'$  in  $L_1, R_1$  respectively. If  $c$  and  $c'$  have common neighbors in both  $L_1$  and  $R_1$ , then  $G$  contains a square, a contradiction. Hence, we may assume that  $l \neq l'$  and that  $cl', c'l \notin E(G)$ .

If  $c$  and  $c'$  have a common neighbor  $r'' \in R_1$ , then the paths  $r''cl$ ,  $r''c'l'$  and  $r''r'_1Pl'_1$  form a pyramid. So,  $r \neq r'$ ,  $cr' \notin E(G)$  and  $c'r \notin E(G)$ . Hence, the paths  $rcl$ ,  $rc'l'$  and  $P$  form a prism.  $\square$

For  $i \in \{1, 2\}$ , an  $(L, i)$ -viaduct w.r.t.  $(C, H)$  is a path  $Q = u_L \dots u_R$  of  $G$  such that:

- (i)  $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$ ;
- (ii)  $V(Q^*)$  is anticomplete to  $V(H \setminus c_i)$ ;
- (iii)  $C \cap V(Q) = \{u_L\}$ ;
- (iv)  $u_R \in R_i \setminus C$  (so possibly,  $u_R = r_i$ );
- (v) one of the following holds:

- $u_L$  is major w.r.t.  $H$ ,  $N_H(u_L) \subseteq V(H_L)$ ,  $u_L c_i \in E(G)$ ; or
- $u_L \in L_i \cap C$ .

For  $i \in \{1, 2\}$ , an  $(R, i)$ -viaduct w.r.t.  $(C, H)$  is a path  $Q = u_L \dots u_R$  of  $G$  such that:

- (i)  $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$ ;
- (ii)  $V(Q^*)$  is anticomplete to  $V(H \setminus c_i)$ ;
- (iii)  $C \cap V(Q) = \{u_R\}$ ;
- (iv)  $u_L \in L_i \setminus C$  (so possibly,  $u_L = l_i$ );
- (v) one of the following holds:
  - $u_R$  is major w.r.t.  $H$ ,  $N_H(u_R) \subseteq V(H_R)$ ,  $u_R c_i \in E(G)$ ; or
  - $u_R \in R_i \cap C$ .

We call *viaduct* any path that is an  $(L, i)$ -viaduct or an  $(R, i)$ -viaduct for  $i \in \{1, 2\}$ .

**Lemma 4.6** *For every  $i \in \{1, 2\}$ , every  $(L, i)$ -viaduct and  $(R, i)$ -viaduct has length at least 2 and contains an odd number of neighbors of  $c_i$  (at least 3).*

*Proof.* Suppose  $Q = u_L \dots u_R$  is an  $(L, 1)$ -viaduct (the proof is similar for other types of viaducts).

Since  $u_L \in C$  and  $u_R \notin C$ , we have  $u_L \neq u_R$ . So  $Q$  has length at least 1, and suppose for a contradiction that it has length 1. Then clearly  $u_R \neq r_1$  and since  $u_L$  and  $u_R$  are nested, by Lemma 3.9 applied to  $Q$  and  $H$ , and (i) is the only possible outcome. So, one end of  $Q$  is a cap, a contradiction to the definition of viaducts (observe however that if  $u, v$  are vertices like in outcome (i) Lemma 3.9, then  $uvr_1$  is possibly a viaduct of length 2). So  $Q$  has length at least 2.

Observe that  $u_L \notin V(H)$  (while  $u_R$  is either in  $V(H)$  or is a clone of  $r_1$ ). Let  $x$  be the neighbor of  $u_L$  in  $H_L$ , closest to  $c_2$  along  $H_L$ . Consider the hole  $J$  induced by  $V(Q) \cup V(xH_Lc_2) \cup (V(H_R) \setminus \{c_1, r_1\})$  (note that  $r_1$  may be in  $J$ , when  $r_1 = u_R$ ). Now,  $c_1 \notin V(J)$  and  $c_1$  contains two non-adjacent neighbors in  $J$  (namely  $u_L$  and  $u_R$ ), hence, by Lemmas 3.1 and 3.2,  $c$  is a clone or a major vertex w.r.t.  $J$ , and it therefore has an odd number of neighbors in  $J$  (at least 3).  $\square$

The *potential* of  $(C, c_1, c_2)$  is the number of  $(C, c_1, c_2)$ -heavy vertices. The main result of this section is the following.

**Lemma 4.7** *Let  $C$  be a proper separator of a graph  $G \in \mathcal{C}$ . Let  $c_1$  and  $c_2$  be non-adjacent vertices of  $C$ , chosen such that the potential of  $(C, c_1, c_2)$  is maximum. Let  $H$  be a clean  $(C, c_1, c_2)$ -hole. If  $c \in C \setminus \{c_1, c_2\}$ , then one of the following statements holds:*

- (i)  $c$  is  $(C, H)$ -heavy;
- (ii) For some  $i \in \{1, 2\}$ ,  $c$  has a neighbor in  $L_i \setminus C$  and a neighbor in  $R_i \setminus C$ ;
- (iii)  $c$  is the end of some viaduct w.r.t.  $(C, H)$ .

*Proof.* Since  $C$  is a proper separator and  $L$  is connected, there exists a path  $Q_L = c \dots c_L$  such that  $V(Q_L \setminus c) \subseteq L$  and  $c_L$  has neighbors in the interior of  $H_L$  (possibly  $c = c_L$ ). There exists a similar path  $Q_R = c \dots c_R$ . We set  $Q = c_L Q_L c Q_R c_R$  and suppose that  $Q$  is minimal (so  $Q_L$  and  $Q_R$  are shortest paths).

(1) *We may assume that  $Q$  has length at least 1. In particular,  $c_L$  and  $c_R$  are nested w.r.t.  $H$ .*

Otherwise,  $Q = c = c_L = c_R$ . Since  $c$  has a neighbor in  $H_L^*$  and in  $H_R^*$ , it is either a major vertex or a clone. If it is major, then it is heavy w.r.t.  $H$  and (i) holds. If it is a clone, it must be a clone of  $c_1$  or  $c_2$ , so (ii) holds. This proves (1).

(2) *We may assume that  $c_1$  has neighbors in the interior of  $Q$  and  $c_2$  has no neighbors in the interior of  $Q$ .*

Suppose that both  $c_1$  and  $c_2$  have neighbors in the interior of  $Q$ . Then,  $H$  and a shortest path from  $c_1$  to  $c_2$  with interior in the interior of  $Q$  form a theta, a contradiction.

So, suppose that none of  $c_1, c_2$  have neighbors in the interior of  $Q$ . Since by (1)  $c_L$  and  $c_R$  are nested w.r.t.  $H$ , we apply Lemma 3.9 to  $Q$ . Since  $c_L$  has neighbors in the interior of  $H_L$  and  $c_R$  has neighbors in the interior of  $H_R$ , outcomes (ii), (iii) and (iv) cannot hold.

Hence outcome (i) holds. So  $Q = c_L c_R$ ,  $c \in \{c_L, c_R\}$  and exactly one of  $c_1$  or  $c_2$  (say  $c_1$ ) is a common neighbor of  $c_L$  and  $c_R$ , and up to symmetry,  $c_L$  is major or clone of  $l_1$ , and  $c_R$  is a cap. If  $c = c_L$ , then  $c_L c_R r_1$  is an  $(L, 1)$ -viaduct and (iii) holds. If  $c = c_R$ , then since  $H$  is clean,  $c_L$  cannot be major, so it is a clone,  $c_L \in L_1 \setminus C$ ,  $c r_1 \in E(G)$  so (ii) holds.

Hence, we may assume that exactly one of  $c_1$  or  $c_2$  has neighbors in the interior of  $Q$ , and up to symmetry, we may assume that it is  $c_1$ . This proves (2).

(3) *If  $c_L c_1 \notin E(G)$ , then  $N_H(c_L) = \{l_1\}$ .*

Let  $x_L$  be the neighbor of  $c_1$  in  $Q$ , closest to  $c_L$  along  $Q$  ( $x_L$  exists by (2) and  $x_L \neq c_L$  by assumption). Since  $c_L$  and  $x_L$  are nested (because  $x_L$  has no neighbor in the interior of  $H_L$ ), we may apply Lemma 3.9 to  $c_L Q x_L$ . Since  $c_L$  and  $x_L$  have no common neighbor on  $H$ , (i), (ii) and (iii) do not hold. Hence (iv) holds and  $N_H(c_L) = \{l_1\}$ . This proves (3).

(4) *If  $c_L$  is a cap or a pending vertex, then  $\{l_1\} \subseteq N_H(c_L) \subseteq \{l_1, c_1\}$ .*

*If  $c_R$  is a cap or a pending vertex, then  $\{r_1\} \subseteq N_H(c_R) \subseteq \{r_1, c_1\}$ .*

If  $c_L c_1 \notin E(G)$ , then our claim holds by (3). Otherwise,  $c_L c_1 \in E(G)$ ,  $c_L$  must be a cap and  $N_H(c_L) = \{c_1, l_1\}$ . The proof is similar for the claim about  $c_R$ . This proves (4).

(5) *If  $c_L$  is a clone or a major vertex w.r.t.  $H$ , then  $c_1 c_L \in E(G)$  and either  $c_L \in C$  or  $c_L \in L_1 \setminus C$ .*

*The analogous statement holds for  $c_R$ .*

By symmetry, it suffices to prove the statement for  $c_L$ , so assume that  $c_L$  is a clone or a major vertex w.r.t.  $H$ . By (3),  $c_L c_1 \in E(G)$ . If  $c_L$  is major then  $c_L \in C$  because  $H$  is

clean w.r.t.  $C$ . If  $c_L$  is clone and  $c_L \notin C$ , then  $c_L \in L_1 \setminus C$  because  $c_L$  has no neighbors in  $R$ . This proves (5).

(6) We may assume that  $c_L \in L_1 \setminus C$  or  $\{l_1\} \subseteq N_H(c_L) \subseteq \{l_1, c_1\}$ , and  $c_R \in R_1 \setminus C$  or  $\{r_1\} \subseteq N_H(c_R) \subseteq \{r_1, c_1\}$ .

By symmetry it suffices to prove the statement about  $c_L$ . If  $c_L$  is a pending vertex or a cap, then the result follows by (4). So, suppose that  $c_L$  is a clone or a major vertex. By (5),  $c_1 c_L \in E(G)$  and either  $c_L \in C$  or  $c_L \in L_1 \setminus C$ . We may assume that  $c_L \in C$ , and hence  $c_R \notin C$ . Note that by (1) and since  $c_1 c_L \in E(G)$ , if  $c_L$  is a clone w.r.t.  $H$ , then it is a clone of  $l_1$ , and if it is major then  $N_H(c_L) \subseteq V(H_L)$ .

If  $c_R$  is pending or cap w.r.t.  $H$ , then by (4),  $\{r_1\} \subseteq N(c_R) \cap V(H) \subseteq \{r_1, c_1\}$  and hence  $c_L Q c_R r_1$  is an  $(L, 1)$ -viaduct and (iii) holds. So, we may assume that  $c_R$  is a clone or a major vertex w.r.t.  $H$ . By (5) and since  $c_R \notin C$ , it follows that  $c_R \in R_1$ . But then  $Q$  is an  $(L, 1)$ -viaduct and (iii) holds. This proves (6).

By (6),  $\{l_1\} \subseteq N_H(c_L) \subseteq \{l'_1, l_1, c_1\}$  and  $\{r_1\} \subseteq N_H(c_R) \subseteq \{r'_1, r_1, c_1\}$ . So,  $V(Q) \cup V(H) \setminus \{c_1\}$  contains a hole that contains  $Q$  and  $c_2$ , which we denote by  $J$ . Note that  $J$  is a  $(C, c, c_2)$ -hole.

(7)  $J$  is a clean w.r.t.  $C$ .

Otherwise, let  $d \notin C$  be a vertex that is major w.r.t.  $J$ . By symmetry, we may assume that  $d \in L$ .

Since  $d$  is major w.r.t.  $J$  and not major w.r.t.  $H$  (since  $H$  is clean),  $d$  must have a neighbor in  $Q$ . Let  $d_L$  (resp.  $d_R$ ) be the neighbor of  $d$  in  $Q$  that is closest to  $c_L$  (resp.  $c_R$ ) along  $Q$ . Note that  $d_L, d_R \in V(c_L Q c)$  since  $d \in L$ . If  $d_L Q d_R$  is of length greater than 2, then  $V(Q_L) \cup \{d\}$  contains a path from  $c$  to  $c_L$  that is shorter than  $Q_L$ , contradicting the minimality of  $Q_L$ . So,  $d_L Q d_R$  is of length at most 2.

Since  $d$  is major w.r.t.  $J$ , it follows that  $N_J(d) \not\subseteq V(Q)$ . Suppose that  $N_J(d) \subseteq V(Q) \cup \{c_2\}$ . So  $d$  is adjacent to  $c_2$ . By Lemma 3.1, it follows that  $d$  has exactly three neighbors in  $J$  that are furthermore pairwise non-adjacent, namely  $d_L$ ,  $d_R$  and  $c_2$ . But then  $d_L Q d_R$  and  $d$  form a square. Therefore  $d$  has a neighbor in  $H_L \setminus \{c_1, c_2\}$ . By minimality of  $Q_L$ , it follows that  $N_Q(d) \subseteq \{c_L, c'_L\}$ , where  $c'_L$  is the neighbor of  $c_L$  in  $Q$ . By Lemma 3.1 applied to  $d$  and  $J$ ,  $d$  has two non-adjacent neighbors in  $J \setminus Q$ . Since  $V(J \setminus Q) \subseteq V(H)$  and  $d$  is not major w.r.t.  $H$ , it follows that  $d$  is a clone of some vertex  $d'$  w.r.t.  $H$ , where  $d'$  is an internal vertex of  $J \setminus Q$  (so  $d' \notin \{c_1, c_2, l_1\}$  and  $d' = l'_1$  is possibly only when  $c_L$  is not a clone). So, by Lemma 3.1,  $N_Q(d) = \{c_L, c'_L\}$ . If  $c_L \in L_1$ , then  $(H_{c_L \setminus l_1}, d)$  is an even wheel. So  $N_H(c_L) \subseteq \{l_1, c_1\}$ . Note that by minimality of  $Q$ , no internal vertex of  $Q$  has a neighbor in  $H \setminus \{c_1, c_2\}$ . Let  $c'_1$  be the neighbor of  $c_1$  in the interior of  $Q$  that is closest to  $c_L$  along  $Q$  (it exists by (2)). If  $N_H(c_L) = \{l_1\}$ , then  $H_{d \setminus d'}$  and  $c'_L Q c'_1$  form a theta from  $d$  to  $c_1$ . So  $N_H(c_L) = \{l_1, c_1\}$ . Let  $D$  be the path from  $d$  to  $l_1$  contained in  $(H \setminus \{d', c_2\}) \cup \{d\}$ . Then  $D$  and  $c_L Q c'_1$  form an even wheel with center  $c_L$ . This proves (7).



(8) Let  $d \in C \setminus \{c, c_1, c_2\}$ . If  $d$  is  $(C, H)$ -heavy, then  $d$  is  $(C, J)$ -heavy.

For suppose that  $d$  is  $(C, H)$ -heavy but not  $(C, J)$ -heavy. So  $d$  is major w.r.t.  $H$  and has neighbors in both  $H_L^*$  and  $H_R^*$ .

Suppose that  $d$  does not have a neighbor in  $J_L^*$ . Then  $c_L \in L_1 \setminus C$ ,  $\{l_1\} \subseteq N_{H_L}(d) \subseteq \{l_1, c_1\}$ , and  $d$  is not adjacent to  $c_L$ . But then since  $d$  is major w.r.t.  $H$ , by Lemma 3.1, it follows that  $H_{c_L \setminus l_1}$  and  $d$  form either an even wheel with center  $d$  or a theta. So  $d$  has a neighbor in  $J_L^*$ , and by symmetry  $d$  has a neighbor in  $J_R^*$ . Since  $d$  is not major w.r.t.  $J$ , it follows that  $d$  is a clone of  $c$  or  $c_2$  w.r.t.  $J$ .

If  $d$  is a clone of  $c$  w.r.t.  $J$ , then  $N_H(d) \subseteq \{l_1, c_1, r_1\}$  contradicting the assumption that  $d$  is major w.r.t.  $H$ . So  $d$  is a clone of  $c_2$  w.r.t.  $J$ .

Since  $H$  is of length greater than 4, w.l.o.g.  $H_L$  is of length greater than 2. In particular,  $J$  contains  $l_2$ . Since  $d$  is major w.r.t.  $H$  and  $H$  contains  $l_2$  and  $c_2$ , by Lemma 3.1,  $d$  has at least five neighbors in  $H$ . It follows that  $d$  is adjacent to  $l_1$  or  $r_1$ . If  $d$  is adjacent to  $l_1$ , then  $c_L \in L_1 \setminus C$  and  $d$  is not adjacent to  $c_L$ . But then  $(H_{c_L \setminus l_1}, d)$  is an even wheel. So  $d$  is not adjacent to  $l_1$  and hence  $N_H(d) = \{l_2, c_2, r_2, r_1, c_1\}$ . In particular,  $r_1 \neq r_2$ , i.e.  $H_R$  is of length greater than 2. But then, we get a contradiction by a symmetric argument. This proves (8).

To conclude the proof, by (7)  $J$  is clean w.r.t.  $C$ . Also,  $c_1$  has neighbor in  $J_L^*$  and  $J_R^*$ , and is therefore major w.r.t.  $J$  or a clone of  $c$ . In this last case,  $c_L \in L_1 \setminus C$ ,  $c_R \in R_1 \setminus C$  and  $Q$  has length 2, so (ii) holds. Hence, we may assume that  $c_1$  is major w.r.t.  $J$ .

Note that  $c$  is not major w.r.t.  $H$ . By (8), we see that the number of  $(C, J)$ -heavy vertices is greater than the number of  $(C, H)$ -heavy vertices. So, by Lemma 4.3, the potential of  $(C, c, c_2)$  is greater than then potential of  $(C, c_1, c_2)$ , a contradiction to the choice of  $c_1$  and  $c_2$ .  $\square$

**Lemma 4.8** For  $i \in \{1, 2\}$ , there does not exist both an  $(L, i)$ -viaduct and an  $(R, i)$ -viaduct. In particular, at least one of  $L_i, R_i$  contains no vertex of  $C$ .

*Proof.* Suppose there exists an  $(L, 1)$ -viaduct  $P = u_L \dots u_R$  and an  $(R, 1)$ -viaduct  $Q = v_L \dots v_R$  (the case where  $i = 2$  is similar). Then,  $u_L \in C$ ,  $u_R \in R_1$ ,  $v_L \in L_1$  and  $v_R \in C$ . Note that  $N_H(u_L) \subseteq V(H_L)$  and  $N_H(v_R) \subseteq V(H_R)$ .

(1)  $V(P \setminus u_L)$  and  $V(Q \setminus v_R)$  are disjoint and anticomplete. Moreover,  $u_L v_R \notin E(G)$ .

The first claim is because  $L$  and  $R$  are connected components of  $G \setminus C$  and  $V(P \setminus u_L) \subseteq R$  and  $V(Q \setminus v_R) \subseteq L$ .

Since  $u_L$  and  $v_R$  are nested and both major or clones w.r.t.  $H$ ,  $u_L v_R \notin E(G)$  follows from Lemma 3.5. This proves (1).

Let  $x_L$  be the neighbor of  $u_L$  in  $v_L l'_1 H_L c_2$ , closest to  $v_L$  along this path. Let  $y_L$  be the neighbor of  $u_L$  in  $v_L l'_1 H_L c_2$ , closest to  $c_2$  along this path. Note that  $x_L$  and  $y_L$  exist and are distinct from the definition of viaducts and Lemma 4.6. (but possibly,  $x_L = v_L$ ,

$y_L = l'_1$  and  $x_L y_L \in E(G)$  when  $u_L$  is a clone w.r.t.  $H$ ). Let  $x_R$  be the neighbor of  $v_R$  in  $u_R r'_1 H_{RC2}$ , closest to  $u_R$  along this path. Let  $y_R$  be the neighbor of  $v_R$  in  $u_R r'_1 H_{RC2}$ , closest to  $c_2$  along this path.

If  $x_L \neq v_L$ , we set  $S_L = u_L x_L H_L l'_1 v_L$ . If  $x_L = v_L$  we set  $S_L = u_L v_L$ . If  $x_R \neq u_R$ , we set  $S_R = v_R x_R H_L r'_1 u_R$ . If  $x_R = u_R$  we set  $S_R = v_R u_R$ .

By (1),  $J = u_L S_L v_L Q v_R S_R u_R P u_L$  is a cycle whose only possible chords are edges from  $u_L$  to  $Q \setminus v_L$  and from  $v_R$  to  $P \setminus u_R$ . And such chords exist for otherwise,  $J$  is a hole and by Lemma 4.6,  $c_1$  has an even number of neighbors in  $J$ .

Up to the symmetry between  $P$  and  $Q$ , we suppose that  $v_R$  has a neighbor in  $P \setminus u_R$  and let  $p$  be the neighbor of  $v_R$  in  $P$  closest to  $u_L$  along  $P$  (note that by (1),  $p \neq u_L$ ).

If  $u_L$  has a neighbor in  $Q \setminus v_L$ , then let  $q$  be the neighbor of  $u_L$  in  $Q$  closest to  $v_R$  along  $Q$  (note that by (1),  $q \neq v_R$ ). We see that the three paths  $v_R p P u_L$ ,  $v_R q Q u_L$  and  $v_R y_R H_{RC2} H_L y_L u_L$  form theta, a contradiction. Hence,  $u_L$  has no neighbor in  $Q \setminus v_L$ .

If  $x_L y_L \notin E(G)$ , then the three paths  $v_R p P u_L$ ,  $v_R Q v_L S_L x_L u_L$  and  $v_R y_R H_{RC2} H_L y_L u_L$  form a theta. If  $x_L y_L \in E(G)$ , then  $x_L = v_L$  and  $y_L = l'_1$ , so the three paths  $v_R p P u_L$ ,  $v_R Q v_L$  and  $v_R y_R H_{RC2} H_L l'_1$  form a pyramid. In every case, there is a contradiction.  $\square$

**Lemma 4.9** *Let  $c_1, c_2$ , and  $H$  be as in Lemma 4.7. Let  $i \in \{1, 2\}$ .*

- *Suppose  $R_i \cap C = \emptyset$ . Then there exists a vertex  $c \in C_i$  such that for all  $x \in C_i$ ,  $x \in C$  if and only if  $N_{L_i}(c) \subseteq N_{L_i}(x)$ .*
- *Suppose  $L_i \cap C = \emptyset$ . Then there exists a vertex  $c \in C_i$  such that for all  $x \in C_i$ ,  $x \in C$  if and only if  $N_{R_i}(c) \subseteq N_{R_i}(x)$ .*

*Proof.* By symmetry, it is enough to prove the first claim for  $i = 1$ , so suppose  $R_1 \cap C = \emptyset$ . Note that  $C_1 \cap C \neq \emptyset$  since  $c_1 \in C_1$ . Let  $c \in C_1 \cap C$  be such that  $N_{L_1}(c)$  is minimal (inclusion wise). By Lemma 4.5,  $N_H(c) \subseteq \{c_1, l_1, r_1\}$ , and hence by Lemma 4.7 applied to  $c$ ,  $c$  has a neighbor  $l$  in  $L_1 \setminus C$  and a neighbor  $r$  in  $R_1 \setminus C$ . Let  $x \in C_1$ .

If  $N_{L_1}(c) \subseteq N_{L_1}(x)$ , then  $x$  is adjacent to  $l \in L_1 \setminus C \subseteq L$  and to some vertex of  $R_1 = R_1 \setminus C \subseteq R$ . Hence,  $x \in C$ .

Conversely, suppose that  $x \notin C$  and  $N_{L_1}(c) \not\subseteq N_{L_1}(x)$ . This means that  $c$  has a neighbor  $y$  in  $L_1$  that is not adjacent to  $x$ . If  $x$  has a neighbor  $z$  in  $L_1 \setminus N(c)$ , then  $xzyca$  is a square by Lemma 4.5, a contradiction. Hence,  $N_{L_1}(x) \subsetneq N_{L_1}(c)$ , contradicting the choice of  $c$ .  $\square$

## 5 The main proof

We describe two algorithms  $\mathcal{A}_{L,L}$  and  $\mathcal{A}_{L,R}$  that enumerate some proper separators of an input graph  $G$ . Note that these algorithms can be applied to any graph. Algorithm  $\mathcal{A}_{L,L}$

is described in Table 1. Algorithm  $\mathcal{A}_{L,R}$  is very similar to  $\mathcal{A}_{L,L}$ , only steps 9.–14. slightly differ (the roles of  $L_2$  and  $R_2$  are swapped). In Table 2 we indicate what are these steps.

**Lemma 5.1** *Let  $G$  be a graph in  $\mathcal{C}$ . If we run the two algorithms  $\mathcal{A}_{L,L}$  and  $\mathcal{A}_{L,R}$  on  $G$ , then the output is the list of all proper separators of  $G$  and the running time is at most  $O(|V(G)|^{10})$ .*

*Proof.* Because of step 14., the algorithm obviously outputs a list of proper separators of  $G$ . Conversely, consider a proper separator  $D$ , and let us check that at least one of  $\mathcal{A}_{L,L}$  or  $\mathcal{A}_{L,R}$  outputs  $D$ .

Let  $c_1, c_2$  be vertices in  $D$  such that the potential of  $(D, c_1, c_2)$  is maximum. At some point, in step 1., the algorithm considers the pair of vertices  $(c_1, c_2)$  and correctly puts  $c_1$  and  $c_2$  in  $C$ . At this step,  $C \subseteq D$ .

Let  $J$  be a shortest  $(D, c_1, c_2)$ -hole. Note that by Lemma 4.2,  $J$  is clean. We use the notation  $l_1, r_1, l_2, r_2$  as in section 4. At some point in step 2., the algorithm considers the 4-tuple  $(l_1, r_1, l_2, r_2)$ .

In step 3., we claim that all  $(D, c_1, c_2)$ -heavy vertices are put in  $C$ . Indeed, let  $v$  be such a  $(D, c_1, c_2)$ -heavy vertex. Note that by Lemma 4.3,  $v$  is  $(D, J)$ -heavy and hence  $c_1$  and  $c_2$  do not belong to the same  $v$ -sector of  $J$ . By Theorem 2.4 applied to  $J$  and  $v$ , vertices  $c_1$  and  $c_2$  are in different components of  $G \setminus (N[v] \setminus \{c_1, c_2\})$ . Note that all  $(D, J)$ -heavy vertices are obviously in  $D$ , since they have neighbors in both  $L$  and  $R$ . Hence, at this step, we have  $C \subseteq D$ . Note that if  $l_1$  say is put in  $C$  at this step, then  $l_1$  is adjacent to some vertex of  $R$ , a contradiction, so it is correct to discard  $C$ . There is a similar argument for  $r_1, l_2, r_2$ .

We claim that in step 4. a clean  $(D, c_1, c_2)$ -hole is computed. Indeed, by Lemma 4.7 applied to  $J$ , all vertices of  $D$  are either heavy (and these are already in  $C$ ), or adjacent to  $c_1$  or  $c_2$  from definitions of viaducts and by Lemma 4.5. So, when we compute the paths, all vertices of  $D$  are removed. Note that the paths exist, because of  $J$ . Hence,  $V(H_L^*) \subseteq L$  and  $V(H_R^*) \subseteq R$  showing that  $H$  is a hole. In particular, it is correct to discard  $C$  when  $H$  is not a hole. Clearly,  $H$  is a  $(D, c_1, c_2)$ -hole. Since  $H$  and  $J$  both go through  $c_1, l_1, r_1, c_2, l_2, r_2$  and  $J$  is a shortest  $(D, c_1, c_2)$ -hole, the length of  $H$  is the same as the length of  $J$ , and hence  $H$  is also a shortest  $(D, c_1, c_2)$ -hole. So, by Lemma 4.2,  $H$  is clean w.r.t.  $D$ . Note that  $H$  and  $J$  are potentially different holes, but they both go through the same vertices  $c_1, l_1, r_1, c_2, l_2, r_2$  and they have the same heavy vertices by Lemma 4.3.

Since  $H$  is clean, in step 5. it is correct to put in  $C$  every vertex that is major w.r.t.  $H$ . We still have  $C \subseteq D$ .

In step 8., we know by Lemma 4.5 (and since  $H$  is not a square) that discarding  $C$  is correct whenever we have to do so.

Now, by Lemma 4.8 and the symmetry between  $L$  and  $R$  we may assume that  $R_1 \cap D = \emptyset$ . More precisely, if  $R_1 \cap D \neq \emptyset$  then  $L_1 \cap D = \emptyset$ , so at some other step of enumeration, the 4-tuple  $(r_1, l_1, r_2, l_2)$  (and not  $(l_1, r_1, l_2, r_2)$ ) is considered, so that  $R_1 \cap D = \emptyset$ .

1. Enumerate all pairs of distinct and non-adjacent vertices  $(c_1, c_2)$  of  $G$ .  
Set  $C = \{c_1, c_2\}$ .
2. Enumerate all 4-tuple of vertices  $(l_1, r_1, l_2, r_2)$  such that  $\{l_1, c_1, r_1\}$  and  $\{l_2, c_2, r_2\}$  both induce a path of length 2 and  $\{l_1, l_2\}$  is anticomplete to  $\{r_1, r_2\}$ . Note that possibly  $l_1 = l_2$  or  $r_1 = r_2$ .
3. Add to  $C$  every vertex  $v$  such that  $c_1$  and  $c_2$  are in two distinct connected components of  $G \setminus (N[v] \setminus \{c_1, c_2\})$ . If  $\{l_1, r_1, l_2, r_2\} \cap C \neq \emptyset$ , discard  $C$ .
4. In  $G \setminus ((C \cup N(c_1) \cup N(c_2)) \setminus \{l_1, r_1, l_2, r_2\})$ , compute a shortest path  $H_L$  from  $l_1$  to  $l_2$  and a shortest path  $H_R$  from  $r_1$  to  $r_2$ . Set  $H = c_1 r_1 H_R r_2 c_2 l_2 H_L l_1 c_1$ . If  $H$  is not a hole, discard  $C$ .
5. Add to  $C$  every vertex that is major w.r.t.  $H$ .
6. Compute the set  $L_1$  of clones of  $l_1$  w.r.t.  $H$  and add  $l_1$  to  $L_1$ . Compute similar sets  $R_1$ ,  $L_2$  and  $R_2$ .
7. Compute the set  $C_1$  of vertices that have neighbors in both  $L_1$  and  $R_1$  and that are not major w.r.t.  $H$ . Note that  $c_1 \in C_1$ . Compute a similar set  $C_2$ .
8. Check that  $L_1, C_1, R_1, L_2, C_2, R_2$  are disjoint cliques, except that possibly exactly one of the equalities  $L_1 = L_2$  and  $R_1 = R_2$  holds. If this check fails, discard  $C$ .
9. Enumerate all pairs of vertices  $c'_1 \in C_1, c'_2 \in C_2$ .
10. Add to  $C$  all vertices  $x$  from  $C_1$  such that  $N_{L_1}(c'_1) \subseteq N_{L_1}(x)$ .
11. Add to  $C$  all vertices  $x$  from  $C_2$  such that  $N_{L_2}(c'_2) \subseteq N_{L_2}(x)$ .
12. Add to  $C$  every vertex  $c \in L_1$  such that there exists a path  $Q = c \dots v$  with the following properties:  $v \in R_1$ ,  $V(Q) \cap C = \emptyset$ ,  $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$  and  $Q^*$  is anticomplete to  $H \setminus c_1$ .
13. Add to  $C$  every vertex  $c \in L_2$  such that there exists a path  $Q = c \dots v$  with the following properties:  $v \in R_2$ ,  $V(Q) \cap C = \emptyset$ ,  $V(Q^*) \cap (V(H) \cup L_2 \cup R_2) = \emptyset$  and  $Q^*$  is anticomplete to  $H \setminus c_2$ .
14. Check whether  $C$  is proper separator of  $G$ . If not, discard  $C$ . Otherwise return  $C$ , and go to the next step of enumeration (so step 1., 2. or 9.).

Table 1: Algorithm  $\mathcal{A}_{L,L}$

9. Enumerate all pairs of vertices  $c'_1 \in C_1, c'_2 \in C_2$ .
10. Add to  $C$  all vertices  $x$  from  $C_1$  such that  $N_{L_1}(c'_1) \subseteq N_{L_1}(x)$ .
11. Add to  $C$  all vertices  $x$  from  $C_2$  such that  $N_{R_2}(c'_2) \subseteq N_{R_2}(x)$ .
12. Add to  $C$  every vertex  $c \in L_1$  such that there exists a path  $Q = c \dots v$  with the following properties:  $v \in R_1, V(Q) \cap C = \emptyset, V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$  and  $Q^*$  is anticomplete to  $H \setminus c_1$ .
13. Add to  $C$  every vertex  $c \in R_2$  such that there exists a path  $Q = c \dots v$  with the following properties:  $v \in L_2, V(Q) \cap C = \emptyset, V(Q^*) \cap (V(H) \cup L_2 \cup R_2) = \emptyset$  and  $Q^*$  is anticomplete to  $H \setminus c_2$ .
14. Check whether  $C$  is proper separator of  $G$ . If not, discard  $C$ . Otherwise return  $C$ , and go to the next step of enumeration (so step 1., 2. or 9.).

Table 2: Algorithm  $\mathcal{A}_{L,R}$  (only steps 9.–14. are described)

By Lemma 4.8, we may consider two cases:

- Case 1:  $R_1 \cap D$  and  $R_2 \cap D$  are empty (algorithm  $\mathcal{A}_{L,L}$ ).
- Case 2:  $R_1 \cap D$  and  $L_2 \cap D$  are empty (algorithm  $\mathcal{A}_{L,R}$ ).

We shall prove that in each case, the algorithm that is indicated above outputs  $D$ . The cases being similar, we just handle Case 1, so we suppose that  $R_1 \cap D$  and  $R_2 \cap D$  are empty. Hence, by Lemma 4.9 there exists a vertex  $c'_1 \in C_1$  such that for all  $x \in C_1, x \in D$  if and only if  $N_{L_1}(c'_1) \subseteq N_{L_1}(x)$ , and there exists a similar vertex  $c'_2 \in C_2$ . At some point, in step 9., the vertices  $c'_1, c'_2$  will be considered. And by Lemma 4.9, steps 10. and 11. correctly put in  $C$  the sets  $C_1 \cap D$  and  $C_2 \cap D$ .

At this step of the algorithm, all major vertices and vertices of  $C_1 \cap D$  and  $C_2 \cap D$  are in  $C$ . The only vertices in  $D \setminus C$  are therefore in  $L_1$  and  $L_2$ . Let  $c$  be a vertex in  $C \cap L_1$ . By Lemma 4.7, there exists a viaduct with end  $c$ , so that in step 12. a path  $Q$  is detected and  $c$  is correctly added to  $C$ . Conversely, if some path  $Q$  is detected in step 12., then  $c \in D$ . For otherwise, since  $R_1 \cap D = \emptyset$ ,  $Q$  is a path from  $L$  to  $R$ , so it must contain a vertex of  $D$ . Since all vertices of  $D \setminus (L_1 \cup L_2)$  are in  $C$  and therefore not in  $Q$ , they are not used by  $Q$ , so there is a contradiction.

Similarly, in step 13.,  $D \cap L_2$  is put in  $C$ .

Now,  $C = D$ . Hence, in step 14.,  $C$  is detected as a proper separator and the algorithm outputs  $D$  as claimed.

### Complexity analysis

The enumeration of all vertices takes time  $O(n^8)$ , and for each of them, all the computations that we do rely on connectivity checks that can be implemented to run in time  $O(n^2)$  with BFS. The total running time is therefore  $O(n^{10})$ .  $\square$

We can now prove Theorem 2.3, restated below.

**Theorem 2.3** *Every graph in  $\mathcal{C}$  on  $n$  vertices contains at most  $O(n^8)$  minimal separators. There is an algorithm of complexity  $O(n^{10})$  that enumerates them. Consequently, there exists a polynomial time algorithm for the Maximum Weighted Independent Set restricted to  $\mathcal{C}$ .*

*Proof.* For each of the 8-tuple of vertices that is considered by algorithms  $\mathcal{A}_{L,L}$  and  $\mathcal{A}_{L,L}$ , each algorithm outputs at most one proper separator. Hence, there is at most  $O(n^8)$  such separators. As explained at the beginning of Section 4, non-proper minimal separators are all clique separators, and there are at most  $O(n)$  and they can be enumerated in time  $O(n^3)$ . In Section 2, it is explained why this implies that the Maximum Weighted Independent Set restricted to  $\mathcal{C}$  can be solved in polynomial time.  $\square$

### Complexity of MWIS in $\mathcal{C}$

We do not recall here the definition of a *potential maximal clique*, see [4]. A potential maximal clique in a graph  $G$  is a subset of  $V(G)$  with special properties. We denote by  $m$  the number of edges in  $G$ , by  $p$  the number of potential maximal cliques in  $G$  and by  $s$  be the number of minimal separators in  $G$ . In [4], it is proved that  $p \leq O(ns^2 + ns + 1)$  (Proposition 22) and that, given the list of minimal separators, the potential maximal cliques of  $G$  can be listed in time  $O(n^2ms^2)$  (Theorem 23). In [9], based on [8], it is proved that, given the list of potential maximal cliques, the MWIS problem can be solved in time  $O(n^5mp)$  in any graph (Proposition 1). By Theorem 2.3,  $s \leq O(n^8)$ , so  $p \leq O(n^{17})$ . Hence, in  $\mathcal{C}$ , the MWIS problem can be solved in time  $O(n^{24})$ .

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